Bour's minimal surface in three dimensional Lorentz-Minkowski space

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Introduction

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- 1744 with the Swedish Mathematician Leonhard Euler's (1707-1783) paper,

- and to the 1760 French Mathematician Joseph Louis Lagrange's (1736-1813) paper.

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- This is a definition of Lagrange, who first defined minimal surface in 1760.

Introduction

Brief History of the *Classical Minimal Surfaces:*

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11. Richmond's (1863-1948) surface (?)

Introduction

Almost a hundred years later...

Introduction

1980s – 90s.

- Chen-Gackstatter's surface (1981)

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- Jorge-Meeks's surface (1983)
- Hoffman, Meeks, Karcher, Kusner, Rosenberg, Lopez, Ros, Rossman, Miyaoka, Sato, ...

Introduction

2000s – ...

- Fujimori, Shoda, Traizet, Weber, ...

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- He gave a well known theorem about *the* **helicoidal and rotational surfaces**.
- And also the **Bour-Enneper equation** (today called the *sine-Gordon wave equation*) used in *soliton theory* and *quantum field theories* in Physics was first set down by Bour.

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- These surfaces have been called $\mathfrak{B}_m$ (following J. Haag) to emphasize the value of m.

Introduction

papers dealing with the $\mathfrak{B}_m$ in the literature:

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- Bour, E. Theorie de la deformation des surfaces. Journal de l'École Imperiale Polytechnique, tome 22, cahier 39 (1862), pp. 99-109.

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- Ribaucour, A. Etude sur les elassoides ou surfaces a courbure moyenne nulle. Memoires Couronnes de l'Academie Royale de Belgique, vol. XLIV (1882), chapter XX, pp. 215-224.

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- Demoulin, A. Bulletin des Sciences Mathematiques (2), vol. XXI (1897), pp. 244-252.

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- Stübler, E. Mathematische Annalen, vol. 75 (1914), pp. 148-176.

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- Stübler, E. Mathematische Annalen, vol. 75 (1914), pp. 148-176.
- Whittemore, J. K. Minimal surfaces applicable to surfaces of revolution. Ann. of Math. (2) 19 (1917), no. 1, 1–20.

Introduction

- All real minimal surfaces applicable to rotational surfaces setting

$$\mathfrak{F}(s) = C\ s^{m-2}$$

in the Weierstrass representation equations, where $s, C \in \mathbb{C}$, $m \in \mathbb{R}$, and $\mathfrak{F}(s)$ is an analytic function.

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  - For $C = 1$, $m = 0$ we obtain the Catenoid,
  - $C = i$, $m = 0$, the right Helicoid,
  - $C = 1$, $m = 2$, Enneper's surface (see, also [2,4,16]).

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- Alfred Gray [4] gave the complex forms of the Bour's curve and surface of value *m* in 1997.
- Moreover, Bour's surface has not been studied up till now in three dimensional Minkowski space $\mathbb{L}^3$.

Introduction

Ikawa [10, 11] shows that a generalized helicoid is isometric to a rotational surface by Bour's theorem in the Euclidean and Minkowski 3-spaces. In addition, he determine these surfaces, with the additional conditions that they are minimal and have the same Gauss map.

Introduction

Güler [5, 7] shows that a generalized helicoid with lightlike profile curve is isometric to a rotational surface with lightlike profile curve, by Bour's theorem in the Minkowski 3-space.

Introduction

Güler, Yaylı and Hacısalihoğlu establish some relations between the Laplace-Beltrami operator and the curvatures of helicoidal surfaces in 3-Euclidean space. In addition, Bour's theorem on the Gauss map, and some special examples are given in [6]. Some geometric properties of the timelike rotational surfaces with lightlike profile curve of (S,L), (T,L) and (L,L)-types is shown in Minkowski 3-space in [7,8,9].

Introduction

We will give Bour's minimal surfaces in $\mathbb{E}^3$ and $\mathbb{L}^3$.

Euclidean case

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  - the surfaces will be smooth,
  - and simply connected.

Euclidean case

Let $\mathbb{E}^3$ be a three dimensional Euclidean space with natural metric

$$\langle ., . \rangle_0 = dx^2 + dy^2 + dz^2.$$

Euclidean case

In 1818, at age 31, C.F. Gauss (1777-1855) contracted to undertake a geodetic survey, for the German state of Hanover, in order to link up with the existing Danish grid.

Euclidean case

- With the help of this surveying, he invented the "*heliotrope*" (an instrument used in geodetic surveying for making long distance observations by means of the sun's rays throwing from a mirror).

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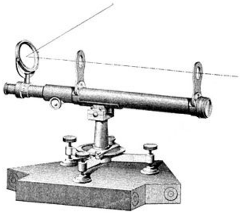
- 

Euclidean case

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- Infinitesimal squares were mapped by map X to infinitesimal squares on surface.

- He obtained a map, and called **conformal** if satisfy

$$\begin{aligned} \langle X_u, X_u \rangle_0 &= \langle X_v, X_v \rangle_0 , \\ \langle X_u, X_v \rangle_0 &= 0, \end{aligned}$$

where u, v are local isothermic parameters.

Euclidean case

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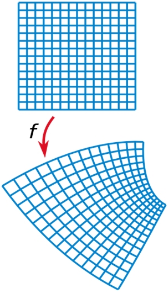
- 

Figure 0 A conformal mapping

Euclidean case

- An important family of examples of conformal maps comes from complex analysis.

Euclidean case

- An important family of examples of conformal maps comes from complex analysis.
- If $\mathcal{U}$ is an open subset of the complex plane $\mathbb{C}$, then a function $f : \mathcal{U} \longrightarrow \mathbb{C}$ is conformal iff it is **holomorphic** (or complex differentiable) and its derivative is everywhere non-zero on $\mathcal{U}$.

Euclidean case

- Let $\mathcal{U}$ be an open subset of $\mathbb{C}$. A **minimal** (or *isotropic*) **curve** is an analytic function $\Psi : \mathcal{U} \to \mathbb{C}^n$ such that

$$\left(\Psi'(z)\right)^2 = 0,$$

where $z \in \mathcal{U}$, and $\Psi' := \frac{\partial \Psi}{\partial z}$.

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$$\begin{aligned} \left\langle \Psi', \overline{\Psi'} \right\rangle_0 &= \left|\Psi'\right|^2 \\ &\neq 0, \end{aligned}$$

Ψ is a **regular minimal curve**.

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Ψ is a **regular minimal curve**.

- A minimal surface is the associated family of a minimal curve.

Euclidean case

Now, we give the *Weierstrass Representation Theorem* for minimal surfaces in $\mathbb{E}^3$ [15], discovered by K. Weierstrass (1815-1897) in 1866 (also see [1, 16], for details).

Euclidean case

Theorem

Let $\mathfrak{F}$ and $\mathcal{G}$ be two holomorphic functions defined on a simply connected open subset U of $\mathbb{C}$ such that $\mathfrak{F}$ does not vanish on U. Then the map

$$\mathbf{x}(u,v) = \operatorname{Re} \int^z \begin{pmatrix} \mathfrak{F}\left(1-\mathcal{G}^2\right) \\ i\,\mathfrak{F}\left(1+\mathcal{G}^2\right) \\ 2\mathfrak{F}\mathcal{G} \end{pmatrix} dz$$

is a minimal, conformal immersion of U into $\mathbb{E}^3$, and $\mathbf{x}$ is called the Weierstrass patch, determined by $\mathfrak{F}(z)$ and $\mathcal{G}(z)$.

Euclidean case

Lemma

Let $\Psi : \mathcal{U} \to \mathbb{C}^3$ *minimal curve and write* $\Psi' = (\varphi_1, \varphi_2, \varphi_3)$. *Then*

$$\mathfrak{F} = \frac{\varphi_1 - i\varphi_2}{2} \quad \text{and} \quad \mathcal{G} = \frac{\varphi_3}{\varphi_1 - i\varphi_2}$$

give rise to the Weierstrass representation of Ψ. *That is*

$$\Psi' = \left(\mathfrak{F}\left(1 - \mathcal{G}^2\right), i\mathfrak{F}\left(1 + \mathcal{G}^2\right), 2\mathfrak{F}\mathcal{G}\right).$$

Euclidean case

Lemma

The **Bour's curve** *of value m*

$$\left(\frac{z^{m-1}}{m-1}-\frac{z^{m+1}}{m+1},\ i\left(\frac{z^{m-1}}{m-1}+\frac{z^{m+1}}{m+1}\right),\ 2\frac{z^{m}}{m}\right) \tag{1}$$

is a minimal curve in $\mathbb{E}^3$, *where* $m \in \mathbb{R}-\{-1,0,1\}$, $z \in \mathcal{U} \subset \mathbb{C}$, $i=\sqrt{-1}$.

Euclidean case

Proof.

Using differential z of the Bour's curve of value m, we have

$$\Omega(z) = \left(z^{m-2} - z^m,\ i\left(z^{m-2} + z^m\right),\ 2z^{m-1}\right). \qquad (2)$$

Hence we get

$$(\Omega)^2 = 0.$$

□

Euclidean case

The Bour's minimal curve of value 3 (see Fig. 0.1) is intersects itself three times along three straight rays, which meet an angle $2\pi/3$ at the origin in $\mathbb{E}^3$.

Euclidean case

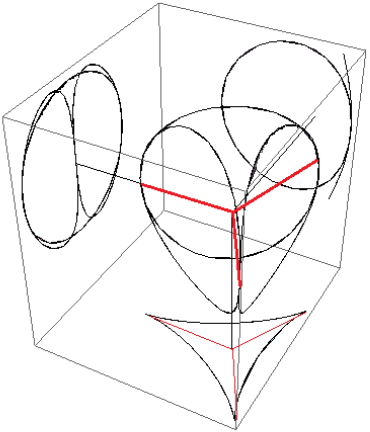

Figure 0.1 Bour's minimal curve and its shadows

Euclidean case

- Bour's minimal surface of value m is the associated family of Bour's minimal curve.

Euclidean case

Lemma

The Weierstrass patch determined by the functions

$$\mathfrak{F}(z) = z^{m-2} \quad \text{and} \quad \mathcal{G}(z) = z$$

is a representation of the **Bour's minimal surface** *of value* $m \in \mathbb{R}$ *in* $\mathbb{E}^3$.

Euclidean case

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- $$\mathfrak{B}_m(u,v) = \mathrm{Re}\int \Phi(z)\,dz, \qquad (3)$$
- where $m \in \mathbb{R}$, (u,v) are coordinates on the surface, $z = u + iv$ is the corresponding complex coordinate,

$$\Phi(z) = \left(\frac{z^{m-1}}{m-1} - \frac{z^{m+1}}{m+1}, i\left(\frac{z^{m-1}}{m-1} + \frac{z^{m+1}}{m+1} \right), 2\frac{z^m}{m} \right),$$

$(\Phi)^2 = 0$, and Φ is an analytic function.

Euclidean case

- For $z = re^{i\theta}$, Im part of the $\mathfrak{B}_m(r,\theta)$ is a conjugate surface, where (r,θ) is polar coordinates.

Euclidean case

- For $z = re^{i\theta}$, Im part of the $\mathfrak{B}_m\left(r,\theta\right)$ is a conjugate surface, where $\left(r,\theta\right)$ is polar coordinates.
- The conjugate surface of the Bour's surface of value m is

$$\begin{aligned}\mathfrak{B}^*_m\left(r,\theta\right) &= -\mathrm{Re}\int i\Phi \\ &= \mathrm{Re}\int e^{-i\pi/2}\Phi.\end{aligned}$$

Euclidean case

- The associated family is thus described by

$$\begin{aligned} \mathfrak{B}_m(r,\theta;\alpha) &= \operatorname{Re}\int e^{-i\alpha}\Phi \\ &= \cos(\alpha)\operatorname{Re}\int\Phi + \sin(\alpha)\operatorname{Im}\int\Phi \\ &= \cos(\alpha)\ \mathfrak{B}_m(r,\theta) + \sin(\alpha)\ \mathfrak{B}_m^*(r,\theta). \end{aligned}$$

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- When $\alpha = 0$, (resp., $\alpha = \pi/2$) we have the Bour's surface of value m (resp., the conjugate surface).

Euclidean case

Theorem

Bour's surface of value m

$$\mathfrak{B}_m(r,\theta) = \begin{pmatrix} r^{m-1}\frac{\cos[(m-1)\theta]}{m-1} - r^{m+1}\frac{\cos[(m+1)\theta]}{m+1} \\ -r^{m-1}\frac{\sin[(m-1)\theta]}{m-1} - r^{m+1}\frac{\sin[(m+1)\theta]}{m+1} \\ 2r^m\frac{\cos(m\theta)}{m} \end{pmatrix} \quad (4)$$

is a minimal surface in $\mathbb{E}^3$*, where* $m \in \mathbb{R} - \{-1, 0, 1\}$ *, in* (r, θ) *coordinates.*

Euclidean case

Proof.

- The coefficients of the first fundamental form of the Bour's surface are

$$
\begin{aligned}
E &= r^{2m-4}\left(1+r^2\right)^2, \\
F &= 0, \\
G &= r^{2m-2}\left(1+r^2\right)^2,
\end{aligned}
$$

Euclidean case

Proof.

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- So, we have

$$\det I = r^{4m-6}\left(1+r^2\right)^4.$$

Euclidean case

Proof. (Cont.)
The Gauss map of the surface is

$$e = \frac{1}{1+r^2}\begin{pmatrix} 2r\cos(\theta) \\ 2r\sin(\theta) \\ r^2-1 \end{pmatrix}.$$

Euclidean case

Proof. (Cont.)

- The coefficients of the second fundamental form of the Bour's surface are

Euclidean case

Proof. (Cont.)

- The coefficients of the second fundamental form of the Bour's surface are
- $$\begin{aligned} L &= -2r^{m-2}\cos(m\theta), \\ M &= 2r^{m-1}\sin(m\theta), \\ N &= 2r^{m}\cos(m\theta). \end{aligned}$$

Euclidean case

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- We have
$$\det II = -4r^{2m-2}.$$

Euclidean case

Proof. (Cont.)

- Hence, the mean and the Gaussian curvatures of the Bour's surface of value m, respectively, are

Euclidean case

Proof. (Cont.)

- Hence, the mean and the Gaussian curvatures of the Bour's surface of value m, respectively, are
- $$H = 0, \;\; K = -\left(\frac{2r^{2-m}}{(1+r^2)^2}\right)^2.$$

Euclidean case

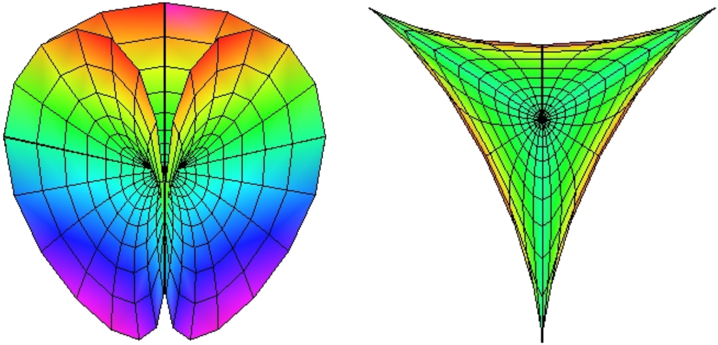
Example

If we take $m = 3$ in $\mathfrak{B}_m(r, \theta)$, then we have the **Bour's minimal surface** (see Fig. 1)

$$\mathfrak{B}_3(r, \theta) = \begin{pmatrix} \frac{r^2}{2}\cos(2\theta) - \frac{r^4}{4}\cos(4\theta) \\ -\frac{r^2}{2}\sin(2\theta) - \frac{r^4}{4}\sin(4\theta) \\ \frac{2}{3}r^3\cos(3\theta) \end{pmatrix}, \tag{5}$$

where $r \in [-1, 1]$, $\theta \in [0, \pi]$. When $r = 1$, and $z = 0$, we have *deltoid curve*, which is a 3-cusped hypocycloid (Steiner's hypocycloid (1856)), also called tricuspoid, discovered by Euler in 1745, on plane xy in Fig. 0.1.

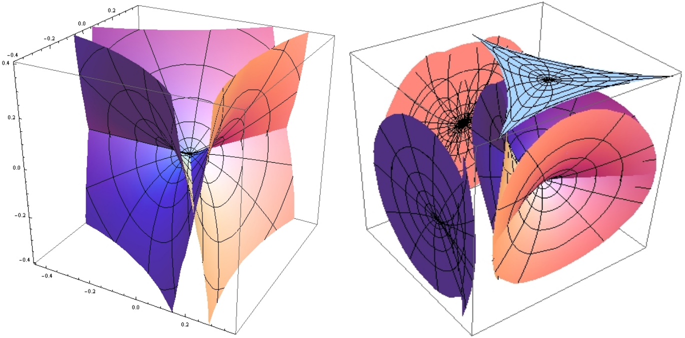
Euclidean case



(a) (b)

Figure 1 Bour's minimal surface of value 3, $\mathfrak{B}_3(r, \theta)$

Euclidean case



(*c*) (*d*)

Figure 1 Bour's minimal surface of value 3, $\mathfrak{B}_3\left(r,\theta\right)$

Euclidean case

The coefficients of the first fundamental form of the Bour's surface of value 3 are

$$E = r^2 \left(1 + r^2\right)^2, \; F = 0, \; G = r^4 \left(1 + r^2\right)^2.$$

So,

$$\det I = r^6 \left(1 + r^2\right)^4.$$

Euclidean case

The Gauss map of the surface $\mathfrak{B}_3$ is

$$e = \frac{1}{1+r^2}\left(2r\cos(\theta), 2r\sin(\theta), r^2 - 1\right).$$

Euclidean case

The coefficients of the second fundamental form of the surface are

$$L = -2r\cos(3\theta), \quad M = 2r^2\sin(3\theta), \quad N = 2r^3\cos(3\theta).$$

Then,

$$\det II = -4r^4.$$

Euclidean case

The mean and the Gaussian curvatures of the Bour's minimal surface of value 3 are, respectively,

$$H = 0, \quad K = -\frac{4}{r^2 \left(1 + r^2\right)^4}.$$

Euclidean case

The Weierstrass patch determined by the functions

$$(\mathfrak{F}, \mathcal{G}) = (z, z)$$

is a representation of the Bour's minimal surface of value 3.

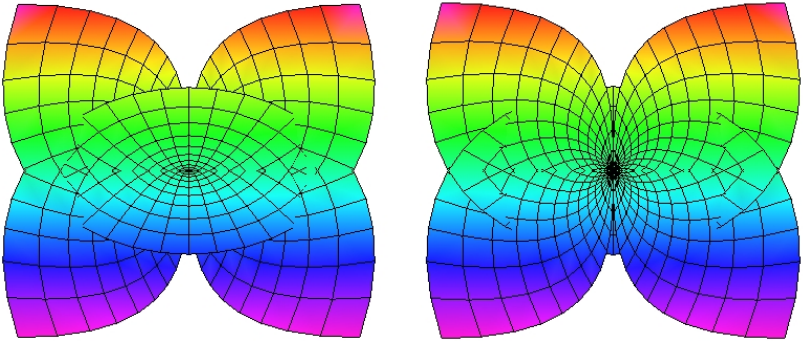
Euclidean case

The parametric form of the surface (see Fig. 2) is

$$\mathfrak{B}_3(u,v)=\begin{pmatrix} -\frac{u^4}{4}-\frac{v^4}{4}+\frac{3}{2}u^2v^2+\frac{u^2}{2}-\frac{v^2}{2} \\ -u^3v-uv^3-uv \\ \frac{2}{3}u^3-2uv^2 \end{pmatrix}, \tag{6}$$

where $u,v\in\mathbb{R}$.

Euclidean case



(a) (b)

Figure 2 Surface of $\mathfrak{B}_3(u, v)$, $u, v \in [-1, 1]$

Euclidean case

The coefficients of the first fundamental form of the Bour's surface of value 3 in u, v coordinates are

$$E = \left(u^2 + v^2\right)\left(1 + u^2 + v^2\right)^2 = G, \quad F = 0,$$

So,

$$\det I = \left(u^2 + v^2\right)^2\left(1 + u^2 + v^2\right)^4.$$

Euclidean case

The Gauss map of the surface $\mathfrak{B}_3$ is

$$e = \frac{1}{1+u^2+v^2}\left(2u, 2v, u^2+v^2-1\right).$$

Euclidean case

The coefficients of the second fundamental form of the surface are

$$L = -2u, \;\; M = 2v, \;\; N = 2u.$$

Then,

$$\det II = -4\left(u^2 + v^2\right).$$

Euclidean case

The mean and the Gaussian curvatures of the Bour's minimal surface of value 3 are, respectively,

$$H = 0, \quad K = -\frac{4}{\left(u^2+v^2\right)\left(1+u^2+v^2\right)^4}.$$

Euclidean case, some remarks

In some literature, however, the Weierstrass representation of the Bour's minimal surface is known as $(\mathfrak{F},\mathcal{G}) = (1, \zeta^{1/2})$. That is, in polar coordinates, the surface is described by (see Figure 2.1)

$$\begin{array}{rcl} x & = & r\cos(\theta) - \frac{1}{2}r^2\cos(2\theta), \\ y & = & -r\sin(\theta) - \frac{1}{2}r^2\sin(2\theta), \\ z & = & \frac{4}{3}r^{3/2}\cos\left(\frac{3\theta}{2}\right), \end{array} \tag{7}$$

where $r \in [-1/2, 1/2]$, $\theta \in [0, 4\pi]$.

Euclidean case, some remarks

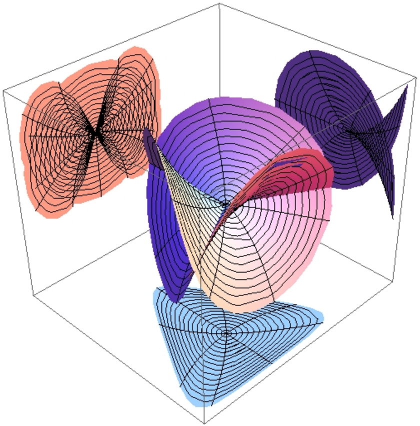


Figure 2.1 Minimal surface, $(\mathfrak{F},\mathcal{G}) = (1, \zeta^{1/2})$

Euclidean case, some remarks

But this is not Bour's surface, and these equations are incorrect. Since Enneper's family $\mathfrak{E}_m$ is defined by $(\mathfrak{F},\mathcal{G}) = (1, \zeta^m)$, then the surface belongs to Enneper's family, and it is the surface $\mathfrak{E}_{1/2}$ (see Figure 2.2).

Euclidean case, some remarks

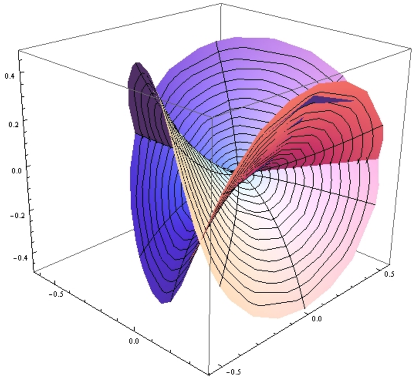


Figure 2.2 The surface $\mathfrak{E}_{1/2}$

Euclidean case

Theorem

(K. Weierstrass, 1903) Assume that the function $w = f(\zeta)$, *where* $\zeta = \xi + i\eta$ *and* $w = u + iv$, *is analytic in* $|\zeta - \zeta_0| < r$ *and satisfies a real algebraic relation* $P(\xi, \eta, u) = 0$. *Then* $f(\zeta)$ *is an algebraic function of its argument.*

Euclidean case

- An **algebraic curve** over a field K is an equation $f(x, y) = 0$, where $f(x, y)$ is a polynomial in x and y with coefficients in K.

Euclidean case

- An **algebraic curve** over a field K is an equation $f(x, y) = 0$, where $f(x, y)$ is a polynomial in x and y with coefficients in K.
- The set of roots of a polynomial $f(x, y, z) = 0$. An **algebraic surface** is said to be of degree (order) $n = max(i + j + k)$, where n is the maximum sum of powers of all terms $a_m x^{i_m} y^{j_m} z^{k_m}$.

Euclidean case

Integral free form of the Weierstrass representation (obtained by K. Weierstrass, 1903) is

$$\begin{aligned}\begin{pmatrix} x \\ y \\ z \end{pmatrix} &= \operatorname{Re}\begin{pmatrix} \left(1-w^2\right)\phi''(w)+2w\phi'(w)-2\phi(w) \\ i\left[\left(1+w^2\right)\phi''(w)-2w\phi'(w)+2\phi(w)\right] \\ 2\left[w\phi''(w)-\phi'(w)\right] \end{pmatrix} \\ &\equiv \operatorname{Re}\begin{pmatrix} f_1(w) \\ f_2(w) \\ f_3(w) \end{pmatrix},\end{aligned}$$

Euclidean case

where $\phi(w)$ (algebraic function) and the functions $f_i(w)$ are connected by the relation

$$\phi(w) = \frac{1}{4}\left(w^2 - 1\right) f_1(w) - \frac{i}{4}\left(w^2 + 1\right) f_2(w) - \frac{1}{2} w f_3(w).$$

Euclidean case

- Integral free form formulas are suitable for algebraic minimal surfaces.

Euclidean case

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- For instance, $\phi(w) = \frac{1}{6}w^3$ give rise to Enneper's minimal surface $\mathfrak{E} := \mathfrak{B}_2$ (see, also [16]).

Euclidean case

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- For instance, $\phi(w) = \frac{1}{6}w^3$ give rise to Enneper's minimal surface $\mathfrak{E} := \mathfrak{B}_2$ (see, also [16]).
- We obtain the function

$$\phi(w) = \frac{1}{24}w^4$$

leads to Bour's minimal surface $\mathfrak{B} := \mathfrak{B}_3$.

Euclidean case

- Integral free form formulas are suitable for algebraic minimal surfaces.
- For instance, $\phi(w) = \frac{1}{6}w^3$ give rise to Enneper's minimal surface $\mathfrak{E} := \mathfrak{B}_2$ (see, also [16]).
- We obtain the function

$$\phi(w) = \frac{1}{24}w^4$$

  leads to Bour's minimal surface $\mathfrak{B} := \mathfrak{B}_3$.
- And also, it is clear that

$$\phi'_{\mathfrak{B}} = \phi_{\mathfrak{E}}.$$

Euclidean case

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- if $m \in \mathbb{Z} \quad \Rightarrow \quad degree\left(\mathfrak{B}_m\right) = \left(m+1\right)^2$.
- He also shows that
- if $m < 1 \quad \Rightarrow \quad class\left(\mathfrak{B}_m\right) = degree\left(\mathfrak{B}_m\right)$,
- if $m > 1 \quad \Rightarrow \quad class\left(\mathfrak{B}_m\right) < degree\left(\mathfrak{B}_m\right)$.

Euclidean case

That is,

- $cl\left(\mathfrak{B}_2\right) = 6\ (\text{Enneper})\,,$

Euclidean case

That is,

- $cl\left(\mathfrak{B}_2\right) = 6\ \left(\text{Enneper}\right),$
- $cl\left(\mathfrak{B}_3\right) = 8,\ \left(\text{Bour}\right),$

Euclidean case

That is,

- $cl\left(\mathfrak{B}_2\right) = 6$ (Enneper) ,
- $cl\left(\mathfrak{B}_3\right) = 8,$ (Bour) ,
- $cl\left(\mathfrak{B}_4\right) = 10,$

Euclidean case

That is,

- $cl\left(\mathfrak{B}_2\right) = 6$ (Enneper),
- $cl\left(\mathfrak{B}_3\right) = 8$, (Bour),
- $cl\left(\mathfrak{B}_4\right) = 10$,
- $cl\left(\mathfrak{B}_5\right) = 12$,

Euclidean case

That is,

- $cl\left(\mathfrak{B}_2\right) = 6\ (\text{Enneper})$,
- $cl\left(\mathfrak{B}_3\right) = 8,\ (\text{Bour})$,
- $cl\left(\mathfrak{B}_4\right) = 10,$
- $cl\left(\mathfrak{B}_5\right) = 12,$
- ...

Euclidean case

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- $cl\left(\mathfrak{B}_2\right) = 6$ (Enneper),
- $cl\left(\mathfrak{B}_3\right) = 8$, (Bour),
- $cl\left(\mathfrak{B}_4\right) = 10$,
- $cl\left(\mathfrak{B}_5\right) = 12$,
- ...
- $cl\left(\mathfrak{B}_m\right) = 2q\left(p+q\right)$.

Euclidean case

That is,

- $cl\left(\mathfrak{B}_2\right) = 6 \ (\text{Enneper})\,,$
- $cl\left(\mathfrak{B}_3\right) = 8, \ (\text{Bour})\,,$
- $cl\left(\mathfrak{B}_4\right) = 10,$
- $cl\left(\mathfrak{B}_5\right) = 12,$
- ...
- $cl\left(\mathfrak{B}_m\right) = 2q\left(p+q\right).$

- $\deg\left(\mathfrak{B}_2\right) = 9 \ (\text{Enneper})\,,$

Euclidean case

That is,

- $cl\left(\mathfrak{B}_2\right) = 6\ (\text{Enneper})$,
- $cl\left(\mathfrak{B}_3\right) = 8,\ (\text{Bour})$,
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- ...
- $cl\left(\mathfrak{B}_m\right) = 2q\left(p+q\right).$

- $\deg\left(\mathfrak{B}_2\right) = 9\ (\text{Enneper})$,
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- ...
- $cl\left(\mathfrak{B}_m\right) = 2q\left(p+q\right)$.

- $\deg\left(\mathfrak{B}_2\right) = 9 \ (\text{Enneper})$,
- $\deg\left(\mathfrak{B}_3\right) = 16 \ (\text{Bour})$,
- $\deg\left(\mathfrak{B}_4\right) = 25$,

Euclidean case

That is,

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- $cl\left(\mathfrak{B}_4\right) = 10$,
- $cl\left(\mathfrak{B}_5\right) = 12$,
- ...
- $cl\left(\mathfrak{B}_m\right) = 2q\left(p+q\right)$.

- $\deg\left(\mathfrak{B}_2\right) = 9$ (Enneper) ,
- $\deg\left(\mathfrak{B}_3\right) = 16$ (Bour) ,
- $\deg\left(\mathfrak{B}_4\right) = 25$,
- $\deg\left(\mathfrak{B}_5\right) = 36$,

Euclidean case

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- $cl\left(\mathfrak{B}_5\right) = 12,$
- ...
- $cl\left(\mathfrak{B}_m\right) = 2q\left(p+q\right).$

- $\deg\left(\mathfrak{B}_2\right) = 9\ \left(\text{Enneper}\right),$
- $\deg\left(\mathfrak{B}_3\right) = 16\ \left(\text{Bour}\right),$
- $\deg\left(\mathfrak{B}_4\right) = 25,$
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- ...

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That is,

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- $cl\left(\mathfrak{B}_5\right) = 12$,
- ...
- $cl\left(\mathfrak{B}_m\right) = 2q\left(p+q\right)$.

- $\deg\left(\mathfrak{B}_2\right) = 9\ (\text{Enneper})$,
- $\deg\left(\mathfrak{B}_3\right) = 16\ (\text{Bour})$,
- $\deg\left(\mathfrak{B}_4\right) = 25$,
- $\deg\left(\mathfrak{B}_5\right) = 36$,
- ...
- $\deg\left(\mathfrak{B}_m\right) = \left(m+1\right)^2$.

Euclidean case

- We calculate the implicit equations, classes, and degrees of the surfaces $\mathfrak{B}_2, \mathfrak{B}_3.\mathfrak{B}_4, \mathfrak{B}_5, \mathfrak{B}_6$ using Sylvester and Gröbner eliminate methods by the help of Maple programme.

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- Our findings agree with Ribaucour's.

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- Our findings agree with Ribaucour's.
- And we give the following table

Euclidean case

$\mathfrak{B}_m(u,v)$	$\deg(x)$	$\deg(y)$	$\deg(z)$	$cl(\mathfrak{B}_m)$	$\deg(\mathfrak{B}_m)$	$Syl(x,y,u)$	$Syl(F,G,v)$
$\mathfrak{B}_2$	3	3	2	6	9	5×5	11×11
$\mathfrak{B}_3$	4	4	3	8	16	7×7	18×18
$\mathfrak{B}_4$	5	5	4	10	25	9×9	29×29
$\mathfrak{B}_5$	6	6	5	12	36	11×11	40×40
$\mathfrak{B}_6$	7	7	6	14	49	13×13	55×55
$\mathfrak{B}_7$	8	8	7	16	64	15×15	70×70
$\mathfrak{B}_8$	9	9	8	18	81	17×17	89×89
$\mathfrak{B}_9$	10	10	9	20	100	19×19	108×108
$\mathfrak{B}_{10}$	11	11	10	22	121	21×21	131×131
...	...	...	...	...	...	...	...
$\mathfrak{B}_m$	$m+1$	$m+1$	m	$2m+2$	$(m+1)^2$	$(2m+1) \times (2m+1)$	$(m+1)^2+m$ ...if m even $(m+1)^2+m-1$ ...if m odd

degree and class of $\mathfrak{B}_m(u,v)$

Euclidean case

- We also show the relations between the **degree of the algebraic function** $\phi^2(w)$ in the integral free form formulas and the **class of the surfaces**. We know

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- $\mathfrak{B}_2 : \phi_{\mathfrak{B}_2}(w) = \frac{1}{6}w^3$ (Enneper's minimal surface),
- $\mathfrak{B}_3 : \phi_{\mathfrak{B}_3}(w) = \frac{1}{24}z^4$ (Bour's minimal surface),

Euclidean case

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- $\mathfrak{B}_3 : \phi_{\mathfrak{B}_3}(w) = \frac{1}{24}z^4$ (Bour's minimal surface),
- $\mathfrak{B}_4 : \phi_{\mathfrak{B}_4}(w) = \frac{1}{120}w^5$,
- ...

Euclidean case

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- $\mathfrak{B}_2 : \phi_{\mathfrak{B}_2}(w) = \frac{1}{6}w^3$ (Enneper's minimal surface),
- $\mathfrak{B}_3 : \phi_{\mathfrak{B}_3}(w) = \frac{1}{24}z^4$ (Bour's minimal surface),
- $\mathfrak{B}_4 : \phi_{\mathfrak{B}_4}(w) = \frac{1}{120}w^5$,
- ...
- $\mathfrak{B}_m : \phi_{\mathfrak{B}_m}(w) = \frac{1}{(m+1)!}w^{m+1}$.

Euclidean case

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Euclidean case

- Then we can see
- $\deg\left(\phi^2_{\mathfrak{B}_2}\right) = 6 = cl\left(\mathfrak{B}_2\right),$
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Euclidean case

- Then we can see
- $\deg\left(\phi^2_{\mathfrak{B}_2}\right) = 6 = cl\left(\mathfrak{B}_2\right)$,
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Euclidean case

- Then we can see
- $\deg\left(\phi^2_{\mathfrak{B}_2}\right) = 6 = cl\left(\mathfrak{B}_2\right)$,
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- $\deg\left(\phi^2_{\mathfrak{B}_4}\right) = 10 = cl\left(\mathfrak{B}_4\right)$,
- ...
- $\deg\left(\phi^2_{\mathfrak{B}_m}\right) = 2m + 2 = cl\left(\mathfrak{B}_m\right)$.

Euclidean case

- Bour's minimal surface $\mathfrak{B}_3$ and its conjugate are as follow

$$\mathfrak{B}_3\left(u,v\right)=\left(\begin{array}{c} -\frac{u^4}{4}-\frac{v^4}{4}+\frac{3}{2}u^2v^2+\frac{u^2}{2}-\frac{v^2}{2} \\ -u^3v-uv^3-uv \\ \frac{2}{3}u^3-2uv^2 \end{array}\right),$$

$$\mathfrak{B}_3^*\left(u,v\right)=\left(\begin{array}{c} -u^3v+uv^3+uv \\ \frac{u^4}{4}+\frac{v^4}{4}-\frac{3}{2}u^2v^2+\frac{u^2}{2}-\frac{v^2}{2} \\ -\frac{2}{3}v^3+2u^2v \end{array}\right),$$

Euclidean case

- Bour's minimal surface $\mathfrak{B}_3$ and its conjugate are as follow

$$\mathfrak{B}_3(u,v) = \begin{pmatrix} -\frac{u^4}{4} - \frac{v^4}{4} + \frac{3}{2}u^2v^2 + \frac{u^2}{2} - \frac{v^2}{2} \\ -u^3v - uv^3 - uv \\ \frac{2}{3}u^3 - 2uv^2 \end{pmatrix},$$

$$\mathfrak{B}_3^*(u,v) = \begin{pmatrix} -u^3v + uv^3 + uv \\ \frac{u^4}{4} + \frac{v^4}{4} - \frac{3}{2}u^2v^2 + \frac{u^2}{2} - \frac{v^2}{2} \\ -\frac{2}{3}v^3 + 2u^2v \end{pmatrix},$$

- and then we can see the Cauchy-Riemann equations hold

$$(\mathfrak{B}_3)_u = (\mathfrak{B}_3^*)_v, \quad (\mathfrak{B}_3)_v = -(\mathfrak{B}_3^*)_u.$$

Euclidean case

- We know $\mathfrak{B}_3\left(u,v\right)=\mathrm{Re}\int\Phi dz$, $X:=\mathfrak{B}_3$,

$$X_{uu}+X_{vv}=0,$$

(i.e. $\mathfrak{B}_3$ minimal),

$$\begin{aligned}\left\langle X_u,X_u\right\rangle &= \left\langle X_v,X_v\right\rangle, \\ \left\langle X_u,X_v\right\rangle &= 0,\end{aligned}$$

(i.e. $\mathfrak{B}_3$ conformal).

Euclidean case

- We know $\mathfrak{B}_3(u,v) = \operatorname{Re}\int \Phi dz$, $X := \mathfrak{B}_3$,

$$X_{uu} + X_{vv} = 0,$$

(i.e. $\mathfrak{B}_3$ minimal),

$$\begin{aligned} \langle X_u, X_u \rangle &= \langle X_v, X_v \rangle, \\ \langle X_u, X_v \rangle &= 0, \end{aligned}$$

(i.e. $\mathfrak{B}_3$ conformal).

- Then we can see

$$\begin{aligned} \Phi &= X_u - i\, X_v \\ &= \left(z - z^3, i\left(z + z^3\right), 2z^2\right), \end{aligned}$$

and $(\Phi)^2 = 0$, Φ analytic (in each component), and also it can be seen for $Y := \mathfrak{B}_3^*$.

Euclidean case

Problems.

1. Find the $\mathfrak{B}_3$ algebraic or not,

Euclidean case

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1. Find the $\mathfrak{B}_3$ algebraic or not,
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3. degree,

Euclidean case

Problems.

1. Find the $\mathfrak{B}_3$ algebraic or not,
2. the cartesian equation of $\mathfrak{B}_3$,
3. degree,
4. and class.

Euclidean case

4. Hint. The tangent plane at a point (u, v) on Bour's surface $\mathfrak{B}_3$ is given in terms of running coordinates x, y, z by

$$X(u, v)x + Y(u, v)y + Z(u, v)z + P(u, v) = 0.$$

For the inhomogeneous tangential coordinates $\overline{u} = X/P$, $\overline{v} = Y/P$, and $\overline{w} = Z/P$. By eliminating u and v, obtain the equation for the surface $\mathfrak{B}_3$ in tangential coordinates. Maximum degree of the equation gives **class** of Bour's surface $\mathfrak{B}_3$.

Euclidean case

We compute the irreducible implicit equation of surface $\mathfrak{B}_3$ using Sylvester and Gröbner eliminate methods by software programmes:

Euclidean case

$$
\begin{aligned}
&-859963392\,x^4 z^6 - 764411904\,y^2 x^4 z^4 - 1719926784\,y^2 x^2 z^6 + 509607936\,y^4 x^2 z^4 - 1934917632\,z^{10} - 2579890176\,x^2 z^8 \\
&\quad - 859963392\,z^6 y^4 - 84934656\,z^4 y^6 - 2579890176\,z^8 y^2 + 1632586752\,z^{12} + 268435456\,y^{12} - 28991029248\,x^6 y^6 \\
&\quad + 31340888064\,x^6 z^6 - 3877393536\,z^{12} y^2 + 37650272256\,z^8 y^4 - 3654844416\,z^8 y^6 + 38985007104\,z^6 y^6 \\
&\quad + 14511882240\,z^{10} y^2 - 7255941120\,z^{10} y^4 + 3623878656\,z^2 y^{10} + 17836277760\,z^4 y^8 - 14834368512\,z^8 x^4 y^2 \\
&\quad - 6115295232\,x^7 z^4 y^2 - 56396611584\,y^6 x^3 z^4 - 10192158720\,x^5 y^4 z^4 + 5435817984\,x^9 z^4 - 3009871872\,x^6 z^8 \\
&\quad + 21743271936\,y^4 x^8 - 22932357120\,x^5 z^6 y^2 + 119757864960\,x^6 y^2 z^4 + 3057647616\,x^7 z^4 + 9459597312\,x^5 z^6 \\
&\quad + 7309688832\,x^3 z^8 + 272097792\,z^{12} x^3 + 37650272256\,x^4 z^8 + 14511882240\,z^{10} x^2 + 10037385216\,x^3 z^{10} \\
&\quad + 29023764480\,x^5 z^8 + 8153726976\,x^8 z^4 - 9965666304\,x^5 z^2 y^4 - 58047528960\,z^8 x^3 y^2 - 18919194624\,x^3 y^2 z^6 \\
&\quad + 43486543872\,x^6 y^4 z^2 - 7255941120\,x^4 z^{10} + 22932357120\,x^7 z^6 + 77970014208\,x^4 z^4 y^4 - 3057647616\,x^5 z^4 y^2 \\
&\quad - 3877393536\,x^2 z^{12} - 459165024\,z^{14} + 43046721\,z^{16} + 14495514624\,y^8 x^4 - 3221225472\,x^2 y^{10} - 21929066496\,x z^8 y^2 \\
&\quad + 75300544512\,x^2 z^8 y^2 - 905969664\,y^8 x z^2 - 14495514624\,y^8 x^2 z^2 - 15288238080\,y^4 x^3 z^4 + 48157949952\,y^4 x^2 z^6 \\
&\quad - 28378791936\,y^4 z^6 x - 14269022208\,y^8 z^4 x + 162819735552\,y^2 x^4 z^6 + 2717908992\,y^2 x^7 z^2 + 32614907904\,x^8 z^2 y^2 \\
&\quad + 5737807872\,x^3 y^6 z^2 - 114661785600\,x^3 y^4 z^6 - 7247757312\,x^4 y^6 z^2 - 15797846016\,y^6 x^2 z^4 - 14511882240\,x^2 z^{10} y^2 \\
&\quad - 68797071360\,x z^6 y^6 - 30112155648\,x z^{10} y^2 - 9172942848\,x z^4 y^6 - 87071293440\,x z^8 y^4 - 5159780352\,x^2 z^8 y^4 \\
&\quad - 816293376\,x z^{12} y^2 = 0
\end{aligned}
$$

Implicit equation of $\mathfrak{B}_3$, degree$(\mathfrak{B}_3)$=16

Euclidean case

Answers.

1. $\mathfrak{B}_3$ is an algebraic minimal surface.

Euclidean case

Answers.

1. $\mathfrak{B}_3$ is an algebraic minimal surface.
2. We find the irreducible implicit equation of $\mathfrak{B}_3$.

Euclidean case

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3. Degree $(\mathfrak{B}_3) = 16$.

Euclidean case

Answers.

1. $\mathfrak{B}_3$ is an algebraic minimal surface.
2. We find the irreducible implicit equation of $\mathfrak{B}_3$.
3. Degree $(\mathfrak{B}_3) = 16$.
4. Class $(\mathfrak{B}_3) = 8$.

Euclidean case

Answer (4). We find $P(u,v) = \frac{(u^2+v^2+2)(3uv^2-u^3)}{6(u^2+v^2+1)}$, and the inhomogeneous tangential coordinates

$$\begin{aligned} \overline{u} &= \frac{12u}{(u^2+v^2+2)(3uv^2-u^3)}, \\ \overline{v} &= \frac{12v}{(u^2+v^2+2)(3uv^2-u^3)}, \\ \overline{w} &= \frac{6(u^2+v^2-1)}{(u^2+v^2+2)(3uv^2-u^3)}. \end{aligned}$$

By eliminating u and v, we obtain the equation for the surface $\mathfrak{B}_3$ in tangential coordinates. Maximum degree of the equation gives **class=8** of Bour's surface $\mathfrak{B}_3$.

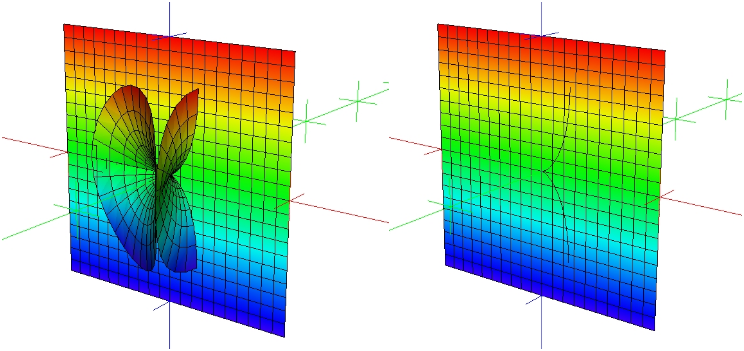
Euclidean case

$$9\overline{u}^8 + 72\overline{u}^7 + 144\overline{u}^6 + 288\overline{u}^5\overline{w}^2 + 192\overline{u}^3\overline{w}^4 + 8\overline{u}^6\overline{w}^2$$
$$-48\overline{u}^4\overline{v}^2\overline{w}^2 - 576\overline{uv}^2\overline{w}^4 + 81\overline{u}^2\overline{v}^6 + 432\overline{u}^4\overline{v}^2 - 45\overline{u}^6\overline{v}^2$$
$$-72\overline{u}^5\overline{v}^2 + 432\overline{u}^2\overline{v}^4 - 360\overline{u}^3\overline{v}^4 - 216\overline{uv}^6 + 27\overline{u}^4\overline{v}^4$$
$$+144\overline{v}^6 - 576\overline{u}^3\overline{v}^2\overline{w}^2 + 72\overline{u}^2\overline{v}^4\overline{w}^2 - 864\overline{uv}^4\overline{w}^2 = 0$$

.

Implicit equation of $\mathfrak{B}_3$ in tangential coordinates, class($\mathfrak{B}_3$)=8

Euclidean case



(a) (b)

Figure 30 $\mathfrak{B}_3\left(r,\theta\right)$ and its curve $\gamma\left(r\right)$ on plane xz

Euclidean case

5. Find the implicit equation of the curve $\gamma(r) = \left(\frac{r^2}{2} - \frac{r^4}{4}, 0, \frac{2}{3}r^3\right)$ (see Fig. 30) on plane xz, and its degree.

Theorem

(L. Henneberg, 1876) A plane intersects an algebraic minimal surface in an algebraic curve [16].

Euclidean case

5. Find the implicit equation of the curve $\gamma(r) = \left(\frac{r^2}{2} - \frac{r^4}{4}, 0, \frac{2}{3}r^3\right)$ (see Fig. 30) on plane xz, and its degree.

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Euclidean case

5. Find the implicit equation of the curve $\gamma(r) = \left(\frac{r^2}{2} - \frac{r^4}{4}, 0, \frac{2}{3}r^3\right)$ (see Fig. 30) on plane xz, and its degree.

- **Answer**. The implicit equation of γ is
- $1024x^2 + 864xz^2 = z^2(288 - 81z^2)$, degree$(\gamma) = 4$.

Theorem

(L. Henneberg, 1876) A plane intersects an algebraic minimal surface in an algebraic curve [16].

Euclidean case

Total curvature of $\mathfrak{B}_m$ is

$$\begin{aligned}\mathfrak{C}\left(\mathfrak{B}_m\right) &= \iint K dA \\ &= \iint -\frac{4}{\left(1+u^2+v^2\right)^2} dudv \\ &= -4\pi.\end{aligned}$$

Applications

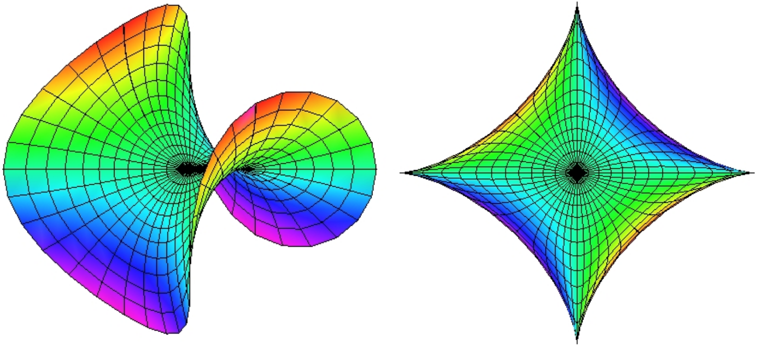
Example

If take $m = 2$, we have **Enneper's minimal surface** (see Fig. 3)

$$\mathfrak{B}_2\left(r,\theta\right) = \left(\begin{array}{c} r\cos\left(\theta\right) - \frac{r^3}{3}\cos\left(3\theta\right) \\ -r\sin\left(\theta\right) - \frac{r^3}{3}\sin\left(3\theta\right) \\ r^2\cos\left(2\theta\right) \end{array}\right),$$

where $r \in [-1,1]$, $\theta \in [0,\pi]$.

Applications



(a) (b)

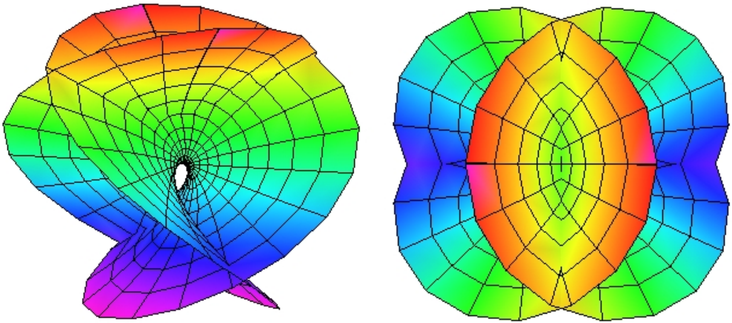
Figure 3 Bour's minimal surface of value 2

Applications

Example

If $m = 2$, we have **Enneper's minimal surface** (see Fig. 4) $\mathfrak{B}_2\left(r, \theta\right)$, where $r \in [-3, 3]$, $\theta \in [0, \pi]$.

Applications



(a) (b)

Figure 4 Bour's minimal surface of value 2

Applications

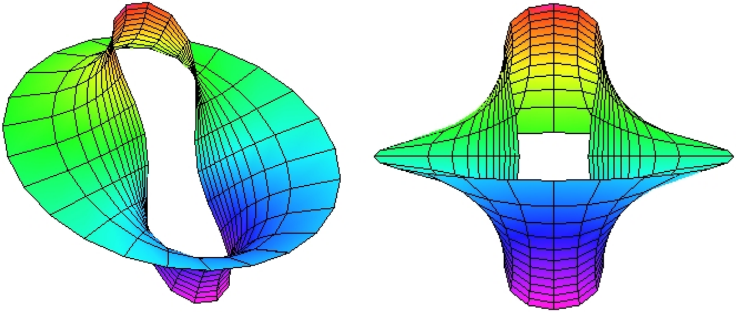
Example

If take $m = \frac{1}{2}$, we have **Richmond's-like minimal surface** (see Fig. 5)

$$\left(\begin{array}{c} -2r^{-1/2}\cos\left(\frac{\theta}{2}\right) - \frac{2}{3}r^{3/2}\cos\left(\frac{3\theta}{2}\right) \\ -2r^{-1/2}\sin\left(\frac{\theta}{2}\right) - \frac{2}{3}r^{3/2}\sin\left(\frac{3\theta}{2}\right) \\ 4r^{1/2}\cos\left(\frac{\theta}{2}\right) \end{array} \right),$$

where $r \in [-1, 1]$, $\theta \in [-2\pi, 2\pi]$.

Applications



(*a*) (*b*)

Figure 5 Bour's minimal surface of value 1/2

Applications

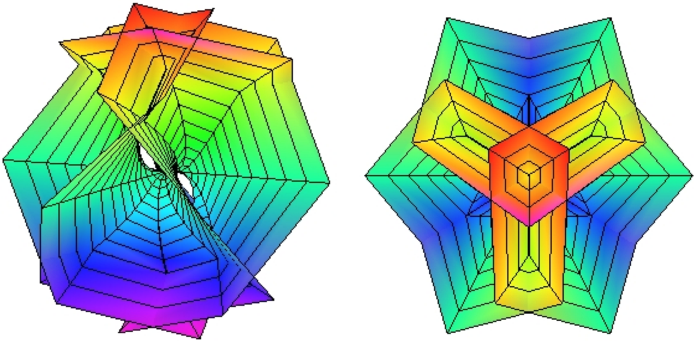
Example

If $m = \frac{3}{2}$, we have (see Fig. 6)

$$\begin{pmatrix} 2r^{1/2}\cos\left(\frac{\theta}{2}\right) - \frac{2}{5}r^{5/2}\cos\left(\frac{5\theta}{2}\right) \\ -2r^{1/2}\sin\left(\frac{\theta}{2}\right) - \frac{2}{5}r^{5/2}\sin\left(\frac{5\theta}{2}\right) \\ \frac{4}{3}r^{3/2}\cos\left(\frac{3\theta}{2}\right) \end{pmatrix},$$

where $r \in [-3, 3]$, $\theta \in [-2\pi, 2\pi]$.

Applications



(a) (b)

Figure 6 Bour's minimal surface of value 3/2

Applications

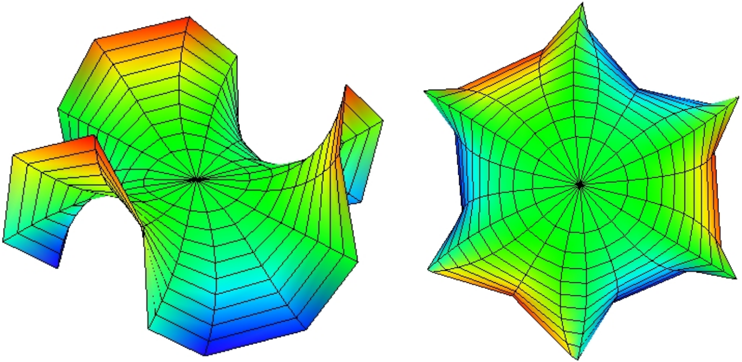
Example

If $m = \frac{3}{2}$, we have (see Fig. 7)

$$\begin{pmatrix} 2r^{1/2}\cos\left(\frac{\theta}{2}\right) - \frac{2}{5}r^{5/2}\cos\left(\frac{5\theta}{2}\right) \\ -2r^{1/2}\sin\left(\frac{\theta}{2}\right) - \frac{2}{5}r^{5/2}\sin\left(\frac{5\theta}{2}\right) \\ \frac{4}{3}r^{3/2}\cos\left(\frac{3\theta}{2}\right) \end{pmatrix},$$

where $r \in [-1, 1]$, $\theta \in [-2\pi, 2\pi]$.

Applications



(a) (b)

Figure 7 Bour's minimal surface of value 3/2

Applications

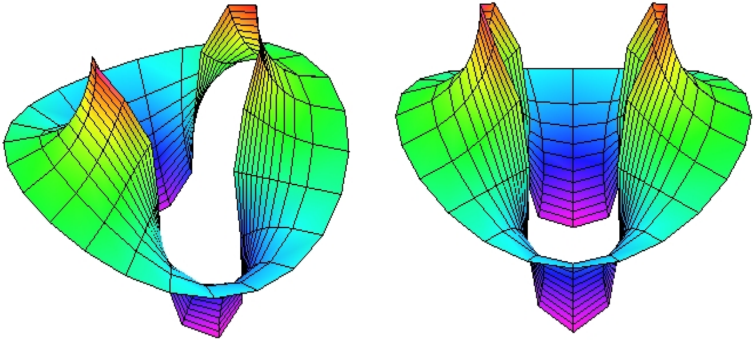
Example

If $m = \frac{2}{3}$, we have (see Fig. 8)

$$\begin{pmatrix} -3r^{-1/3}\cos\left(\frac{\theta}{3}\right) - \frac{3}{5}r^{5/3}\cos\left(\frac{5\theta}{3}\right) \\ -3r^{-1/3}\sin\left(\frac{\theta}{3}\right) - \frac{3}{5}r^{5/3}\sin\left(\frac{5\theta}{3}\right) \\ 3r^{2/3}\cos\left(\frac{2\theta}{3}\right) \end{pmatrix},$$

where $r \in [-1, 1]$, $\theta \in [-3\pi, 3\pi]$.

Applications



(a) (b)

Figure 8 Bour's minimal surface of value $2/3$

Applications

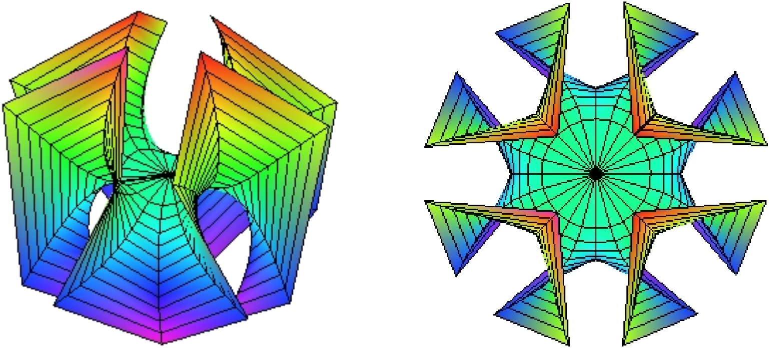
Example

If $m = \frac{4}{3}$, we have (see Fig. 9)

$$\left(\begin{array}{c} 3r^{1/3}\cos\left(\frac{\theta}{3}\right) - \frac{3}{7}r^{7/3}\cos\left(\frac{7\theta}{3}\right) \\ -3r^{1/3}\sin\left(\frac{\theta}{3}\right) - \frac{3}{7}r^{7/3}\sin\left(\frac{7\theta}{3}\right) \\ \frac{3}{2}r^{4/3}\cos\left(\frac{4\theta}{3}\right) \end{array} \right),$$

where $r \in [-2, 2]$, $\theta \in [-3\pi, 3\pi]$.

Applications



(*a*) (*b*)

Figure 9 Bour's minimal surface of value 4/3

Applications

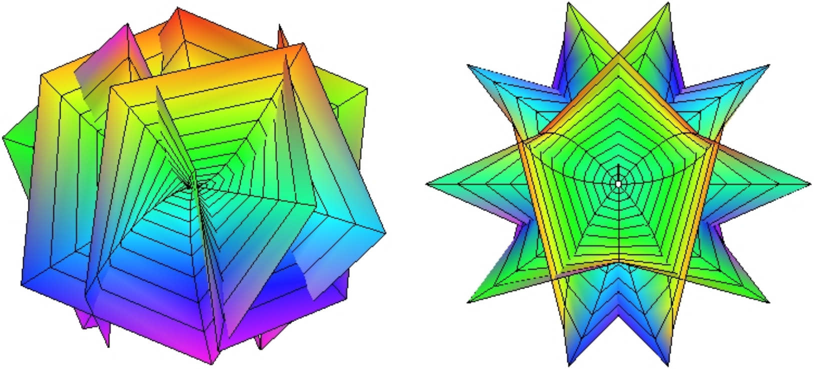
Example

If $m = \frac{5}{2}$, we have (see Fig. 10)

$$\begin{pmatrix} \frac{2}{3}r^{3/2}\cos\left(\frac{3\theta}{2}\right) - \frac{2}{7}r^{7/2}\cos\left(\frac{7\theta}{2}\right) \\ -\frac{2}{3}r^{3/2}\sin\left(\frac{3\theta}{2}\right) - \frac{2}{7}r^{7/2}\sin\left(\frac{7\theta}{2}\right) \\ \frac{4}{5}r^{5/2}\cos\left(\frac{5\theta}{2}\right) \end{pmatrix},$$

where $r \in [-1, 1]$, $\theta \in [-2\pi, 2\pi]$.

Applications



(a) (b)

Figure 10 Bour's minimal surface of value 5/2

Applications

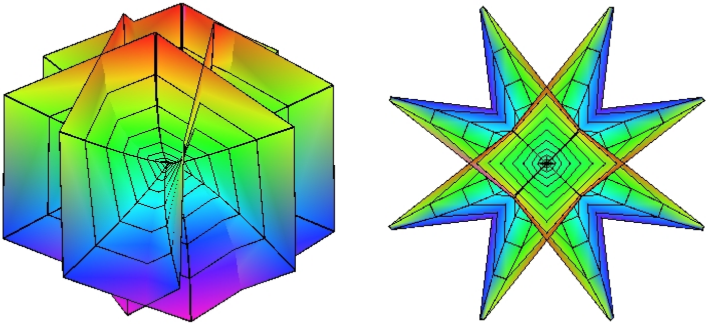
Example

If $m = 4$, we have (see Fig. 11)

$$\begin{pmatrix} \frac{1}{3}r^3\cos(3\theta) - \frac{1}{5}r^5\cos(5\theta) \\ -\frac{1}{3}r^3\sin(3\theta) - \frac{1}{5}r^5\sin(5\theta) \\ \frac{1}{2}r^4\cos(4\theta) \end{pmatrix},$$

where $r \in [-1, 1]$, $\theta \in [0, 2\pi]$.

Applications



(a) (b)

Figure 11 Bour's minimal surface of value 4

Now, we will see the definite and indefinite cases of the Bour's minimal surface.

Definite case

Let $\mathbb{L}^3$ be a 3-dimensional Minkowski space with natural Lorentzian metric

$$\langle .,. \rangle_1 = dx^2 + dy^2 - dz^2.$$

Definite case

- A *vector* w in $\mathbb{L}^3$ is called

Definite case

- A *vector* w in $\mathbb{L}^3$ is called
  - **spacelike** if $\langle w, w \rangle_1 > 0$ or $w = 0$,

Definite case

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  - **spacelike** if $\langle w, w \rangle_1 > 0$ or $w = 0$,
  - **timelike** if $\langle w, w \rangle_1 < 0$,

Definite case

- A *vector* w in $\mathbb{L}^3$ is called
  - **spacelike** if $\langle w, w\rangle_1 > 0$ or $w = 0$,
  - **timelike** if $\langle w, w\rangle_1 < 0$,
  - **lightlike** if $w \neq 0$ satisfies $\langle w, w\rangle_1 = 0$.

Definite case

- A *surface* in $\mathbb{L}^3$ is called a **spacelike** (resp. *timelike*, *degenere* (lightlike)) if the induced metric on the surface is a **positive definite Riemannian** (resp. Lorentzian, degenere) metric.

Definite case

- A *surface* in $\mathbb{L}^3$ is called a **spacelike** (resp. *timelike*, *degenere* (lightlike)) if the induced metric on the surface is a **positive definite Riemannian** (resp. Lorentzian, degenere) metric.
- A space-like surface with vanishing mean curvature is called a **maximal surface**.

Definite case

Theorem

(Weierstrass representation for maximal surfaces in $\mathbb{L}^3$). Let $\mathfrak{F}$ and $\mathcal{G}$ be two holomorphic functions defined on a simply connected open subset U of $\mathbb{C}$ such that $\mathfrak{F}$ does not vanish and $|\mathcal{G}| \neq 1$ on U. Then the map

$$\mathbf{x}(u,v) = \mathrm{Re} \int^z \begin{pmatrix} \mathfrak{F}\left(1+\mathcal{G}^2\right) \\ i\,\mathfrak{F}\left(1-\mathcal{G}^2\right) \\ 2\mathfrak{F}\mathcal{G} \end{pmatrix} dz$$

is a conformal immersion of U into $\mathbb{L}^3$ whose image is a maximal surface [1, 13, 15].

Definite case

Lemma

The Weierstrass patch determined by the functions

$$(\mathfrak{F}(z), \mathcal{G}(z)) = \left(z^{m-2}, z\right)$$

is a representation of the Bour's surface of value $m \in \mathbb{R}$ in $\mathbb{L}^3$.

Definite case

Theorem

Bour's surface of value m

$$\mathfrak{B}_m(r,\theta) = \begin{pmatrix} \frac{r^{m-1}}{m-1}\cos\left[(m-1)\theta\right] + \frac{r^{m+1}}{m+1}\cos\left[(m+1)\theta\right] \\ -\frac{r^{m-1}}{m-1}\sin\left[(m-1)\theta\right] + \frac{r^{m+1}}{m+1}\sin\left[(m+1)\theta\right] \\ 2\frac{r^m}{m}\cos(m\theta) \end{pmatrix} \tag{8}$$

is a maximal surface in $\mathbb{L}^3$, *where* $m \in \mathbb{R} - \{-1, 0, 1\}$.

Definite case

Proof.

- The coefficients of the first fundamental form of the surface $\mathfrak{B}_m$ are

$$\begin{aligned} E &= r^{2m-4}\left(1-r^2\right)^2, \\ F &= 0, \\ G &= r^{2m-2}\left(1-r^2\right)^2. \end{aligned}$$

Definite case

Proof.

- The coefficients of the first fundamental form of the surface $\mathfrak{B}_m$ are

$$\begin{aligned} E &= r^{2m-4}\left(1-r^2\right)^2, \\ F &= 0, \\ G &= r^{2m-2}\left(1-r^2\right)^2. \end{aligned}$$

- We have

$$\det I = \left[r^{2m-3}\left(1-r^2\right)^2\right]^2.$$

Definite case

Proof.

- The coefficients of the first fundamental form of the surface $\mathfrak{B}_m$ are

$$\begin{aligned} E &= r^{2m-4}\left(1-r^2\right)^2, \\ F &= 0, \\ G &= r^{2m-2}\left(1-r^2\right)^2. \end{aligned}$$

- We have

$$\det I = \left[r^{2m-3}\left(1-r^2\right)^2\right]^2.$$

- So, $\mathfrak{B}_m$ is a spacelike surface.

Definite case

Proof. (Cont.)
The Gauss map of the surface is

$$e = \frac{1}{r^2 - 1} \begin{pmatrix} 2r\cos(\theta) \\ 2r\sin(\theta) \\ r^2 + 1 \end{pmatrix}.$$

Definite case

Proof. (Cont.)

- The coefficients of the second fundamental form of the Bour's surface are

$$\begin{aligned} L &= 2r^{m-2}\cos(m\theta), \\ M &= -2r^{m-1}\sin(m\theta), \\ N &= -2r^{m}\cos(m\theta). \end{aligned}$$

Definite case

Proof. (Cont.)

- The coefficients of the second fundamental form of the Bour's surface are

$$\begin{aligned} L &= 2r^{m-2}\cos(m\theta), \\ M &= -2r^{m-1}\sin(m\theta), \\ N &= -2r^{m}\cos(m\theta). \end{aligned}$$

- Then, we have

$$\det II = -4r^{2m-2}.$$

Definite case

Proof. (Cont.)

- In spacelike case, the Gaussian curvature is defined by

$$K = \epsilon \, \frac{\det II}{|\det I|},$$

where $\epsilon := \langle e, e \rangle_1 = -1$ in $\mathbb{L}^3$.

Definite case

Proof. (Cont.)

- In spacelike case, the Gaussian curvature is defined by

$$K = \epsilon \frac{\det II}{|\det I|},$$

where $\epsilon := \langle e, e \rangle_1 = -1$ in $\mathbb{L}^3$.

- Hence, the Gaussian curvature and the mean curvature of the Bour's surface of value m, respectively, are

$$K = \left(\frac{2r^{2-m}}{(1-r^2)^2} \right)^2, \quad H = 0.$$

Definite case

Proof. (Cont.)

- In spacelike case, the Gaussian curvature is defined by

$$K = \epsilon \frac{\det II}{|\det I|},$$

where $\epsilon := \langle e, e \rangle_1 = -1$ in $\mathbb{L}^3$.

- Hence, the Gaussian curvature and the mean curvature of the Bour's surface of value m, respectively, are

$$K = \left(\frac{2r^{2-m}}{(1-r^2)^2} \right)^2, \quad H = 0.$$

- So, the $\mathfrak{B}_m$ is a maximal surface in $\mathbb{L}^3$.

Definite case

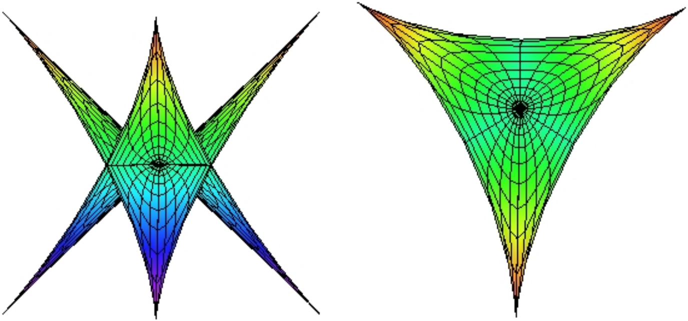
Example

If take $m = 3$ in $\mathfrak{B}_m(r,\theta)$, we have **Bour's maximal surface** (see Fig. 12)

$$\mathfrak{B}_3(r,\theta) = \begin{pmatrix} \frac{r^2}{2}\cos(2\theta) + \frac{r^4}{4}\cos(4\theta) \\ -\frac{r^2}{2}\sin(2\theta) + \frac{r^4}{4}\sin(4\theta) \\ \frac{2}{3}r^3\cos(3\theta) \end{pmatrix} \quad (9)$$

in Minkowski 3-space, where $r \in [-1,1]$, $\theta \in [0,\pi]$.

Definite case



(a) (b)

Figure 12 Bour's maximal surface $\mathfrak{B}_3\,(r,\theta)\,,\,(\mathfrak{F},\mathcal{G})=(z,z)$

Definite case

The coefficients of the first fundamental form of the Bour's maximal surface of value 3 are

$$E = r^2 \left(1 - r^2\right)^2, \; F = 0, \; G = r^4 \left(1 - r^2\right)^2.$$

So,

$$\det I = r^6 \left(1 - r^2\right)^4.$$

Definite case

The Gauss map of the surface is

$$e = \frac{1}{r^2 - 1} \left(2r\cos(\theta), 2r\sin(\theta), 1 + r^2 \right).$$

Definite case

The coefficients of the second fundamental form of the surface are

$$L = 2r\cos(3\theta), \quad M = -2r^2\sin(3\theta), \quad N = -2r^3\cos(3\theta).$$

Then,

$$\det II = -4r^4.$$

Definite case

The mean and the Gaussian curvatures of the Bour's maximal surface of value 3 are, respectively,

$$H = 0, \quad K = \frac{4}{r^2 \left(1 - r^2\right)^4}.$$

Definite case

The Weierstrass patch determined by the functions

$$(\mathfrak{F}, \mathcal{G}) = (z, z)$$

is a representation of the Bour's maximal surface of value 3 in $\mathbb{L}^3$.

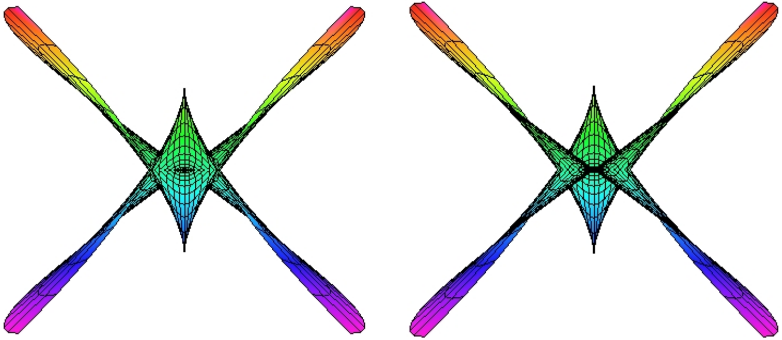
Definite case

The parametric form of the surface (see Fig. 13) is

$$\mathfrak{B}_3(u,v)=\begin{pmatrix} \frac{u^4}{4}+\frac{v^4}{4}-\frac{3}{2}u^2v^2+\frac{u^2}{2}-\frac{v^2}{2} \\ u^3v-uv^3-uv \\ \frac{2}{3}u^3-2uv^2 \end{pmatrix}, \qquad (10)$$

where $u, v \in \mathbb{R}$.

Definite case



(a) (b)

Figure 13 Maximal surface $\mathfrak{B}_3\,(u, v)\,,\ u, v \in [-1, 1]$

Applications of the definite case

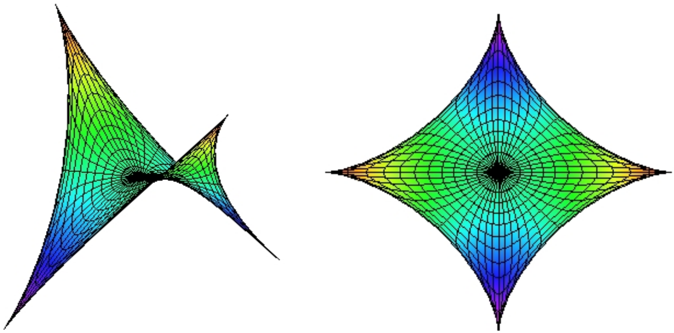
Example

If take $m = 2$, we have **Enneper's maximal surface** (see Fig. 14)

$$\mathfrak{B}_2(r,\theta) = \begin{pmatrix} r\cos(\theta) + \frac{r^3}{3}\cos(3\theta) \\ -r\sin(\theta) + \frac{r^3}{3}\sin(3\theta) \\ r^2\cos(2\theta) \end{pmatrix}$$

in $\mathbb{L}^3$, where $r \in [-1,1]$, $\theta \in [0,\pi]$.

Applications of the definite case



(a) (b)

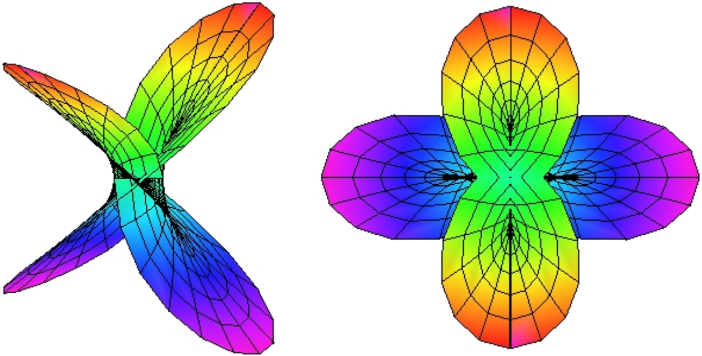
Figure 14 Maximal surface $\mathfrak{B}_2$, $(\mathfrak{F}, \mathcal{G}) = (1, z)$

Applications of the definite case

Example

If take $m = 2$, we have $\mathfrak{B}_2\left(r, \theta\right)$ (see Fig. 15) in $\mathbb{L}^3$, where $r \in [-3, 3]$, $\theta \in [0, \pi]$.

Applications of the definite case



(a) (b)

Figure 15 Maximal surface $\mathfrak{B}_2$, $(\mathfrak{F}, \mathcal{G}) = (1, z)$

Applications of the definite case

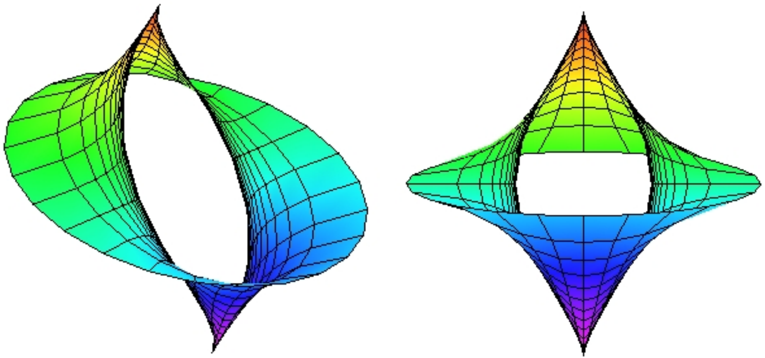
Example

If take $m = \frac{1}{2}$, we have (see Fig. 16)

$$\begin{pmatrix} -2r^{-1/2}\cos\left(\frac{\theta}{2}\right) + \frac{2}{3}r^{3/2}\cos\left(\frac{3\theta}{2}\right) \\ -2r^{-1/2}\sin\left(\frac{\theta}{2}\right) + \frac{2}{3}r^{3/2}\sin\left(\frac{3\theta}{2}\right) \\ 4r^{1/2}\cos\left(\frac{\theta}{2}\right) \end{pmatrix}$$

in $\mathbb{L}^3$.

Applications of the definite case



(a) (b)

Figure 16 Maximal surface $\mathfrak{B}_{1/2}$, $(\mathfrak{F}, \mathcal{G}) = \left(z^{-3/2}, z\right)$

Applications of the definite case

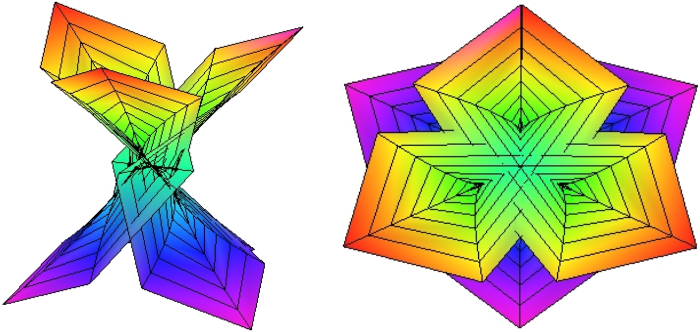
Example

If $m = \frac{3}{2}$, we have (see Fig. 17)

$$\begin{pmatrix} 2r^{-1/2}\cos\left(\frac{\theta}{2}\right) + \frac{2}{5}r^{5/2}\cos\left(\frac{5\theta}{2}\right) \\ -2r^{-1/2}\sin\left(\frac{\theta}{2}\right) + \frac{2}{5}r^{5/2}\sin\left(\frac{5\theta}{2}\right) \\ \frac{4}{3}r^{3/2}\cos\left(\frac{3\theta}{2}\right) \end{pmatrix}$$

in $\mathbb{L}^3$.

Applications of the definite case



(a) (b)

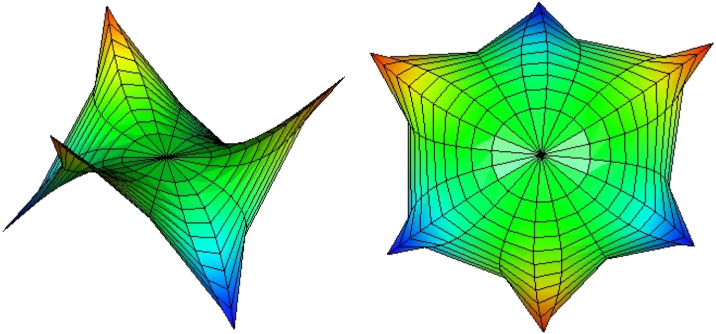
Figure 17 Maximal surface $\mathfrak{B}_{3/2}$, $(\mathfrak{F}, \mathcal{G}) = \left(z^{-1/2}, z\right)$

Applications of the definite case

Example

If $m = \frac{3}{2}$, we have $\mathfrak{B}_{3/2}(r, \theta)$ (see Fig. 18) in $\mathbb{L}^3$.

Applications of the definite case



(a) (b)

Figure 18 Maximal surface $\mathfrak{B}_{3/2}$, $(\mathfrak{F}, \mathcal{G}) = (z^{-1/2}, z)$

Applications of the definite case

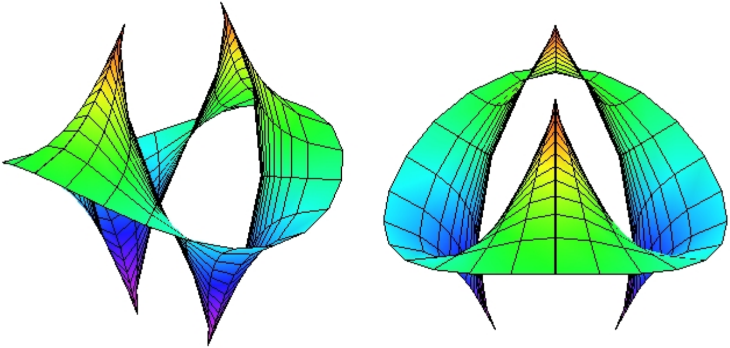
Example

If $m = \frac{2}{3}$, we have (see Fig. 19)

$$\begin{pmatrix} -3r^{-1/3}\cos\left(\frac{\theta}{3}\right) + \frac{3}{5}r^{5/3}\cos\left(\frac{5\theta}{3}\right) \\ -3r^{-1/3}\sin\left(\frac{\theta}{3}\right) + \frac{3}{5}r^{5/3}\sin\left(\frac{5\theta}{3}\right) \\ 3r^{2/3}\cos\left(\frac{2\theta}{3}\right) \end{pmatrix}$$

in $\mathbb{L}^3$.

Applications of the definite case



(a) (b)

Figure 19 Maximal surface $\mathfrak{B}_{2/3}$, $(\mathfrak{F}, \mathcal{G}) = \left(z^{-4/3}, z\right)$

Applications of the definite case

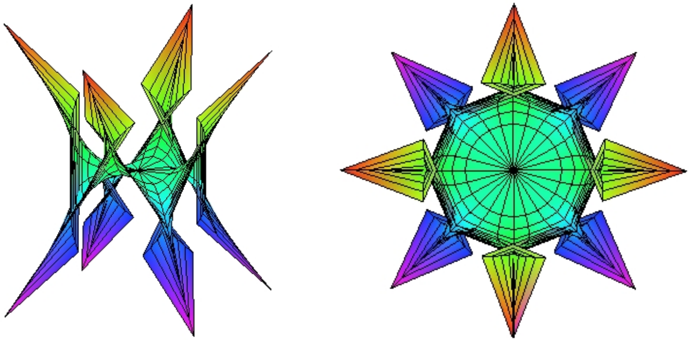
Example

If $m = \frac{4}{3}$, then we have (see Fig. 20)

$$\left(\begin{array}{c} 3r^{1/3}\cos\left(\frac{\theta}{3}\right) + \frac{3}{7}r^{7/3}\cos\left(\frac{7\theta}{3}\right) \\ -3r^{1/3}\sin\left(\frac{\theta}{3}\right) + \frac{3}{7}r^{7/3}\sin\left(\frac{7\theta}{3}\right) \\ \frac{3}{2}r^{4/3}\cos\left(\frac{4\theta}{3}\right) \end{array} \right),$$

in $\mathbb{L}^3$.

Applications of the definite case



(a) (b)

Figure 20 Maximal surface $\mathfrak{B}_{4/3}$, $(\mathfrak{F}, \mathcal{G}) = \left(z^{-2/3}, z\right)$

Applications of the definite case

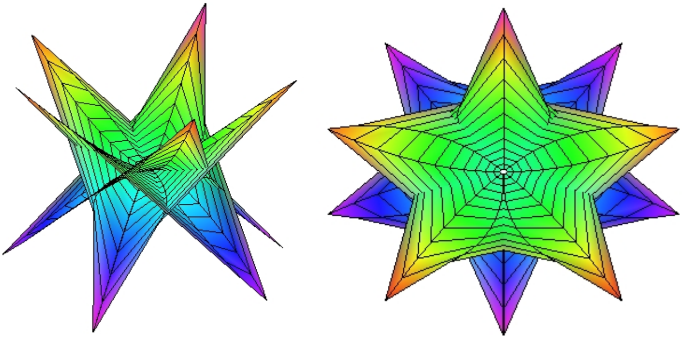
Example

If $m = \frac{5}{2}$, then we have (see Fig. 21)

$$\begin{pmatrix} \frac{2}{3} r^{3/2} \cos\left(\frac{3\theta}{2}\right) + \frac{2}{7} r^{7/2} \cos\left(\frac{7\theta}{2}\right) \\ -\frac{2}{3} r^{3/2} \sin\left(\frac{3\theta}{2}\right) + \frac{2}{7} r^{7/2} \sin\left(\frac{7\theta}{2}\right) \\ \frac{4}{5} r^{5/2} \cos\left(\frac{5\theta}{2}\right) \end{pmatrix},$$

in $\mathbb{L}^3$.

Applications of the definite case



(a) (b)

Figure 21 Maximal surface $\mathfrak{B}_{5/2}$, $(\mathfrak{F}, \mathcal{G}) = \left(z^{1/2}, z\right)$

Applications of the definite case

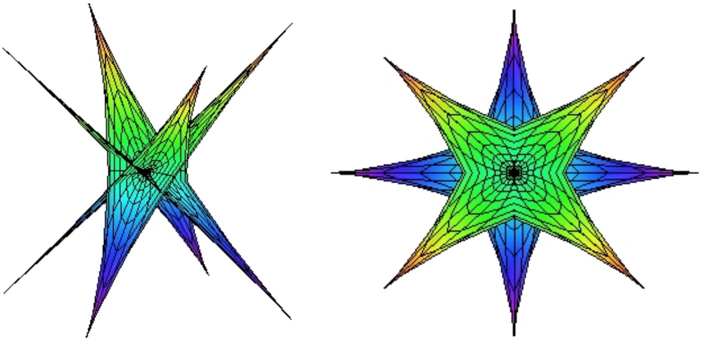
Example

If $m = 4$, then we have (see Fig. 22)

$$\begin{pmatrix} \frac{1}{3}r^3\cos(3\theta) + \frac{1}{5}r^5\cos(5\theta) \\ -\frac{1}{3}r^3\sin(3\theta) + \frac{1}{5}r^5\sin(5\theta) \\ \frac{1}{2}r^4\cos(4\theta) \end{pmatrix}$$

in $\mathbb{L}^3$.

Applications of the definite case



(a) (b)

Figure 22 Maximal surface $\mathfrak{B}_4$, $(\mathfrak{F}, \mathcal{G}) = (z^2, z)$

Indefinite case

Let $\mathbb{L}^2 = \left(\mathbb{R}^2, -dx^2 + dy^2\right)$ be Minkowski plane, and $\mathbb{L}^3$ be a 3-dimensional Minkowski space with natural Lorentzian metric

$$\langle ., . \rangle_1 = -dx^2 + dy^2 + dz^2.$$

Indefinite case

Theorem

(Weierstrass representation for timelike minimal surfaces in $\mathbb{L}^3$) *Let* $\mathbf{x} : \mathbf{M} \rightarrow \mathbb{L}^3$ *be a timelike surface parametrized by null coordinates* (u, v), *where* $u := -x + y$, $v := x + y$. *Timelike minimal surface is represented by*

$$\mathbf{x}(u,v) = \int^u \begin{pmatrix} -f\left(1+g^2\right) \\ f\left(1-g^2\right) \\ 2fg \end{pmatrix} du + \int^v \begin{pmatrix} \mathfrak{f}\left(1+\mathfrak{g}^2\right) \\ \mathfrak{f}\left(1-\mathfrak{g}^2\right) \\ 2\mathfrak{f}\mathfrak{g} \end{pmatrix} dv. \tag{11}$$

Indefinite case

The functions $f(u)$, $g(u)$, $\mathfrak{f}(v)$ and $\mathfrak{g}(v)$ are defined by

$$f = \frac{-\phi_1 + \phi_2}{2}, \;\; g = \frac{\phi_3}{-\phi_1 + \phi_2},$$

$$\mathfrak{f} = \frac{\mu_1 + \mu_2}{2}, \;\; \mathfrak{g} = \frac{\mu_3}{\mu_1 + \mu_2},$$

and $\phi = (\phi_1, \phi_2, \phi_3)$, $\mu = (\mu_1, \mu_2, \mu_3)$ vector valued functions, $\phi(u) := \mathbf{x}_u$, $\mu(v) := \mathbf{x}_v$ satisfy

$$(\phi)^2 = 0, \;\; (\mu)^2 = 0.$$

Indefinite case

Hence, the timelike minimal surface has the form

$$\begin{aligned}\mathbf{x}(u,v) &= \int^u \phi(u)\,du + \int^v \mu(v)\,dv \\ &= \Omega(u) + \Psi(v),\end{aligned}$$

and its conjugate

$$\mathbf{x}^*(u,v) = \Omega(u) - \Psi(v),$$

where $\phi(u)$ and $\mu(v)$ are linearly independent, $\Omega(u)$ and $\Psi(v)$ are null curves in $\mathbb{L}^3$.

Indefinite case

Weierstrass formula for the timelike minimal surfaces obtained by M. Magid [14] in 1991 (see [12], for details).

Indefinite case

Lemma

The Weierstrass patch determined by the functions

$$\left(f(u), g(u)\right) = \left(u^{m-2}, u\right) \quad \text{and} \quad \left(\mathfrak{f}(v), \mathfrak{g}(v)\right) = \left(v^{m-2}, v\right)$$

is a representation of the Bour's timelike minimal surface of value m in $\mathbb{L}^3$, where $m \in \mathbb{R}$.

Indefinite case

- Bour's timelike minimal surface of value *m* is

Indefinite case

- Bour's timelike minimal surface of value m is
- $$\int^u \begin{pmatrix} -u^{m-2}\left(1+u^2\right) \\ u^{m-2}\left(1-u^2\right) \\ 2u^{m-1} \end{pmatrix} du + \int^v \begin{pmatrix} v^{m-2}\left(1+v^2\right) \\ v^{m-2}\left(1-v^2\right) \\ 2v^{m-1} \end{pmatrix} dv,$$

Indefinite case

- Bour's timelike minimal surface of value m is
- $$\int^u \begin{pmatrix} -u^{m-2}\left(1+u^2\right) \\ u^{m-2}\left(1-u^2\right) \\ 2u^{m-1} \end{pmatrix} du + \int^v \begin{pmatrix} v^{m-2}\left(1+v^2\right) \\ v^{m-2}\left(1-v^2\right) \\ 2v^{m-1} \end{pmatrix} dv,$$
- and it has the form

$$\mathfrak{B}_m(u,v) = \begin{pmatrix} -\frac{1}{m-1}\left(u^{m-1}-v^{m-1}\right) - \frac{1}{m+1}\left(u^{m+1}-v^{m+1}\right) \\ \frac{1}{m-1}\left(u^{m-1}+v^{m-1}\right) - \frac{1}{m+1}\left(u^{m+1}+v^{m+1}\right) \\ 2\frac{1}{m}\left(u^m+v^m\right) \end{pmatrix}. \tag{12}$$

Indefinite case

Therefore, $\mathfrak{B}_m(r,\theta)$ is

$$
\begin{aligned}
x &= -\frac{r^{m-1}}{m-1}\left(\cos^{(m-1)}(\theta)-\sin^{(m-1)}(\theta)\right)\\
&\quad -\frac{r^{m+1}}{m+1}\left(\cos^{(m+1)}(\theta)-\sin^{(m+1)}(\theta)\right),
\end{aligned}
$$

$$
\begin{aligned}
y &= \frac{r^{m-1}}{m-1}\left(\cos^{(m-1)}(\theta)+\sin^{(m-1)}(\theta)\right)\\
&\quad -\frac{r^{m+1}}{m+1}\left(\cos^{(m+1)}(\theta)+\sin^{(m+1)}(\theta)\right),
\end{aligned}
$$

$$
z = 2\frac{r^m}{m}\left(\cos^m(\theta)+\sin^m(\theta)\right).
$$

Indefinite case

Theorem

Bour's surface $\mathfrak{B}_m(r,\theta)$ *is a timelike minimal surface in* $\mathbb{L}^3$, *where* $m \in \mathbb{R} - \{-1,0,1\}$.

Indefinite case

Proof.

- The coefficients of the first fundamental form of the $\mathfrak{B}_m$ are

Indefinite case

Proof.

- The coefficients of the first fundamental form of the $\mathfrak{B}_m$ are
- $$\begin{aligned} E &= 4r^{2m-4}\left(\sin\theta\cos\theta\right)^{m-1}\left(1+r^2\sin\theta\cos\theta\right)^2, \\ F &= 2r^{2m-3}\left(\sin\theta\cos\theta\right)^{m-2}\left(1+r^2\sin\theta\cos\theta\right)^2\cos\left(2\theta\right), \\ G &= -4r^{2m-2}\left(\sin\theta\cos\theta\right)^{m-1}\left(1+r^2\sin\theta\cos\theta\right)^2. \end{aligned}$$

Indefinite case

Proof. (Cont.)

- Then we have

Indefinite case

Proof. (Cont.)

- Then we have
- $$\det I = -\left[2r^{2m-3}\left(\sin\theta\cos\theta\right)^{m-2}\left(1+r^2\sin\theta\cos\theta\right)^2\right]^2.$$

Indefinite case

Proof. (Cont.)

- Then we have
- $$\det I = -\left[2r^{2m-3}\left(\sin\theta\cos\theta\right)^{m-2}\left(1+r^2\sin\theta\cos\theta\right)^2\right]^2.$$
- So, $\mathfrak{B}_m$ is a timelike surface.

Indefinite case

Proof. (Cont.)

- The Gauss map is

Indefinite case

Proof. (Cont.)

- The Gauss map is
- $$e = \frac{1}{1 + r^2 \sin\theta\cos\theta} \begin{pmatrix} -r\left(\sin\theta - \cos\theta\right) \\ r\left(\sin\theta + \cos\theta\right) \\ r^2 \sin\theta\cos\theta - 1 \end{pmatrix}.$$

Indefinite case

Proof. (Cont.)

- The coefficients of the second fundamental form of the surface are

Indefinite case

Proof. (Cont.)

- The coefficients of the second fundamental form of the surface are
-

$$\begin{aligned} L &= -2r^{m-2}(\sin^m(\theta) + \cos^m(\theta)), \\ M &= 2r^{m-1}(\sin(\theta)\cos^{m-1}(\theta) - \cos(\theta)\sin^{m-1}(\theta)), \\ N &= -2r^m(\sin^2(\theta)\cos^{m-2}(\theta) + \cos^2(\theta)\sin^{m-2}(\theta)). \end{aligned}$$

Indefinite case

Proof. (Cont.)

- The coefficients of the second fundamental form of the surface are
- $$\begin{aligned} L &= -2r^{m-2}(\sin^m(\theta)+\cos^m(\theta)), \\ M &= 2r^{m-1}(\sin(\theta)\cos^{m-1}(\theta)-\cos(\theta)\sin^{m-1}(\theta)), \\ N &= -2r^m(\sin^2(\theta)\cos^{m-2}(\theta)+\cos^2(\theta)\sin^{m-2}(\theta)). \end{aligned}$$
- We have
$$\det II = -4r^{2m-2}(\sin\theta\cos\theta)^{m-2}.$$

Indefinite case

Proof. (Cont.)

- Hence, the Gaussian curvature and the mean curvature, respectively, are

$$K = (\sin\theta\cos\theta)^{2-m}\left(\frac{r^{2-m}}{(1+r^2\sin\theta\cos\theta)^2}\right)^2,$$

Indefinite case

Proof. (Cont.)

- Hence, the Gaussian curvature and the mean curvature, respectively, are

$$K = (\sin\theta\cos\theta)^{2-m}\left(\frac{r^{2-m}}{(1+r^2\sin\theta\cos\theta)^2}\right)^2,$$

- and

$$H = 0.$$

Indefinite case

Proof. (Cont.)

- Hence, the Gaussian curvature and the mean curvature, respectively, are

$$K = (\sin\theta\cos\theta)^{2-m}\left(\frac{r^{2-m}}{(1+r^2\sin\theta\cos\theta)^2}\right)^2,$$

- and

$$H = 0.$$

- So, the $\mathfrak{B}_m$ is a timelike minimal surface in $\mathbb{L}^3$.

Indefinite case

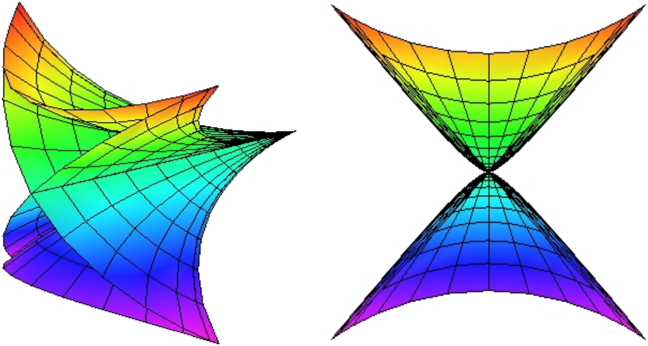
Example

If take $m = 3$ in $\mathfrak{B}_m(r, \theta)$, we have **Bour's timelike minimal surface** (see Fig. 23)

$$\mathfrak{B}_3(r, \theta) = \begin{pmatrix} \left(-\frac{r^2}{2} - \frac{r^4}{4}\right)\cos(2\theta) \\ \frac{r^2}{2} - \frac{r^4}{4}\left(\cos^4(\theta) + \sin^4(\theta)\right) \\ 2\frac{r^3}{3}\left(\cos^3(\theta) + \sin^3(\theta)\right) \end{pmatrix} \tag{13}$$

in Minkowski 3-space, where $r \in [-1, 1]$, $\theta \in [0, \pi]$.

Indefinite case



(a) (b)

Figure 23 Bour's timelike minimal surface $\mathfrak{B}_3\,(r,\theta)$

Indefinite case

- The coefficients of the first fundamental form of the Bour's timelike minimal surface of value 3 are

Indefinite case

- The coefficients of the first fundamental form of the Bour's timelike minimal surface of value 3 are
- $$\begin{aligned} E &= 4r^2 (\sin\theta\cos\theta)^2 \left(1 + r^2 \sin\theta\cos\theta\right)^2, \\ F &= r^3 \sin 2\theta \left(1 + r^2 \sin\theta\cos\theta\right)^2 \cos(2\theta), \\ G &= -4r^4 (\sin\theta\cos\theta)^2 \left(1 + r^2 \sin\theta\cos\theta\right)^2. \end{aligned}$$

Indefinite case

- The coefficients of the first fundamental form of the Bour's timelike minimal surface of value 3 are
-
$$\begin{aligned} E &= 4r^2 (\sin\theta\cos\theta)^2 \left(1 + r^2 \sin\theta\cos\theta\right)^2, \\ F &= r^3 \sin 2\theta \left(1 + r^2 \sin\theta\cos\theta\right)^2 \cos(2\theta), \\ G &= -4r^4 (\sin\theta\cos\theta)^2 \left(1 + r^2 \sin\theta\cos\theta\right)^2. \end{aligned}$$

- Then

$$\det I = -4r^6 (\sin\theta\cos\theta)^2 \left(1 + r^2 \sin\theta\cos\theta\right)^4.$$

Indefinite case

- The coefficients of the second fundamental form of the surface are

Indefinite case

- The coefficients of the second fundamental form of the surface are
- $$\begin{aligned} L &= -2r(\sin^3(\theta) + \cos^3(\theta)), \\ M &= 2r^2(\sin(\theta)\cos^2(\theta) - \cos(\theta)\sin^2(\theta)), \\ N &= -2r^3(\sin^2(\theta)\cos(\theta) + \cos^2(\theta)\sin(\theta)). \end{aligned}$$

Indefinite case

- The coefficients of the second fundamental form of the surface are
-
$$\begin{aligned} L &= -2r(\sin^3(\theta) + \cos^3(\theta)), \\ M &= 2r^2(\sin(\theta)\cos^2(\theta) - \cos(\theta)\sin^2(\theta)), \\ N &= -2r^3(\sin^2(\theta)\cos(\theta) + \cos^2(\theta)\sin(\theta)). \end{aligned}$$
- So,
$$\det II = -4r^4 \sin\theta\cos\theta.$$

Indefinite case

The mean and the Gaussian curvatures of the Bour's minimal surface of value 3 are, respectively,

$$H = 0, \quad K = \frac{1}{r^2 \sin\theta \cos\theta \left(1 + r^2 \sin\theta \cos\theta\right)^4}.$$

Indefinite case

- The Weierstrass patch determined by the functions

Indefinite case

- The Weierstrass patch determined by the functions
- $$(f, g) = (u, u) \quad \text{and} \quad (\mathfrak{f}, \mathfrak{g}) = (v, v)$$

  in $\mathbb{L}^3$.

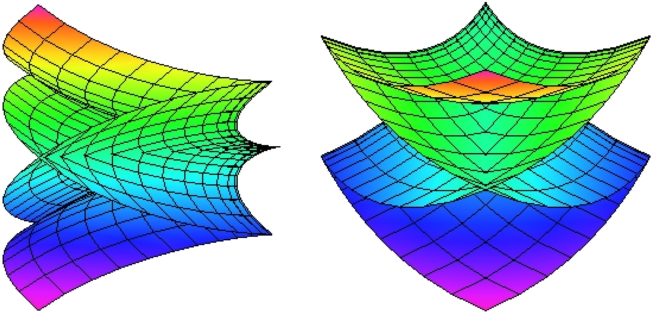
Indefinite case

The parametric form of the surface (see Fig. 24) is

$$\mathfrak{B}_3(u,v) = \begin{pmatrix} -\frac{1}{2}\left(u^2 - v^2\right) - \frac{1}{4}\left(u^4 - v^4\right) \\ \frac{1}{2}\left(u^2 + v^2\right) - \frac{1}{4}\left(u^4 + v^4\right) \\ \frac{2}{3}\left(u^3 + v^3\right) \end{pmatrix}, \qquad (14)$$

where $u, v \in I \subset \mathbb{R}$.

Indefinite case



(*a*) (*b*)

Figure 24 Timelike minimal surface $\mathfrak{B}_3\left(u,v\right)$, $u,v\in\left[-1,1\right]$

Applications of the indefinite case

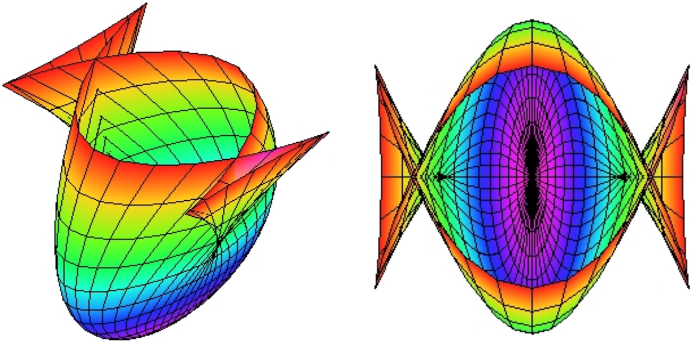
Example

If take $m = 2$, we have $\mathfrak{B}_2(r,\theta)$ (see Fig. 25)

$$\begin{pmatrix} -r(\cos(\theta)-\sin(\theta)) - \frac{r^3}{3}\left(\cos^3(\theta)-\sin^3(\theta)\right) \\ r(\cos(\theta)+\sin(\theta)) - \frac{r^3}{3}\left(\cos^3(\theta)+\sin^3(\theta)\right) \\ r^2 \end{pmatrix}$$

in $\mathbb{L}^3$, where $r \in [-2,2]$, $\theta \in [-\pi/2, \pi/2]$.

Applications of the indefinite case



(a) (b)

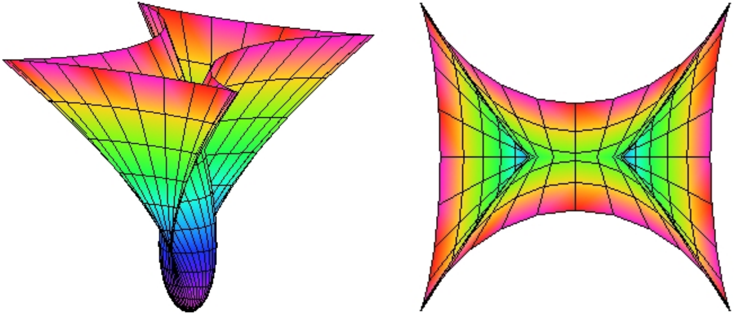
Figure 25 Bour's timelike minimal surface $\mathfrak{B}_2(r, \theta)$

Applications of the indefinite case

Example

If take $m = 2$, we have $\mathfrak{B}_2(r, \theta)$ (see Fig. 26) in $\mathbb{L}^3$.

Applications of the indefinite case



(a) (b)

Figure 26 Bour's timelike minimal surface $\mathfrak{B}_2\,(r,\theta)$

Applications of the indefinite case

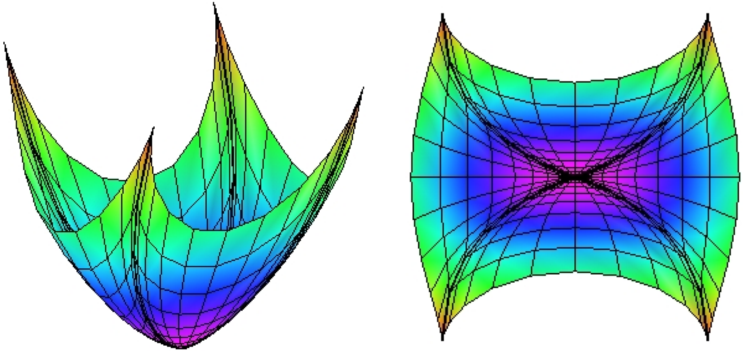
Example

If take $m = 4$, we have $\mathfrak{B}_4\left(r, \theta\right)$ (see Fig. 27)

$$\begin{pmatrix} -\frac{r^3}{3}\left(\cos^3\left(\theta\right) - \sin^3\left(\theta\right)\right) - \frac{r^5}{5}\left(\cos^5\left(\theta\right) - \sin^5\left(\theta\right)\right) \\ \frac{r^3}{3}\left(\cos^3\left(\theta\right) + \sin^3\left(\theta\right)\right) - \frac{r^5}{5}\left(\cos^5\left(\theta\right) + \sin^5\left(\theta\right)\right) \\ \frac{r^4}{4}\left(\cos^4\left(\theta\right) + \sin^4\left(\theta\right)\right) \end{pmatrix}$$

in $\mathbb{L}^3$.

Applications of the indefinite case



(a) (b)

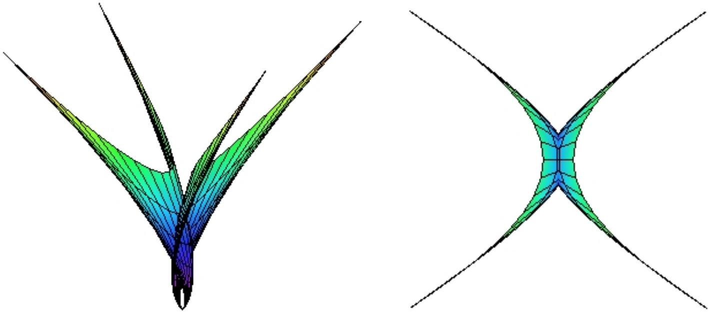
Figure 27 Bour's timelike minimal surface $\mathfrak{B}_4\,(r,\theta)$

Applications of the indefinite case

Example

If take $m = 4$, we have $\mathfrak{B}_2\left(r, \theta\right)$ (see Fig. 28) in $\mathbb{L}^3$.

Applications of the indefinite case



(a) (b)

Figure 28 Bour's timelike minimal surface $\mathfrak{B}_4(r,\theta)$

Applications of the indefinite case

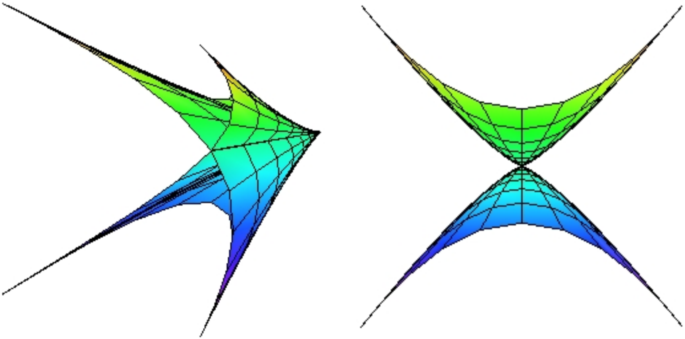
Example

If take $m = 5$, we have $\mathfrak{B}_5(r, \theta)$ (see Fig. 29)

$$\begin{pmatrix} -\frac{r^4}{4}\left(\cos^4(\theta) - \sin^4(\theta)\right) - \frac{r^6}{6}\left(\cos^6(\theta) - \sin^6(\theta)\right) \\ \frac{r^4}{4}\left(\cos^4(\theta) + \sin^4(\theta)\right) - \frac{r^6}{6}\left(\cos^6(\theta) + \sin^6(\theta)\right) \\ \frac{r^5}{5}\left(\cos^5(\theta) + \sin^5(\theta)\right) \end{pmatrix}$$

in $\mathbb{L}^3$.

Applications of the indefinite case



(a) (b)

Figure 29 Bour's timelike minimal surface $\mathfrak{B}_5(r, \theta)$

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merci

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(1832 – 1866)

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