



**4th INTERNATIONAL CONFERENCE ON
ANALYSIS AND ITS APPLICATIONS**

September 11-14, 2018 Kirşehir / TURKEY



Extended Abstract Book

Editors

Vatan KARAKAYA-Necip ŞİMŞEK

Mohammad MURSALEEN-Qamrul HASAN ANSARI

FOREWORDS

Dear Conference Participants,

First of all, I would like to thank you for coming today to participate at this opening ceremony and I wish to welcome you to Turkey and Kırşehir. I hope we will have a good time during this conference.

Kırşehir Ahi Evran University was founded on March 17, 2006 in Kırşehir which is located in the center of Turkey. With approximately 22 thousand students and more than 800 academicians, our university is a young and dynamic university. Our university consists of 39 academic units, including 8 faculties, 3 institutions, 5 schools of higher education, 7 vocational schools of higher education and 16 research and application centers, and 7 different campuses in which education maintains.

Kırşehir Ahi Evran University, by the President of the Republic of Turkey Mr. Recep Tayyip Erdogan made a statement on 18.01.2016, is selected as one of five pilot universities in the field of agriculture and geothermal by within the scope of "Regional Development Focused Mission Differentiation and Specialization Project". Ahi Evran University is also the first and only university to be awarded the ISO 9001: 2015 Quality Management System Certificate by successfully completing the ISO 9001: 2015 Quality Management Standard External Audit Process with all its units.

The projects of our university are

- Geothermal Welding and Transmission to Project Fields
- Clustering Project in Thermal Sera
- Roughage Production Project
- Walnut-Focused Development and Development Project
- Geothermal Rehabilitation Center Project
- Sportsman Health Research, Application and Thermal Rehabilitation Center Project
- Training and Promotion Project of Pilot University Projects.

The fundamental duty of universities is to produce information. The means of these are conferences, symposiums, workshops, etc. In this sense, we will discuss problems related to mathematics and reach to solutions in this conference.

The purpose of this conference is to bring together experts and young analysts from all over the world working in different fields of mathematics and its applications to present their researches, exchange new ideas, discuss challenging issues, foster future collaborations and interact with each other.

This conference allows the participation of many prominent experts from different countries who will present works on different fields of mathematics, especially fixed point theory, approximation theory, nonlinear analysis, variational analysis, optimization, summability theory, sequence spaces, dynamical systems and their applications, and also algebra, geometry.

It bring together more than 130 participants from countries of different part of the world for example Algeria, Ajarian, Azerbaijan, Egypt, Congo, Yemen, Korea, China, India, Iran, Sudan, Morocco, Saudi Arabia, Tunisia, Ghana, Turkey, Uzbekistan, United States of America, Jordan, out of which 124 are contributing to the meeting with oral and 3 with poster presentations, including five plenary talks.

We also thank plenary speaker distinguished Prof. Mohammad MURSALEEN, distinguished Prof. Zuhair NASHED, distinguished Prof. Jong Kyu KIM, distinguished Prof. Qamrul Hasan ANSARI and distinguished Prof. Bayram ŞAHİN for contribution to the our symposium.

We hope to promote collaborative and networking opportunities among senior scholars and graduate students in order to advance new perspectives. The additional emphasis at ICAA-2018 is to put importance on applications in related areas, as well as other science, such as natural science, economics, computer science and various engineering sciences.

The papers presented in this conference will be considered in the journals listed on the conference websites and below:

- Journal of Nonlinear and Convex Analysis (SCI-Exp.)
- Carpathian Journal of Mathematics (SCI-Exp.)
- Bulletin of Mathematical Analysis and Applications (E-SCI)
- Journal of Inequalities and Special Functions (E-SCI)
- Creative Mathematics and Informatics
- Istanbul Commerce University Journal of Science
- Nonlinear Functional Analysis and Applications (SCOPUS)

This booklet contains the titles and extended abstracts of some contributed talks at the 4th International Conference on Analysis and Its Applications. Only some abstracts were not available at the time of printing the booklet. They will be made available on the conference website icaa2018.ahievran.edu.tr when the organizers receive them. All talks will take place in Faculty of Arts and Sciences in Ahi Evran University, Baębaşı Campus, Kırşehir/Turkey.

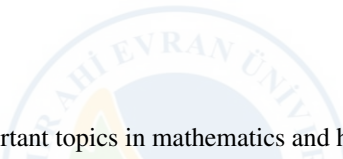
Finally, we thank you for your participation and wish you a productive time during conference in Kırşehir, Turkey.

Prof. Vatan KARAKAYA
On Behalf of Organizing Committee
Chairman
(Rector of Kırşehir Ahi Evran University)

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Analysis is one of the most important topics in mathematics and has been a focus of attention of all great mathematicians. There are many areas comes under this topic. However, this conference mainly devoted to some selected topics from analysis, mainly, Theory of Summability and Approximation, Fixed Point Theory, Fourier Analysis, Wavelet and Harmonic Analysis, Variational Analysis, Convex Analysis and Optimization, Geometry of Banach Spaces, Sequence Spaces and Matrix Transformations. During the last half century, nonlinear and variational analysis have been developed very rapidly because of their numerous applications to optimization, control theory, economics, engineering, management, medical sciences and other disciplines. On the other hand, the modern summability theory plays a very important role in linking theory of sequence spaces and matrix transformations with measures of noncompactness. Measures of noncompactness are widely used tools in fixed point theory, differential equations, functional equations, integral and integro-differential equations, optimization, etc. In the recent years, measures of noncompactness have also been used in defining geometric properties of Banach spaces as well as in characterizing compact operators between sequence spaces. We expect the participation of many prominent experts from different countries who will present their current research work and will also mention some hot topics for further research.

Prof. Qamrul Hasan ANSARI

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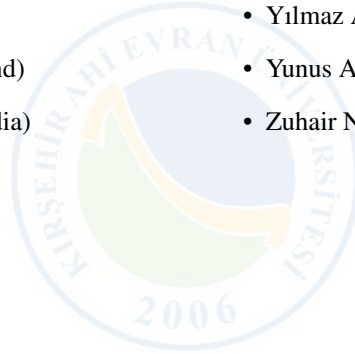
It was a great moment of excitement when Prof. Vatan Karakaya, Rector, Ahi Evran University, discussed with me the matter of organizing the “International Conference on Analysis and Its Applications (ICAA-2018)” at Ahi Evran University, Kırşehir. Now it is a matter of great pleasure that the matter of holding this conference is finally materialized. This conference is in the sequel of the first one which was held during December 19-21, 2015 (ICAA-2015) in Aligarh Muslim University, India with over 100 participants, the second one which was held during July 12-15, 2016 (ICAA-2016) in Ahi Evran University, Kırşehir, Turkey with over 300 participants and third one was held during November 20-22, 2017 (ICAA-2017) in Aligarh Muslim University, India, with over 100 participants. Being one of the Co-Chairmen of the conference, I feel privileged and delighted to welcome all delegates, eminent mathematicians, speakers and young researchers in this international event. It is expected that the delegates and the participants will be benefitted by the experience of this conference and the legacy of knowledge dissemination will be continued. I wish all of you to have a nice and enjoyable participation in the conference.

Prof. Mohammad MURSALEEN

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**CONTRIBUTED
TALKS**
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On Applications Of Multidimensional Affine Transform In The Supply Of Missing Data

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Keywords:

Image Processing,
Multivariate Data,
Shearlet,
Supply of Missing Data.

MSC:

Abstract: Supply of missing data is one of the image processing applications. Shearlet transform, which is an important affine transformation, can be used for multivariate data analysis. In this study, we used shearlet transform in the supply of missing data and we compared the obtained results.

1. Introduction

In imaging science, image processing is any form of signal processing for which the input is an image, such as a photograph or video frame; the output of image processing may be either an image or a set of characteristics or parameters related to the image. Most image-processing techniques involve treating the image as a two-dimensional signal and applying standard signal-processing techniques to it.

In almost any research performed, there is the potential for missing or incomplete data. Incomplete or missing data can occur for many reasons. Incomplete data are a common occurrence and can have a significant effect on the conclusions that can be drawn from the data. It can reduce the representativeness of the image and can therefore distort inferences about the images. If it is possible, preventing data from missingness before the actual data gathering takes place is useful. However, this technique may not be practical especially working with medical data. In situations where incomplete data are likely to occur, the researcher is often advised to plan to use robust methods of data analysis. Two recent papers related to supply of missing data (inpainting) problem are interesting [2,3]. In [2] mathematical theory of shearlet and wavelet transforms related to inpainting problem are studied and numerical application results are compared. In [3] several inpainting methods applied to images with different maskings.

2. Basics Shearlet Transform

Shearlets has emerged in recent years with many successful applications; some related work can be listed as [1,4,6,9]. The emergence of wavelets about 30 years ago represents a milestone in the development of efficient encoding of piecewise regular signals. Wavelet Transform decomposes an image by projecting onto several dilated and translated version of one single function, namely the mother wavelet. The key property enabling such a unified treatment of the continuum and digital setting is a Multiresolution Analysis, which allows a direct transition between the realms of real variable functions and digital signals. Despite their success, wavelets unfortunately have a very limited ability to resolve edges and other distributed discontinuities which usually occur in multidimensional data. Wavelet representations are optimal for approximating data with pointwise singularities only and cannot handle equally well distributed singularities such as singularities along curves. However, in dimensions two and higher, distributed discontinuities such as edges of surface boundaries are usually present or even dominant, and as a result wavelets are far from optimal in dealing with multivariate data. The limitations of wavelets and traditional multiscale systems have stimulated a flurry of activity involving

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mathematicians, engineers, and applied scientists. In 2006, shearlets were introduced by Guo, Kutyniok, Labate, Lim, and Weiss. This approach was derived within a larger class of affine-like systems the so-called composite wavelets as a truly multivariate extension of the wavelet framework. One of the distinctive features of shearlets is the use of shearing to control directional selectivity, in contrast to rotation used by curvelets. This is a fundamentally different concept, since it allows shearlet systems to be derived from a single or finite set of generators, and it also ensures a unified treatment of the continuum and digital world due to the fact that the shear matrix preserves the integer lattice. The shearlet transform is a non-isotropic version of the wavelet transform. It was shown that the asymptotic decay rate of the continuous shearlet transform, for fine scales, can be used to find both the location and the orientation of the edges of an image, and that the coefficients of large magnitude will come from edges. Also, the decay rate across scales can be used to distinguish between noise spikes and edges.

3. The Supply of Missing Data (Image Inpainting)

The main inpainting methods are primarily divided into three categories: sparsity-based, variational, and patch-based. Sparsity-based methods involve a combination of harmonic analysis with convex optimization (see, for example, [7, 8, 10]). Recently the compressed sensing methodology, namely exact recovery of sparse or sparsified data from highly incomplete linear nonadaptive measurements by l_1 minimization or thresholding, has been very effectively applied to this problem. The pioneering paper is [7] which uses curvelets as sparsifying system for inpainting. Also, some work has been done to compare variational approaches with those built on l_1 minimization [11, 12]. It also prohibits a deep understanding of why directional representation systems such as shearlets outperform wavelets when inpainting images strongly governed by curvilinear structures such as seismic images. Variational methods have been used on a number of papers in image processing literature. A few of these are [13-16]. The main idea of variational-based inpainting is that information is propagated from the boundary of the holes along isophotes (edges) in the image to fill them in. Many of the methods are inspired by real physical processes, like diffusion, osmosis, and uid dynamics. In patch based or exemplar based inpainting, information is also propagated from the edge(s) of the missing data inward. However, in contrast to the variational approaches, the hole is iteratively filled using patches or averages of patches from other parts of the image [3]. Some examples of exemplar based inpainting are [17-20].

4. Numerical Results

In this study, we want to present the results of our approach with two medical images. This two images, obtained through Medical School's Hospital at Kocaeli University in Turkey, are vessel contour image shown in Fig. 1 (a) and chest X-ray image in Fig. 1 (b).

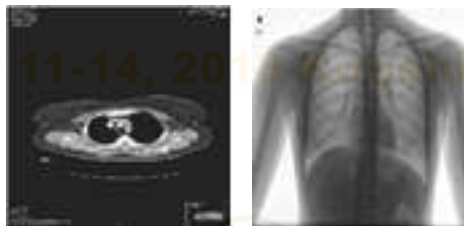


Fig. 1. (a) Vessel contour image; (b) Chest X-Ray image

Our contribution is two-fold. We will present a horizontal masking of arbitrary height, a circular masking of arbitrary radius and an elliptic masking of arbitrary semi-major and semi-minor axis lengths. Then we will describe the missing traces recovery as an image inpainting problem using shearlets with iterative thresholding for medical images. Also we will give the PSNR values for all mask situations.

Horizontal masking code is written for arbitrary height, Circular masking code is written for arbitrary radius and elliptic masking code is written for arbitrary semi-major and semi-minor axis lengths in Matlab. A horizontal masking is shown in Figure 2 (a), a circular masking is shown in Figure 2 (b) and an elliptic masking is shown in Figure 2 (c). Obtained all maskings are applied to two images, see Figures 3(b), 4(b), 5(b), 6(b) and 7(b), 8(b). Masked images are inpainted by shearlets shown in Figures 3(c), 4(c), 5(c), 6(c) and 7(c), 8(c). For shearlets, some part of the shearlet program codes are obtained at the shearlet web site [5].

The algorithms for both horizontal masking and shearlet inpainting problem are shown in Table 1 and Table 2, respectively. The algorithms for other masking situations are shown similarly. The phrase peak signal-to-noise ratio, often abbreviated PSNR, is an engineering term for the ratio between the maximum possible power of a signal and the power of corrupting noise that affects the fidelity of its representation. Because many signals have a very wide dynamic range, PSNR is usually expressed in terms of

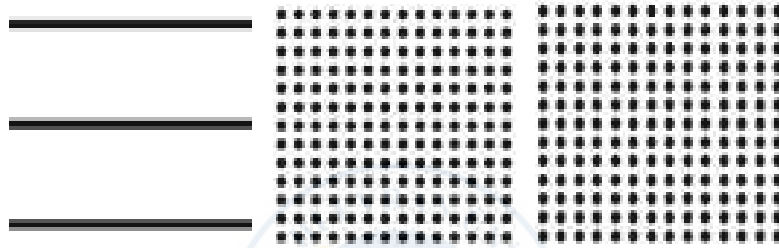


Fig. 2. (a) Horizontal Masking; (b)Circular Masking; (c) Elliptic Masking



Fig. 3. (a) Vessel Contour image; (b) Vessel Contour image after horizontal masking; (c) inpainting of Vessel Contour image with shearlet transformation

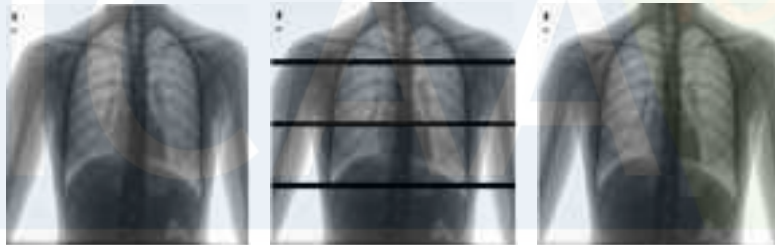


Fig. 4. (a) Chest X-ray image; (b) Chest X-ray image after horizontal masking; (c) inpainting of Chest X-ray image with shearlet transformation



Fig. 5. (a) Vessel Contour image; (b) Vessel Contour image after circular masking; (c) inpainting of Vessel Contour image with shearlet transformation



Fig. 6. (a) Chest X-ray image; (b) Chest X-ray image after circular masking; (c) inpainting of Chest X-ray image with shearlet transformation

the logarithmic decibel scale. As a performance measure, computation of PSNR values can be calculated as $PSNR = 10 \log_{10} \left(\frac{MAX_I^2}{MSE} \right)$.

Here, MAX_I is the maximum possible pixel value of the image. The mean squared error (MSE) which for two $m \times n$ monochrome images I and K where one of the images is considered a noisy approximation of the



Fig. 7. (a) Vessel Contour image; (b) Vessel Contour image after elliptic masking; (c) inpainting of Vessel Contour image with shearlet transformation



Fig. 8. (a) Chest X-ray image; (b) Chest X-ray image after elliptic masking; (c) inpainting of Chest X-ray image with shearlet transformation

Table 1. Horizontal Mask Pseudocode

<p><i>Parameters:</i> Height parameter h. Image size parameters. Zero matrix with the same size as the image matrix. Step size.</p>
<p><i>Algorithm:</i> Obtaining horizontal mask with height h. Flooring horizontal masks with height h in the whole image.</p>
<p><i>Output:</i> Horizontal mask matrix.</p>

Table 2. Inpainting Algorithm Pseudocode with Shearlet Transformation

<p><i>Parameters:</i> Image. Horizontal mask. Iteration parameter. Shearlet transformation filters.</p>
<p><i>Algorithm:</i> 1. Applying horizontal mask algorithm to image. 2. Determining threshold value using iterative thresholding. 3. Obtaining image with shearlet transformation. 4. Calculating PSNR values.</p>
<p><i>Output:</i> Horizontal masked image. Inpainted image. PSNR values.</p>

other is defined as:

$$MSE = \frac{1}{mn} \sum_{i=0}^{m-1} \sum_{j=0}^{n-1} (I(i,j) - K(i,j))^2.$$

In Table 3, it can be seen PSNR values of shearlet inpainting method with horizontal masking, In Table 4, it can be seen PSNR values of shearlet inpainting method with circular masking and In Table 5, it can be seen PSNR values of shearlet inpainting method with elliptic masking.

Table 3. Comparison the PSNR values of shearlet inpainting method for the medical images with horizontal masking

Image	h=5	h=6	h=7	h=8
Vessel contour	32,8797dB	31.5355dB	31.3063dB	30.9861dB
Chest X-ray	46,9872dB	45.8738dB	45.2786dB	43.8633dB

Table 4. Comparison the PSNR values of shearlet inpainting method the medical images with circular masking

Image	r=8	r=10	r=12	r=15
Vessel contour	34.5548dB	31.2077dB	28.7652dB	25.5612dB
Chest X-ray	39.7342dB	38.0187dB	36.8234dB	33.7826dB

Table 5. Comparison the PSNR values of shearlet inpainting method for the medical images with elliptic masking

Image	a=10, b=5	a=10, b=7	a=9, b=12	a=10,b=15	a=14,b=15
Vessel contour	34.7668dB	33.1736dB	31.2363dB	29.0471dB	25.2422dB
Chest X-ray	35.9250dB	34.6950dB	31.1084dB	29.0614dB	27.2058dB

5. Conclusions

In this paper we develop and use a horizontal masking algorithm with arbitrary height for medical images, we develop and use a circular masking algorithm with arbitrary radius for medical images and we develop and use an elliptic masking algorithm for arbitrary semi-major and semi-minor axis lengths. Also we observed that, we apply shearlet image inpainting to recover horizontal masked data including two medical images with 20% of the test data masked, we apply shearlet image inpainting to recover circular masked data including two medical images with 22% of the test data masked and we apply shearlet image inpainting to recover elliptic masked data including two medical images with 23.5% of the test data masked.

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Asymptotically \mathcal{I}_2 -Invariant Equivalence of Double Sequences and Some Properties

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Abstract: In this paper, we give definitions of asymptotically ideal equivalent, asymptotically invariant equivalent and strongly asymptotically invariant equivalent for double sequences. Also, we give some properties and examine the existence relationships among these new type equivalence concepts.

1. Introduction and Background

Let σ be a mapping of the positive integers into themselves. A continuous linear functional ϕ on ℓ_∞ , the space of real bounded sequences, is said to be an invariant mean or a σ -mean if it satisfies following conditions:

1. $\phi(x) \geq 0$, when the sequence $x = (x_n)$ has $x_n \geq 0$ for all n ,
2. $\phi(e) = 1$, where $e = (1, 1, 1, \dots)$ and
3. $\phi(x_{\sigma(n)}) = \phi(x_n)$ for all $x \in \ell_\infty$.

The mappings σ are assumed to be one-to-one and such that $\sigma^m(n) \neq n$ for all positive integers n and m , where $\sigma^m(n)$ denotes the m th iterate of the mapping σ at n . Thus, ϕ extends the limit functional on c , the space of convergent sequences, in the sense that $\phi(x) = \lim x$ for all $x \in c$.

In the case σ is translation mappings $\sigma(n) = n + 1$, the σ -mean is often called a Banach limit and the space V_σ , the set of bounded sequences all of whose invariant means are equal, is the set of almost convergent sequences \hat{c} . It can be shown that

$$V_\sigma = \left\{ x = (x_n) \in \ell_\infty : \lim_{m \rightarrow \infty} \frac{1}{m} \sum_{k=1}^m x_{\sigma^k(n)} = L, \text{ uniformly in } n \right\}.$$

Several authors have studied invariant convergent sequences (see, [11–15, 19–21, 23–25]). The concept of strongly σ -convergence was defined by Mursaleen in [12]:

A bounded sequence $x = (x_k)$ is said to be strongly σ -convergent to L if

$$\lim_{m \rightarrow \infty} \frac{1}{m} \sum_{k=1}^m |x_{\sigma^k(n)} - L| = 0,$$

uniformly in n . It is denoted by $x_k \rightarrow L[V_\sigma]$.

By $[V_\sigma]$, we denote the set of all strongly σ -convergent sequences.

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In the case $\sigma(n) = n + 1$, the space $[V_\sigma]$ is the set of strongly almost convergent sequences $[\hat{c}]$.

The concept of strongly σ -convergence was generalized by Savaş [20] as below:

$$[V_\sigma]_p = \left\{ x = (x_k) : \lim_{m \rightarrow \infty} \frac{1}{m} \sum_{k=1}^m |x_{\sigma^k(n)} - L|^p = 0, \text{ uniformly in } n \right\},$$

where $0 < p < \infty$.

If $p = 1$, then $[V_\sigma]_p = [V_\sigma]$. It is known that $[V_\sigma]_p \subset \ell_\infty$.

The idea of statistical convergence was introduced by Fast [6] and studied by many authors. The concept of σ -statistically convergent sequence was introduced by Savaş and Nuray in [23]. The idea of \mathcal{I} -convergence was introduced by Kostyrko et al. [8] as a generalization of statistical convergence which is based on the structure of the ideal \mathcal{I} of subset of the set of natural numbers \mathbb{N} . Similar concepts can be seen in [7, 14].

A family of sets $\mathcal{I} \subseteq 2^{\mathbb{N}}$ is called an ideal if and only if (i) $\emptyset \in \mathcal{I}$, (ii) For each $A, B \in \mathcal{I}$ we have $A \cup B \in \mathcal{I}$, (iii) For each $A \in \mathcal{I}$ and each $B \subseteq A$ we have $B \in \mathcal{I}$.

An ideal is called non-trivial if $\mathbb{N} \notin \mathcal{I}$ and non-trivial ideal is called admissible if $\{n\} \in \mathcal{I}$ for each $n \in \mathbb{N}$.

Recently, the concepts of σ -uniform density of subset A of the set \mathbb{N} and corresponding \mathcal{I} - σ -convergence for real number sequences was introduced by Nuray et al. [14]. Marouf [10] presented definitions for asymptotically equivalent sequences and asymptotic regular matrices. Then, the concept of asymptotically equivalence has been developed by many other researchers (see, [16, 17, 22]).

Two nonnegative sequences $x = (x_k)$ and $y = (y_k)$ are said to be asymptotically equivalent if $\lim_{k \rightarrow \infty} \frac{x_k}{y_k} = 1$. It is denoted by $x \sim y$.

Convergence and \mathcal{I} -convergence of double sequences in a metric space and some properties of this convergence, and similar concepts which are noted following can be seen in [1, 2, 9, 18].

A double sequence $x = (x_{kj})$ is said to be bounded if $\sup_{k,j} x_{kj} < \infty$. The set of all bounded double sequences of sets will be denoted by ℓ_∞^2 .

A nontrivial ideal \mathcal{I}_2 of $\mathbb{N} \times \mathbb{N}$ is called strongly admissible ideal if $\{i\} \times \mathbb{N}$ and $\mathbb{N} \times \{i\}$ belong to \mathcal{I}_2 for each $i \in \mathbb{N}$.

It is evident that a strongly admissible ideal is admissible also.

Let (X, ρ) be a metric space and \mathcal{I}_2 be a strongly admissible ideal in $\mathbb{N} \times \mathbb{N}$. A sequence $x = (x_{mn})$ in X is said to be \mathcal{I}_2 -convergent to $L \in X$, if for any $\varepsilon > 0$

$$A(\varepsilon) = \{(m, n) \in \mathbb{N} \times \mathbb{N} : \rho(x_{mn}, L) \geq \varepsilon\} \in \mathcal{I}_2.$$

It is denoted by $\mathcal{I}_2 - \lim_{m,n \rightarrow \infty} x_{mn} = L$.

Let $A \subseteq \mathbb{N} \times \mathbb{N}$ and

$$s_{mn} := \min_{k,j} |A \cap \{(\sigma(k), \sigma(j)), (\sigma^2(k), \sigma^2(j)), \dots, (\sigma^m(k), \sigma^n(j))\}|$$

and

$$S_{mn} := \max_{k,j} |A \cap \{(\sigma(k), \sigma(j)), (\sigma^2(k), \sigma^2(j)), \dots, (\sigma^m(k), \sigma^n(j))\}|.$$

If the following limits exists

$$\underline{V}_2(A) := \lim_{m,n \rightarrow \infty} \frac{s_{mn}}{mn} \quad \text{and} \quad \overline{V}_2(A) := \lim_{m,n \rightarrow \infty} \frac{S_{mn}}{mn},$$

then they are called a lower and an upper σ -uniform density of the set A , respectively. If $\underline{V}_2(A) = \overline{V}_2(A)$, then $V_2(A) = \underline{V}_2(A) = \overline{V}_2(A)$ is called the σ -uniform density of A .

Denote by \mathcal{I}_2^σ the class of all $A \subseteq \mathbb{N} \times \mathbb{N}$ with $V_2(A) = 0$.

Throughout the paper we let $\mathcal{I}_2^\sigma \subset 2^{\mathbb{N} \times \mathbb{N}}$ be a strongly admissible ideal.

Dündar et al. [3] studied the concepts of invariant convergence, strongly invariant convergen, p -strongly invariant convergen and ideal invariant convergence of double sequences.

A double sequence $x = (x_{kj})$ is said to be \mathcal{I}_2^σ -invariant convergent or \mathcal{I}_2^σ -convergent to L if for every $\varepsilon > 0$

$$A(\varepsilon) = \{(k, j) : |x_{kj} - L| \geq \varepsilon\} \in \mathcal{I}_2^\sigma,$$

that is, $V_2(A(\varepsilon)) = 0$. In this case, we write $\mathcal{I}_2^\sigma - \lim x = L$ or $x_{kj} \rightarrow L(\mathcal{I}_2^\sigma)$. The set of all \mathcal{I}_2 -invariant convergent double sequences will be denoted by \mathfrak{I}_2^σ . A double sequence $x = (x_{kj})$ is said to be strongly invariant convergent to L if

$$\lim_{m,n \rightarrow \infty} \frac{1}{mn} \sum_{k,j=1,1}^{m,n} |x_{\sigma^k(s), \sigma^j(t)} - L| = 0,$$

uniformly in s, t . In this case, we write $x_{kj} \rightarrow L([V_\sigma^2])$. A double sequence $x = (x_{kj})$ is said to be p -strongly invariant convergent to L , if

$$\lim_{m,n \rightarrow \infty} \frac{1}{mn} \sum_{k,j=1,1}^{m,n} |x_{\sigma^k(s), \sigma^j(t)} - L|^p = 0,$$

uniformly in s, t , where $0 < p < \infty$. In this case, we write $x_{kj} \rightarrow L([V_\sigma^2]_p)$.

The set of all p -strongly invariant convergent double sequences will be denoted by $[V_\sigma^2]_p$.

Hazarika [4] introduced the notion of asymptotically \mathcal{I} -equivalent sequences and investigated some properties of it. Definitions of P -asymptotically equivalence, asymptotically statistical equivalence and asymptotically \mathcal{I}_2 -equivalence of double sequences were presented by Hazarika and Kumar [5] as following:

Two nonnegative double sequences $x = (x_{kl})$ and $y = (y_{kl})$ are said to be P -asymptotically equivalent if

$$P - \lim_{k,l} \frac{x_{kl}}{y_{kl}} = 1,$$

denoted by $x \sim^P y$.

Two nonnegative double sequences $x = (x_{kl})$ and $y = (y_{kl})$ are said to be asymptotically statistical equivalent of multiple L provided that for every $\varepsilon > 0$

$$P - \lim_{m,n} \frac{1}{mn} \left| \left\{ k \leq m, l \leq n : \left| \frac{x_{kl}}{y_{kl}} - L \right| \geq \varepsilon \right\} \right| = 0,$$

denoted by $x \sim^{\mathcal{I}_2^L} y$ and simply asymptotically statistical equivalent if $L = 1$.

Two nonnegative double sequences $x = (x_{kl})$ and $y = (y_{kl})$ are said to be asymptotically \mathcal{I}_2 -equivalent of multiple L provided that for every $\varepsilon > 0$

$$\left\{ (k, l) \in \mathbb{N} \times \mathbb{N} : \left| \frac{x_{kl}}{y_{kl}} - L \right| \geq \varepsilon \right\} \in \mathcal{I}_2.$$

denoted by $x \sim^{\mathcal{I}_2^L} y$ and simply asymptotically \mathcal{I}_2 -equivalent if $L = 1$.

2. Asymptotically \mathcal{I}_2^σ -Equivalence

Definition 2.1 Two nonnegative double sequences $x = (x_{kl})$ and $y = (y_{kl})$ are said to be asymptotically invariant equivalent or asymptotically σ_2 -equivalent of multiple L if

$$\lim_{m,n \rightarrow \infty} \frac{1}{mn} \sum_{k,l=1,1}^{m,n} \frac{x_{\sigma^k(s), \sigma^l(t)}}{y_{\sigma^k(s), \sigma^l(t)}} = L,$$

uniformly in s, t . In this case, we write $x \overset{V_\sigma^2}{\sim} y$ and simply σ_2 -asymptotically equivalent, if $L = 1$.

Definition 2.2 Two nonnegative double sequences $x = (x_{kl})$ and $y = (y_{kl})$ are said to be asymptotically \mathcal{I}_2^σ -equivalent of multiple L if for every $\varepsilon > 0$,

$$A_\varepsilon := \left\{ (k, l) \in \mathbb{N} \times \mathbb{N} : \left| \frac{x_{kl}}{y_{kl}} - L \right| \geq \varepsilon \right\} \in \mathcal{I}_2^\sigma,$$

i.e., $V_2(A_\varepsilon) = 0$. In this case, we write $x \overset{\mathcal{I}_2^\sigma}{\sim} y$ and simply asymptotically \mathcal{I}_2^σ -equivalent, if $L = 1$.

The set of all asymptotically \mathcal{I}_2^σ -equivalent of multiple L sequences will be denoted by $\mathfrak{I}_{2(L)}^\sigma$.

Theorem 2.3 Suppose that $x = (x_{kl})$ and $y = (y_{kl})$ are bounded double sequences. If x and y are asymptotically \mathcal{I}_2^σ -equivalent of multiple L , then these sequences are σ_2 -asymptotically equivalent of multiple L .

Proof. Let $m, n, s, t \in \mathbb{N}$ be arbitrary and $\varepsilon > 0$. Now, we calculate

$$u(m, n, s, t) := \left| \frac{1}{mn} \sum_{k,l=1,1}^{m,n} \frac{x_{\sigma^k(s), \sigma^l(t)}}{y_{\sigma^k(s), \sigma^l(t)}} - L \right|.$$

We have

$$u(m, n, s, t) \leq u^{(1)}(m, n, s, t) + u^{(2)}(m, n, s, t),$$

where

$$u^{(1)}(m, n, s, t) := \frac{1}{mn} \sum_{k,l=1,1}^{m,n} \left| \frac{x_{\sigma^k(s), \sigma^l(t)}}{y_{\sigma^k(s), \sigma^l(t)}} - L \right|$$

$$\left| \frac{x_{\sigma^k(s), \sigma^l(t)}}{y_{\sigma^k(s), \sigma^l(t)}} - L \right| \geq \varepsilon$$

and

$$u^{(2)}(m, n, s, t) := \frac{1}{mn} \sum_{k,l=1,1}^{m,n} \left| \frac{x_{\sigma^k(s), \sigma^l(t)}}{y_{\sigma^k(s), \sigma^l(t)}} - L \right|$$

$$\left| \frac{x_{\sigma^k(s), \sigma^l(t)}}{y_{\sigma^k(s), \sigma^l(t)}} - L \right| < \varepsilon$$

We get $u^{(2)}(m, n, s, t) < \varepsilon$, for every $s, t = 1, 2, \dots$. The boundedness of $x = (x_{kl})$ and $y = (y_{kl})$ implies that there exists a $M > 0$ such that

$$\left| \frac{x_{\sigma^k(s), \sigma^l(t)}}{y_{\sigma^k(s), \sigma^l(t)}} - L \right| \leq M,$$

for $k, l = 1, 2, \dots, s, t = 1, 2, \dots$. Then, this implies that

$$u^{(1)}(m, n, s, t) \leq \frac{M}{mn} \left| \left\{ 1 \leq k \leq m, 1 \leq l \leq n : \left| \frac{x_{\sigma^k(s), \sigma^l(t)}}{y_{\sigma^k(s), \sigma^l(t)}} - L \right| \geq \varepsilon \right\} \right|$$

$$\leq M \frac{\max_{s,t} \left| \left\{ 1 \leq k \leq m, 1 \leq l \leq n : \left| \frac{x_{\sigma^k(s), \sigma^l(t)}}{y_{\sigma^k(s), \sigma^l(t)}} - L \right| \geq \varepsilon \right\} \right|}{mn} = M \frac{S_{mn}}{mn},$$

hence x and y are σ_2 -asymptotically equivalent to multiple L . \square

The converse of Theorem 2.3 does not hold. For example, $x = (x_{kl})$ and $y = (y_{kl})$ are the double sequences defined by following;

$$x_{kl} := \begin{cases} 2 & , \text{ if } k+l \text{ is an even integer,} \\ 0 & , \text{ if } k+l \text{ is an odd integer.} \end{cases}$$

$$y_{kl} := 1$$

When $\sigma(m) = m + 1$ and $\sigma(n) = n + 1$, this sequences are asymptotically σ_2 -equivalent but they are not asymptotically \mathcal{I}_2^σ -equivalent.

Definition 2.4 Two nonnegative double sequence $x = (x_{kl})$ and $y = (y_{kl})$ are said to be strongly asymptotically invariant equivalent or strongly asymptotically σ_2 -equivalent of multiple L if

$$\lim_{m,n \rightarrow \infty} \frac{1}{mn} \sum_{k,l=1,1}^{m,n} \left| \frac{x_{\sigma^k(s), \sigma^l(t)}}{y_{\sigma^k(s), \sigma^l(t)}} - L \right| = 0,$$

uniformly in s, t . In this case, we write $x \overset{[V_{2(L)}^\sigma]}{\sim} y$ and simply strongly asymptotically σ_2 -equivalent if $L = 1$.

Definition 2.5 Let $0 < p < \infty$. Two nonnegative double sequence $x = (x_{kl})$ and $y = (y_{kl})$ are said to be p -strongly asymptotically invariant equivalent or p -strongly asymptotically σ_2 -equivalent of multiple L if

$$\lim_{m,n \rightarrow \infty} \frac{1}{mn} \sum_{k,l=1,1}^{m,n} \left| \frac{x_{\sigma^k(s), \sigma^l(t)}}{y_{\sigma^k(s), \sigma^l(t)}} - L \right|^p = 0,$$

uniformly in s, t . In this case, we write $x \overset{[V_{2(L)}^\sigma]^p}{\sim} y$ and simply p -strongly asymptotically σ_2 -equivalent if $L = 1$.

The set of all p -strongly asymptotically σ_2 -equivalent of multiple L sequences will be denoted by $[\mathcal{I}_{2(L)}^\sigma]_p$.

Theorem 2.6 Let $0 < p < \infty$. Then, $x \overset{[\mathcal{I}_{2(L)}^\sigma]^p}{\sim} y \Rightarrow x \overset{\mathcal{I}_{2(L)}^\sigma}{\sim} y$.

Proof. Let $x \overset{[\mathcal{I}_{2(L)}^\sigma]^p}{\sim} y$ and given $\varepsilon > 0$. Then, for every $s, t \in \mathbb{N}$ we have

$$\begin{aligned} \sum_{k,l=1}^{m,n} \left| \frac{x_{\sigma^k(s), \sigma^l(t)}}{y_{\sigma^k(s), \sigma^l(t)}} - L \right|^p &\geq \sum_{k,l=1}^{m,n} \left| \frac{x_{\sigma^k(s), \sigma^l(t)}}{y_{\sigma^k(s), \sigma^l(t)}} - L \right|^p \\ &\quad \left| \frac{x_{\sigma^k(s), \sigma^l(t)}}{y_{\sigma^k(s), \sigma^l(t)}} - L \right| \geq \varepsilon \\ &\geq \varepsilon^p \left| \left\{ 1 \leq k \leq m, 1 \leq l \leq n : \left| \frac{x_{\sigma^k(s), \sigma^l(t)}}{y_{\sigma^k(s), \sigma^l(t)}} - L \right| \geq \varepsilon \right\} \right| \\ &\geq \varepsilon^p \max_{s,t} \left| \left\{ 1 \leq k \leq m, 1 \leq l \leq n : \left| \frac{x_{\sigma^k(s), \sigma^l(t)}}{y_{\sigma^k(s), \sigma^l(t)}} - L \right| \geq \varepsilon \right\} \right| \end{aligned}$$

and

$$\begin{aligned} \frac{1}{mn} \sum_{k,l=1}^{m,n} \left| \frac{x_{\sigma^k(s), \sigma^l(t)}}{y_{\sigma^k(s), \sigma^l(t)}} - L \right|^p &\geq \frac{\max_{s,t} \left| \left\{ 1 \leq k \leq m, 1 \leq l \leq n : \left| \frac{x_{\sigma^k(s), \sigma^l(t)}}{y_{\sigma^k(s), \sigma^l(t)}} - L \right| \geq \varepsilon \right\} \right|}{mn} \\ &= \varepsilon^p \frac{S_{mn}}{mn} \end{aligned}$$

for every $s, t = 1, 2, \dots$. This implies $\lim_{m,n \rightarrow \infty} \frac{S_{mn}}{mn} = 0$ and so $x \overset{\mathcal{I}_{2(L)}^\sigma}{\sim} y$. □

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Helicoidal Hypersurfaces of Dini-Type in the Four Dimensional Minkowski Space

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Abstract: We define Ulisse Dini-type helicoidal hypersurfaces with spacelike axis in Minkowski 4-space \mathbb{E}_1^4 . We compute the Gaussian and the mean curvature of the hypersurface. Moreover, we obtain some special symmetries to the curvatures when they are flat and maximal.

1. Introduction

After Chen and Piccini [5] the submanifolds theory of finite type in space forms has been studied by many geometers [1, 2], [4]-[9], [13]-[23], [25], [28]-[30], [33]

General rotational surfaces as a source of examples of surfaces in the four dimensional Euclidean space were introduced by Moore [26, 27]. Ganchev and Milousheva [14] considered the analogue of these surfaces in the Minkowski 4-space. They classified completely the minimal general rotational surfaces and the general rotational surfaces consisting of parabolic points. Verstraelen, Valrave and Yaprak [32] studied the minimal translation surfaces in \mathbb{E}^n for arbitrary dimension n .

In classical surface geometry in Euclidean space, it is well known that the right helicoid (resp. catenoid) is the only ruled (resp. rotational surface) which is minimal. If we focus on the ruled (helicoid) and rotational characters, we have Bour's theorem in [3].

Do Carmo and Dajczer [11] proved that there exists a two-parameter family of helicoidal surfaces isometric to a given helicoidal surface using a result of Bour [3] in Euclidean 3-space \mathbb{E}^3 .

There are only a few works in the literature about Italian Matematician Ulisse Dini's helicoidal surface [10] in \mathbb{E}^3 .

In this paper, we consider the Ulisse Dini-type helicoidal hypersurface with spacelike axis in Minkowski 4-space \mathbb{E}_1^4 . We indicate basic notions of 4-dimensional Minkowskian geometry, and define helicoidal hypersurface in section 2. Moreover, we obtain the Ulisse Dini-type helicoidal hypersurface, and then calculate its curvatures with some interesting symmetric results in the last section.

2. Helicoidal hypersurface with spacelike axis in Minkowski 4-space

A rotational hypersurface $M \subset \mathbb{E}_1^n$ generated by a curve C around an axis ℓ that does not meet C is obtained by taking the orbit of C under those orthogonal transformations of \mathbb{E}_1^n that leave ℓ pointwise fixed (See [12, Remark 2.3]).

Suppose that when a curve C rotates around the axis ℓ , it simultaneously displaces parallel lines orthogonal to the axis ℓ , so that the speed of displacement is proportional to the speed of rotation. Then the resulting hypersurface is called the *helicoidal hypersurface* with axis ℓ and pitches $a, b \in \mathbb{R} \setminus \{0\}$.

Consider the particular case $n = 4$ and let C be the curve parametrized by

$$\gamma(u) = (\varphi(u), f(u), 0, 0). \quad (2.1)$$

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If ℓ is the spacelike x_1 -axis, then an orthogonal transformation of \mathbb{E}_1^n that leaves ℓ pointwise fixed has the form $Z(v, w)$ as follows

$$Z(v, w) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cosh w & 0 & \sinh w \\ 0 & \sinh v \sinh w & \cosh v & \cosh w \sinh v \\ 0 & \cosh v \sinh w & \sinh v & \cosh v \cosh w \end{pmatrix}, \quad (2.2)$$

where

$$Z^T \varepsilon Z = Z \varepsilon Z^T = \varepsilon, \quad Z \ell = \ell, \quad \det Z = 1, \quad \varepsilon = \text{diag}(1, 1, 1, -1),$$

$v, w \in \mathbb{R}$. Therefore, the parametrization of the rotational hypersurface generated by a curve C around an axis ℓ is

$$\mathbf{H}(u, v, w) = Z(v, w) \gamma(u)^T + (av + bw)(1, 0, 0, 0)^T, \quad (2.3)$$

where $u \in I$, $v, w \in [0, 2\pi]$, $a, b \in \mathbb{R} \setminus \{0\}$.

Clearly, we write an helicoidal hypersurface with spacelike axis as follows:

$$\mathbf{H}(u, v, w) = \begin{pmatrix} \varphi(u) + av + bw \\ f(u) \sinh w \\ f(u) \sinh v \cosh w \\ f(u) \cosh v \cosh w \end{pmatrix}. \quad (2.4)$$

When $w = 0$, we have an helicoidal surface with spacelike axis in \mathbb{E}_1^4 .

In the rest of this paper, we shall identify a vector (a, b, c, d) with its transpose $(a, b, c, d)^T$. Let $\mathbf{M} = \mathbf{M}(u, v, w)$ be an isometric immersion of a hypersurface M^3 in \mathbb{E}_1^4 . The Minkowski inner product of $\vec{x} = (x_1, x_2, x_3, x_4)$, $\vec{y} = (y_1, y_2, y_3, y_4)$ is defined as follows

$$\vec{x} \cdot \vec{y} = \sum_{i=1}^3 x_i y_i - x_4 y_4,$$

and triple Minkowski vector product of $\vec{x} \times \vec{y} \times \vec{z}$ is defined as follows

$$\begin{aligned} & (x_2 y_3 z_4 - x_2 y_4 z_3 - x_3 y_2 z_4 + x_3 y_4 z_2 + x_4 y_2 z_3 - x_4 y_3 z_2, \\ & -x_1 y_3 z_4 + x_1 y_4 z_3 + x_3 y_1 z_4 - x_3 z_1 y_4 - y_1 x_4 z_3 + x_4 y_3 z_1, \\ & x_1 y_2 z_4 - x_1 y_4 z_2 - x_2 y_1 z_4 + x_2 z_1 y_4 + y_1 x_4 z_2 - x_4 y_2 z_1, \\ & x_1 y_2 z_3 - x_1 y_3 z_2 - x_2 y_1 z_3 + x_2 y_3 z_1 + x_3 y_1 z_2 - x_3 y_2 z_1). \end{aligned}$$

For a hypersurface \mathbf{M} in \mathbb{E}_1^4 , the first fundamental form matrix is as follows $\mathbf{I} = (g_{ij})_{3 \times 3}$, and $\det \mathbf{I} = \det (g_{ij})$, and then, the second fundamental form matrix is $\mathbf{II} = (h_{ij})_{3 \times 3}$, and $\det \mathbf{II} = \det (h_{ij})$, where $1 \leq i, j \leq 3$,

$$\begin{aligned} g_{11} &= \mathbf{M}_u \cdot \mathbf{M}_u, \quad g_{12} = \mathbf{M}_u \cdot \mathbf{M}_v, \dots, \quad g_{33} = \mathbf{M}_w \cdot \mathbf{M}_w, \\ h_{11} &= \mathbf{M}_{uu} \cdot \mathbf{G}, \quad h_{12} = \mathbf{M}_{uv} \cdot \mathbf{G}, \dots, \quad h_{33} = \mathbf{M}_{ww} \cdot \mathbf{G}, \end{aligned}$$

" \cdot " means Lorentzian dot product, and some partial differentials that we represent are $\mathbf{M}_u = \frac{\partial \mathbf{M}}{\partial u}$, $\mathbf{M}_{uw} = \frac{\partial^2 \mathbf{M}}{\partial u \partial w}$,

$$\mathbf{G} = \frac{\mathbf{M}_u \times \mathbf{M}_v \times \mathbf{M}_w}{\|\mathbf{M}_u \times \mathbf{M}_v \times \mathbf{M}_w\|}$$

is the Gauss map (i.e. the unit normal vector). The product matrices $(g_{ij})^{-1} \cdot (h_{ij})$, gives the matrix of the shape operator (i.e. Weingarten map) \mathbf{S} as follows $\mathbf{S} = \frac{1}{\det \mathbf{I}} (s_{ij})_{3 \times 3}$, where $\frac{s_{ij}}{\det \mathbf{I}} = g_{ij}^{-1} \cdot h_{ij}$. So, we get the formulas of the Gaussian curvature and the mean curvature, respectively, as follows

$$K = \det(\mathbf{S}) = \frac{\det \mathbf{II}}{\det \mathbf{I}}, \quad (2.5)$$

and

$$H = \frac{1}{3} \text{tr}(\mathbf{S}). \quad (2.6)$$

3. Dini-Type Helicoidal Hypersurface with Spacelike Axis

Taking $f(u) = \sinh u$ in (2.4), we get Dini-type helicoidal hypersurface with spacelike axis in \mathbb{E}_1^4 as follows

$$\mathfrak{D}(u, v, w) = \begin{pmatrix} \varphi(u) + av + bw \\ \sinh u \sinh w \\ \sinh u \sinh v \cosh w \\ \sinh u \cosh v \cosh w \end{pmatrix}, \quad (3.1)$$

where $u \in \mathbb{R} \setminus \{0\}$ and $0 \leq v, w < 2\pi$.

Using the first differentials of (3.1) with respect to u, v, w , we get the first quantities

$$\mathbf{I} = \begin{pmatrix} \varphi'^2 - \cosh^2 u & a\varphi' & b\varphi' \\ a\varphi' & \sinh^2 u \cosh^2 w + a^2 & ab \\ b\varphi' & ab & \sinh^2 u + b^2 \end{pmatrix},$$

and have

$$\det \mathbf{I} = \{ \varphi'^2 \sinh^2 u \cosh^2 w - [a^2 + (b^2 + \sinh^2 u) \cosh^2 w] \cosh^2 u \} \sinh^2 u,$$

where $\varphi = \varphi(u)$, $\varphi' = \frac{d\varphi}{du}$. By using the second differentials we have the second quantities

$$\mathbf{II} = \begin{pmatrix} \frac{\sinh^2 u \cosh w (-\varphi'' \cosh u + \varphi' \sinh u)}{\sqrt{|\det \mathbf{I}|}} & \frac{a \sinh u \cosh^2 u \cosh w}{\sqrt{|\det \mathbf{I}|}} & \frac{b \sinh u \cosh^2 u \cosh w}{\sqrt{|\det \mathbf{I}|}} \\ \frac{a \sinh u \cosh^2 u \cosh w}{\sqrt{|\det \mathbf{I}|}} & \frac{\sinh^2 u \cosh^2 w (\varphi' \sinh u \cosh w - b \cosh u \sinh w)}{\sqrt{|\det \mathbf{I}|}} & \frac{a \sinh^2 u \cosh u \sinh w}{\sqrt{|\det \mathbf{I}|}} \\ \frac{b \sinh u \cosh^2 u \cosh w}{\sqrt{|\det \mathbf{I}|}} & \frac{a \sinh^2 u \cosh u \sinh w}{\sqrt{|\det \mathbf{I}|}} & \frac{\varphi' \sinh^3 u \cosh w}{\sqrt{|\det \mathbf{I}|}} \end{pmatrix},$$

and we get

$$\det \mathbf{II} = \frac{\begin{pmatrix} -\varphi'^2 \varphi'' \sinh^8 u \cosh u \cosh^5 w + b\varphi' \varphi'' \sinh^7 u \cosh^2 u \sinh w \cosh^4 w \\ + a^2 \varphi'' \sinh^6 u \cosh^3 u \sinh^2 w \cosh w + \varphi'^3 \cosh^5 w \sinh^9 u \\ - b\varphi'^2 \sinh^8 u \cosh u \cosh^4 w \sinh w \\ - [a^2 \sinh^2 u \sinh^2 w + (a^2 + b^2 \cosh^2 w) \cosh^2 u \cosh^2 w] \varphi' \sinh^5 u \cosh^2 u \cosh w \\ + b(2a^2 + b^2 \cosh^2 w) \sinh^4 u \cosh^5 u \sinh w \cosh^2 w \end{pmatrix}}{(\det \mathbf{I})^{3/2}}.$$

The Gauss map of the hypersurface is given by

$$e_{\mathfrak{D}} = \frac{1}{\sqrt{\det \mathbf{I}}} \begin{pmatrix} -\sinh^2 u \cosh u \cosh w \\ (-\varphi' \sinh u \sinh w - b \cosh u \cosh w) \sinh u \cosh w \\ (-\varphi' \sinh u \sinh v \cosh^2 w + a \cosh u \cosh v + b \cosh u \sinh v \sinh w \cosh w) \sinh u \\ (-\varphi' \sinh u \cosh v \cosh^2 w + a \cosh u \sinh v + b \cosh u \cosh v \sinh w \cosh w) \sinh u \end{pmatrix}. \quad (3.2)$$

Finally, we calculate the Gaussian curvature of the helicoidal hypersurface with spacelike axis as follows

$$K = \frac{\alpha_1 \varphi'^2 \varphi'' + \alpha_2 \varphi' \varphi'' + \alpha_3 \varphi'' + \alpha_4 \varphi'^3 + \alpha_5 \varphi'^2 + \alpha_6 \varphi' + \alpha_7}{(\det \mathbf{I})^{5/2}},$$

where

$$\alpha_1 = -\sinh^8 u \cosh u \cosh^5 w,$$

$$\alpha_2 = b \sinh^7 u \cosh^2 u \sinh w \cosh^4 w,$$

$$\alpha_3 = a^2 \sinh^6 u \cosh^3 u \sinh^2 w \cosh w,$$

$$\alpha_4 = \sinh^9 u \cosh^5 w,$$

$$\alpha_5 = -b \sinh^8 u \cosh u \cosh^4 w \sinh w,$$

$$\alpha_6 = -[a^2 \sinh^7 u \sinh^2 w + (a^2 + b^2 \cosh^2 w) \sinh^5 u \cosh^2 u \cosh^2 w] \cosh^2 u \cosh w,$$

$$\alpha_7 = b(2a^2 + b^2 \cosh^2 w) \sinh^4 u \cosh^5 u \sinh w \cosh^2 w.$$

Then we calculate the mean curvature of the helicoidal hypersurface with spacelike axis as follows

$$H = \frac{\beta_1 \varphi'' + \beta_2 \varphi'^3 + \beta_3 \varphi'^2 + \beta_4 \varphi' + \beta_5}{3(\det \mathbf{I})^{3/2}},$$

where

$$\beta_1 = -[\sinh^6 u \cosh^3 w + (a^2 + b^2 \cosh^2 w) \sinh^4 u \cosh w] \cosh u$$

$$\begin{aligned}
\beta_2 &= +2 \sinh^5 u \cosh^3 w, \\
\beta_3 &= -b \sinh^4 u \cosh u \sinh w \cosh^2 w, \\
\beta_4 &= \sinh^7 u \cosh^3 w + (a^2 + b^2 \cosh^2 w) \sinh^5 u \cosh w - 2 \sinh^5 u \cosh^2 u \cosh^3 w \\
&\quad - 3(a^2 + b^2 \cosh^2 w) \sinh^3 u \cosh^2 u \cosh w, \\
\beta_5 &= +b \sinh^4 u \cosh^3 u \cosh^2 w \sinh w + b(2a^2 + b^2 \cosh^2 w) \sinh^2 u \cosh^3 u \sinh w.
\end{aligned}$$

Theorem 1. Let $\mathcal{D} : M^3 \rightarrow \mathbb{E}_1^4$ be an immersion given by (3.1). Then M^3 is flat if and only if

$$\alpha_1 \varphi'^2 \varphi'' + \alpha_2 \varphi' \varphi'' + \alpha_3 \varphi'' + \alpha_4 \varphi'^3 + \alpha_5 \varphi'^2 + \alpha_6 \varphi' + \alpha_7 = 0.$$

Theorem 2. Let $\mathcal{D} : M^3 \rightarrow \mathbb{E}_1^4$ be an immersion given by (3.1). Then M^3 is maximal if and only if

$$\beta_1 \varphi'' + \beta_2 \varphi'^3 + \beta_3 \varphi'^2 + \beta_4 \varphi' + \beta_5 = 0.$$

Finding solutions of these two equations are attracted problems.

Proposition 1. If \mathcal{D} is Dini-type maximal helicoidal hypersurface with spacelike axis (i.e. $H = 0$) in Minkowski 4-space, taking (as in Dini helicoidal surface in Euclidean 3-space)

$$\varphi(u) = \cosh u + \log \left(\tanh \frac{u}{2} \right),$$

then we get

$$\sum_{i=0}^6 A_i \tanh^i \left(\frac{u}{2} \right) = 0, \quad (3.3)$$

where

$$\begin{aligned}
A_6 &= -\beta_2, \\
A_5 &= 2\beta_1 + 6\beta_2 \sinh u + 2\beta_3, \\
A_4 &= (3 - 12 \sinh^2 u) \beta_2 - 8\beta_3 \sinh u - 4\beta_4, \\
A_3 &= 8\beta_1 \cosh u + (8 \sinh^3 u - 12 \sinh u) \beta_2 + (8 \sinh^2 u - 4) \beta_3 + 8\beta_4 \sinh u + 8\beta_5, \\
A_2 &= (-3 + 12 \sinh^2 u) \beta_2 + 8\beta_3 \sinh u + 4\beta_4, \\
A_1 &= -2\beta_1 + 6\beta_2 \sinh u + 2\beta_3, \\
A_0 &= \beta_2.
\end{aligned} \quad (3.4)$$

Proposition 2. If \mathcal{D} is Dini-type flat hypersurface with spacelike axis (i.e. $K = 0$) in Minkowski 4-space, taking (as in Dini helicoidal surface in Euclidean 3-space)

$$\varphi(u) = \cosh u + \log \left(\tanh \frac{u}{2} \right),$$

then we get

$$\sum_{i=0}^8 B_i \tanh^i \left(\frac{u}{2} \right) = 0, \quad (3.5)$$

where

$$\begin{aligned}
B_8 &= \alpha_1, \\
B_7 &= -4\alpha_1 \sinh u - 2\alpha_2 - 2\alpha_4, \\
B_6 &= (-2 + 4 \sinh^2 u + 4 \cosh u) \alpha_1 + 4\alpha_2 \sinh u + 4\alpha_3 + 12\alpha_4 \sinh u + 4\alpha_5, \\
B_5 &= (4 \sinh u - 16 \cosh u \sinh u) \alpha_1 + (2 - 8 \cosh u) \alpha_2 + (6 - 24 \sinh^2 u) \alpha_4 \\
&\quad - 16\alpha_5 \sinh u - 8\alpha_6, \\
B_4 &= (-8 \cosh u + 16 \cosh u \sinh^2 u) \alpha_1 + 16\alpha_2 \cosh u \sinh u + 16\alpha_3 \cosh u \\
&\quad + (16 \sinh^3 u - 24 \sinh u) \alpha_4 + (16 \sinh^2 u - 8) \alpha_5 + 16\alpha_6 \sinh u + 16\alpha_7, \\
B_3 &= (4 \sinh u + 16 \cosh u \sinh u) \alpha_1 + 2\alpha_2 + 8\alpha_2 \cosh u + (-6 + 24 \sinh^2 u) \alpha_4 \\
&\quad + 16\alpha_5 \sinh u + 8\alpha_6, \\
B_2 &= (2 - 4 \sinh^2 u + 4 \cosh u) \alpha_1 - 4\alpha_2 \sinh u - 4\alpha_3 + 12\alpha_4 \sinh u + 4\alpha_5, \\
B_1 &= -4\alpha_1 \sinh u - 2\alpha_2 + 2\alpha_4, \\
B_0 &= -\alpha_1.
\end{aligned} \quad (3.6)$$

Corollary 1. In Proposition 1, and Proposition 2, we see the following special symmetries, respectively:

$$A_6 \sim A_0, A_5 \sim A_1, A_4 \sim A_2,$$

and

$$B_8 \sim B_0, B_7 \sim B_1, B_6 \sim B_2, B_5 \sim B_3,$$

where " \sim " means the α_i ($i = 1, 2, \dots, 7$) and β_j ($j = 1, 2, \dots, 5$) term coefficients which ignored signs, respectively, are equal.

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Torus Hypersurface in 4-Space

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Abstract:

We consider torus hypersurface in the four dimensional Euclidean space \mathbb{E}^4 . We give some basic notions of \mathbb{E}^4 , then we define rotational hypersurface. Finally, we define torus hypersurface, and calculate its curvatures with some results.

1. Introduction

General rotational surfaces in the 4-dimensional Euclidean space were introduced by Moore [37, 38]. Then the topic has been studied by many geometers such as [1]-[36], [39]-[45].

Ganchev and Milousheva [24] considered the analogue of these surfaces in the Minkowski 4-space. Magid, Scharlach and Vrancken [36] introduced the affine umbilical surfaces in 4-space. considered hypersurfaces in \mathbb{E}^4 with harmonic mean curvature vector field. Scharlach [40] studied on affine geometry of surfaces and hypersurfaces in 4-space. Cheng and Wan [14] considered complete hypersurfaces of 4-space with constant mean curvature.

Güler, Magid and Yaylı [29] studied Laplace Beltrami operator of a helicoidal hypersurface in \mathbb{E}^4 . Güler, Hacısalihoğlu and Kim [26] worked on the Gauss map and the third Laplace-Beltrami operator of rotational hypersurface in \mathbb{E}^4 . Güler, Kaimakamis and Magid [27] introduced the helicoidal hypersurfaces in Minkowski 4-space \mathbb{E}_1^4 . Güler and Turgay [30] studied Cheng-Yau operator and Gauss map of rotational hypersurfaces in \mathbb{E}^4 . Moreover; Güler, Turgay and Kim [31] considered L_2 operator and Gauss map of rotational hypersurfaces in \mathbb{E}^5 . Some relations among the Laplace-Beltrami operator and curvatures of the helicoidal surfaces were shown by Güler, Yaylı and Hacısalihoğlu [32].

In this paper, we study the Torus hypersurface in Euclidean 4-space \mathbb{E}^4 . We give some basic notions of four dimensional Euclidean geometry in section 2. In section 3, we define rotational hypersurface. Moreover, we obtain Torus hypersurface, and calculate its curvatures in the last section.

2. Preliminaries

We introduce the first and second fundamental forms, matrix of the shape operator S , Gaussian curvature K , and the mean curvature H of hypersurface $\mathbf{M} = \mathbf{M}(u, v, w)$ in Euclidean 4-space \mathbb{E}^4 . In the rest of this paper, we shall identify a vector (a, b, c, d) with its transpose $(a, b, c, d)^t$.

Let \mathbf{M} be an isometric immersion of a hypersurface M^3 in \mathbb{E}^4 . The triple vector product of $\vec{x} = (x_1, x_2, x_3, x_4)$, $\vec{y} = (y_1, y_2, y_3, y_4)$, $\vec{z} = (z_1, z_2, z_3, z_4)$ on \mathbb{E}^4 is defined as follows

$$\vec{x} \times \vec{y} \times \vec{z} = \det \begin{pmatrix} e_1 & e_2 & e_3 & e_4 \\ x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \\ z_1 & z_2 & z_3 & z_4 \end{pmatrix}.$$

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For a hypersurface \mathbf{M} in \mathbb{E}^4 we have

$$\begin{aligned}\det I &= \det \begin{pmatrix} E & F & A \\ F & G & B \\ A & B & C \end{pmatrix} \\ &= (EG - F^2)C - A^2G + 2ABF - B^2E,\end{aligned}$$

and

$$\begin{aligned}\det II &= \det \begin{pmatrix} L & M & P \\ M & N & T \\ P & T & V \end{pmatrix} \\ &= (LN - M^2)V - P^2N + 2PTM - T^2L,\end{aligned}$$

where

$$A = \mathbf{M}_u \cdot \mathbf{M}_w, B = \mathbf{M}_v \cdot \mathbf{M}_w, C = \mathbf{M}_w \cdot \mathbf{M}_w,$$

$$P = \mathbf{M}_{uw} \cdot e, T = \mathbf{M}_{vw} \cdot e, V = \mathbf{M}_{ww} \cdot e,$$

e is the Gauss map (i.e. the unit normal vector field). We compute

$$\begin{pmatrix} E & F & A \\ F & G & B \\ A & B & C \end{pmatrix}^{-1} \begin{pmatrix} L & M & P \\ M & N & T \\ P & T & V \end{pmatrix},$$

and it gives the matrix of the shape operator \mathbf{S} as follows

$$\mathbf{S} = \frac{1}{\det I} \begin{pmatrix} s_{11} & s_{12} & s_{13} \\ s_{21} & s_{22} & s_{23} \\ s_{31} & s_{32} & s_{33} \end{pmatrix}, \quad (2.1)$$

where

$$\begin{aligned}s_{11} &= ABM - CFM - AGP + BFP + CGL - B^2L, \\ s_{12} &= ABN - CFN - AGT + BFT + CGM - B^2M, \\ s_{13} &= ABT - CFT - AGV + BFV + CGP - B^2P, \\ s_{21} &= ABL - CFL + AFP - BPE + CME - A^2M, \\ s_{22} &= ABM - CFM + AFT - BTE + CNE - A^2N, \\ s_{23} &= ABP - CFP + AFV - BVE + CTE - A^2T, \\ s_{31} &= -AGL + BFL + AFM - BME + GPE - F^2P, \\ s_{32} &= -AGM + BFM + AFN - BNE + GTE - F^2T, \\ s_{33} &= -AGP + BFP + AFT - BTE + GVE - F^2V.\end{aligned}$$

So, we get the following formulas of the Gaussian and the mean curvatures

$$\begin{aligned}K &= \det(\mathbf{S}) = \frac{\det II}{\det I} \\ &= \frac{(LN - M^2)V + 2MPT - P^2N - T^2L}{(EG - F^2)C + 2ABF - A^2G - B^2E},\end{aligned}$$

and

$$\begin{aligned}H &= \frac{1}{3} \text{tr}(\mathbf{S}) \\ &= \frac{1}{3 \det I} [(EN + GL - 2FM)C + (EG - F^2)V \\ &\quad - A^2N - B^2L - 2(APG + BTE - ABM - ATF - BPF)].\end{aligned}$$

A hypersurface \mathbf{M} is minimal if $H = 0$ identically on \mathbf{M} .

3. Rotational Hypersurface

For an open interval $I \subset \mathbb{R}$, let $\gamma: I \rightarrow \Pi$ be a curve in a plane Π in \mathbb{E}^4 , and let ℓ be a straight line in Π . A *rotational hypersurface* in \mathbb{E}^4 is defined as a hypersurface rotating a curve γ around a line ℓ (are called the *profile curve* and the *axis*, respectively).

We may suppose that ℓ is the line spanned by the vector $(0,0,0,1)^t$. The orthogonal matrix which fixes the above vector is

$$Z(v, w) = \begin{pmatrix} \cos v \cos w & -\sin v & -\cos v \sin w & 0 \\ \sin v \cos w & \cos v & -\sin v \sin w & 0 \\ \sin w & 0 & \cos w & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad (3.1)$$

where $v, w \in \mathbb{R}$. The matrix Z supplies the following equations

$$Z\ell = \ell, \quad Z^t Z = ZZ^t = I_4, \quad \det Z = 1,$$

simultaneously. When the axis of rotation is ℓ , there is an Euclidean transformation by which the axis is ℓ transformed to the x_4 -axis of \mathbb{E}^4 . Profile curve is given by

$$\gamma(u) = (f(u), 0, 0, \varphi(u)),$$

where $f(u), \varphi(u) : I \subset \mathbb{R} \rightarrow \mathbb{R}$ are C^∞ functions for all $u \in I$. So, the rotational hypersurface spanned by the vector $(0,0,0,1)$ is as follows

$$\mathbf{R}(u, v, w) = Z(v, w)\gamma(u)^t,$$

where $u \in I, v, w \in [0, 2\pi)$. Clearly, we write rotational hypersurface as follows

$$\mathbf{R}(u, v, w) = \begin{pmatrix} f(u) \cos v \cos w \\ f(u) \sin v \cos w \\ f(u) \sin w \\ g(u) \end{pmatrix}. \quad (3.2)$$

4. Torus Hypersurface

In \mathbb{E}^4 , taking profile curve

$$\gamma(u) = (a + r \cos u, 0, 0, r \sin u)$$

with the orthogonal matrix Z , then we obtain torus hypersurface as follows

$$\mathfrak{T}(u, v, w) = \begin{pmatrix} (a + r \cos u) \cos v \cos w \\ (a + r \cos u) \sin v \cos w \\ (a + r \cos u) \sin w \\ r \sin u \end{pmatrix}, \quad (4.1)$$

where $a, u \in \mathbb{R} \setminus \{0\}$ and $0 \leq v, w \leq 2\pi$.

By using the first differentials of (4.1) with respect to u, v, w , we get the first quantities as follows

$$I = \begin{pmatrix} r^2 & 0 & 0 \\ 0 & \beta \cos^2 w & 0 \\ 0 & 0 & \beta \end{pmatrix},$$

where $\beta = r^2 \cos^2 u + 2ar \cos u + a^2$. Taking the second differentials with respect to u, v, w , we have the second quantities as follows

$$II = \begin{pmatrix} -r & 0 & 0 \\ 0 & -\lambda \cos^2 w & 0 \\ 0 & 0 & -\lambda \end{pmatrix}.$$

where $\lambda = (a + r \cos u) \cos u$. The Gauss map of the torus hypersurface is

$$e_{\mathfrak{T}} = \begin{pmatrix} \cos u \cos v \cos w \\ \cos u \sin v \cos w \\ \cos u \sin w \\ \sin u \end{pmatrix}. \quad (4.2)$$

Finally, the Gaussian curvature of the torus hypersurface is as follows

$$K = -\frac{\cos^2 u}{r(a + r \cos u)^2},$$

and the mean curvature is as follows

$$H = -\frac{a + 3r \cos u}{3r(a + r \cos u)},$$

Corollary 1. Let $\varpi : M^3 \rightarrow \mathbb{E}^4$ be an immersion given by (4.1). Then M^3 is minimal if and only if

$$a + 3r \cos u = 0.$$

Proof. Solutions of the above eq. are as follows

$$\begin{cases} \delta_1 \cup \delta_2 & \text{if } r \neq 0, \\ \mathbb{C} & \text{if } a = 0 \wedge r = 0, \\ \emptyset & \text{if } a \neq 0 \wedge r = 0. \end{cases}$$

where $\delta_1 = \{\pi - \arccos(\frac{a}{3r}) + 2\pi k \mid k \in \mathbb{Z}\}$, $\delta_2 = \{-\pi + \arccos(\frac{a}{3r}) + 2\pi k \mid k \in \mathbb{Z}\}$.

Corollary 2. Let $\varpi : M^3 \rightarrow \mathbb{E}^4$ be an immersion given by (4.1). Then M^3 is flat hypersurface (i.e. $K = 0$) if and only if

$$u = \left\{ \frac{1}{2}\pi + \pi k \mid k \in \mathbb{Z} \right\}.$$

Corollary 3. Let $\varpi : M^3 \rightarrow \mathbb{E}^4$ be an immersion given by (4.1). Then M^3 has following relation

$$3H \cos^2 u - (a + r \cos u)(a + 3r \cos u)K = 0.$$

Proof. Solutions of a for the above eq. are as follows

$$\begin{cases} \emptyset & \text{if } 3H \cos^2 u \neq 0 \wedge K = 0, \\ \pm \left(\frac{\sqrt{K^2 r^2 + 3HK} - 2Kr}{K} \right) \cos u & \text{if } K \neq 0. \end{cases}$$

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On Quasi-Lacunary Invariant Convergence of Sequences of Sets

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Abstract: In this study, we give definitions of Wijsman quasi-lacunary invariant convergence, Wijsman strongly quasi-lacunary invariant convergence and Wijsman quasi-lacunary invariant statistically convergence for sequences of sets. We also examine the existence of some relations among these definitions and some convergence types for sequences of sets given in [7, 14], too.

1. INTRODUCTION AND BACKGROUNDS

The concept of statistical convergence was firstly introduced by Fast [4] and this concept has been studied by Šalát [18], Fridy [5] and many others, too.

A sequence $x = (x_k)$ is statistically convergent to L if for every $\varepsilon > 0$

$$\lim_{n \rightarrow \infty} \frac{1}{n} \left| \{k \leq n : |x_k - L| \geq \varepsilon\} \right| = 0,$$

where the vertical bars indicate the number of elements in the enclosed set.

By a lacunary sequence we mean an increasing integer sequence $\theta = \{k_r\}$ such that $k_0 = 0$ and $h_r = k_r - k_{r-1} \rightarrow \infty$ as $r \rightarrow \infty$. Throughout this study the intervals determined by θ will be denoted by $I_r = (k_{r-1}, k_r]$.

Then, Fridy and Orhan [6] defined lacunary statistical convergence of a sequence using the lacunary sequence concept as follows:

Let $\theta = \{k_r\}$ be a lacunary sequence. A sequence $x = (x_k)$ is lacunary statistically convergent to L if for every $\varepsilon > 0$,

$$\lim_{r \rightarrow \infty} \frac{1}{h_r} \left| \{k \in I_r : |x_k - L| \geq \varepsilon\} \right| = 0.$$

Several authors have studied on the concepts of invariant mean and invariant convergent (see, [9–11, 17, 19, 22]). Let σ be a mapping of the positive integers into themselves. A continuous linear functional ϕ on ℓ_∞ , the space of real bounded sequences, is said to be an invariant mean or a σ -mean if it satisfies following conditions:

1. $\phi(x) \geq 0$, when the sequence $x = (x_n)$ has $x_n \geq 0$ for all n ,
2. $\phi(e) = 1$, where $e = (1, 1, 1, \dots)$ and
3. $\phi(x_{\sigma(n)}) = \phi(x_n)$ for all $x \in \ell_\infty$.

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The mappings σ are assumed to be one-to-one and such that $\sigma^m(n) \neq n$ for all positive integers n and m , where $\sigma^m(n)$ denotes the m th iterate of the mapping σ at n . Thus, ϕ extends the limit functional on c , the space of convergent sequences, in the sense that $\phi(x) = \lim x$ for all $x \in c$. In the case σ is translation mappings $\sigma(n) = n + 1$, the σ -mean is often called a Banach limit.

The space of lacunary strong σ -convergent sequences L_θ was defined by Savaş [20] as below:

$$L_\theta = \left\{ x = (x_k) : \lim_{r \rightarrow \infty} \frac{1}{h_r} \sum_{k \in I_r} |x_{\sigma^k(m)} - L| = 0, \text{ uniformly in } m \right\}.$$

Pancaroglu and Nuray [15] introduced the concept of lacunary invariant summability as follows:

Let $\theta = \{k_r\}$ be a lacunary sequence. A sequence $x = (x_k)$ is said to be lacunary invariant summable to L if

$$\lim_{r \rightarrow \infty} \frac{1}{h_r} \sum_{k \in I_r} x_{\sigma^k(m)} = L,$$

uniformly in m .

The concept of lacunary σ -statistically convergent sequence was defined by Savaş and Nuray in [21] as below:

Let $\theta = \{k_r\}$ be a lacunary sequence. A sequence $x = (x_k)$ is $S_{\sigma\theta}$ -convergent to L if for every $\varepsilon > 0$

$$\lim_{r \rightarrow \infty} \frac{1}{h_r} \left| \{k \in I_r : |x_{\sigma^k(m)} - L| \geq \varepsilon\} \right| = 0,$$

uniformly in m .

Let X be any non-empty set and \mathbb{N} be the set of natural numbers. The function

$$f : \mathbb{N} \rightarrow P(X)$$

is defined by $f(k) = A_k \in P(X)$ for each $k \in \mathbb{N}$, where $P(X)$ is power set of X . The sequence $\{A_k\} = (A_1, A_2, \dots)$, which is the range's elements of f , is said to be sequences of sets.

Let (X, ρ) be a metric space. For any point $x \in X$ and any non-empty subset A of X , the distance from x to A is defined by

$$d(x, A) = \inf_{a \in A} \rho(x, a).$$

Throughout the paper we take (X, ρ) as a metric space and A, A_k as any non-empty closed subsets of X . There are different convergence notions for sequence of sets. One of them handled in this paper is the concept of Wijsman convergence (see, [1–3, 12, 16, 25, 26]).

A sequence $\{A_k\}$ is said to be Wijsman convergent to A if for each $x \in X$,

$$\lim_{k \rightarrow \infty} d(x, A_k) = d(x, A)$$

and denoted by $A_k \xrightarrow{w} A$.

A sequence $\{A_k\}$ is said to be bounded if for each $x \in X$, $\sup_k \{d(x, A_k)\} < \infty$.

The set of all bounded sequences of sets is denoted by L_∞ .

The concepts of Wijsman lacunary summability, Wijsman strongly lacunary summability and Wijsman lacunary statistical convergence were introduced by Ulusu and Nuray [23, 24].

Using the invariant mean concept, the concepts of Wijsman lacunary invariant convergence, Wijsman strongly lacunary invariant convergence and Wijsman lacunary invariant statistical convergence were also defined by Pancaroglu and Nuray [16] as follows:

Let $\theta = \{k_r\}$ be a lacunary sequence. A sequence $\{A_k\}$ is said to be Wijsman lacunary invariant convergent to A if for each $x \in X$,

$$\lim_{r \rightarrow \infty} \frac{1}{h_r} \sum_{k \in I_r} d(x, A_{\sigma^k(m)}) = d(x, A)$$

uniformly in m .

A sequence $\{A_k\}$ is said to be Wijsman strongly lacunary invariant convergent to A if for each $x \in X$,

$$\lim_{r \rightarrow \infty} \frac{1}{h_r} \sum_{k \in I_r} |d(x, A_{\sigma^k(m)}) - d(x, A)| = 0$$

uniformly in m .

A sequence $\{A_k\}$ is said to be Wijsman lacunary invariant statistically convergent to A if for every $\varepsilon > 0$ and each $x \in X$,

$$\lim_{r \rightarrow \infty} \frac{1}{h_r} \left| \{k \in I_r : |d(x, A_{\sigma^k(m)}) - d(x, A)| \geq \varepsilon\} \right| = 0$$

uniformly in m .

The idea of quasi-almost convergence in a normed space was introduced by Hajduković [8]. Then, Nuray [13] studied concepts of quasi-invariant convergence and quasi-invariant statistical convergence in a normed space. Recently, Gülle and Ulusu [7] introduced the concept of Wijsman strongly quasi-invariant convergence for sequences of sets as below:

A sequence $\{A_k\}$ is said to be Wijsman strongly quasi-invariant convergent to A if for each $x \in X$,

$$\lim_{p \rightarrow \infty} \frac{1}{p} \sum_{k=0}^{p-1} |d_x(A_{\sigma^k(np)}) - d_x(A)| = 0$$

uniformly in n where $d_x(A_{\sigma^k(np)}) = d(x, A_{\sigma^k(np)})$ and $d_x(A) = d(x, A)$. It is denoted by $A_k \xrightarrow{[WQV\sigma]} A$.

2. MAIN RESULTS

In this study, we give definitions of Wijsman quasi-lacunary invariant convergence, Wijsman strongly quasi-lacunary invariant convergence and Wijsman quasi-lacunary invariant statistically convergence for sequences of sets. We also examine the existence of some relations among these definitions and some convergence types for sequences of sets given in [7, 14], too.

Definition 2.1 Let $\theta = \{k_r\}$ be a lacunary sequence. A sequence $\{A_k\}$ is said to be Wijsman quasi-lacunary invariant convergent to A if for each $x \in X$,

$$\lim_{r \rightarrow \infty} \left| \frac{1}{h_r} \sum_{k \in I_r} d_x(A_{\sigma^k(nr)}) - d_x(A) \right| = 0$$

uniformly in n . In this case, we write $A_k \xrightarrow{WQV_{\sigma}^{\theta}} A$.

Theorem 2.2 If a sequence $\{A_k\}$ is Wijsman lacunary invariant convergent to A , then $\{A_k\}$ is Wijsman quasi-lacunary invariant convergent to A .

Proof. Suppose that the sequence $\{A_k\}$ is Wijsman lacunary invariant convergent to A . Then, for each $x \in X$ and every $\varepsilon > 0$ there exists an integer $r_0 > 0$ such that for all $r > r_0$

$$\left| \frac{1}{h_r} \sum_{k \in I_r} d_x(A_{\sigma^k(m)}) - d_x(A) \right| < \varepsilon,$$

for all m . If m is taken as $m = nr$, then we have

$$\left| \frac{1}{h_r} \sum_{k \in I_r} d_x(A_{\sigma^k(nr)}) - d_x(A) \right| < \varepsilon,$$

for all n . Since $\varepsilon > 0$ is an arbitrary, the limit is taken for $r \rightarrow \infty$ we can write

$$\left| \frac{1}{h_r} \sum_{k \in I_r} d_x(A_{\sigma^k(nr)}) - d_x(A) \right| \rightarrow 0$$

for all n . That is, the sequence $\{A_k\}$ is Wijsman quasi-lacunary invariant convergent to A . \square

Definition 2.3 Let $\theta = \{k_r\}$ be a lacunary sequence. A sequence $\{A_k\}$ is Wijsman quasi-lacunary invariant statistically convergent to A if for every $\varepsilon > 0$ and each $x \in X$,

$$\lim_{r \rightarrow \infty} \frac{1}{h_r} \left| \left\{ k \in I_r : |d_x(A_{\sigma^k(nr)}) - d_x(A)| \geq \varepsilon \right\} \right| = 0$$

uniformly in n . In this case, we write $A_k \xrightarrow{WQS_{\sigma}^{\theta}} A$.

Theorem 2.4 If a sequence $\{A_k\}$ is Wijsman lacunary invariant statistically convergent to A , then $\{A_k\}$ is Wijsman quasi-lacunary invariant statistically convergent to A .

Proof. Suppose that the sequence $\{A_k\}$ is Wijsman lacunary invariant statistically convergent to A . In this case, when $\delta > 0$ is given, for each $x \in X$ and for every $\varepsilon > 0$ there exists an integer $r_0 > 0$ such that for all $r > r_0$

$$\frac{1}{h_r} \left| \left\{ k \in I_r : |d_x(A_{\sigma^k(m)}) - d_x(A)| \geq \varepsilon \right\} \right| < \delta,$$

for all m .

If m is taken as $m = nr$, then we have

$$\frac{1}{h_r} \left| \{k \in I_r : |d_x(A_{\sigma^k(nr)}) - d_x(A)| \geq \varepsilon\} \right| < \delta,$$

for all n . Since $\delta > 0$ is an arbitrary, we have

$$\lim_{r \rightarrow \infty} \frac{1}{h_r} \left| \{k \in I_r : |d_x(A_{\sigma^k(nr)}) - d_x(A)| \geq \varepsilon\} \right| = 0,$$

for all n which means that $\{A_k\}$ is Wijsman quasi-lacunary invariant statistically convergent to A . \square

Definition 2.5 Let $\theta = \{k_r\}$ be a lacunary sequence. A sequence $\{A_k\}$ is Wijsman strongly quasi-lacunary invariant convergent to A if for each $x \in X$,

$$\lim_{r \rightarrow \infty} \frac{1}{h_r} \sum_{k \in I_r} |d_x(A_{\sigma^k(nr)}) - d_x(A)| = 0$$

uniformly in n . In this case, we write $A_k \xrightarrow{[WQV_{\sigma\theta}]} A$.

Theorem 2.6 For any lacunary sequence $\theta = \{k_r\}$,

$$A_k \xrightarrow{[WQV_{\sigma\theta}]} A \Leftrightarrow A_k \xrightarrow{[WQV_{\sigma}]} A.$$

Proof. Let $A_k \xrightarrow{[WQV_{\sigma\theta}]} A$ and $\varepsilon > 0$ is given. Then, there exists an integer r_0 such that for each $x \in X$

$$\frac{1}{h_r} \sum_{k=0}^{h_r-1} |d_x(A_{\sigma^k(nr)}) - d_x(A)| < \varepsilon$$

for $r \geq r_0$ and $nr = k_{r-1} + 1 + w$, $w \geq 0$. Let $p \geq h_r$. Thus, p can be written as $p = \alpha \cdot h_r + \theta$ where $0 \leq \theta \leq h_r$ and α is an integer. Since $p \geq h_r$, $\alpha \geq 1$. Then,

$$\begin{aligned} \frac{1}{p} \sum_{k=0}^{p-1} |d_x(A_{\sigma^k(np)}) - d_x(A)| &\leq \frac{1}{p} \sum_{k=0}^{(\alpha+1)h_r-1} |d_x(A_{\sigma^k(nr)}) - d_x(A)| \\ &= \frac{1}{p} \sum_{j=0}^{\alpha} \sum_{k=jh_r}^{(j+1)h_r-1} |d_x(A_{\sigma^k(nr)}) - d_x(A)| \\ &\leq \frac{1}{p} \varepsilon h_r (\alpha + 1) \\ &\leq \frac{2\alpha h_r \varepsilon}{p} \quad (\alpha \geq 1). \end{aligned}$$

For $\frac{h_r}{p} \leq 1$ and since $\frac{\alpha h_r}{p} \leq 1$

$$\frac{1}{p} \sum_{k=0}^{p-1} |d_x(A_{\sigma^k(np)}) - d_x(A)| \leq 2\varepsilon,$$

that is, $A_k \xrightarrow{[WQV_{\sigma}]} A$.

Let $A_k \xrightarrow{[WQV_{\sigma}]} A$ and $\varepsilon > 0$ is given. Then, there exists $P > 0$ such that for each $x \in X$

$$\frac{1}{p} \sum_{k=0}^{p-1} |d_x(A_{\sigma^k(np)}) - d_x(A)| < \varepsilon$$

for all $p > P$. Since $\theta = \{k_r\}$ is a lacunary sequence, a number $R > 0$ can be chosen such that $h_r > P$ where $r \geq R$. Thereby

$$\frac{1}{h_r} \sum_{k \in I_r} |d_x(A_{\sigma^k(nr)}) - d_x(A)| < \varepsilon,$$

that is, $A_k \xrightarrow{[WQV_{\sigma\theta}]} A$. The proof of theorem is completed. \square

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On Inclusion Theorems For Absolute Cesàro Summability Methods

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Abstract:

In this study, we give necessary and sufficient conditions in order that $|C, -1|_k \Rightarrow |C, \alpha|$, $k > 1$ for the case $\alpha > -1$, so we also complete some open problems in this concept.

1. Introduction

Let Σx_n be an infinite series with partial sum s_n , and by (σ_n^α) and (u_n^α) we denote the n -th Cesàro means of order α with $\alpha > -1$ of the sequences (s_n) and (nx_n) , respectively, i.e.,

$$\sigma_n^\alpha = \frac{1}{A_n^\alpha} \sum_{v=0}^n A_{n-v}^{\alpha-1} s_v$$

and

$$u_n^\alpha = \frac{1}{A_n^\alpha} \sum_{v=0}^n A_{n-v}^{\alpha-1} v x_v \quad (1.1)$$

where $A_0^\alpha = 1$, $A_n^\alpha = \binom{\alpha+n}{n}$, $A_n^\alpha = 0$, $n \geq 1$. The series Σx_n is said to be summable $|C, \alpha|_k$, $k \geq 1$, if (see [1])

$$\sum_{n=1}^{\infty} n^{k-1} |\sigma_n^\alpha - \sigma_{n-1}^\alpha|^k < \infty. \quad (1.2)$$

On the other hand, by the well known identity $u_n^\alpha = n(\sigma_n^\alpha - \sigma_{n-1}^\alpha)$ [4], the condition (1.2) can be stated by

$$\sum_{n=1}^{\infty} \frac{1}{n} |u_n^\alpha|^k < \infty.$$

Note that Thorpe [12] gave the Cesàro summability for $\alpha = -1$ as follows. If the series to sequence transformation

$$T_n = \sum_{v=0}^{n-1} x_v + (n+1)x_n \quad (1.3)$$

tends to s as n tends to infinity, then the series Σx_n is summable by Cesàro summability $(C, -1)$ to the number s [12].

Also, by the definition of Sarigöl [9] and Thorpe [12], the series Σx_n is said to be summable $|C, -1|_k$ (see [2]) if

$$\sum_{n=1}^{\infty} n^{k-1} |T_n - T_{n-1}|^k < \infty.$$

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In this context the series spaces $|C_\alpha|_k, k \geq 1$, have been defined as the set of all series summable by the absolute Cesàro summability method $|C, \alpha|_k$ in [7] and [2] for $\alpha > -1$ and $\alpha = -1$, respectively.

If A and B are methods of summability, B is said to include A (written $A \Rightarrow B$) if every series summable by the method A is also summable by the method B . A and B are said to be equivalent (written $A \Leftrightarrow B$) if each methods includes the other.

Problems on inclusion dealing absolute Cesàro mean summability were investigated in detail by several authors [3,5,6,8,10,11], and some well known results have recently been extended by Sarigöl [8] and Sarigöl & Hazar [3].

In this study, we give necessary and sufficient conditions in order that $|C, -1|_k \Rightarrow |C, \alpha|, k > 1$ for the case $\alpha > -1$, which completes some open problems in literature.

We require the following lemmas for our investigations.

Throughout this paper, k^* denote the conjugate of $k > 1$, i.e., $1/k + 1/k^* = 1$, and $1/k^* = 0$ for $k = 1$.

Lemma 1.1. Let $1 < k < \infty$. Then, $A(x) \in \ell$ whenever $x \in \ell_k$ if and only if

$$\sum_{v=0}^{\infty} \left(\sum_{n=0}^{\infty} |a_{nv}| \right)^{k^*} < \infty,$$

where $\ell_k = \{x = (x_v) : \Sigma |x_v|^k < \infty\}$ [8].

2. Main Results

The aim of this study is to prove the following theorem.

Theorem 2.1. Let $k > 1$. Then, $|C, -1|_k \Rightarrow |C, \alpha|$ if and only if

$$\sum_{r=1}^{\infty} \left(\sum_{n=r}^{\infty} \left| \frac{r^{1/k}}{nA_n^\alpha} \sum_{v=r}^n \frac{A_{n-v}^{\alpha-1}}{v+1} \right| \right)^{k^*} < \infty \quad (2.1)$$

Proof. Let define u_n^α and T_n by (1.1) and (1.3) respectively. Using the definition u_n^α and T_n , we define the sequences $y = (y_n)$ and $\tilde{y} = (\tilde{y}_n)$ by

$$y_n = n^{1/k^*} ((n+1)x_n - (n-1)x_{n-1}), n \geq 1 \text{ and } y_0 = x_0 \quad (2.2)$$

and

$$\tilde{y}_n = \frac{u_n^\alpha}{n} = \frac{1}{nA_n^\alpha} \sum_{v=1}^n A_{n-v}^{\alpha-1} v x_v, n \geq 1 \text{ and } y_0 = x_0$$

respectively.

Then, $|C, -1|_k \Rightarrow |C, \alpha|$ iff $\tilde{y} \in \ell$ whenever $y \in \ell_k$. By inversion of (2.2), we write for $n \geq 1$

$$x_n = \frac{1}{n(n+1)} \sum_{v=1}^n v^{1/k} y_v \quad (2.3)$$

Hence, by (2.3) we get for $n \geq 1$

$$\begin{aligned} \tilde{y}_n &= \frac{1}{nA_n^\alpha} \sum_{v=1}^n A_{n-v}^{\alpha-1} v x_v = \frac{1}{nA_n^\alpha} \sum_{v=1}^n A_{n-v}^{\alpha-1} v \frac{1}{v(v+1)} \sum_{r=1}^v r^{1/k} y_r \\ &= \frac{1}{nA_n^\alpha} \sum_{r=1}^n \left(\sum_{v=r}^n \frac{A_{n-v}^{\alpha-1}}{(v+1)} \right) r^{1/k} y_r \\ &= \sum_{r=1}^n c_{nr} y_r \end{aligned}$$

where

$$c_{nr} = \begin{cases} \frac{r^{1/k}}{nA_n^\alpha} \sum_{v=r}^n \frac{A_{n-v}^{\alpha-1}}{v+1}, & 1 \leq r \leq n \\ 0, & r > n. \end{cases}$$

So $\tilde{y} \in \ell$ whenever $y \in \ell_k$ if and only if

$$\sum_{r=1}^{\infty} \left(\sum_{n=r}^{\infty} \left| \frac{r^{1/k}}{nA_n^\alpha} \sum_{v=r}^n \frac{A_{n-v}^{\alpha-1}}{v+1} \right| \right)^{k^*} < \infty.$$

by Lemma 1.1 or, equivalently, (2.1) holds. Thus the proof is completed.

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On Mizoguchi-Takahashi's Type Set Valued $(\alpha-\theta)$ Contractions

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Abstract:

We introduce a new concept for set-valued contractions as Mizoguchi-Takahashi's type set-valued $(\alpha-\theta)$ contractions and present some fixed point results. Our results extend and generalize many fixed point theorems in the literature.

1. Introduction and Preliminaries

Let (X, d) be a metric space. We denote by $\mathcal{C}(X)$ the family of all nonempty closed subsets of X , by $\mathcal{K}(X)$ the family of all nonempty compact subsets of X and by $\mathcal{CB}(X)$ the family of all nonempty, closed and bounded subsets of X . It is well known that $\mathcal{K}(X) \subseteq \mathcal{CB}(X) \subseteq \mathcal{C}(X)$. Let H be the Pompeiu-Hausdorff metric on $\mathcal{CB}(X)$, that is, for $A, B \in \mathcal{CB}(X)$

$$H(A, B) = \max \left\{ \sup_{x \in A} D(x, B), \sup_{y \in B} D(y, A) \right\}$$

where $D(x, B) = \inf \{d(x, y) : y \in B\}$. H also is called generalized Pompeiu-Hausdorff distance on $\mathcal{C}(X)$

Taking into account the Pompeiu-Hausdorff metric, Nadler [19] in 1969 initiated the idea for multivalued contraction mapping and extended the Banach contraction principle to multivalued mappings and proved the following:

Theorem 1.1 (Nadler [19]) *Let (X, d) be a complete metric space and $T : X \rightarrow \mathcal{CB}(X)$ multivalued contraction, that is, there exists $L \in [0, 1)$ such that*

$$H(Tx, Ty) \leq Ld(x, y)$$

for all $x, y \in X$. Then T has a fixed point in X .

Later on, several researches were conducted on a variety of generalizations, extensions and applications of this result of Nadler (see[3, 5, 6, 13, 17]). Furthermore, the following theorem was proved by Mizoguchi and Takahashi [17] that is, in fact, a partial answer of question of Reich [22]:

Theorem 1.2 ([17]) *Let (X, d) be a complete metric space and $T : X \rightarrow \mathcal{CB}(X)$ is a mapping such that*

$$H(Tx, Ty) \leq k(d(x, y))d(x, y),$$

for all $x, y \in X$, $x \neq y$, where $k : (0, \infty) \rightarrow [0, 1)$ is a function that satisfies

$$\limsup_{t \rightarrow s^+} k(t) < 1 \text{ for all } s \geq 0.$$

Then T has a fixed point in X .

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We can find both a simple proof of Theorem 1.2 and an example showing that it is real generalization of Nadler's in [24]. We can also find a lot of generalizations of Mizoguchi-Takahashi's fixed point theorem in the literature [3, 4, 6].

Moreover, an attracted generalization of the Banach contraction principle given by Jleli and Samet [12], introduced a new type of contractive condition, which throughout this study, we shall call it as θ -contraction. Now, we recall basic definitions, relevant notions and some related results concerning θ -contraction. Let $\theta : (0, \infty) \rightarrow (1, \infty)$ be a function. Next we will consider the following properties for θ :

- (θ_1) θ is nondecreasing;
- (θ_2) For each sequence $\{t_n\} \subset (0, \infty)$, $\lim_{n \rightarrow \infty} \theta(t_n) = 1$ and $\lim_{n \rightarrow \infty} t_n = 0^+$ are equivalent;
- (θ_3) There exist $r \in (0, 1)$ and $l \in (0, \infty]$ such that $\lim_{t \rightarrow 0^+} \frac{\theta(t)-1}{t^r} = l$;
- (θ_4) $\theta(\inf A) = \inf \theta(A)$ for all $A \subset (0, \infty)$ with $\inf A > 0$.

We denote by Θ and Ω be the set of all functions θ satisfying (θ_1)-(θ_3) and (θ_1)-(θ_4), respectively. It is clear that $\Omega \subset \Theta$. Some examples of the functions belonging Ω are $\theta_1(t) = e^{\sqrt{t}}$ and $\theta_2(t) = e^{\sqrt{te^t}}$. If we define as $\theta_3(t) = \begin{cases} e^{\sqrt{t}} & , t < 1 \\ 9 & , t \geq 1 \end{cases}$, then we can see $\theta_3 \in \Theta \setminus \Omega$. Note that, if a function θ satisfies (θ_1), then it satisfies (θ_4) if and only if it is right continuous.

By considering the conditions (θ_1)-(θ_3), Jleli and Samet [12] introduced the concept of θ -contraction, which is more general than Banach contraction. Let (X, d) be a metric space and $\theta \in \Theta$. A mapping $T : X \rightarrow X$ is said to be a θ -contraction if there exists a constant $k \in [0, 1)$ such that

$$\theta(d(Tx, Ty)) \leq [\theta(d(x, y))]^k, \quad (1.1)$$

for all $x, y \in X$ with $d(Tx, Ty) > 0$. As a real generalization of Banach contraction principle, Jleli and Samet proved that every θ -contraction on a complete metric space has a unique fixed point. In addition, from (θ_1) and (1.1), it is easy to concluded that every θ -contraction T is a contractive mapping, i.e., $d(Tx, Ty) < d(x, y)$ for all $x, y \in X$ with $Tx \neq Ty$. Thus, every θ -contraction mapping on a metric space is continuous. Then, Hançer et al. [10] extended the concept of θ -contraction to set-valued case and Minak and Altun [16] introduced the nonlinear case of it as follows: Let (X, d) be a metric space and $T : X \rightarrow \mathcal{CB}(X)$ be given a mapping. Then, (i) T is said to be a multivalued θ -contraction with $\theta \in \Theta$ if there exists $k \in (0, 1)$ such that

$$\theta(H(Tx, Ty)) \leq [\theta(d(x, y))]^k,$$

for all $x, y \in X$ with $H(Tx, Ty) > 0$.

(ii) T is said to be a multivalued nonlinear θ -contraction with $\theta \in \Theta$ if there exists a function $k : (0, \infty) \rightarrow [0, 1)$ such that

$$\limsup_{t \rightarrow s^+} k(t) < 1, \quad \forall s \geq 0,$$

satisfying

$$\theta(H(Tx, Ty)) \leq [\theta(d(x, y))]^{k(d(x, y))},$$

for all $x, y \in X$ with $H(Tx, Ty) > 0$.

Therefore, considering the class Ω , the following theorems are provided.

Theorem 1.3 ([10]) *Let (X, d) be a complete metric space, and $T : X \rightarrow \mathcal{CB}(X)$ be a multivalued θ -contraction with $\theta \in \Omega$. Then T has a fixed point in X .*

Theorem 1.4 ([16]) *Let (X, d) be a complete metric space, and $T : X \rightarrow \mathcal{CB}(X)$ be a multivalued nonlinear θ -contraction with $\theta \in \Omega$. Then T has a fixed point in X .*

Furthermore, the fixed point results for these type mappings are given several researches (see [1, 7, 8]).

On the other hand, Samet et al [23] introduced the concept of α - ψ -contractive and α -admissible mappings and established various fixed point theorems for such mappings on complete metric spaces. Asl et al [2] also defined the notion of α -admissibility and α_* -admissibility for multivalued mappings as follows: Let (X, d) be a metric space, $T : X \rightarrow \mathcal{P}(X)$ be a mapping and $\alpha : X \times X \rightarrow [0, \infty)$ be a function. We say that T is an α -admissible mapping whenever for each $x \in X$ and $y \in Tx$ with $\alpha(x, y) \geq 1$ implies $\alpha(y, z) \geq 1$ for all $z \in Ty$ and T is an α_* -admissible mapping whenever for each $x \in X$ and $y \in Tx$ with $\alpha(x, y) \geq 1$ implies $\alpha_*(Tx, Ty) \geq 1$, where $\alpha_*(Tx, Ty) = \inf\{\alpha(a, b) : a \in Tx, b \in Ty\}$. It is clear that α_* -admissible mapping is also α -admissible, but the converse may not be true as shown in Example 15 of [14]. We say that α has (B) property whenever $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$ and $x_n \rightarrow x$, then $\alpha(x_n, x) \geq 1$ for all $n \in \mathbb{N}$.

Consider the collection Ψ of nondecreasing functions $\psi : [0, \infty) \rightarrow [0, \infty)$ such that $\sum_{n=1}^{\infty} \psi^n(t) < \infty$ for all $t > 0$, where ψ^n is the n th iterate of ψ . It is clear that for each $\psi \in \Psi$, we have $\psi(t) < t$ for all $t > 0$ and $\psi(0) = 0$. Let $T : X \rightarrow \mathcal{CB}(X)$. Then,

i) T is said to be multivalued α - ψ -contractive whenever

$$\alpha(x, y)H(Tx, Ty) \leq \psi(d((x, y)))$$

for all $x, y \in X$,

ii) T is said to be multivalued α_* - ψ -contractive whenever

$$\alpha_*(Tx, Ty)H(Tx, Ty) \leq \psi(d((x, y)))$$

for all $x, y \in X$.

Asl, Rezapour and Shahzad [2] and Mohammadi, Rezapour and Shahzad [18] presented the following fixed point theorems for multivalued α - ψ -contractive and multivalued α_* - ψ -contractive mappings.

Theorem 1.5 ([2]) *Let (X, d) be a complete metric space, $\alpha : X \times X \rightarrow [0, \infty)$ be a function, $\psi \in \Psi$ be a strictly increasing map and $T : X \rightarrow \mathcal{CB}(X)$ be an α -admissible and α - ψ -contractive multifunction. Suppose that there exist $x_0 \in X$ and $x_1 \in Tx_0$ such that $\alpha(x_0, x_1) \geq 1$. If T is continuous or α has (B) property, then T has a fixed point.*

Theorem 1.6 ([18]) *Let (X, d) be a complete metric space, $\alpha : X \times X \rightarrow [0, \infty)$ be a function, $\psi \in \Psi$ be a strictly increasing map and $T : X \rightarrow \mathcal{CB}(X)$ be an α_* -admissible and α_* - ψ -contractive multifunction. Suppose that there exist $x_0 \in X$ and $x_1 \in Tx_0$ such that $\alpha(x_0, x_1) \geq 1$. If T is continuous or α has (B) property, then T has a fixed point.*

In this paper, taking into account the above results, we present some new fixed point results for multivalued θ -contractions, by considering the α -admissibility and α_* -admissibility of a multivalued mappings on complete metric spaces.

2. Main Results

Before we give our main results, we recall the following: Let X and Y be two topological spaces. Then, a multivalued mapping $T : X \rightarrow \mathcal{P}(Y)$ is said to be upper semicontinuous (lower semicontinuous) if the inverse image of closed sets (open sets) is closed (open). A multivalued mapping is continuous if it is upper as well as lower semicontinuous.

Lemma 2.1 ([11]) *Let (X, d) be a metric space and $T : X \rightarrow \mathcal{P}(X)$ be an upper semicontinuous mapping such that Tx is closed for all $x \in X$. If $x_n \rightarrow x_0$, $y_n \rightarrow y_0$ and $y_n \in Tx_n$, then $y_0 \in Tx_0$.*

Let (X, d) be a metric space, $T : X \rightarrow \mathcal{CB}(X)$ and $\alpha : X \times X \rightarrow [0, \infty)$ be two mappings. Define a set

$$S_{T, \alpha} = \{(x, y) : \alpha(x, y) \geq 1 \text{ and } H(Tx, Ty) > 0\} \subset X \times X.$$

Given $\theta \in \Theta$, we say that T is a \mathcal{MT} -type multivalued $(\alpha\text{-}\theta)$ -contraction if there exists a function $k : (0, \infty) \rightarrow [0, 1)$ satisfying

$$\limsup_{t \rightarrow s^+} k(t) < 1, \text{ for all } s \geq 0$$

such that

$$\theta(H(Tx, Ty)) \leq [\theta(d(x, y))]^{k(d(x, y))} \quad (2.1)$$

for all $(x, y) \in S_{T, \alpha}$.

Now we present our main result.

Theorem 2.2 *Let (X, d) be a complete metric space and $T : X \rightarrow \mathcal{H}(X)$ be an α -admissible and \mathcal{MT} -type multivalued $(\alpha\text{-}\theta)$ -contraction. Suppose that there exist $x_0 \in X$ and $x_1 \in Tx_0$ such that $\alpha(x_0, x_1) \geq 1$. If T is upper semicontinuous or α has (B) property, then T has a fixed point.*

Proof. Suppose that T has no fixed point. Then for all $x \in X$, $D(x, Tx) > 0$. Let x_0 and x_1 be as mentioned in the hypothesis, then $H(Tx_0, Tx_1) > 0$ (otherwise $D(x_1, Tx_1) = 0$, this is a contradiction). Therefore $(x_0, x_1) \in S_{T, \alpha}$, thus we can use the condition (2.1) for x_0 and x_1 . Then considering $(\theta)_1$, we have

$$\theta(D(x_1, Tx_1)) \leq \theta(H(Tx_0, Tx_1)) \leq [\theta(d(x_0, x_1))]^{k(d(x_0, x_1))}. \quad (2.2)$$

Since Tx_1 is compact, there exists $x_2 \in Tx_1$ such that $d(x_1, x_2) = D(x_1, Tx_1)$. From (2.2),

$$\theta(d(x_1, x_2)) \leq \theta(H(Tx_0, Tx_1)) \leq [\theta(d(x_0, x_1))]^{k(d(x_0, x_1))}.$$

Also, since T is an α -admissible mapping $\alpha(x_1, x_2) \geq 1$. Again, since $x_2 \in Tx_1$, then $H(Tx_1, Tx_2) > 0$. Therefore $(x_1, x_2) \in S_{T, \alpha}$, thus we can use (2.1) for x_1 and x_2 . Then

$$\theta(D(x_2, Tx_2)) \leq \theta(H(Tx_1, Tx_2)) \leq [\theta(d(x_1, x_2))]^{k(d(x_1, x_2))}.$$

Since Tx_2 is compact, there exists $x_3 \in Tx_2$ such that $d(x_2, x_3) = D(x_2, Tx_2)$. Therefore, we have

$$\theta(d(x_2, x_3)) \leq \theta(H(Tx_1, Tx_2)) \leq [\theta(d(x_1, x_2))]^{k d(x_1, x_2)}.$$

By induction, we can find a sequence $\{x_n\}$ in X such that $x_{n+1} \in Tx_n$, $(x_n, x_{n+1}) \in S_{T, \alpha}$ and

$$\theta(d(x_n, x_{n+1})) \leq [\theta(d(x_n, x_{n-1}))]^{k d(x_{n-1}, x_n)} \quad (2.3)$$

for all $n \in \mathbb{N}$. Thus, from (θ_1) the sequence $\{d(x_n, x_{n+1})\}$ is decreasing and hence convergent. From (2.3), there exists $b \in (0, 1)$ and $n_0 \in \mathbb{N}$ such that $k(d(x_n, x_{n+1})) < b$ for all $n \geq n_0$. Thus, we obtain for all $n \geq n_0$,

$$\begin{aligned} 1 &< \theta(d(x_n, x_{n+1})) \\ &\leq [\theta(d(x_{n-1}, x_n))]^{k d(x_{n-1}, x_n)} \\ &\leq [\theta(d(x_{n-2}, x_{n-1}))]^{k d(x_{n-1}, x_n) k d(x_{n-1}, x_n)} \\ &\quad \vdots \\ &\leq [\theta(d(x_0, x_1))]^{k d(x_0, x_1) \cdots k d(x_{n-1}, x_n) k d(x_{n-1}, x_n)} \\ &= [\theta(d(x_0, x_1))]^{k d(x_0, x_1) \cdots k d(x_{n_0-1}, x_{n_0}) k d(x_{n_0}, x_{n_0+1}) \cdots k d(x_{n-1}, x_n) k d(x_{n-1}, x_n)} \\ &\leq [\theta(d(x_0, x_1))]^{k d(x_{n_0}, x_{n_0+1}) \cdots k d(x_{n-1}, x_n) k d(x_{n-1}, x_n)} \\ &\leq [\theta(d(x_0, x_1))]^{b^{(n-n_0)}}. \end{aligned}$$

Hence, we obtain

$$1 < \theta(d(x_n, x_{n+1})) \leq [\theta(d(x_0, x_1))]^{b^{(n-n_0)}} \quad (2.4)$$

for all $n \geq n_0$. Letting $n \rightarrow \infty$ in (2.4), we have

$$\lim_{n \rightarrow \infty} \theta(d(x_n, x_{n+1})) = 1. \quad (2.5)$$

From (θ_2) , $\lim_{n \rightarrow \infty} d(x_n, x_{n+1}) = 0^+$ and so from (θ_3) there exist $r \in (0, 1)$ and $l \in (0, \infty]$ such that

$$\lim_{n \rightarrow \infty} \frac{\theta(d(x_n, x_{n+1})) - 1}{[d(x_n, x_{n+1})]^r} = l.$$

Suppose that $l < \infty$. In this case, let $B = \frac{1}{2} > 0$. From the definition of the limit, there exists $n_0 \in \mathbb{N}$ such that, for all $n \geq n_0$,

$$\left| \frac{\theta(d(x_n, x_{n+1})) - 1}{[d(x_n, x_{n+1})]^r} - l \right| \leq B.$$

This implies that, for all $n \geq n_0$,

$$\frac{\theta(d(x_n, x_{n+1})) - 1}{[d(x_n, x_{n+1})]^r} \geq l - B = B.$$

Then, for all $n \geq n_0$,

$$n [d(x_n, x_{n+1})]^r \leq An [\theta(d(x_n, x_{n+1})) - 1],$$

where $A = 1/B$.

Suppose now that $l = \infty$. Let $B > 0$ be an arbitrary positive number. From the definition of the limit, there exists $n_0 \in \mathbb{N}$ such that, for all $n \geq n_0$,

$$\frac{\theta(d(x_n, x_{n+1})) - 1}{[d(x_n, x_{n+1})]^r} \geq B.$$

This implies that, for all $n \geq n_0$,

$$n [d(x_n, x_{n+1})]^r \leq An [\theta(d(x_n, x_{n+1})) - 1],$$

where $A = 1/B$.

Thus, in all cases, there exist $A > 0$ and $n_0 \in \mathbb{N}$ such that, for all $n \geq n_0$,

$$n [d(x_n, x_{n+1})]^r \leq An [\theta(d(x_n, x_{n+1})) - 1].$$

Using (2.4), we obtain, for all $n \geq n_0$,

$$n [d(x_n, x_{n+1})]^r \leq An \left[[\theta(d(x_0, x_1))]^{b^{(n-n_0)}} - 1 \right].$$

Letting $n \rightarrow \infty$ in the above inequality, we obtain

$$\lim_{n \rightarrow \infty} n [d(x_n, x_{n+1})]^r = 0.$$

Thus, there exists $n_1 \in \mathbb{N}$ such that $n [d(x_n, x_{n+1})]^r \leq 1$ for all $n \geq n_1$. So, we have, for all $n \geq n_1$

$$d(x_n, x_{n+1}) \leq \frac{1}{n^{1/r}}. \quad (2.6)$$

In order to show that $\{x_n\}$ is a Cauchy sequence, consider $m, n \in \mathbb{N}$ such that $m > n \geq n_1$. Using the triangular inequality for the metric and from (2.6), we have

$$\begin{aligned} d(x_n, x_m) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{m-1}, x_m) \\ &= \sum_{i=n}^{m-1} d(x_i, x_{i+1}) \leq \sum_{i=n}^{m-1} \frac{1}{i^{1/r}} \leq \sum_{i=n}^{\infty} \frac{1}{i^{1/r}}. \end{aligned}$$

By the convergence of the series $\sum_{i=1}^{\infty} \frac{1}{i^{1/r}}$, letting to limit $n \rightarrow \infty$, we get $d(x_n, x_m) \rightarrow 0$. This yields that $\{x_n\}$ is a Cauchy sequence in (X, d) . Since (X, d) is a complete metric space, there exists $z \in X$ such that $\lim_{n \rightarrow \infty} x_n = z$. If T is upper semicontinuous, then by Lemma (2.1) we have $z \in Tz$, which is a contradiction. Now assume that α has (B) property. Since $\lim_{n \rightarrow \infty} x_n = z$ and $D(z, Tz) > 0$, then there exists $n_0 \in \mathbb{N}$ such that $D(x_{n+1}, Tz) > 0$ for all $n \geq n_0$. Therefore for all $n \geq n_0$

$$H(Tx_n, Tz) > 0,$$

thus $(x_n, z) \in S_{T, \alpha}$ for all $n \geq n_0$. From (2.1) and (θ_1) , we have

$$\begin{aligned} \theta(D(x_{n+1}, Tz)) &\leq \theta(H(Tx_n, Tz)) \\ &\leq [\theta(d(x_n, z))]^{k(d(x_n, z))} \end{aligned}$$

and so

$$D(x_{n+1}, Tz) \leq d(x_n, z)$$

for all $n \geq n_0$. Passing to limit $n \rightarrow \infty$, we obtain $D(z, Tz) = 0$, which is a contradiction. Therefore T has a fixed point in X . \square

We cannot extend the range of T to $\mathcal{CB}(X)$ in Theorem 2.2 with the same conditions. Example 1 in [10] shows this fact. However, we can take $\mathcal{CB}(X)$ instead of $\mathcal{H}(X)$ by adding condition (θ_4) on θ .

Theorem 2.3 *Let (X, d) be a complete metric space and $T : X \rightarrow \mathcal{CB}(X)$ be an α -admissible and \mathcal{MT} -type multivalued $(\alpha-\theta)$ -contraction with $\theta \in \Omega$. Suppose there exist $x_0 \in X$ and $x_1 \in Tx_0$ such that $\alpha(x_0, x_1) \geq 1$. If T is upper semicontinuous or α has (B) property, then T has a fixed point.*

Proof. We begin as in the proof of Theorem 2.2. Considering the condition (θ_4) , we can write

$$\theta(D(x_1, Tx_1)) = \inf_{y \in Tx_1} \theta(d(x_1, y)).$$

Thus from

$$\theta(D(x_1, Tx_1)) \leq \theta(H(Tx_0, Tx_1))$$

we have

$$\begin{aligned} \inf_{y \in Tx_1} \theta(d(x_1, y)) &\leq [\theta(d(x_0, x_1))]^{k(d(x_0, x_1))} \\ &< [\theta(d(x_0, x_1))]^{\frac{k(d(x_0, x_1))+1}{2}} \end{aligned}$$

Therefore there exists $x_2 \in Tx_1$ such that

$$\theta(d(x_1, x_2)) \leq [\theta(d(x_0, x_1))]^{k(d(x_0, x_1))}.$$

The rest of the proof can be completed as in the proof of Theorem 2.2. \square

Remark 2.4 If we take $\alpha(x, y) = 1$ in Theorem 2.3, we obtain Theorem 1.3.

Remark 2.5 By taking $\alpha(x, y) = 1$ and $\theta(t) = e^{\sqrt{t}}$ in Theorem 2.3, we obtain the famous Mizoguchi-Takahashi's fixed point theorem.

Since α_* -admissible mapping is also α -admissible, we can obtain following corollary.

Corollary 2.6 *Let (X, d) be a complete metric space and $T : X \rightarrow \mathcal{H}(X)$ be an α_* -admissible and \mathcal{MT} -multivalued $(\alpha-\theta)$ -contraction. Suppose that there exist $x_0 \in X$ and $x_1 \in Tx_0$ such that $\alpha(x_0, x_1) \geq 1$. If T is upper semicontinuous or α has (B) property, then T has a fixed point.*

3. Application in partially ordered metric spaces

Recently, there have been so many interesting developments in fixed point theory in metric spaces endowed with a partial order. The first result in this direction was given by Ran and Reurings [21] where they extended the Banach contraction principle in partially ordered sets with some application to a matrix equation. Later, many important results have been obtained in this direction (see [15, 20]). In this section, we will present some results about this direction. In 2004, Feng and Liu [9] defined relations between two sets. Let X be a nonempty set and \preceq be a partial order on X . Let A, B be two nonempty subsets of X , the relations between A and B are defined as follows:

- (a) $A \prec_1 B \Leftrightarrow$ for every $a \in A$, there exists $b \in B$ such that $a \preceq b$,
- (b) $A \prec_2 B \Leftrightarrow$ for every $b \in B$, there exists $a \in A$ such that $a \preceq b$,
- (c) $A \prec B \Leftrightarrow A \prec_1 B$ and $A \prec_2 B$.

\prec_1 and \prec_2 are different relations between A and B . For example, let $X = \mathbb{R}$, $A = [\frac{1}{2}, 1]$, $B = [0, 1]$, \preceq be usual order on X , then $A \prec_1 B$ but $A \not\prec_2 B$; if $A = [0, 1]$, $B = [0, \frac{1}{2}]$, then $A \prec_2 B$ while $A \not\prec_1 B$. \prec_1 , \prec_2 and \prec are reflexive and transitive, but are not antisymmetric. For instance, let $X = \mathbb{R}$, $A = [0, 3]$, $B = [0, 1] \cup [2, 3]$, \preceq be usual order on X , then $A \prec B$ and $B \prec A$, but $A \neq B$. Hence, they are not partial orders. Note that if A is a nonempty subset of X with $A \prec_1 A$, then A is singleton. (see [9]).

Theorem 3.1 Let (X, \preceq) be a partially ordered set and suppose that there exist a metric d in X such that (X, d) is complete metric space. Let $T : X \rightarrow \mathcal{CB}(X)$ (resp. $\mathcal{X}(X)$) be an upper semicontinuous multivalued mapping such that

$$\theta(H(Tx, Ty)) \leq [\theta(d(x, y))]^{k(d(x, y))}$$

for all $(x, y) \in S_{\preceq}$, where $\theta \in \Omega$ (resp. $\theta \in \Theta$) and $k : (0, \infty) \rightarrow [0, 1)$ be a function satisfying

$$\limsup_{t \rightarrow s^+} k(t) < 1, \text{ for all } s \geq 0$$

and $S_{\preceq} = \{(x, y) \in X \times X : x \preceq y \text{ and } H(Tx, Ty) > 0\}$. Assume that for each $x \in X$ and $y \in Tx$ with $x \preceq y$, we have $y \preceq z$ for all $z \in Ty$ and there exist $x_0 \in X$, $x_1 \in Tx_0$ such that $\{x_0\} \prec_1 Tx_0$, then T has a fixed point.

Proof. Define a mapping $\alpha : X \times X \rightarrow [0, \infty)$ by

$$\alpha(x, y) = \begin{cases} 1 & , \quad x \preceq y \\ 0 & , \quad \text{otherwise} \end{cases}$$

Then $S_{\preceq} = S_{T, \alpha}$. Therefore T is \mathcal{MT} -multivalued $(\alpha-\theta)$ -contraction. Also, since $\{x_0\} \prec_1 Tx_0$, then there exists $x_1 \in Tx_0$ such that $x_0 \preceq x_1$ and so $\alpha(x_0, x_1) \geq 1$. Now let $x \in X$ and $y \in Tx$ with $\alpha(x, y) \geq 1$, then $x \preceq y$ and so by the hypotheses we have $y \preceq z$ for all $z \in Ty$. Therefore, $\alpha(y, z) \geq 1$ for all $z \in Ty$. This shows that T is α -admissible. Therefore, from Theorem 2.3 (resp. Theorem 2.2), T has a fixed point in X . \square

Remark 3.2 We can give similar result using \prec_2 instead of \prec_1 .

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Novel Contour Surfaces To The (2+1)-Dimensional Date-Jimbo-Kashiwara-Miwa Equation

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Abstract:

In this manuscript, improved Bernoulli sub-equation function method based on the Bernoulli differential method is considered. This method is based on the converting the (2+1)-dimensional Date-Jimbo-Kashiwara-Miwa equation into ordinary differential equation. Some new solutions such as complex and exponential are obtained. To better understanding of physical meanings of model are introduced by plotting two- and three-dimensional surfaces along with contour simulations. Finally, a conclusion is presented by mentioning important acquisitions founded in this study.

1. Introduction

Developing of the computational programs and tools, the works on the nonlinear media and applications have been taken attentions of scientists and experts. Some important models have been submitted to the literature. Moreover, many methods have been revised or improved for solving these special models. For example, the generalized Bernoulli sub-ODE method has been studied by B. Zheng [1]. Newly modified Riccati-Bernoulli equation method has been firstly submitted by X.F. Yang and at al [2]. Hirota method and auxiliary variable, and so on have been used to find new results [3].

This paper is constructed by the following sections. In Section 2, we give a brief introduction to the powerful newly improved Bernoulli sub-equation function method (IBSEFM). The complex travelling wave solutions of the (2+1)-dimensional Date-Jimbo-Kashiwara-Miwa equation (DJKME), which it is defined as

$$u_{xxxxy} + 4u_{xy}u_x + 2u_{xxx}u_y + 6u_{xy}u_{xx} + u_{yyy} - 2u_{xxt} = 0, \quad (1)$$

are been obtained in section 3 [4]. Finally, a comprehensively conclusion are given in section 4.

2. Description of IBSEFM

IBSEFM formed by extending the Bernoulli sub-equation function method [5-16] will be given in this subsection. Therefore, we consider the follows.

Step 1. We consider the following partial differential equation;

$$P(u, u_x, u_y, u_t, \dots) = 0. \quad (2)$$

and take the wave transformation;

$$u(x, y, t) = U(\eta), \eta = \mu(x + \alpha y - kt). \quad (3)$$

where μ, α, k are constants and can be determined later. By substituting Eq.(3), Eq.(2) converts a nonlinear ordinary differential equation (NODE) as following;

$$N(U, U', U'', U''', \dots) = 0. \quad (4)$$

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Step 2. Considering trial equation of solution in Eq.(4), it can be written as following;

$$U(\eta) = \frac{\sum_{i=0}^n a_i F^i(\eta)}{\sum_{j=0}^m b_j F^j(\eta)} = \frac{a_0 + a_1 F(\eta) + a_2 F^2(\eta) + \dots + a_n F^n(\eta)}{b_0 + b_1 F(\eta) + b_2 F^2(\eta) + \dots + b_m F^m(\eta)}. \quad (5)$$

According to the Bernoulli theory, we can consider the general form of Bernoulli differential equation for F' as following;

$$F' = wF + \lambda F^M, w \neq 0, \lambda \neq 0, M \in \mathbb{R} - \{0, 1, 2\}. \quad (6)$$

where $F = F(\eta)$ is Bernoulli differential polynomial. Substituting above relations in Eq.(4), it yields us an equation of polynomial $\Omega(F)$ of F as following;

$$\Omega(F) = \rho_s F^s + \dots + \rho_1 F + \rho_0 = 0. \quad (7)$$

According to the balance principle, we can determine the relationship between n , m and M .

Step 3. The coefficients of $\Omega(F)$ all be zero will yield us an algebraic system of equations;

$$\rho_i = 0, i = 0, \dots, s. \quad (8)$$

Solving this system, we will specify the values of a_0, a_1, \dots, a_n and b_0, b_1, \dots, b_n .

Step 4. When we solve nonlinear Bernoulli differential equation Eq.(2.6), we obtain the following two situations according to b and d ,

$$F(\eta) = \left[\frac{-\lambda}{w} + \frac{E}{e^{w(M-1)\eta}} \right]^{\frac{1}{1-M}}, w \neq \lambda. \quad (9)$$

$$F(\eta) = \left[\frac{(E-1) + (E+1)\tanh(w(1-M)\frac{\eta}{2})}{1 - \tanh(w(1-M)\frac{\eta}{2})} \right], w = \lambda, E \in \mathbb{R}. \quad (10)$$

Using a complete discrimination system for polynomial of F , we solve this system with the help of computer programming and classify the exact solutions to Eq.(4).

3. Implementations of the Method

In this subsection of manuscript, we apply the method to the DJKME.

Example-1 First of all, if we perform travelling wave transformation into the Eq.(1) in the following manner;

$$u(x, y, t) = U(\xi), \xi = \mu x + \alpha \mu y - \mu k t, \quad (11)$$

where μ, α, k are real constants, we get the following nonlinear equation;

$$a\mu^5 U^{(5)} + 6a\mu^4 \frac{d^3 U}{d\xi^3} \frac{dU}{d\xi} + (a^3 + 2k)\mu^3 \frac{d^3 U}{d\xi^3} + 6a\mu^4 \left(\frac{d^2 U}{d\xi^2} \right)^2 = 0. \quad (12)$$

After some simplifications and calculations along with integrations, we can reach the following model

$$a\mu^5 \frac{d^3 U}{d\xi^3} + 3a\mu^4 \left(\frac{dU}{d\xi} \right)^2 + (a^3 + 2k)\mu^3 \frac{dU}{d\xi} = 0. \quad (13)$$

For simplicity, if we reconsider in Eq.(13)

$$V = \frac{dU}{d\xi}, \quad (14)$$

we can rewrite Eq.(13) along with some easily calculation as following;

$$a\mu^2 \frac{d^2 V}{d\xi^2} + 3a\mu V^2 + (a^3 + 2k)V = 0. \quad (15)$$

With balance principle, we obtain following relationship for m, n and M ;

$$2M + m = n + 2. \quad (16)$$

This resolution procedure is applied and we obtain results as follows;

Case 1.

If we take $M = 3, m = 2$ and $n = 6$ in Eq.(16), then, we can write following trial solutions form as;

$$V = \frac{a_0 + a_1F + a_2F^2 + a_3F^3 + a_4F^4 + a_5F^5 + a_6F^6}{b_0 + b_1F + b_2F^2} = \frac{\Upsilon}{\Psi}, \quad (17)$$

$$\frac{dV}{d\xi} = \frac{\frac{d\Upsilon}{d\xi}\Psi - \Upsilon\frac{d\Psi}{d\xi}}{\Psi^2},$$

and

$$\frac{d^2V}{d\xi^2} = \frac{\frac{d^2\Upsilon}{d\xi^2}\Psi - \Upsilon\frac{d^2\Psi}{d\xi^2}}{\Psi^2} - \dots, \quad (18)$$

⋮

where $\frac{dF}{d\xi} = wF + \lambda F^3, a_6 \neq 0, b_2 \neq 0$. When we use Eqs.(17,19) in the Eq.(15), we get a system of algebraic equations from the coefficients of polynomial of F . By solving this system of equations with the help of some computational programs such as Mathematica, Matlab and Maple, it yields us the following coefficients;

Case.1.1. For $w \neq \lambda$ we can consider the coefficients as following;

$$\begin{aligned} a_0 &= \frac{-(a^3 + 2k)b_0}{3a\mu}, a_1 = \frac{-(a^3 + 2k)b_1}{3a\mu}, a_2 = \frac{-4\sqrt{(a^3 + 2k)\lambda}b_0}{\sqrt{a}} - \frac{(a^3 + 2k)b_2}{3a\mu}, \\ a_3 &= \frac{-4\sqrt{(a^3 + 2k)\lambda}b_1}{\sqrt{a}}, a_4 = -8\lambda^2\mu b_0 - \frac{4\sqrt{(a^3 + 2k)\lambda}b_2}{\sqrt{a}} \\ a_5 &= -8\lambda^2\mu b_1, a_6 = -8\lambda^2\mu b_2, w = \frac{\sqrt{(a^3 + 2k)}}{2\mu\sqrt{a}}. \end{aligned} \quad (19)$$

Substituting Eq.(19) into Eq.(17) along with Eq.(14), we obtain the following exponential function solution to Eq.(1) in the following form;

$$\begin{aligned} V_1(\xi) &= \frac{a^3 + 2k}{3a\mu} + 8\lambda^2\mu \left(ce^{-\frac{\sqrt{a^3+2k}}{\sqrt{a\mu}}\xi} - \frac{2\mu\lambda\sqrt{a}}{\sqrt{a^3+2k}} \right)^{-2} + \frac{4\lambda\sqrt{a}\sqrt{a^3+2k}}{3a} \\ &\quad - \left(ce^{-\frac{\sqrt{a^3+2k}}{\sqrt{a\mu}}\xi} - \frac{2\mu\lambda\sqrt{a}}{\sqrt{a^3+2k}} \right)^{-1}. \end{aligned} \quad (20)$$

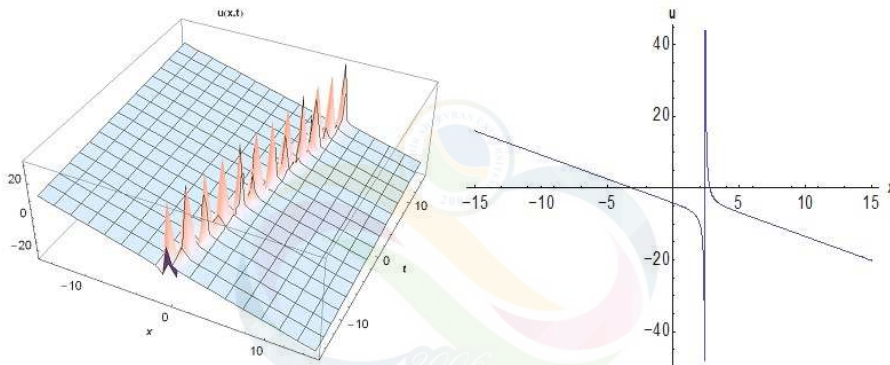


Fig. 1. The 2D and 3D surfaces of Eq.(20) for $a = 0.1, c = 3, d = 4, \mu = 0.1, \lambda = k = 0.2, y = 2, -15 < x < 15, -15 < t < 15$ and $t = 0.85$ for the 2D graphic.

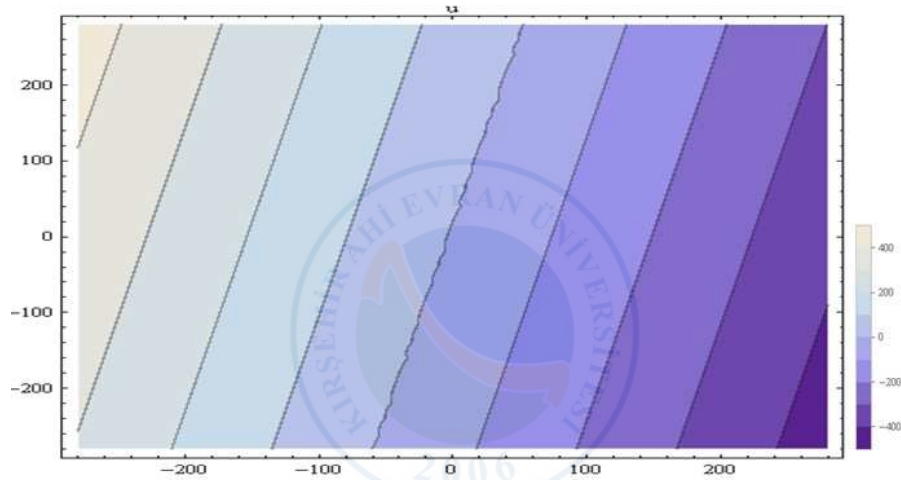


Fig. 2. Contour surfaces of Eq.(20) for $a = 0.1, c = 3, d = 4, \mu = 0.1, \lambda = k = 0.2, y = 2, -280 < x < 280, -280 < t < 280$.

Case.1.2. For $w \neq \lambda$ we can consider the coefficients as following;

$$a_0 = a_1 = 0, a_2 = \frac{4i\sqrt{(a^3+2k)}\lambda b_0}{\sqrt{a}}, a_3 = \frac{4i\sqrt{a^3+2k}\lambda b_1}{\sqrt{a}}, w = \frac{-i\sqrt{a^3+2k}}{2\mu\sqrt{a}}$$

$$a_4 = 4\lambda(-2\lambda\mu b_0 + \frac{i\sqrt{a^3+2k}\lambda b_2}{\sqrt{a}}), a_5 = -8\lambda^2\mu b_1, a_6 = -8\lambda^2\mu b_2, \quad (21)$$

Taking Eq.(21) into Eq.(17) along with Eq.(14), we find the following complex exponential function solution to Eq.(1);

$$V_2(\xi) = -8\lambda^2\mu \left(ce^{\frac{i\sqrt{a^3+2k}}{\sqrt{a\mu}}\xi} - \frac{2i\mu\lambda\sqrt{a}}{\sqrt{a^3+2k}} \right)^{-2} + \frac{4i\lambda\sqrt{a}\sqrt{a^3+2k}}{\sqrt{a}} \left(ce^{\frac{i\sqrt{a^3+2k}}{\sqrt{a\mu}}\xi} - \frac{2i\mu\lambda\sqrt{a}}{\sqrt{a^3+2k}} \right)^{-1}, \quad (22)$$

where λ, μ, c, a, k are real constant and non-zero.

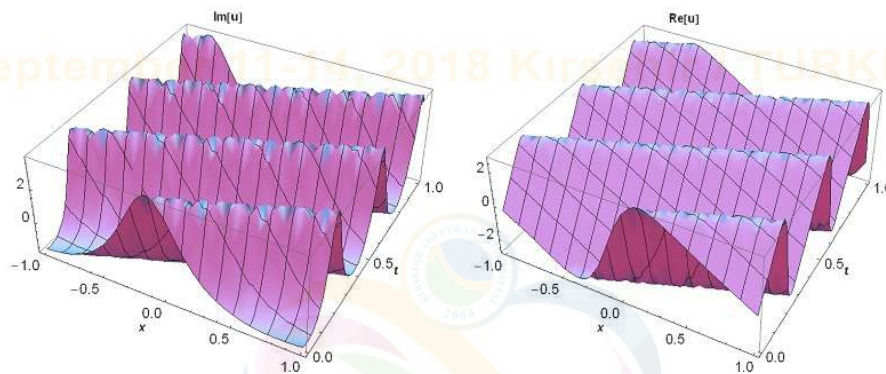


Fig. 3. The 3D surfaces of Eq.(22) for $a = 1, c = 2, \mu = 3, \lambda = 0.4, k = 5, y = 0.6, -1 < x < 1, 0 < t < 1$

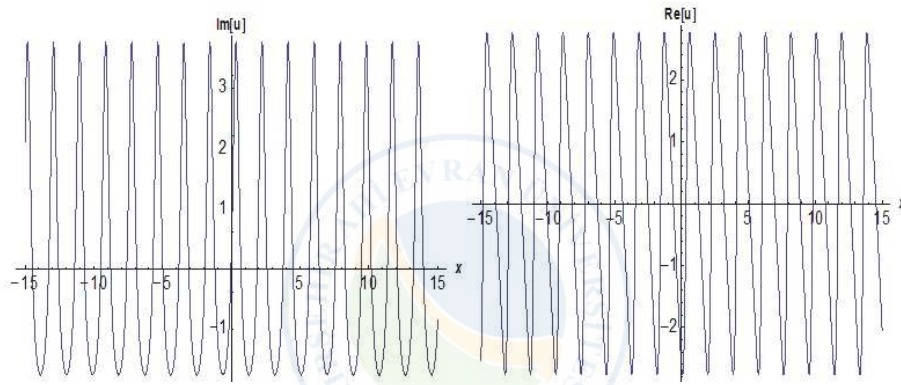


Fig. 4. 2D graphs of Eq.(22) for $a = 1, c = 2, \mu = 3, \lambda = 0.4, k = 5, y = 0.6, t = 0.85, -15 < x < 15$.

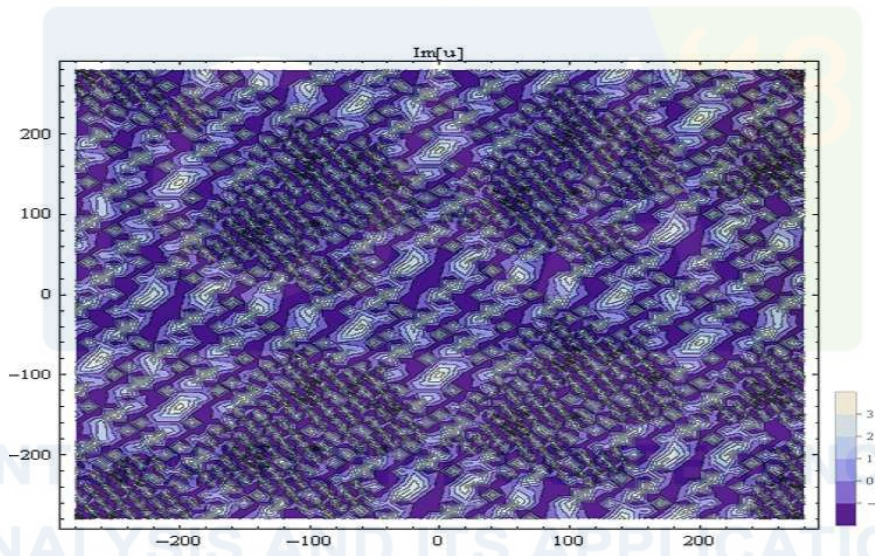


Fig. 5. Contour surfaces of imaginary part of Eq.(22) for $a = 1, c = 2, \mu = 3, \lambda = 0.4, k = 5, y = 0.6, -280 < x < 280, -280 < t < 280$.

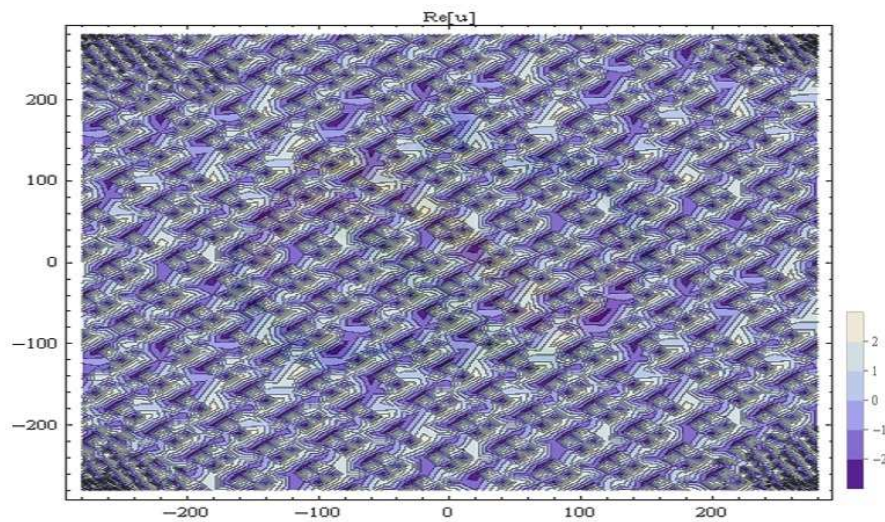


Fig. 6. Contour surfaces of real part of Eq.(22) for $a = 1, c = 2, \mu = 3, \lambda = 0.4, k = 5, y = 0.6, -280 < x < 280, -280 < t < 280$.

4. Conclusions

In this paper, we take use of the IBSEFM to obtain several results of DJKME with the help of some computational programs such as Maple and Mathematica. These travelling wave solutions such as exponential and complex function solutions have been obtained. Two- and three-dimensional surfaces along with the contour simulations for both results have been also plotted by the same program. It can be seen that the IBSEFM is a simple, powerful and original mathematical tool to find the exact solutions of the nonlinear system, and it can be also extended to solve other nonlinear models, especially higher dimension nonlinear models and the coupled nonlinear partial differential equations.

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On The Roots of An Evolution Equation

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Bernoulli sub-equation
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S-Integrable evolution
equation,
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Exponential roots..

MSC: 00A05, 00A69,
00A72, 12H20, 34A25

Abstract: In this paper, we apply Bernoulli sub-equation function method to the model which reads as 'S-Integrable' evolution equation. Complex and exponential functional roots are obtained. Plotting two- and three-dimensional surfaces along with contour simulations give rise to more sophisticated information about the model. At the end of the paper, we present a conclusion by giving vital informations about the surfaces of roots.

1. Introduction

With the more sophisticated and deeper investigated tools, the studies conducted on the nonlinear evolution equations (NLEEs) often leads to new findings in applied science. In recent decade, many sophisticated NLEEs have been submitted literature. In particular, Kundu A. has proposed a higher-order nonlinear system [1]. G. Zhang and his team have established a new mathematical structure of acoustic wake effect in aerosol acoustic agglomeration [2]. Another novel model arising in natural gas science and engineering is belong to the R. Ming and et al [3]. They have newly developed a model for developing the prediction of liquid loading in horizontal gas wells. Moreover, they have conducted deeper investigation along with case study. Recently, Jun Bi and his team have presented a new model to determine the thermal conductivity of fine-grained soils [4]. They have studied on a three-parameter model to calculate the thermal conductivity. Thermal conductivity problems, one of the most important real world problems arising in environment, earth science, and engineering applications, include vital parameters for mankind. The world has witnessed a giant natural disease being Tsunami generated by earthquakes in 2011 in Japan. In this sense, Chunga K and Toulkeridis T have presented a scientific work for the first evidence of tsunami as a major historic event [5]. When such real world problems are modelled, mankind takes advantage of these diseases by converting it useful one. This is only possible by investigating more and deeper in real bases. In this sense, all natural problems can be symbolized by using various NLEEs. Moreover, many tools different studies to obtain the roots and to the better understanding physical properties such mathematical models have been developed [6-22]. This paper includes the following sections. We give general properties of Bernoulli sub-equation function method (BSEFM) in section 2. The complex and exponential travelling wave solutions to the 'S-Integrable' evolution equation (SIEE), which it is defined as [23]

$$u_t - a_1 u_{xxx} + a_2 u_{xxxx} = -6a_1 \epsilon u u_x + 10a_2 (\epsilon u u_{xxx} + 2\epsilon u_x u_{xx} - 3\epsilon^2 u^2 u_x), \quad (1)$$

are obtained in section 3. Finally, a comprehensively conclusion are given in section 4.

2. Description of BSEFM

BSEFM will be given in this sub-section [24, 25]. Therefore, we consider the following steps.

Step 1. We consider the following partial differential equation;

$$P(u, u_x, u_{xx}, u_t, \dots) = 0. \quad (2)$$

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and take the wave transformation;

$$u(x,t) = U(\eta), \eta = kx - ct. \quad (3)$$

where c, k are constants and can be determined later. By substituting Eq.(3), Eq.(2) converts a nonlinear ordinary differential equation (NODE) as following;

$$N(U, U', U'', U''', \dots) = 0. \quad (4)$$

Step 2. Considering trial equation of solution in Eq.(4), it can be written as following;

$$U(\eta) = \sum_{i=0}^m b_i F^i(\eta) = b_0 + b_1 F(\eta) + b_2 F^2(\eta) + \dots + b_m F^m(\eta). \quad (5)$$

According to the Bernoulli theory, we can consider the general form of Bernoulli differential equation for F' as following;

$$F' = wF + \lambda F^M, w \neq 0, \lambda \neq 0, M \in \mathbb{R} - \{0, 1, 2\}. \quad (6)$$

where $F = F(\eta)$. Substituting above relations in Eq.(4), yields an equation of polynomia $\Omega(F)$ of F as following;

$$\Omega(F) = \rho_s F^s + \dots + \rho_1 F + \rho_0 = 0. \quad (7)$$

According to the balance principle, we can determine the relationship between m and M .

Step 3. Let the coefficients of $\Omega(F)$ all be zero will yield us an algebraic system of equations;

$$\rho_i = 0, i = 0, \dots, s. \quad (8)$$

Solving this system, we will specify the values of b_0, b_1, \dots, b_m .

Step 4. When we solve nonlinear Bernoulli differential equation Eq.(6), we obtain the following two situations according to w and λ ,

$$F(\eta) = \left[\frac{-\lambda}{w} + \frac{E}{e^{w(M-1)\eta}} \right]^{\frac{1}{1-M}}, w \neq \lambda. \quad (9)$$

$$F(\eta) = \left[\frac{(E-1) + (E+1)\tanh(w(1-M)\frac{\eta}{2})}{1 - \tanh(w(1-M)\frac{\eta}{2})} \right]^{\frac{1}{1-M}}, w = \lambda, E \in \mathbb{R}. \quad (10)$$

Using a complete discrimination system for polynomial of F , we solve this system with the help of computer programming and classify the exact solutions to Eq.(4).

3. Application of BSEFM

In this subsection of manuscript, we apply the method to the SIEE.

Example-1 Conducting the travelling wave transformation into the Eq.(1) in the following manner;

$$u(x,t) = U(\xi), \xi = kx - ct, \quad (11)$$

where c, k are real constants, it can be obtained the following nonlinear equation;

$$\begin{aligned} a_2 k^5 U^{(5)} - a_1 k^3 \frac{d^3 U}{d\xi^3} + 6a_1 \varepsilon k U \frac{dU}{d\xi} - 10a_2 \varepsilon k^3 U \frac{d^3 U}{d\xi^3} \\ - 20a_2 \varepsilon k^3 \frac{dU}{d\xi} \frac{d^2 U}{d\xi^2} + 30a_2 k \varepsilon^2 U^2 \frac{dU}{d\xi} - c \frac{dU}{d\xi} = 0. \end{aligned} \quad (12)$$

When we integrate Eq.(12) along with ξ and getting to the zero of integration constant, we can find the following model

$$a_2 k^5 U^{(4)} - a_1 k^3 \frac{d^2 U}{d\xi^2} - 10a_2 \varepsilon k^3 U \frac{d^2 U}{d\xi^2} - 5a_2 \varepsilon k^3 \left(\frac{dU}{d\xi} \right)^2 + 10a_2 k \varepsilon^2 U^3 + 3a_1 k \varepsilon U^2 - cU = 0. \quad (13)$$

With the help of balance, it can be written the following relationship for m and M ;

$$2M = m + 2. \quad (14)$$

Case 1.

If we take $M = 3$ and $m = 4$ then, we can write the following trial solution form as;

$$U = b_0 + b_1F + b_2F^2 + b_3F^3 + b_4F^4$$

$$\frac{dU}{d\xi} = \dots$$

$$\vdots$$

$$\frac{d^4U}{d\xi^4} = \dots$$
(15)

where $\frac{dF}{d\xi} = wF + \lambda F^3, b_4 \neq 0$. When we use Eqs.(15) in the Eq.(13), we get a system of algebraic equations from the coefficients of polynomial of F . By solving this system of equations with the help of some computational programs such as Mathematica, Matlab and Maple, it yields us the following coefficients;

Case.1.1. For $w \neq \lambda$ we can consider the coefficients as following;

$$b_0 = \frac{-(15 + \sqrt{30})a_1}{100\epsilon a_2}, b_1 = 0, b_2 = \frac{-80i\sqrt{10}c\lambda\sqrt{b_2}}{13\sqrt{3}\epsilon a_1^{\frac{3}{2}}}, b_3 = 0, k = \frac{-200ca_2}{39a_1^2},$$

$$b_4 = \frac{320000c^2\lambda^2a_1^2}{1521\epsilon a_1^4}, w = \frac{-39i\sqrt{3}a_1^{\frac{5}{2}}}{400\sqrt{10}ca_2^{\frac{3}{2}}}$$
(16)

Substituting Eq.(16) into Eq.(15), we obtain the following new complex exponential function solution to Eq.(1);

$$u_1(x,t) = \frac{a_1(-15 - \sqrt{30})}{100\epsilon a_2} + \frac{a_1}{\epsilon a_2} \frac{2400c\lambda}{4000c\lambda + \frac{39i\sqrt{30}\epsilon a_1^{\frac{5}{2}}}{a_2^{\frac{3}{2}}} e^{\frac{(kx-ct)}{200c\sqrt{10}a_2^{\frac{3}{2}}}}}$$

$$- \frac{280000c^2\lambda^2a_2^2a_1}{\epsilon} (400\sqrt{30}c\lambda a_2^{\frac{3}{2}} + 117i\epsilon a_1^{\frac{5}{2}} e^{\frac{(kx-ct)}{200c\sqrt{10}a_2^{\frac{3}{2}}}})^{-2}$$
(17)

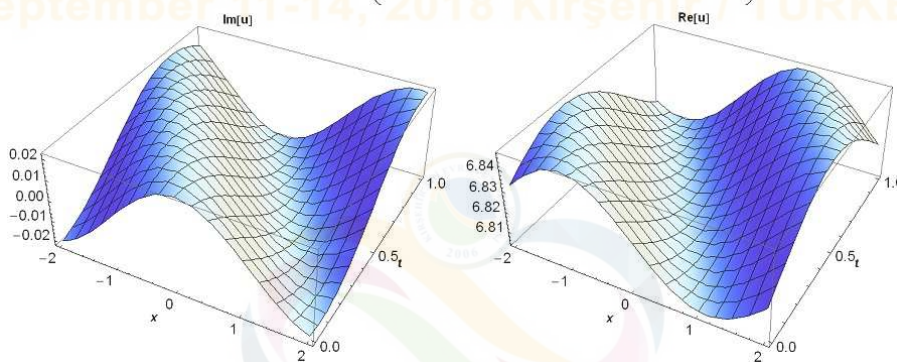


Fig. 1. The 3D surfaces of Eq.(17) for $a_1 = 2, \epsilon = -0.2, a_2 = 0.3, c = 0.4, \lambda = 0.01, E = 2, -2 < x < 2, 0 < t < 1$.

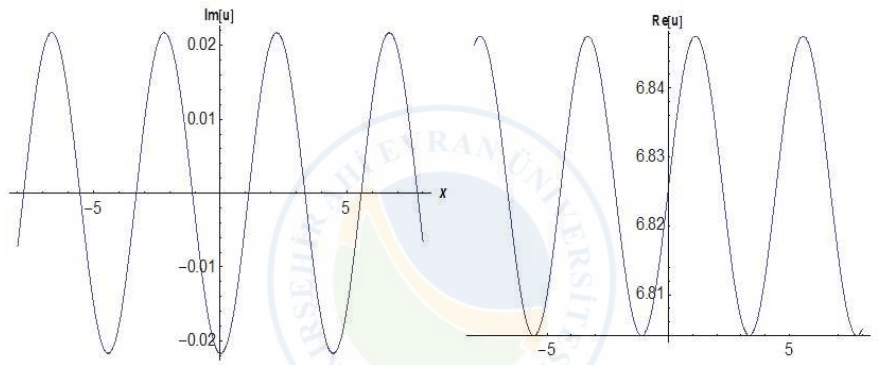


Fig. 2. The 2D surfaces of Eq.(17) for $a_1 = 2, \varepsilon = -0.2, a_2 = 0.3, c = 0.4, \lambda = 0.01, E = 2, t = 0.85, -8 < x < 8$.

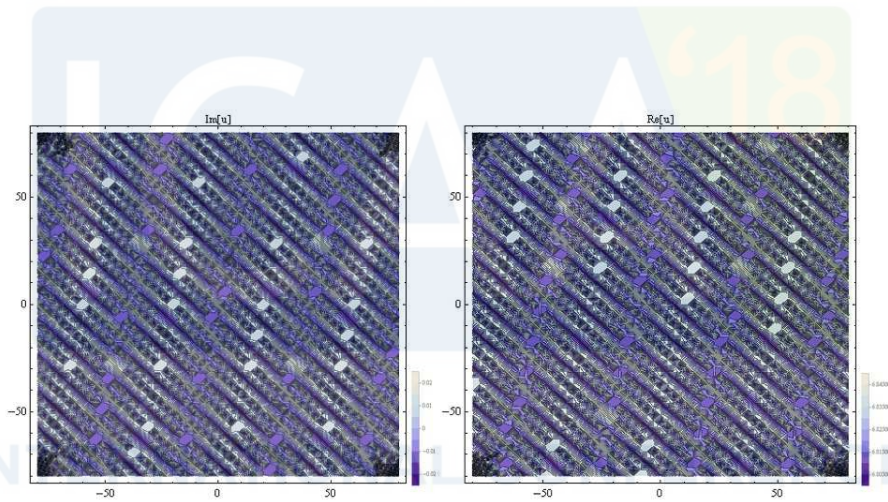


Fig. 3. Contour surfaces of Eq.(17) for $a_1 = 2, \varepsilon = -0.2, a_2 = 0.3, c = 0.4, \lambda = 0.01, E = 2, -80 < x < 80, -80 < t < 80$.

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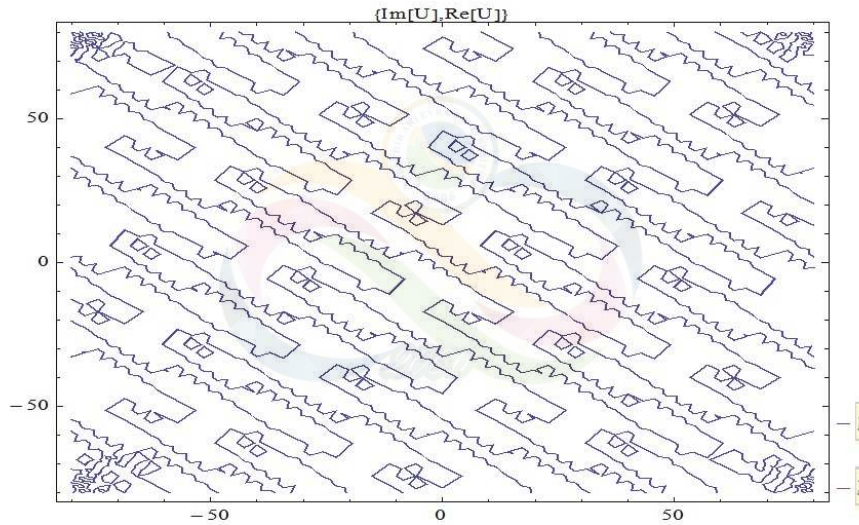


Fig. 4. Contour surfaces of combination of both sides of Eq.(17) for $a_1 = 2, \varepsilon = -0.2, a_2 = 0.3, c = 0.4, \lambda = 0.01, E = 2, -80 < x < 80, -80 < t < 80$.

Case.1.2. For $w \neq \lambda$ we can consider another coefficients as following;

$$b_0 = \frac{-(15 + \sqrt{30})a_1}{100\epsilon a_2}, b_1 = 0, b_2 = \frac{80i\sqrt{10}c\lambda\sqrt{a_2}}{13\sqrt{3}\epsilon a_1^{\frac{3}{2}}}, b_3 = 0, k = \frac{-200ca_2}{39a_1^2},$$

$$b_4 = \frac{320000c^2\lambda^2 a_2^2}{1521\epsilon a_1^4}, w = \frac{39i\sqrt{3}a_1^{\frac{5}{2}}}{400\sqrt{10}ca_2^{\frac{3}{2}}}, \quad (18)$$

Substituting Eq.(18) into Eq.(15), we obtain the following another new complex exponential function solution to Eq.(1);

$$u_2(x,t) = \frac{a_1}{100\epsilon a_2} \left(-15 - \sqrt{30} + \frac{2808000i\sqrt{30}cE\lambda a_1^{\frac{5}{2}} a_2^{\frac{3}{2}} e^{(kx-ct) \frac{39i\sqrt{3}a_1^{\frac{5}{2}}}{200c\sqrt{10}a_2^{\frac{3}{2}}}}}{(117iEa_1^{\frac{5}{2}} + 400i\sqrt{30}c\lambda a_2^{\frac{3}{2}} e^{(kx-ct) \frac{39i\sqrt{3}a_1^{\frac{5}{2}}}{200c\sqrt{10}a_2^{\frac{3}{2}}}})^2} \right). \quad (19)$$

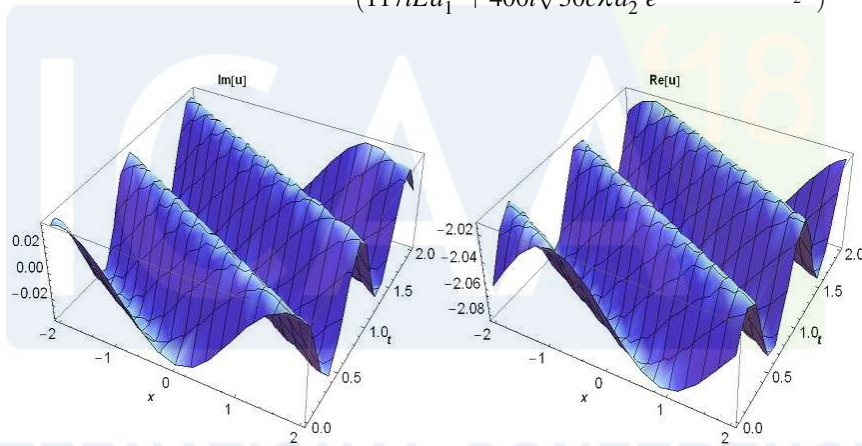


Fig. 5. The 3D graphs of Eq.(19) for $a_1 = 2, \epsilon = 1, a_2 = 0.2, c = 0.4, \lambda = 0.1, E = -2, -2 < x < 2, -2 < t < 2$.

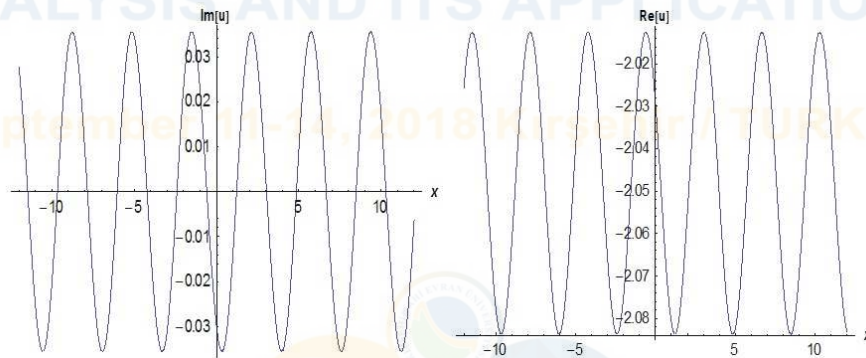


Fig. 6. The 2D graphs of Eq.(19) for $a_1 = 2, \epsilon = 1, a_2 = 0.2, c = 0.4, \lambda = 0.1, E = -2, t = 0.85, -12 < x < 12$.

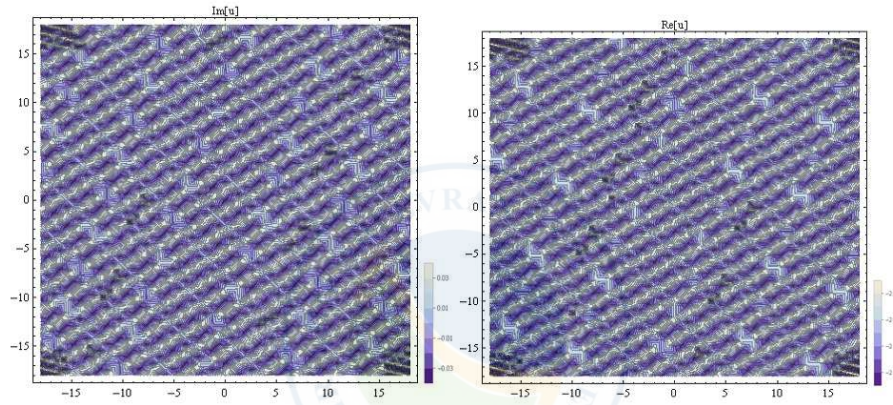


Fig. 7. The Contour graphs of Eq.(19) for $a_1 = 2, \varepsilon = 1, a_2 = 0.2, c = 0.4, \lambda = 0.1, E = -2, -18 < x < 18, -18 < t < 18$.

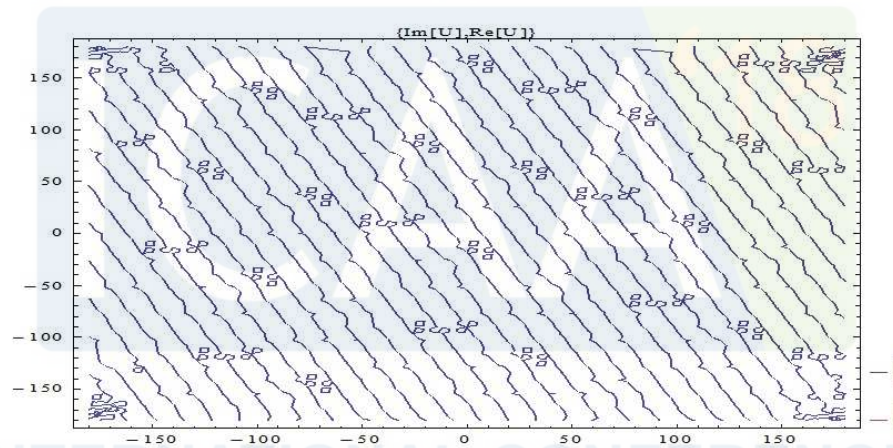


Fig. 8. Contour surfaces of combination of both sides of Eq.(19) for $a_1 = 2, \varepsilon = 1, a_2 = 0.2, c = 0.4, \lambda = 0.1, E = -2, -18 < x < 18, -18 < t < 18$.

4. Conclusions

In this paper, we have applied the BSEFM into the SIEE. Several complex travelling wave solutions have been obtained. Two- and three-dimensional surfaces along with the contour simulations for both results have been also plotted by using computational programs such as Maple and Mathematica. Contour surfaces of combination of both sides of results have been also drawn. These solutions have been newly presented to the literature. To the best of our knowledge, the application of BSEFM to the SIEE has been not submitted beforehand.

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Investigation of Suborbital Graphs of the Type

$$\Gamma_0(L, M) = \begin{pmatrix} a & bM \\ cL & d \end{pmatrix}$$

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Abstract:

In this study, the definition of suborbital graphs of the type $\begin{pmatrix} a & bM \\ cL & d \end{pmatrix}$, its transitive and invariant state on the set of generalized rational numbers, basic congruence groups, number of basic congruence groups and some of its other features have been tried to be examined with the help of definitions and theories of basic graph theories. $\Gamma_0(n) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma : c \equiv 0 \pmod{n} \right\}$ is connected to the congruential basis n and $\Gamma_0(L, M) = \left\{ \begin{pmatrix} a & bM \\ cL & d \end{pmatrix} \in \Gamma : a, b, c, d \in \mathbb{Z}, ad - bc(LM) = 1 \right\}$ is connected to L . Furthermore, it has been shown that $\Gamma_0(L, M)$ moves on \mathbb{Q} in an imprimitive way, thus, by defining the G-invariant equivalence relation, a different equation relation on $\Gamma_0(L, M)$ is defined which is different from universal and identity equivalence relations.

4th INTERNATIONAL CONFERENCE ON ANALYSIS AND ITS APPLICATIONS

1. Introduction and Preliminaries

In graph theory, $\Gamma = \{F : z \rightarrow \frac{az+b}{cz+d} : a, b, c, d \in \mathbb{Z}, ad - bc = 1\}$ is defined as a subgroup of modular group.

More specifically, this subgroup can be written as $\Gamma = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{Z}, ad - bc = 1 \right\}$. And this

subgroup moves on transitively on generalized rational numbers ($\widehat{\mathbb{Q}} = \mathbb{Q} \cup \infty$). For example, If $\frac{a}{c}$ and $\frac{b}{d} \in \mathbb{Q}$

are taken as $\sigma\left(\frac{a}{c}\right) = \frac{b}{d}$, you get $\sigma \in \Gamma$. Because, for $\gamma, \varphi \in \Gamma$,

from the equation of $\gamma(\infty) = \frac{a}{c} \Rightarrow \infty = \gamma^{-1}\left(\frac{a}{c}\right)$ and $\varphi(\infty) = \frac{b}{d} \Rightarrow \varphi\gamma^{-1}\left(\frac{a}{c}\right) = \frac{b}{d}$

, you get $(\varphi\gamma^{-1})\frac{a}{c} = \frac{b}{d} \Rightarrow \sigma\left(\frac{a}{c}\right) = \frac{b}{d}$.

Then $\frac{a}{c} \in \widehat{\mathbb{Q}}$ in reduced form, it is $(a, c) = 1$ and for $\exists x, y \in \mathbb{Z}, ax - cy = 1$.

So, it is $g = \begin{pmatrix} a & x \\ c & y \end{pmatrix} \in \Gamma$ and $g(\infty) = \begin{pmatrix} a & x \\ c & y \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a \\ c \end{pmatrix}$. In other words, $\frac{a}{c}$ orbits on ∞ and Γ moves transitively on \mathbb{Q} .

Any point on Γ has infinite periods. Suppose that, $\Omega = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ equals to Ω for $\Omega(\infty) = \infty$.

$$\infty = \frac{1}{0} \Rightarrow \Omega(\infty) = \begin{pmatrix} a & x \\ c & y \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a \\ c \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$\Rightarrow a = 1, c = 0$. From this definition, we get $\det \Omega = 1, ad - bc = 1 \Rightarrow d = 1, \forall b \in \mathbb{Z}$.

So, it is $\Omega = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \in \Gamma_\infty \subset \Gamma, \forall b \in \mathbb{Z}$. Thus, Γ_∞ is a group with infinite periods produced

by $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

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Basic congruence subgroup for Γ is defined by $\Gamma(n) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a \equiv d \equiv 1, b \equiv c \equiv 0 \right\}$. In addition to this, some the other subgroups containing the basic congruence subgroup is defined in the following way.

$$\Gamma_1(n) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma : a \equiv d \equiv 1, c \equiv 0 \pmod{n} \right\}$$

$$\Gamma_0(n) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma : c \equiv 0 \pmod{n} \right\}$$

$$\Gamma^0(n) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma : b \equiv 0 \pmod{n} \right\}$$

$$\Gamma_0^0(n) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma : b \equiv c \equiv 0 \pmod{n} \right\}$$

Among these equivalence groups, there is an order as $\Gamma(n) \leq \Gamma_1(n) \leq \Gamma_0^0 \leq \Gamma_0(n) \leq \Gamma^0(n)$. The number of basic congruence groups is defined by $\Psi(n) = \prod_{p|n} (1 + \frac{1}{p})$. For $\forall \alpha \in \mathbb{Q}$ and g, g' ; " \approx " equivalence relation is well-defined by $g(\alpha) \approx g'(\alpha) \Leftrightarrow g' \in gH$.

Assume that $\Gamma_0(n) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma : c \equiv 0 \pmod{n} \right\}$ is an equivalence subgroup and it is $v, w \in \mathbb{Q}$, $g =$

$$\begin{pmatrix} a & * \\ c & * \end{pmatrix}, g' = \begin{pmatrix} x & * \\ y & * \end{pmatrix} \in \Gamma \Rightarrow v = g(\infty) = g \begin{pmatrix} a & * \\ c & * \end{pmatrix}(\infty) \text{ and } w = g(\infty) = g' \begin{pmatrix} x & * \\ y & * \end{pmatrix}(\infty).$$

$$v \approx w \Leftrightarrow g^{-1}g' \in H = \Gamma_0(n)$$

$$g^{-1} = \begin{pmatrix} * & * \\ -c & a \end{pmatrix} \Rightarrow g^{-1}g' = \begin{pmatrix} * & * \\ -c & a \end{pmatrix} \begin{pmatrix} x & * \\ y & * \end{pmatrix} = \begin{pmatrix} * & * \\ ay - cx & * \end{pmatrix} \in H = \Gamma_0(n)$$

2. Main Results

We have given some features of $\Gamma = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{Z}, ad - bc = 1 \right\}$ in the introduction section. In

this section, the same features will investigated for suborbital graphs of the type $\Gamma_0(L, M) = \left(\begin{array}{cc} a & bM \\ cL & d \end{array} \right)$

2.1. Proposition

Let $\Gamma_0(L, M) = \left(\begin{array}{cc} a & bM \\ cL & d \end{array} \right)$ be a suborbital graph.

1. $\Gamma_0(L, M)$ moves transitively on $\widehat{\mathbb{Q}}$,

2. Any point has infinite periods in $\Gamma_0(L, M)$.

Proof:(1.) If it is $\frac{a}{cL}$ and $\frac{bM}{d} \in \mathbb{Q}$ as $\sigma(\frac{a}{cL}) = \frac{bM}{d}$, it equals to $\sigma \in \Gamma_0(L, M)$.

As for $\gamma, \varphi \in \Gamma$, it is $\gamma(\infty) = \frac{a}{cL} \Rightarrow \infty = \gamma^{-1}(\frac{a}{cL})$ and $\varphi(\infty) = \frac{bM}{d} \Rightarrow \varphi\gamma^{-1}(\frac{a}{cL}) = \frac{bM}{d}$

$$(\varphi\gamma^{-1})\frac{a}{cL} = \frac{bM}{d} \Rightarrow \sigma(\frac{a}{cL}) = \frac{bM}{d}.$$

Then we get $\frac{a}{cL} \in \widehat{\mathbb{Q}}$ in reduced form, $(a, cL) = 1$ and for $\exists x, y \in \mathbb{Z}$, $ax - cLy = 1$.

So $g = \begin{pmatrix} a & x \\ cL & y \end{pmatrix} \in \Gamma_0(L, M)$ and $g(\infty) = \begin{pmatrix} a & x \\ cL & y \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a \\ cL \end{pmatrix}$. $\frac{a}{cL}$ orbit on ∞ and $\Gamma_0(L, M)$ moves on transitively on $\widehat{\mathbb{Q}}$.

(2.) Assume that $\Omega = \begin{pmatrix} a & bM \\ cL & d \end{pmatrix} \in \Gamma_0(L, M)$ and Ω for $\Omega(\infty) = \infty$ and similarly, $\infty = \frac{1}{0} \Rightarrow \Omega(\infty) =$

$$\begin{pmatrix} a & x \\ cL & y \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a \\ cL \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$\Rightarrow a = 1, cL = 0$. From this definition, we get $\det \Omega = 1$, $ad - bc(LM) = 1 \Rightarrow d = 1$ and $\forall b \in \mathbb{Z}$.

So it is $\Omega = \begin{pmatrix} a & bM \\ cL & d \end{pmatrix} = \begin{pmatrix} 1 & bM \\ 0 & 1 \end{pmatrix} \in \Gamma_\infty(L, M) \subset \Gamma_0(L, M)$, $\forall b \in \mathbb{Z}$. Thus, $\Gamma_\infty(L, M)$ is a group with

infinite periods produced by $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

2.2. Definition

$\Gamma_0(L, M) = \left\{ \begin{pmatrix} a & bM \\ cL & d \end{pmatrix} \in \Gamma : a, b, c, d, L, M \in \mathbb{Z}, ad - bc(LM) = 1 \right\}$ equivalence obtained from $\Gamma(L, M) = \left\{ \begin{pmatrix} a & bM \\ cL & d \end{pmatrix} \in \Gamma : a, b, c, d, L, M \in \mathbb{Z}, ad - bc(LM) = 1 \right\}$. In addition to, we can write equivalences in the following:

$$\Gamma^0(L, M) = \left\{ \begin{pmatrix} a & bM \\ cL & d \end{pmatrix} \in \Gamma(L, M) : a, b, c, d, L, M \in \mathbb{Z}, ad - bc(LM) = 1, bM \equiv 0 \pmod{M} \right\}$$

$$\Gamma_0^0(L, M) = \left\{ \begin{pmatrix} a & bM \\ cL & d \end{pmatrix} \in \Gamma(L, M) : a, b, c, d, L, M \in \mathbb{Z}, ad - bc(LM) = 1, bM \equiv 0 \pmod{M}, cL \equiv 0 \pmod{L} \right\}$$

Among these equivalence groups have $\Gamma(n) \leq \Gamma(L, M) \leq \Gamma_0^0(L, M) \leq \Gamma_0(L, M) \leq \Gamma$ or $\Gamma(n) \leq \Gamma(L, M) \leq \Gamma_0^0(L, M) \leq \Gamma^0(L, M) \leq \Gamma$.

2.3. Proposition

Let number of basic congruence groups of $\Gamma_0(L, M)$ be $\Psi(L, M)$.

$$\Psi(L, M) = |\Gamma : \Gamma_0(L, M)| = L \prod_p \left(1 + \frac{1}{p}\right).$$

2.4. Example

$$\Gamma_0(3, 7) = \left\{ \begin{pmatrix} 11 & 1.7 \\ 1.3 & 2 \end{pmatrix} \in \Gamma_0(L, M) : 11, 1, 2, 3, 7 \in \mathbb{Z}, 11.2 - 1.1(7.3) = 1, 1.3 \equiv 0 \pmod{3} \right\}. \Psi(3, 7) = 8.$$

Solution: $\Psi(3, 7) = |\Gamma : \Gamma_0(3, 7)| = 3 \prod_p \left(1 + \frac{1}{p}\right)$

$$= 3 \prod_{\{1, 3\}} \left(1 + \frac{1}{p}\right)$$

$$= 3 \left(1 + \frac{1}{1}\right) \left(1 + \frac{1}{3}\right) = 3.2. \frac{4}{3} = 8.$$

2.5. Proposition

Let $\Gamma_0(L, M)$ be an suborbital graph and L, M are prime numbers.

$$|\Gamma : \Gamma_0(L, M)| = |\Gamma_0(L, M) : \Gamma_0^0(L, M)| \Rightarrow L = 1 + \frac{1}{1+\frac{1}{L}} \text{ and } M = 1 + \frac{1}{1+\frac{1}{M}}.$$

Proof. Let $\Gamma_0(L, M) = \left\{ \begin{pmatrix} a & bM \\ cL & d \end{pmatrix} \in \Gamma : a, b, c, d, L, M \in \mathbb{Z}, ad - bc(LM) = 1 \right\}$ and

$$\Gamma_0^0(L, M) = \left\{ \begin{pmatrix} a & bM \\ cL & d \end{pmatrix} \in \Gamma(L, M) : a, b, c, d, L, M \in \mathbb{Z}, ad - bc(LM) = 1, bM \equiv 0 \pmod{M}, cL \equiv 0 \pmod{L} \right\}.$$

In the $\Psi(L, M) = |\Gamma : \Gamma_0(L, M)|$, the equality depend on L but we let in the $\Psi(L, M) = |\Gamma_{L, M} : \Gamma_0^0(L, M)|$ depend only M , because cL is fixed on both sides of the equation.

$$|\Gamma : \Gamma_0(L, M)| = |\Gamma_0(L, M) : \Gamma_0^0(L, M)|$$

$$L \prod_p \left(1 + \frac{1}{p}\right) = M \prod_k \left(1 + \frac{1}{k}\right)$$

Then K and M and prime $p \in \{1, L\}, k \in \{1, M\}$.

$$L \left(1 + \frac{1}{1}\right) \left(1 + \frac{1}{L}\right) = M \left(1 + \frac{1}{1}\right) \left(1 + \frac{1}{M}\right)$$

$$L.2. \left(1 + \frac{1}{L}\right) = M.2. \left(1 + \frac{1}{M}\right)$$

$$L \left(1 + \frac{1}{L}\right) = M \left(1 + \frac{1}{M}\right)$$

$$\frac{L}{M} = \frac{1 + \frac{1}{M}}{1 + \frac{1}{L}}$$

$$L = 1 + \frac{1}{M} \text{ and } M = 1 + \frac{1}{L}$$

$$L = 1 + \frac{1}{M} = 1 + \frac{1}{1 + \frac{1}{L}} = 1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{L}}} = 1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{L}}}}$$

$$M = 1 + \frac{1}{L} = 1 + \frac{1}{1 + \frac{1}{M}} = 1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{M}}} = 1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{M}}}}$$

2.6. Preposition

For $\forall n \in \mathbb{N}$, $\Gamma_\infty(L, M) \leq \Gamma_0(L, M) \leq \Gamma$ and $n > 1$, $\Gamma_\infty(L, M) < \Gamma_0(L, M) < \Gamma$. So $\Gamma(L, M)$ moves as imprimitive on $\widehat{\mathbb{Q}}$.

Proof. We have to define a equivalence relation which is different from universal and identity relations to show that you move as imprimitive. Let $\Gamma_0(L, M) = \left\{ \begin{pmatrix} a & bM \\ cL & d \end{pmatrix} \in \Gamma : a, b, c, d, L, M \in \mathbb{Z}, ad - bc(LM) = 1, cL \equiv 0(\text{mod}L) \right\}$ is an congruence subgroup and $v, w \in \widehat{\mathbb{Q}}$ equals to $v = g(\infty) = \begin{pmatrix} a & * \\ cL & * \end{pmatrix}(\infty)$ and $w = g'(\infty) = \begin{pmatrix} x & * \\ y & * \end{pmatrix}$, for $g, g' \in \Gamma_0(L, M)$.

Then $v \approx w \Leftrightarrow g^{-1}g' \in H = \Gamma_0(L, M)$ and $g^{-1} = \begin{pmatrix} * & * \\ -cL & a \end{pmatrix}$,

$g^{-1}g' = \begin{pmatrix} * & * \\ -cL & a \end{pmatrix} \begin{pmatrix} x & * \\ y & * \end{pmatrix} = \begin{pmatrix} * & * \\ ay - cLx & * \end{pmatrix} \in H = \Gamma_0(L, M)$, $ay - cLx \equiv 0(\text{mod}L)$.

$ay - cLx \equiv 0(\text{mod}L) \Rightarrow ay = cLx \Rightarrow \frac{a}{cL} = \frac{x}{y}$

$\frac{a}{cL} \approx_L \frac{x}{y}$
 $v \approx w \Leftrightarrow \frac{a}{cL} \approx_L \frac{x}{y}$.

3. Conclusion

Transitive and invariant state, basic congruence groups, number of basic congruence groups and some of its other features of Γ are investigated for $\Gamma_0(L, M)$ and are obtained the same conclusions. The conclusions of $\Gamma_0(L, M)$ can be examined for $\Gamma^0(L, M)$ and $\Gamma_0^0(L, M)$. In addition to this, continuous fractions are obtained in the 4.5 Preposition. From the definition of continuous fraction, we get Fibonacci quadartic equation.

$$L = 1 + \frac{1}{M} = 1 + \frac{1}{1 + \frac{1}{L}} = 1 + \frac{L}{L+1} = \frac{2L+1}{L+1}$$

$$L = \frac{2L+1}{L+1} \Rightarrow L^2 + L = 2L + 1 \Rightarrow L^2 - L - 1 = 0$$

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Lacunary Statistical Convergence in Fuzzy n-Normed Spaces

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Abstract: In this paper, we introduced the concept of lacunary statistical summable and lacunary statistical convergence in fuzzy n-normed linear spaces. It also has studied the some properties these concepts.

1. Introduction and Background

The concept of convergence of real sequences has been extended to statistical convergence independently by Fast [10] and Schoenberg [31]. The concept of ordinary convergence of a sequence of fuzzy numbers was firstly introduced by Matloka [14] and proved some basic theorems for sequences of fuzzy numbers. Nanda [18] studied the sequences of fuzzy numbers and showed that the set of all convergent sequences of fuzzy numbers form a complete metric space. Şençimen and Pehlivan [32] introduced the notions of statistically convergent sequence and statistically Cauchy sequence in a fuzzy normed linear space. Reddy and Srinivas [25] studied statistical convergence in fuzzy n-normed linear spaces. Türkmen and Çınar [34] presented analogues in fuzz normed linear spaces of the results given by Fridy and Orhan [13] and Türkmen and Dündar [37] studied lacunary statistical convergence of double sequences in fuzzy normed linear spaces. Recently, Savaş [29, 30] studied on I-lacunary statistical convergence of weight g of fuzzy numbers and on lacunary p-summable convergence of weight g for fuzzy numbers via ideal.

Fuzzy sets are considered with respect to a nonempty base set X of elements of interest. The essential idea is that each element $x \in X$ is assigned a membership grade $u(x)$ taking values in $[0, 1]$, with $u(x) = 0$ corresponding to nonmembership, $0 < u(x) < 1$ to partial membership, and $u(x) = 1$ to full membership. According to Zadeh [38], a fuzzy subset of X is a nonempty subset $\{(x, u(x)) : x \in X\}$ of $X \times [0, 1]$ for some function $u : X \rightarrow [0, 1]$. The function u itself is often used for the fuzzy set.

A fuzzy set u on \mathbb{R} is called a fuzzy number if it has the following properties:

1. u is normal, that is, there exists an $x_0 \in \mathbb{R}$ such that $u(x_0) = 1$;
2. u is fuzzy convex, that is, for $x, y \in \mathbb{R}$ and $0 \leq \lambda \leq 1$, $u(\lambda x + (1 - \lambda)y) \geq \min[u(x), u(y)]$;
3. u is upper semicontinuous;
4. $supp u = cl\{x \in \mathbb{R} : u(x) > 0\}$, or denoted by $[u]_0$, is compact.

Now, we recall the basic definitions and concepts [1, 2, 4, 6–13, 15–17, 20–24, 26–28, 32–36].

Let $L(\mathbb{R})$ be set of all fuzzy numbers. If $u \in L(\mathbb{R})$ and $u(t) = 0$ for $t < 0$, then u is called a non-negative fuzzy number. We write $L^*(\mathbb{R})$ by the set of all non-negative fuzzy numbers. We can say that $u \in L^*(\mathbb{R})$ iff $u_{\alpha}^{-} \geq 0$ for each $\alpha \in [0, 1]$. Clearly we have $\tilde{0} \in L(\mathbb{R})$. For $u \in L(\mathbb{R})$, the α level set of u is defined by

$$[u]_{\alpha} = \begin{cases} \{x \in \mathbb{R} : u(x) \geq \alpha\}, & \text{if } \alpha \in (0, 1] \\ supp u, & \text{if } \alpha = 0. \end{cases}$$

A partial order \preceq on $L(\mathbb{R})$ is defined by $u \preceq v$ if $u_{\alpha}^{-} \leq v_{\alpha}^{-}$ and $u_{\alpha}^{+} \leq v_{\alpha}^{+}$ for all $\alpha \in [0, 1]$.

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Arithmetic operation for $t \in \mathbb{R}$, \oplus, \ominus, \odot and \oslash on $L(\mathbb{R}) \times L(\mathbb{R})$ are defined by
 $(u \oplus v)(t) = \sup_{s \in \mathbb{R}} \{u(s) \wedge v(t-s)\}$, $(u \ominus v)(t) = \sup_{s \in \mathbb{R}} \{u(s) \wedge v(s-t)\}$,
 $(u \odot v)(t) = \sup_{s \in \mathbb{R}, s \neq 0} \{u(s) \wedge v(t/s)\}$ and $(u \oslash v)(t) = \sup_{s \in \mathbb{R}} \{u(st) \wedge v(s)\}$.

For $k \in \mathbb{R}^+$, ku is defined as $ku(t) = u(t/k)$ and $0u(t) = \tilde{0}$, $t \in \mathbb{R}$.

Some arithmetic operations for α -level sets are defined as follows:

$u, v \in L(\mathbb{R})$ and $[u]_\alpha = [u_\alpha^-, u_\alpha^+]$ and $[v]_\alpha = [v_\alpha^-, v_\alpha^+]$, $\alpha \in (0, 1]$. Then,

$$[u \oplus v]_\alpha = [u_\alpha^- + v_\alpha^-, u_\alpha^+ + v_\alpha^+],$$

$$[u \ominus v]_\alpha = [u_\alpha^- - v_\alpha^+, u_\alpha^+ - v_\alpha^-],$$

$$[u \odot v]_\alpha = [u_\alpha^- \cdot v_\alpha^-, u_\alpha^+ \cdot v_\alpha^+] \text{ and}$$

$$[\tilde{1} \oslash u]_\alpha = \left[\frac{1}{u_\alpha^+}, \frac{1}{u_\alpha^-} \right], u_\alpha^- > 0.$$

For $u, v \in L(\mathbb{R})$, the supremum metric on $L(\mathbb{R})$ defined as

$$D(u, v) = \sup_{0 \leq \alpha \leq 1} \max \{ |u_\alpha^- - v_\alpha^-|, |u_\alpha^+ - v_\alpha^+| \}.$$

It is known that D is a metric on $L(\mathbb{R})$ and $(L(\mathbb{R}), D)$ is a complete metric space.

A sequence $x = (x_k)$ of fuzzy numbers is said to be convergent to the fuzzy number x_0 , if for every $\varepsilon > 0$ there exists a positive integer k_0 such that $D(x_k, x_0) < \varepsilon$ for $k > k_0$ and a sequence $x = (x_k)$ of fuzzy numbers converges to levelwise to x_0 iff $\lim_{k \rightarrow \infty} [x_k]_\alpha = [x_0]_\alpha^-$ and $\lim_{k \rightarrow \infty} [x_k]_\alpha = [x_0]_\alpha^+$, where $[x_k]_\alpha = [(x_k)_\alpha^-, (x_k)_\alpha^+]$ and $[x_0]_\alpha = [(x_0)_\alpha^-, (x_0)_\alpha^+]$, for every $\alpha \in (0, 1)$.

Let X be a vector space over \mathbb{R} , $\|\cdot\| : X \rightarrow L^*(\mathbb{R})$ and the mappings L, R (respectively, left norm and right norm) $: [0, 1] \times [0, 1] \rightarrow [0, 1]$ be symmetric, nondecreasing in both arguments and satisfy $L(0, 0) = 0$ and $R(1, 1) = 1$. The quadruple $(X, \|\cdot\|, L, R)$ is called fuzzy normed linear space (briefly $(X, \|\cdot\|)$ FNS) and $\|\cdot\|$ a fuzzy norm if the following axioms are satisfied

1. $\|x\| = \tilde{0}$ iff $x = 0$,
2. $\|rx\| = |r| \odot \|x\|$ for $x \in X$, $r \in \mathbb{R}$,
3. For all $x, y \in X$
 - (a) $\|x + y\|(s + t) \geq L(\|x\|(s), \|y\|(t))$, whenever $s \leq \|x\|_1^-, t \leq \|y\|_1^-$ and $s + t \leq \|x + y\|_1^-$,
 - (b) $\|x + y\|(s + t) \leq R(\|x\|(s), \|y\|(t))$, whenever $s \geq \|x\|_1^+, t \geq \|y\|_1^+$ and $s + t \geq \|x + y\|_1^+$.

Let $(X, \|\cdot\|_C)$ be an ordinary normed linear space. Then, a fuzzy norm $\|\cdot\|$ on X can be obtained by

$$\|x\|(t) = \begin{cases} 0 & \text{if } 0 \leq t \leq a\|x\|_C \text{ or } t \geq b\|x\|_C \\ \frac{t}{(1-a)\|x\|_C} - \frac{a}{1-a} & a\|x\|_C \leq t \leq \|x\|_C \\ \frac{t}{(b-1)\|x\|_C} + \frac{b}{b-1} & \|x\|_C \leq t \leq b\|x\|_C \end{cases}$$

where $\|x\|_C$ is the ordinary norm of x ($\neq 0$), $0 < a < 1$ and $1 < b < \infty$. For $x = \theta$, define $\|x\| = \tilde{0}$. Hence, $(X, \|\cdot\|)$ is a fuzzy normed linear space.

Let us consider the topological structure of an FNS $(X, \|\cdot\|)$. For any $\varepsilon > 0$, $\alpha \in [0, 1]$ and $x \in X$, the (ε, α) -neighborhood of x is the set $\mathcal{N}_x(\varepsilon, \alpha) = \{y \in X : \|x - y\|_\alpha^+ < \varepsilon\}$.

Let $(X, \|\cdot\|)$ be an FNS. A sequence $(x_n)_{n=1}^\infty$ in X is convergent to $x \in X$ with respect to the fuzzy norm on X and we denote by $x_n \xrightarrow{FN} x$, provided that $(D) - \lim_{n \rightarrow \infty} \|x_n - x\| = \tilde{0}$; i.e., for every $\varepsilon > 0$ there is an $N(\varepsilon) \in \mathbb{N}$ such that $D(\|x_n - x\|, \tilde{0}) < \varepsilon$ for all $n \geq N(\varepsilon)$. This means that for every $\varepsilon > 0$ there is an $N(\varepsilon) \in \mathbb{N}$ such that

for all $n \geq N(\varepsilon)$, $\sup_{\alpha \in [0, 1]} \|x_n - x\|_\alpha^+ = \|x_n - x\|_0^+ < \varepsilon$.

Let $(X, \|\cdot\|)$ be an FNS. A sequence (x_k) in X is statistically convergent to $L \in X$ with respect to the fuzzy norm on X and we denote by $x_n \xrightarrow{FS} x$, provided that for each $\varepsilon > 0$, we have $\delta(\{k \in \mathbb{N} : D(\|x_k - L\|, \tilde{0}) \geq \varepsilon\}) = 0$. This implies that for each $\varepsilon > 0$, the set

$$K(\varepsilon) = \{k \in \mathbb{N} : \|x_k - L\|_0^+ \geq \varepsilon\}$$

has natural density zero; namely, for each $\varepsilon > 0$, $\|x_k - L\|_0^+ < \varepsilon$ for almost all k .

Let $n \in \mathbb{N}$ and let X be a real linear space of dimension $d \geq n$. A real valued function $\|\cdot, \cdot, \dots, \cdot\|$ on $X \times X \times \dots \times X$ satisfying the following conditions:

- $nN_1 : \|x_1, x_2, \dots, x_n\| = 0$ if and only if x_1, x_2, \dots, x_n are linearly dependent,
- $nN_2 : \|x_1, x_2, \dots, x_n\|$ is invariant under any permutation of x_1, x_2, \dots, x_n ,
- $nN_3 : \|\alpha x_1, x_2, \dots, x_n\| = |\alpha| \|x_1, x_2, \dots, x_n\|$ for all $\alpha \in \mathbb{R}$,

$fnN_4 : \|y + z, x_2, \dots, x_n\| \leq \|y, x_2, \dots, x_n\| + \|z, x_2, \dots, x_n\|$ for all $y, z, x_2, \dots, x_n \in X$, then the function $\|\cdot, \cdot, \dots, \cdot\|$ is called an n -norm on X and pair $(X, \|\cdot, \cdot, \dots, \cdot\|)$ is called n -normed space.

Let X be a real linear space of dimension d , where $2 \leq d < \infty$. Let $\|\cdot, \cdot, \dots, \cdot\| : X^n \rightarrow L^*(\mathbb{R})$ and the mappings L, R (respectively, left norm and right norm) : $[0, 1] \times [0, 1] \rightarrow [0, 1]$ be symmetric, nondecreasing in both arguments and satisfy $L(0, 0) = 0$ and $R(1, 1) = 1$ then the quadruple $(X, \|\cdot, \cdot, \dots, \cdot\|, L, R)$ is called fuzzy n -normed linear space (briefly $(X, \|\cdot, \cdot, \dots, \cdot\|)$ $FnNS$) and $\|\cdot, \cdot, \dots, \cdot\|$ a fuzzy n -norm if the following axioms are satisfied for every $y, x_1, x_2, \dots, x_n \in X$ and $s, t \in \mathbb{R}$

$fnN_1 : \|x_1, x_2, \dots, x_n\| = \tilde{0}$ if and only if x_1, x_2, \dots, x_n are linearly dependent vectors,

$fnN_2 : \|x_1, x_2, \dots, x_n\|$ is invariant under any permutation of x_1, x_2, \dots, x_n ,

$fnN_3 : \|\alpha x_1, x_2, \dots, x_n\| = |\alpha| \|x_1, x_2, \dots, x_n\|$ for all $\alpha \in \mathbb{R}$,

$fnN_4 : \|x_1 + y, x_2, \dots, x_n\|(s + t) \geq L(\|x_1, x_2, \dots, x_n\|(s), \|y, x_2, \dots, x_n\|(t))$ whenever $s \leq \|x_1, x_2, \dots, x_n\|_1^-, t \leq \|y, x_2, \dots, x_n\|_1^-$ and $s + t \leq \|x_1 + y, x_2, \dots, x_n\|_1^-$,

$fnN_5 : \|x_1 + y, x_2, \dots, x_n\|(s + t) \leq R(\|x_1, x_2, \dots, x_n\|(s), \|y, x_2, \dots, x_n\|(t))$ whenever $s \geq \|x_1, x_2, \dots, x_n\|_1^-, t \geq \|y, x_2, \dots, x_n\|_1^-$ and $s + t \geq \|x_1 + y, x_2, \dots, x_n\|_1^-$, where $[\|x_1, x_2, \dots, x_n\|]_\alpha = [\|x_1, x_2, \dots, x_n\|_\alpha^-, \|x_1, x_2, \dots, x_n\|_\alpha^+]$ for $x_1, x_2, \dots, x_n \in X, 0 \leq \alpha \leq 1$ and $\inf_{\alpha \in [0, 1]} \|x_1, x_2, \dots, x_n\|_\alpha^- > 0$. Hence the norm $\|\cdot, \cdot, \dots, \cdot\|$ is called fuzzy

n -norm on X and pair $(X, \|\cdot, \cdot, \dots, \cdot\|)$ is called fuzzy n -normed space.

Let $(X, \|\cdot, \cdot, \dots, \cdot\|)$ be fuzzy n -normed space. A sequence $\{x_k\}$ in X is said to be convergent to an element $x \in X$ with respect to the fuzzy n -norm on X if for every $\varepsilon > 0$ and for every $z_2, z_3, \dots, z_n \neq 0, z_2, z_3, \dots, z_n \in X$, \exists a number $N = N(\varepsilon, z_2, z_3, \dots, z_n)$ such that $D(\|x_k - x, z_2, z_3, \dots, z_n\|, \tilde{0}) < \varepsilon, \forall k \geq N$ or equivalently $(D) -$

$$\lim_{k \rightarrow \infty} \|x_k - x, z_2, z_3, \dots, z_n\| = \tilde{0}.$$

Let $(X, \|\cdot, \cdot, \dots, \cdot\|)$ be fuzzy n -normed space. A sequence $\{x_k\}$ in X is said to be statistically convergent to an element $x \in X$ with respect to the fuzzy n -norm on X if for every $\varepsilon > 0$ and for every $z_2, z_3, \dots, z_n \neq 0, z_2, z_3, \dots, z_n \in X$, we have $\delta\left(\left\{k \in \mathbb{N} : D(\|x_k - x, z_2, z_3, \dots, z_n\|, \tilde{0}) \geq \varepsilon\right\}\right) = 0$.

By a lacunary sequence we mean an increasing integer sequence $\theta = \{k_r\}$ such that $k_0 = 0$ and $h_r = k_r - k_{r-1} \rightarrow \infty$ as $r \rightarrow \infty$. The intervals determined by θ will be denoted by $I_r = (k_{r-1}, k_r]$.

Let $(X, \|\cdot, \cdot, \dots, \cdot\|)$ be an FNS and $\theta = \{k_r\}$ be lacunary sequence. A sequence $x = (x_k)_{k \in \mathbb{N}}$ in X is said to be lacunary summable with respect to fuzzy norm on X if there is an $L \in X$ such that

$$\lim_{r \rightarrow \infty} \frac{1}{h_r} \left(\sum_{k \in I_r} D(\|x_k - L, \tilde{0}\|) \right) = 0.$$

In this case, we can write $x_k \rightarrow L((N_\theta)_{FN})$ or $x_k \xrightarrow{(N_\theta)_{FN}} L$ and

$$(N_\theta)_{FN} = \left\{ x = (x_k) : \lim_{r \rightarrow \infty} \frac{1}{h_r} \left(\sum_{k \in I_r} D(\|x_k - L, \tilde{0}\|) \right) = 0, \text{ for some } L \right\}.$$

A sequence $x = (x_k)$ in X is said to be lacunary statistically convergent or FS_θ -convergent to $L \in X$ with respect to fuzzy norm on X if for each $\varepsilon > 0$

$$\lim_{r \rightarrow \infty} \frac{1}{h_r} |\{k \in I_r : D(\|x_k - L, \tilde{0}\|) \geq \varepsilon\}| = 0$$

where $|A|$ denotes the number of elements of the set $A \subseteq \mathbb{N}$. In this case, we write $x_k \xrightarrow{FS_\theta} L$ or $x_k \rightarrow L(FS_\theta)$ or $FS_\theta - \lim_{k \rightarrow \infty} x_k = L$. This implies that for each $\varepsilon > 0$, the set $K(\varepsilon) = \{k \in I_r : \|x_k - L\|_0^+ \geq \varepsilon\}$ has natural density zero, namely, for each $\varepsilon > 0, \|x_k - L\|_0^+ < \varepsilon$, for almost all k .

2. Main Results

In this section, we introduce the concepts of lacunary summable and lacunary statistically convergence fuzzy n -normed spaces. Also, we investigate some properties and relationships between these concepts.

Throughout the paper, we consider $(X, \|\cdot, \cdot, \dots, \cdot\|)$ be an $FnNS$ and $\theta = (k_r)$ be a lacunary sequence.

Definition 2.1 A sequence $x = (x_m)_{m \in \mathbb{N}}$ in X is said to be lacunary summable with respect to fuzzy n -norm on X if there is an $L \in X$ such that

$$\lim_{r \rightarrow \infty} \frac{1}{h_r} \left(\sum_{m \in I_r} D(\|x_m - L, z_1, z_2, \dots, z_{n-1}\|, \tilde{0}) \right) = 0.$$

In this case, we write $x_m \rightarrow L((N_\theta)_{FN})$ or $x_m \xrightarrow{(N_\theta)_{FN}} L$ and

$$(N_\theta)_{FN} = \left\{ (x_m) : \lim_{r \rightarrow \infty} \frac{1}{h_r} \left(\sum_{m \in I_r} D(\|x_m - L, z_1, z_2, \dots, z_{n-1}\|, \tilde{0}) \right) = 0, \text{ for some } L \right\}$$

Definition 2.2 A sequence $x = (x_m)$ in X is said to be lacunary statistically convergent or FnS_θ -convergent to $L \in X$ with respect to fuzzy n -norm on X if for each $\varepsilon > 0$

$$\lim_{r \rightarrow \infty} \frac{1}{h_r} \left| \{m \in I_r : D(\|x_m - L, z_1, z_2, \dots, z_{n-1}\|, \tilde{0}) \geq \varepsilon\} \right| = 0. \quad (2.1)$$

In this case, we write $x_m \xrightarrow{FnS_\theta} L$ or $x_{mn} \rightarrow L(FnS_\theta)$ or $FnS_\theta - \lim_{m \rightarrow \infty} x_m = L$. This implies that, for each $\varepsilon > 0$, the set $K(\varepsilon) = \{m \in I_r : \|x_m - L, z_1, z_2, \dots, z_{n-1}\|_0^+ \geq \varepsilon\}$ has natural density zero, namely, for each $\varepsilon > 0$, $\|x_m - L, z_1, z_2, \dots, z_{n-1}\|_0^+ < \varepsilon$, for almost all m .

A useful interpretation of the above definition is the following;

$$x_m \xrightarrow{FnS_\theta} L \Leftrightarrow FnS_\theta - \lim \|x_m - L, z_1, z_2, \dots, z_{n-1}\|_0^+ = 0.$$

Note that $Fs_{\theta_2} - \lim \|x_m - L, z_1, z_2, \dots, z_{n-1}\|_0^+ = 0$ implies that

$$FnS_\theta - \lim \|x_m - L, z_1, z_2, \dots, z_{n-1}\|_\alpha^- = FnS_\theta - \lim \|x_m - L, z_1, z_2, \dots, z_{n-1}\|_\alpha^+ = 0,$$

for each $\alpha \in [0, 1]$, since

$$0 \leq \|x_m - L, z_1, z_2, \dots, z_{n-1}\|_\alpha^- \leq \|x_m - L, z_1, z_2, \dots, z_{n-1}\|_\alpha^+ \leq \|x_m - L, z_1, z_2, \dots, z_{n-1}\|_0^+$$

holds for every $m \in \mathbb{N}$ and for each $\alpha \in [0, 1]$.

The set of all lacunary statistically convergent sequence with respect to fuzzy norm on X will be denoted by $FnS_\theta = \{x : \text{for some } L, FnS_\theta - \lim x = L\}$.

Theorem 2.3 We have the following:

(i) $x_m \rightarrow L((N_\theta)_{FnN}) \Rightarrow x_m \rightarrow L(FnS_\theta)$.

(ii) $(N_\theta)_{FnN}$ is a proper subset of FnS_θ .

Proof. (i) If $x_m \rightarrow L((N_\theta)_{FnN})$, then for given $\varepsilon > 0$

$$\begin{aligned} & \sum_{m \in I_r} D(\|x_m - L, z_1, z_2, \dots, z_{n-1}\|, \tilde{0}) \\ & \geq \sum_{m \in I_r} D(\|x_m - L, z_1, z_2, \dots, z_{n-1}\|, \tilde{0}) \\ & \quad D(\|x_m - L, z_1, z_2, \dots, z_{n-1}\|, \tilde{0}) > \varepsilon \\ & \geq \varepsilon \cdot \left| \{m \in I_r : D(\|x_m - L, z_1, z_2, \dots, z_{n-1}\|, \tilde{0}) \geq \varepsilon\} \right|. \end{aligned}$$

Therefore, we have

$$\lim_{r \rightarrow \infty} \frac{1}{h_r} \left| \{m \in I_r : D(\|x_m - L, z_1, z_2, \dots, z_{n-1}\|, \tilde{0}) \geq \varepsilon\} \right| = 0.$$

This implies that $x_m \rightarrow L(FnS_\theta)$.

(ii) In order to indicate that the inclusion $(N_\theta)_{FnN} \subseteq FnS_\theta$ in (i) is proper, let a lacunary sequence θ be given and define a sequence $x = (x_m)$ as follows:

$$x_m = \begin{cases} m & \text{if } k_{r-1} < m < k_{r-1} + [\sqrt{h_r}] \\ & r = 1, 2, \dots \\ 0 & \text{otherwise.} \end{cases}$$

Note that, x is not bounded. We have, for every $\varepsilon > 0$ and for each $x \in X$,

$$\begin{aligned} & \frac{1}{h_r} \left| \{m \in I_r : D(\|x_m - 0, z_1, z_2, \dots, z_{n-1}\|, \tilde{0}) \geq \varepsilon\} \right| \\ & = \frac{[\sqrt{h_r}]}{h_r} \rightarrow 0, \text{ as } r \rightarrow \infty. \end{aligned}$$

That is, $x_m \rightarrow 0(FnS_\theta)$. On the other hand

$$\begin{aligned} \frac{1}{h_r} \sum_{m \in I_r} D(\|x_m - 0, z_1, z_2, \dots, z_{n-1}\|, \tilde{0}) & = \frac{1}{h_r} \sum_{m \in I_r} \|x_m, z_1, z_2, \dots, z_{n-1}\|_0^+ \\ & = \frac{1}{h_r} \cdot \frac{[\sqrt{h_r}] \cdot ([\sqrt{h_r}] + 1)}{2} \\ & \rightarrow \frac{1}{2} \neq 0. \end{aligned}$$

Hence, $x_m \not\rightarrow 0((N_\theta)_{FnN})$. □

Theorem 2.4 Let θ be a lacunary sequence. Then, $x = (x_m) \in L_\infty$ and $x_m \rightarrow L(FnS_\theta) \Rightarrow x_m \rightarrow L((N_\theta)_{FnN})$.

Proof. Suppose that $x \in L_\infty$ and $x_m \rightarrow L(FnS_\theta)$.

Then, we say that $D(\|x_m - L, z_1, z_2, \dots, z_{n-1}\|, \tilde{0}) < M$, for all m . Given $\varepsilon > 0$, we get

$$\begin{aligned} & \frac{1}{h_r} \sum_{m \in I_r} D(\|x_m - L, z_1, z_2, \dots, z_{n-1}\|, \tilde{0}) \\ = & \frac{1}{h_r} \sum_{\substack{m \in I_r \\ D(\|x_m - L, z_1, z_2, \dots, z_{n-1}\|, \tilde{0}) \geq \varepsilon}} D(\|x_m - L, z_1, z_2, \dots, z_{n-1}\|, \tilde{0}) \\ & + \frac{1}{h_r} \sum_{\substack{m \in I_r \\ D(\|x_m - L, z_1, z_2, \dots, z_{n-1}\|, \tilde{0}) < \varepsilon}} D(\|x_m - L, z_1, z_2, \dots, z_{n-1}\|, \tilde{0}) \\ \leq & \frac{M}{h_r} \cdot |\{m \in I_r : D(\|x_m - L, z_1, z_2, \dots, z_{n-1}\|, \tilde{0}) \geq \varepsilon\}| + \varepsilon. \end{aligned}$$

Hence, $x_m \rightarrow L((N_\theta)_{FnN})$. □

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Lacunary \mathcal{I} –convergence in fuzzy normed spaces

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Abstract: In this paper, we have introduced lacunary ideal convergence and condition of being lacunary ideal Cauchy in fuzzy normed linear spaces and study some properties and relations of these concepts.

1. Introduction and Background

The concept of convergence of a sequence of real numbers has been extended to statistical convergence independently by Fast [9] and Schoenberg [32]. A lot of developments have been made in this area after the works of Šalát [28] and Fridy [11]. In general, statistically convergent sequences satisfy many of the properties of ordinary convergent sequences in metric spaces [9, 11, 26]. The idea of \mathcal{I} -convergence was introduced by Kostyrko et al. [15] as a generalization of statistical convergence which is based on the structure of the ideal \mathcal{I} of subset of the set of natural numbers \mathbb{N} . Nurray and Ruckle [22] independently introduced the same with another name generalized statistical convergence. Kostyrko et al. [16] gave some of basic properties of \mathcal{I} -convergence and dealt with extremal \mathcal{I} -limit points. A lot of developments have been made in this area after the works of [17, 27, 34].

The concept of ordinary convergence of a sequence of fuzzy numbers was firstly introduced by Matloka [19] and proved some basic theorems for sequences of fuzzy numbers. Nanda [21] studied the sequences of fuzzy numbers and showed that the set of all convergent sequences of fuzzy numbers form a complete metric space. Şençimen and Pehlivan [33] introduced the notions of statistically convergent sequence and statistically Cauchy sequence in a fuzzy normed linear space. Hazarika [13] studied the concepts of \mathcal{I} -convergence, \mathcal{I}^* -convergence and \mathcal{I} -Cauchy sequence in a fuzzy normed linear space. Türkmen and Çınar [35] studied lacunary statistical convergence in fuzzy normed linear spaces. Recently, Türkmen and Dündar [37] studied lacunary statistical convergence of double sequences and Savaş [30, 31] studied on I-lacunary statistical convergence of weight g of fuzzy numbers and on lacunary p -summable convergence of weight g for fuzzy numbers via ideal. In this paper, we introduce and study the concepts of lacunary \mathcal{I} –convergence, lacunary \mathcal{I}^* –convergence with respect to fuzzy norm where \mathcal{I} denotes the ideal of subsets of \mathbb{N} . Also, we study some properties and relations of them.

Now, we recall the concept of ideal, convergence, statistical convergence, ideal convergence, lacunary convergence, fuzzy normed and some basic definitions (see [1–12, 14, 18, 20, 22–26, 28–30, 33, 35–38])

Fuzzy sets are considered with respect to a nonempty base set X of elements of interest. The essential idea is that each element $x \in X$ is assigned a membership grade $u(x)$ taking values in $[0, 1]$, with $u(x) = 0$ corresponding to nonmembership, $0 < u(x) < 1$ to partial membership, and $u(x) = 1$ to full membership.

According to Zadeh a fuzzy subset of X is a nonempty subset $\{(x, u(x)) : x \in X\}$ of $X \times [0, 1]$ for some function $u : X \rightarrow [0, 1]$. The function u itself is often used for the fuzzy set.

A fuzzy set u on \mathbb{R} is called a fuzzy number if it has the following properties:

i) u is normal, that is, there exists an $x_0 \in \mathbb{R}$ such that $u(x_0) = 1$;

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ii) u is fuzzy convex, that is, for $x, y \in \mathbb{R}$ and $0 \leq \lambda \leq 1$,

$$u(\lambda x + (1 - \lambda)y) \geq \min[u(x), u(y)];$$

iii) u is upper semicontinuous;

iv) $\text{suppu} = \text{cl}\{x \in \mathbb{R} : u(x) > 0\}$, or denoted by $[u]_0$, is compact.

Let $L(\mathbb{R})$ be set of all fuzzy numbers. If $u \in L(\mathbb{R})$ and $u(t) = 0$ for $t < 0$, then u is called a non-negative fuzzy number. We have written $L^*(\mathbb{R})$ by the set of all non-negative fuzzy numbers. We can say that $u \in L^*(\mathbb{R})$ if and only if $u_\alpha^- \geq 0$ for each $\alpha \in [0, 1]$. Clearly we have $\tilde{0} \in L(\mathbb{R})$. For $u \in L(\mathbb{R})$, the α level set of u is defined by

$$[u]_\alpha = \begin{cases} \{x \in \mathbb{R} : u(x) \geq \alpha\}, & \text{if } \alpha \in (0, 1] \\ \text{suppu}, & \text{if } \alpha = 0. \end{cases}$$

Some arithmetic operations for α -level sets are defined as follows: $u, v \in L(\mathbb{R})$ and $[u]_\alpha = [u_\alpha^-, u_\alpha^+]$ and $[v]_\alpha = [v_\alpha^-, v_\alpha^+]$, $\alpha \in (0, 1]$. Then

$$[u \oplus v]_\alpha = [u_\alpha^- + v_\alpha^-, u_\alpha^+ + v_\alpha^+], \quad [u \ominus v]_\alpha = [u_\alpha^- - v_\alpha^+, u_\alpha^+ - v_\alpha^-]$$

$$[u \odot v]_\alpha = [u_\alpha^- \cdot v_\alpha^-, u_\alpha^+ \cdot v_\alpha^+], \quad [\tilde{1} \oslash u]_\alpha = \left[\frac{1}{u_\alpha^+}, \frac{1}{u_\alpha^-} \right], u_\alpha^- > 0.$$

For $u, v \in L(\mathbb{R})$, the supremum metric on $L(\mathbb{R})$ is defined as

$$D(u, v) = \sup_{0 \leq \alpha \leq 1} \max \{ |u_\alpha^- - v_\alpha^-|, |u_\alpha^+ - v_\alpha^+| \}.$$

It is known that D is a metric on $L(\mathbb{R})$, and $(L(\mathbb{R}), D)$ is a complete metric space. A sequence $x = (x_k)$ of fuzzy numbers is said to be convergent to the fuzzy number x_0 if for every $\varepsilon > 0$, there exists a positive integer k_0 such that $D(x_k, x_0) < \varepsilon$ for $k > k_0$. And a sequence $x = (x_k)$ of fuzzy numbers convergens to levelwise to x_0 , if and only if $\lim_{k \rightarrow \infty} [x_k]_\alpha = [x_0]_\alpha^-$ and $\lim_{k \rightarrow \infty} [x_k]_\alpha = [x_0]_\alpha^+$ where $[x_k]_\alpha = [(x_k)_\alpha^-, (x_k)_\alpha^+]$ and $[x_0]_\alpha = [(x_0)_\alpha^-, (x_0)_\alpha^+]$ for every $\alpha \in (0, 1)$.

The statistical converge of fuzzy number defined as follows;

A sequence $X = (X_k)$ of fuzzy numbers is said to be statistically convergent to fuzzy numbers X_0 if every $\varepsilon > 0$,

$$\lim_n \frac{1}{n} |\{k \leq n : \bar{d}(X_k, X_0) \geq \varepsilon\}| = 0.$$

Later, many mathematicians studied statistical convergence of fuzzy numbers and extended to fuzzy normed spaces.

Let X be a vector space over \mathbb{R} , let $\|\cdot\| : X \rightarrow L^*(\mathbb{R})$ and the mappings L, R (respectively, left norm and right norm) $: [0, 1] \times [0, 1] \rightarrow [0, 1]$ be symetric, nondecreasing in both arguments and satisfy $L(0, 0) = 0$ and $R(1, 1) = 1$.

The quadruple $(X, \|\cdot\|, L, R)$ is called fuzzy normed linear space (briefly $(X, \|\cdot\|)$ FNS) and $\|\cdot\|$ a fuzzy norm if the following axioms are satisfied

1) $\|x\| = \tilde{0}$ iff $x = \theta$,

2) $\|rx\| = |r| \odot \|x\|$ for $x \in X, r \in \mathbb{R}$,

3) For all $x, y \in X$

a) $\|x + y\|(s + t) \geq L(\|x\|(s), \|y\|(t))$, whenever $s \leq \|x\|_1^-, t \leq \|y\|_1^-$ and $s + t \leq \|x + y\|_1^-$,

b) $\|x + y\|(s + t) \leq R(\|x\|(s), \|y\|(t))$, whenever $s \geq \|x\|_1^-, t \geq \|y\|_1^-$ and $s + t \geq \|x + y\|_1^-$.

Let $(X, \|\cdot\|_C)$ be an ordinary normed linear space. Then a fuzzy norm $\|\cdot\|$ on X can be obtained

$$\|x\|(t) = \begin{cases} 0 & \text{if } 0 \leq t \leq a\|x\|_C \text{ or } t \geq b\|x\|_C \\ \frac{t}{(1-a)\|x\|_C} - \frac{a}{1-a} & a\|x\|_C \leq t \leq \|x\|_C \\ \frac{t}{(b-1)\|x\|_C} + \frac{b}{b-1} & \|x\|_C \leq t \leq b\|x\|_C \end{cases} \quad (1.1)$$

where $\|x\|_C$ is the ordinary norm of $x (\neq \theta)$,

$0 < a < 1$ and $1 < b < \infty$. For $x = \theta$, define $\|x\| = \tilde{0}$.

Hence, $(X, \|\cdot\|)$ is a fuzzy normed linear space. Şençimen has defined convergence in fuzzy normed spaces as follows;

Let $(X, \|\cdot\|)$ be an fuzzy normed linear space. A sequence $(x_n)_{n=1}^\infty$ in X is convergent to $x \in X$ with respect to the fuzzy norm on X and we denote by $x_n \xrightarrow{FN} x$, provided that $(D) - \lim_{n \rightarrow \infty} \|x_n - x\| = \tilde{0}$; i.e. for every $\varepsilon > 0$ there

is an $N(\varepsilon) \in \mathbb{N}$ such that $D(\|x_n - x\|, \tilde{0}) < \varepsilon$ for all $n > N(\varepsilon)$. This means that for every $\varepsilon > 0$ there is an $N(\varepsilon) \in \mathbb{N}$ such that

$$\sup_{\alpha \in [0, 1]} \|x_n - x\|_\alpha^+ = \|x_n - x\|_0^+ < \varepsilon$$

for all $n \geq N(\varepsilon)$.

Let $(X, \|\cdot\|)$ be an FNS. A sequence (x_k) in X is statistically convergent to $L \in X$ with respect to the fuzzy norm on X and we denote by $x_n \xrightarrow{FS} x$, provided that for each $\varepsilon > 0$, we have $\delta(\{k \in \mathbb{N} : D(\|x_k - L\|, \tilde{0}) \geq \varepsilon\}) = 0$. This implies that for each $\varepsilon > 0$, the set $K(\varepsilon) = \{k \in \mathbb{N} : \|x_k - L\|_0^+ \geq \varepsilon\}$ has natural density zero; namely, for each $\varepsilon > 0$, $\|x_k - L\|_0^+ < \varepsilon$ for almost all k .

By a lacunary sequence we mean an increasing integer sequence $\theta = \{k_r\}$ such that $k_0 = 0$ and $h_r = k_r - k_{r-1} \rightarrow \infty$ as $r \rightarrow \infty$. The intervals determined by θ will be denoted by $I_r = (k_{r-1}, k_r]$.

Let $(X, \|\cdot\|)$ be an FNS and $\theta = \{k_r\}$ be lacunary sequence. A sequence $x = (x_k)_{k \in \mathbb{N}}$ in X is said to be lacunary summable with respect to fuzzy norm on X if there is an $L \in X$ such that

$$\lim_{r \rightarrow \infty} \frac{1}{h_r} \left(\sum_{k \in I_r} D(\|x_k - L\|, \tilde{0}) \right) = 0.$$

In this case, we can write $x_k \rightarrow L ((N_\theta)_{FN})$ or $x_k \xrightarrow{(N_\theta)_{FN}} L$ and

$$(N_\theta)_{FN} = \left\{ x = (x_k) : \lim_{r \rightarrow \infty} \frac{1}{h_r} \left(\sum_{k \in I_r} D(\|x_k - L\|, \tilde{0}) \right) = 0, \text{ for some } L \right\}.$$

A sequence $x = (x_k)$ in X is said to be lacunary statistically convergent or FS_θ -convergent to $L \in X$ with respect to fuzzy norm on X if for each $\varepsilon > 0$

$$\lim_{r \rightarrow \infty} \frac{1}{h_r} |\{k \in I_r : D(\|x_k - L\|, \tilde{0}) \geq \varepsilon\}| = 0$$

where $|A|$ denotes the number of elements of the set $A \subseteq \mathbb{N}$. In this case, we write $x_k \xrightarrow{FS_\theta} L$ or $x_k \rightarrow L (FS_\theta)$ or $FS_\theta - \lim_{k \rightarrow \infty} x_k = L$. This implies that for each $\varepsilon > 0$, the set $K(\varepsilon) = \{k \in I_r : \|x_k - L\|_0^+ \geq \varepsilon\}$ has natural density zero, namely, for each $\varepsilon > 0$, $\|x_k - L\|_0^+ < \varepsilon$, for almost all k .

Let $X \neq \emptyset$. A class \mathcal{I} of subsets of X is said to be an ideal in X provided:

(i) $\emptyset \in \mathcal{I}$, (ii) $A, B \in \mathcal{I}$ implies $A \cup B \in \mathcal{I}$, (iii) $A \in \mathcal{I}$, $B \subset A$ implies $B \in \mathcal{I}$.

\mathcal{I} is called a nontrivial ideal if $X \notin \mathcal{I}$. A nontrivial ideal \mathcal{I} in X is called admissible if $\{x\} \in \mathcal{I}$ for each $x \in X$.

Let $X \neq \emptyset$. A non empty class \mathcal{F} of subsets of X is said to be a filter in X provided:

(i) $\emptyset \notin \mathcal{F}$, (ii) $A, B \in \mathcal{F}$ implies $A \cap B \in \mathcal{F}$, (iii) $A \in \mathcal{F}$, $A \subset B$ implies $B \in \mathcal{F}$.

Let \mathcal{I} is a nontrivial ideal in X , $X \neq \emptyset$, then the class $\mathcal{F}(\mathcal{I}) = \{M \subset X : (\exists A \in \mathcal{I})(M = X \setminus A)\}$ is a filter on X , called the filter associated with \mathcal{I} .

Let $(X, \|\cdot\|)$ be fuzzy normed space. A sequence $x = (x_m)_{m \in \mathbb{N}}$ in X is said to be \mathcal{I} -convergent to $L \in X$ with respect to fuzzy norm on X if for each $\varepsilon > 0$, the set $A(\varepsilon) = \{m \in \mathbb{N} : \|x_m - L\|_0^+ \geq \varepsilon\}$ belongs to \mathcal{I} . In this

case, we write $x_m \xrightarrow{F\mathcal{I}} L$. The element L is called the \mathcal{I} -limit of (x_m) in X .

A sequence (x_m) in X is said to be \mathcal{I}^* -convergent to L in X with respect to the fuzzy norm on X if there exists a set $M \in \mathcal{F}(\mathcal{I})$, $M = \{t_k : t_1 < t_2 < \dots\} \subset \mathbb{N}$ such that $\lim_{k \rightarrow \infty} \|x_{t_k} - L\| = 0$.

2. Main Result

In this section, we gave the definition of lacunary \mathcal{I} -convergence and definition of lacunary \mathcal{I} -Cauchy in fuzzy normed spaces. Also, we investigate some properties these concepts.

Throughout the paper, we let $(X, \|\cdot\|)$ be an FNS and $\mathcal{I} \subset 2^{\mathbb{N}}$ be an admissible ideal.

Definition 2.1 A sequence $x = (x_m)_{m \in \mathbb{N}}$ in X is said to be lacunary \mathcal{I} -convergent to $L_1 \in X$ with respect to fuzzy norm on X if for each $\varepsilon > 0$, the set $\left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{m \in J_r} D(\|x_m - L_1\|, \tilde{0}) \geq \varepsilon \right\}$ belongs to \mathcal{I} . In this case,

we write $x_m \xrightarrow{F\mathcal{I}_\theta} L_1$ or $x_m \rightarrow L_1 (F\mathcal{I}_\theta)$ or $F\mathcal{I}_\theta - \lim_{m \rightarrow \infty} x_m = L_1$. The element L_1 is called the $F\mathcal{I}_\theta$ -limit of (x_m) in X .

Lemma 2.2 Let $(X, \|\cdot\|)$ be a fuzzy normed space, and $x = (x_m)$ be a sequence in X . Then, for every $\varepsilon > 0$, the following statements are equivalent.

a) $F\mathcal{I}_\theta - \lim_{m \rightarrow \infty} x_m = L_1$,

b) $\left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{m \in J_r} \|x_m - L_1\|_0^+ \geq \varepsilon \right\} \in \mathcal{I}$,

$$c) \left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{m \in J_r} \|x_m - L_1\|_0^+ < \varepsilon \right\} \in F(\mathcal{I})$$

$$d) F\mathcal{I}_\theta - \lim_{m \rightarrow \infty} \|x_m - L_1\|_0^+ = 0$$

Theorem 2.3 Let $(X, \|\cdot\|)$ be a fuzzy normed space. If a sequence $x = (x_m)$ in X is lacunary \mathcal{I} -convergent with respect to fuzzy norm on X , then $F\mathcal{I}_\theta - \lim x$ is unique.

Proof. Suppose that $F\mathcal{I}_\theta - \lim x = L_1$ and $F\mathcal{I}_\theta - \lim x = L_2$. Then for any $\varepsilon > 0$, define the following sets;

$$A_1 = \left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{m \in J_r} \|x_m - L_1\|_0^+ \geq \frac{\varepsilon}{2} \right\}$$

$$A_2 = \left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{m \in J_r} \|x_m - L_2\|_0^+ \geq \frac{\varepsilon}{2} \right\}.$$

Since $F\mathcal{I}_\theta - \lim x = L_1$ and $F\mathcal{I}_\theta - \lim x = L_2$, using Lemma 2.2, we have $A_1 \in \mathcal{I}$ and $A_2 \in \mathcal{I}$ for all $\varepsilon > 0$. Now, let $A_3 = A_1 \cup A_2$. Then $A_3 \in \mathcal{I}$. This implies that its complement $(A_3)^c$ is a non-empty set in $F(\mathcal{I})$. Now, if $r \in (A_3)^c$, then we have

$$\frac{1}{h_r} \sum_{m \in J_r} \|x_m - L_1\|_0^+ < \frac{\varepsilon}{2} \text{ and } \frac{1}{h_r} \sum_{m \in J_r} \|x_m - L_2\|_0^+ < \frac{\varepsilon}{2}.$$

Now, clearly, we will get a $p \in \mathbb{N}$ such that

$$\|x_p - L_1\|_0^+ < \frac{1}{h_r} \sum_{m \in J_r} \|x_m - L_1\|_0^+ < \frac{\varepsilon}{2} \text{ and } \|x_p - L_2\|_0^+ < \frac{1}{h_r} \sum_{m \in J_r} \|x_m - L_2\|_0^+ < \frac{\varepsilon}{2}.$$

Then, we have $\|L_1 - L_2\|_0^+ \leq \|x_p - L_1\|_0^+ + \|x_p - L_2\|_0^+ < \varepsilon$

Since $\varepsilon > 0$ is arbitrary, we have $\|L_1 - L_2\|_0^+ = 0$ which implies that $L_1 = L_2$. Therefore, we conclude that $F\mathcal{I}_\theta - \lim x$ is unique. \square

Theorem 2.4 Let $(X, \|\cdot\|)$ be a fuzzy normed space and $(x_m), (y_m)$ be two sequences in X . Then,

i) If $F\mathcal{I}_\theta - \lim x_m = L_1$ and $F\mathcal{I}_\theta - \lim y_m = L_2$, then $F\mathcal{I}_\theta - \lim (x_m \mp y_m) = L_1 \mp L_2$;

ii) If $F\mathcal{I}_\theta - \lim x_m = L_1$ then $F\mathcal{I}_\theta - \lim cx_m = cL_1$ for $c \in \mathbb{R} - \{0\}$.

Proof. i) We shall prove, if $F\mathcal{I}_\theta - \lim x_m = L_1$ and $F\mathcal{I}_\theta - \lim y_m = L_2$, then $F\mathcal{I}_\theta - \lim (x_m + y_m) = L_1 + L_2$, only. The proof of the other part follows similarly. For any $\varepsilon > 0$, define the following sets;

$$A_1 = \left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{m \in J_r} \|x_m - L_1\|_0^+ \geq \frac{\varepsilon}{2} \right\},$$

$$A_2 = \left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{m \in J_r} \|y_m - L_2\|_0^+ \geq \frac{\varepsilon}{2} \right\},$$

Since $F\mathcal{I}_\theta - \lim x_m = L_1$ and $F\mathcal{I}_\theta - \lim y_m = L_2$, using Lemma 2.2, we have $A_1 \in \mathcal{I}$ and $A_2 \in \mathcal{I}$ for all $\varepsilon > 0$.

Now, let $A_3 = A_1 \cup A_2$. Then $A_3 \in \mathcal{I}$. This implies that its complement $(A_3)^c$ is a non-empty set in $F(\mathcal{I})$. We claim that

$$(A_3)^c \subset \left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{m \in J_r} \|x_m - L_1 + y_m - L_2\|_0^+ < \varepsilon \right\}.$$

Let $r \in (A_3)^c$, then we have

$$\frac{1}{h_r} \sum_{m \in J_r} \|x_m - L_1\|_0^+ < \frac{\varepsilon}{2} \text{ and } \frac{1}{h_r} \sum_{m \in J_r} \|y_m - L_2\|_0^+ < \frac{\varepsilon}{2}.$$

Now, we will get a $p \in \mathbb{N}$ such that

$$\|x_p - L_1\|_0^+ < \frac{1}{h_r} \sum_{m \in J_r} \|x_m - L_1\|_0^+ < \frac{\varepsilon}{2} \text{ and } \|y_p - L_2\|_0^+ < \frac{1}{h_r} \sum_{m \in J_r} \|y_m - L_2\|_0^+ < \frac{\varepsilon}{2}.$$

Then, we have $\|x_p - L_1 + y_p - L_2\|_0^+ \leq \|x_p - L_1\|_0^+ + \|y_p - L_2\|_0^+ < \varepsilon$

Hence,

$$(A_3)^c \subset \left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{m \in J_r} \|x_m - L_1 + y_m - L_2\|_0^+ < \varepsilon \right\}.$$

Since $(A_3)^c \in F(\mathcal{I})$, so $\left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{m \in J_r} \|(x_m + y_m) - (L_1 + L_2)\|_0^+ \geq \varepsilon \right\} \in \mathcal{I}$.

Therefore $F\mathcal{I}_\theta - \lim (x_m + y_m) = L_1 + L_2$.

ii) Let $F\mathcal{I}_\theta - \lim x_m = L_1$. Then, for each $\varepsilon > 0$ and $c \in \mathbb{R} - \{0\}$, we define the following set

$$A_1 = \left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{m \in J_r} \|x_m - L_1\|_0^+ < \frac{\varepsilon}{|c|} \right\}.$$

So $A_1 \in F(\mathcal{I})$. Let $r \in A_1$, then we have

$$\frac{1}{h_r} \sum_{m \in J_r} \|x_m - L_1\|_0^+ < \frac{\varepsilon}{|c|}$$

$$\frac{|c|}{h_r} \sum_{m \in J_r} \|x_m - L_1\|_0^+ < |c| \cdot \frac{\varepsilon}{|c|}$$

$$\frac{1}{h_r} \sum_{m \in J_r} |c| \|x_m - L_1\|_0^+ < \varepsilon$$

$$\frac{1}{h_r} \sum_{m \in J_r} \|c \cdot x_m - c \cdot L_1\|_0^+ < \varepsilon$$

Hence,

$$A_1 \subset \left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{m \in J_r} \|c x_m - c L_1\|_0^+ < \varepsilon \right\}$$

and

$$\left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{m \in J_r} \|c x_m - c L_1\|_0^+ < \varepsilon \right\} \in F(\mathcal{I}).$$

Hence $F\mathcal{I}_\theta - \lim c x_m = c L_1$. □

Definition 2.5 Let $(X, \|\cdot\|)$ be a fuzzy normed space. A sequence $x = (x_m)$ in X is said to be $F\mathcal{I}_\theta$ -Cauchy sequence with respect to the fuzzy norm if, for every $\varepsilon > 0$, there exists $n \in \mathbb{N}$ satisfying

$$\left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{m \in J_r} \|x_m - x_n\|_0^+ < \varepsilon \right\} \in F(\mathcal{I}).$$

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On I_2 -Cauchy Double Dequences in Fuzzy Normed Spaces

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Abstract: In this study, we have investigated the concepts of I_2 -Cauchy and I_2 -convergence of double sequences in fuzzy normed spaces. Also, we have investigated some properties and relationships between these concepts.

1. Introduction and Preliminaries

Throughout the paper \mathbb{N} and \mathbb{R} denote the set of all positive integers and the set of all real numbers, respectively. The concept of convergence of a sequence of real numbers has been extended to statistical convergence independently by Fast [14] and Schoenberg [41]. A lot of developments have been made in this area after the various studies of researchers [14, 16, 26, 33].

The idea of I -convergence was introduced by Kostyrko et al. [19] as a generalization of statistical convergence which is based on the structure of the ideal I of subset of the set of natural numbers \mathbb{N} . Das et al. [5] introduced the concept of I -convergence of double sequences in a metric space and studied some properties of this convergence. A lot of developments have been made in this area after the works of [6, 20, 21, 29, 35, 43].

The concept of ordinary convergence of a sequence of fuzzy numbers was firstly introduced by Matloka [23] and proved some basic theorems for sequences of fuzzy numbers. Nanda [28] studied the sequences of fuzzy numbers and showed that the set of all convergent sequences of fuzzy numbers form a complete metric space. Şençimen and Pehlivan [40] introduced the notions of statistically convergent sequence and statistically Cauchy sequence in a fuzzy normed linear space. Hazarika [17] studied the concepts of I -convergence, I^* -convergence and I -Cauchy sequence in a fuzzy normed linear space. Dündar and Talu [11, 12] introduced the concepts of I_2 -convergence, I_2^* -convergence, I_2 -Cauchy sequence for double sequences of fuzzy numbers and studied some properties and relations of them. Hazarika and Kumar introduced the notion of I_2 -convergent and I_2 -Cauchy double sequences in a fuzzy normed linear space. A lot of developments have been made in this area after the various studies of researchers [25, 38, 39, 44, 45, 47].

Now, we recall the concept of ideal, convergence, statistical convergence, ideal convergence of sequence, double sequence and fuzzy normed and some basic definitions (see [1–3, 7–11, 13–16, 24–27, 29–33, 36, 37, 40, 44–46])

Fuzzy sets are considered with respect to a nonempty base set X of elements of interest. The essential idea is that each element $x \in X$ is assigned a membership grade $u(x)$ taking values in $[0, 1]$, with $u(x) = 0$ corresponding to nonmembership, $0 < u(x) < 1$ to partial membership, and $u(x) = 1$ to full membership. According to Zadeh [48], a fuzzy subset of X is a nonempty subset $\{(x, u(x)) : x \in X\}$ of $X \times [0, 1]$ for some function $u : X \rightarrow [0, 1]$. The function u itself is often used for the fuzzy set.

A fuzzy set u on \mathbb{R} is called a fuzzy number if it has the following properties:

1. u is normal, that is, there exists an $x_0 \in \mathbb{R}$ such that $u(x_0) = 1$;

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2. u is fuzzy convex, that is, for $x, y \in \mathbb{R}$ and $0 \leq \lambda \leq 1$, $u(\lambda x + (1 - \lambda)y) \geq \min[u(x), u(y)]$;
3. u is upper semicontinuous;
4. $\text{supp } u = \text{cl}\{x \in \mathbb{R} : u(x) > 0\}$, or denoted by $[u]_0$, is compact.

Let $L(\mathbb{R})$ be set of all fuzzy numbers. If $u \in L(\mathbb{R})$ and $u(t) = 0$ for $t < 0$, then u is called a non-negative fuzzy number. We write $L^*(\mathbb{R})$ by the set of all non-negative fuzzy numbers. We can say that $u \in L^*(\mathbb{R})$ iff $u_{\alpha}^- \geq 0$ for each $\alpha \in [0, 1]$. Clearly we have $\tilde{0} \in L(\mathbb{R})$. For $u \in L(\mathbb{R})$, the α level set of u is defined by

$$[u]_{\alpha} = \begin{cases} \{x \in \mathbb{R} : u(x) \geq \alpha\}, & \text{if } \alpha \in (0, 1] \\ \text{supp } u, & \text{if } \alpha = 0. \end{cases}$$

A partial order \preceq on $L(\mathbb{R})$ is defined by $u \preceq v$ if $u_{\alpha}^- \leq v_{\alpha}^-$ and $u_{\alpha}^+ \leq v_{\alpha}^+$ for all $\alpha \in [0, 1]$.

Arithmetic operation for $t \in \mathbb{R}$, \oplus, \ominus, \odot and \oslash on $L(\mathbb{R}) \times L(\mathbb{R})$ are defined by
 $(u \oplus v)(t) = \sup_{s \in \mathbb{R}} \{u(s) \wedge v(t-s)\}$, $(u \ominus v)(t) = \sup_{s \in \mathbb{R}} \{u(s) \wedge v(s-t)\}$,
 $(u \odot v)(t) = \sup_{s \in \mathbb{R}, s \neq 0} \{u(s) \wedge v(t/s)\}$ and $(u \oslash v)(t) = \sup_{s \in \mathbb{R}} \{u(st) \wedge v(s)\}$.

For $k \in \mathbb{R}^+$, ku is defined as $ku(t) = u(t/k)$ and $0u(t) = \tilde{0}$, $t \in \mathbb{R}$.

Some arithmetic operations for α -level sets are defined as follows:

$u, v \in L(\mathbb{R})$ and $[u]_{\alpha} = [u_{\alpha}^-, u_{\alpha}^+]$ and $[v]_{\alpha} = [v_{\alpha}^-, v_{\alpha}^+]$, $\alpha \in (0, 1]$. Then,
 $[u \oplus v]_{\alpha} = [u_{\alpha}^- + v_{\alpha}^-, u_{\alpha}^+ + v_{\alpha}^+]$, $[u \ominus v]_{\alpha} = [u_{\alpha}^- - v_{\alpha}^+, u_{\alpha}^+ - v_{\alpha}^-]$,
 $[u \odot v]_{\alpha} = [u_{\alpha}^- \cdot v_{\alpha}^-, u_{\alpha}^+ \cdot v_{\alpha}^+]$ and $[\tilde{1} \oslash u]_{\alpha} = [\frac{1}{u_{\alpha}^+}, \frac{1}{u_{\alpha}^-}]$, $u_{\alpha}^- > 0$.

For $u, v \in L(\mathbb{R})$, the supremum metric on $L(\mathbb{R})$ defined as

$$D(u, v) = \sup_{0 \leq \alpha \leq 1} \max \{|u_{\alpha}^- - v_{\alpha}^-|, |u_{\alpha}^+ - v_{\alpha}^+|\}.$$

It is known that D is a metric on $L(\mathbb{R})$ and $(L(\mathbb{R}), D)$ is a complete metric space.

A sequence $x = (x_k)$ of fuzzy numbers is said to be convergent to the fuzzy number x_0 , if for every $\varepsilon > 0$ there exists a positive integer k_0 such that $D(x_k, x_0) < \varepsilon$ for $k > k_0$ and a sequence $x = (x_k)$ of fuzzy numbers convergens to levelwise to x_0 iff $\lim_{k \rightarrow \infty} [x_k]_{\alpha} = [x_0]_{\alpha}^-$ and $\lim_{k \rightarrow \infty} [x_k]_{\alpha} = [x_0]_{\alpha}^+$, where $[x_k]_{\alpha} = [(x_k)_{\alpha}^-, (x_k)_{\alpha}^+]$ and $[x_0]_{\alpha} = [(x_0)_{\alpha}^-, (x_0)_{\alpha}^+]$, for every $\alpha \in (0, 1)$.

Let X be a vector space over \mathbb{R} , $\|\cdot\| : X \rightarrow L^*(\mathbb{R})$ and the mappings $L; R$ (respectively, left norm and right norm) $: [0, 1] \times [0, 1] \rightarrow [0, 1]$ be symetric, nondecreasing in both arguments and satisfy $L(0, 0) = 0$ and $R(1, 1) = 1$. The quadruple $(X, \|\cdot\|, L, R)$ is called fuzzy normed linear space (briefly $(X, \|\cdot\|)$ FNS) and $\|\cdot\|$ a fuzzy norm if the following axioms are satisfied:

1. $\|x\| = \tilde{0}$ iff $x = 0$,
2. $\|rx\| = |r| \odot \|x\|$ for $x \in X$, $r \in \mathbb{R}$,
3. For all $x, y \in X$
 - (a) $\|x+y\|(s+t) \geq L(\|x\|(s), \|y\|(t))$, whenever $s \leq \|x\|_{\tilde{1}}^-$, $t \leq \|y\|_{\tilde{1}}^-$ and $s+t \leq \|x+y\|_{\tilde{1}}^-$,
 - (b) $\|x+y\|(s+t) \leq R(\|x\|(s), \|y\|(t))$, whenever $s \geq \|x\|_{\tilde{1}}^+$, $t \geq \|y\|_{\tilde{1}}^+$ and $s+t \geq \|x+y\|_{\tilde{1}}^+$.

Let $(X, \|\cdot\|_C)$ be an ordinary normed linear space. Then, a fuzzy norm $\|\cdot\|$ on X can be obtained by

$$\|x\|(t) = \begin{cases} 0, & \text{if } 0 \leq t \leq a\|x\|_C \text{ or } t \geq b\|x\|_C \\ \frac{t}{(1-a)\|x\|_C} - \frac{a}{1-a}, & a\|x\|_C \leq t \leq \|x\|_C \\ \frac{-t}{(b-1)\|x\|_C} + \frac{b}{b-1}, & \|x\|_C \leq t \leq b\|x\|_C \end{cases}$$

where $\|x\|_C$ is the ordinary norm of x ($\neq 0$), $0 < a < 1$ and $1 < b < \infty$. For $x = \theta$, define $\|x\| = \tilde{0}$. Hence, $(X, \|\cdot\|)$ is a fuzzy normed linear space.

Let us consider the topological structure of an FNS $(X, \|\cdot\|)$. For any $\varepsilon > 0$, $\alpha \in [0, 1]$ and $x \in X$, the (ε, α) -neighborhood of x is the set $\mathcal{N}_x(\varepsilon, \alpha) = \{y \in X : \|x-y\|_{\alpha}^+ < \varepsilon\}$.

Let $(X, \|\cdot\|)$ be an FNS. A sequence $(x_n)_{n=1}^{\infty}$ in X is convergent to $x \in X$ with respect to the fuzzy norm on X and we denote by $x_n \xrightarrow{FN} x$, provided that $(D) - \lim_{n \rightarrow \infty} \|x_n - x\| = \tilde{0}$; i.e., for every $\varepsilon > 0$ there is an $N(\varepsilon) \in \mathbb{N}$ such that $D(\|x_n - x\|, \tilde{0}) < \varepsilon$ for all $n \geq N(\varepsilon)$. This means that for every $\varepsilon > 0$ there is an $N(\varepsilon) \in \mathbb{N}$ such that for all $n \geq N(\varepsilon)$, $\sup_{\alpha \in [0, 1]} \|x_n - x\|_{\alpha}^+ = \|x_n - x\|_0^+ < \varepsilon$.

Let $(X, \|\cdot\|)$ be an FNS. A sequence (x_k) in X is statistically convergent to $L \in X$ with respect to the fuzzy norm on X and we denote by $x_n \xrightarrow{FS} x$, provided that for each $\varepsilon > 0$, we have $\delta(\{k \in \mathbb{N} : D(\|x_k - L\|, \tilde{0}) \geq \varepsilon\}) = 0$.

This implies that for each $\varepsilon > 0$, the set $K(\varepsilon) = \{k \in \mathbb{N} : \|x_k - L\|_0^+ \geq \varepsilon\}$ has natural density zero; namely, for each $\varepsilon > 0$, $\|x_k - L\|_0^+ < \varepsilon$ for almost all k .

Let $(X, \|\cdot\|)$ be an *FNS*. Then a double sequence (x_{jk}) is said to be convergent to $x \in X$ with respect to the fuzzy norm on X if for every $\varepsilon > 0$ there exist a number $N = N(\varepsilon)$ such that $D(\|x_{jk} - x\|, \tilde{0}) < \varepsilon$, for all $j, k \geq N$.

In this case, we write $x_{jk} \xrightarrow{FN} x$. This means that, for every $\varepsilon > 0$ there exist a number $N = N(\varepsilon)$ such that

$\sup_{\alpha \in [0,1]} \|x_{jk} - x\|_\alpha^+ = \|x_{jk} - x\|_0^+ < \varepsilon$, for all $j, k \geq N$. In terms of neighborhoods, we have $x_{jk} \xrightarrow{FN} x$ provided

that for any $\varepsilon > 0$, there exists a number $N = N(\varepsilon)$ such that $x_{jk} \in \mathcal{N}_x(\varepsilon, 0)$, whenever $j, k \geq N$.

Let $X \neq \emptyset$. A class I of subsets of X is said to be an ideal in X provided:

(i) $\emptyset \in I$, (ii) $A, B \in I$ implies $A \cup B \in I$, (iii) $A \in I, B \subset A$ implies $B \in I$.

I is called a nontrivial ideal if $X \notin I$. A nontrivial ideal I in X is called admissible if $\{x\} \in I$ for each $x \in X$.

A nontrivial ideal I_2 of $\mathbb{N} \times \mathbb{N}$ is called strongly admissible if $\{i\} \times \mathbb{N}$ and $\mathbb{N} \times \{i\}$ belong to I_2 for each $i \in \mathbb{N}$. It is evident that a strongly admissible ideal is also admissible. Throughout the paper we take I_2 as a strongly admissible ideal in $\mathbb{N} \times \mathbb{N}$.

Let $I_2^0 = \{A \subset \mathbb{N} \times \mathbb{N} : (\exists m(A), (i, j) \geq m(A) \Rightarrow (i, j) \notin A)\}$. Then I_2^0 is a nontrivial strongly admissible ideal and clearly an ideal I_2 is strongly admissible if and only if $I_2^0 \subset I_2$.

Let $X \neq \emptyset$. A non empty class \mathcal{F} of subsets of X is said to be a filter in X provided:

(i) $\emptyset \notin \mathcal{F}$, (ii) $A, B \in \mathcal{F}$ implies $A \cap B \in \mathcal{F}$, (iii) $A \in \mathcal{F}, A \subset B$ implies $B \in \mathcal{F}$.

Let I is a nontrivial ideal in $X, X \neq \emptyset$, then the class $\mathcal{F}(I) = \{M \subset X : (\exists A \in I)(M = X \setminus A)\}$ is a filter on X , called the filter associated with I .

Let (X, ρ) be a linear metric space and $I_2 \subset 2^{\mathbb{N} \times \mathbb{N}}$ be a strongly admissible ideal. A double sequence $x = (x_{mn})$ in X is said to be I_2 -convergent to $L \in X$, if for any $\varepsilon > 0$ we have $A(\varepsilon) = \{(m, n) \in \mathbb{N} \times \mathbb{N} : \rho(x_{mn}, L) \geq \varepsilon\} \in I_2$ and we write $I_2 - \lim_{m, n \rightarrow \infty} x_{mn} = L$.

If I_2 is a strongly admissible ideal on $\mathbb{N} \times \mathbb{N}$, then usual convergence implies I_2 -convergence.

Let $(X, \|\cdot\|)$ be fuzzy normed space. A sequence $x = (x_m)_{m \in \mathbb{N}}$ in X is said to be I -convergent to $L \in X$ with respect to fuzzy norm on X if for each $\varepsilon > 0$, the set $A(\varepsilon) = \{m \in \mathbb{N} : \|x_m - L\|_0^+ \geq \varepsilon\}$ belongs to I . In this case, we write $x_m \xrightarrow{I} L$. The element L is called the I -limit of (x_m) in X .

Let $(X, \|\cdot\|)$ be fuzzy normed space. A double sequence $x = (x_{mn})_{(m,n) \in \mathbb{N} \times \mathbb{N}}$ in X is said to be I_2 -convergent to $L_1 \in X$ with respect to fuzzy norm on X if for each $\varepsilon > 0$, the set $A(\varepsilon) = \{(m, n) \in \mathbb{N} \times \mathbb{N} : \|x_{mn} - L_1\|_0^+ \geq \varepsilon\}$ belongs to I_2 . In this case, we write $x_{mn} \xrightarrow{FI_2} L_1$ or $x_{mn} \rightarrow L_1 (FI_2)$ or $FI_2 - \lim_{m, n \rightarrow \infty} x_{mn} = L_1$. The element L_1 is called the FI_2 -limit of (x_{mn}) in X . In terms of neighborhoods, we have $x_{mn} \xrightarrow{FI_2} L_1$ provided that for each $\varepsilon > 0$,

$$\{(m, n) \in \mathbb{N} \times \mathbb{N} : x_{mn} \notin \mathcal{N}_{L_1}(\varepsilon, 0)\} \in I_2.$$

A useful interpretation of the above definition is the following;

$$x_{mn} \xrightarrow{FI_2} L_1 \Leftrightarrow FI_2 - \lim_{m, n \rightarrow \infty} \|x_{mn} - L_1\|_0^+ = 0.$$

Note that $FI_2 - \lim_{m, n \rightarrow \infty} \|x_{mn} - L_1\|_0^+ = 0$ implies that

$$FI_2 - \lim_{\alpha} \|x_{mn} - L_1\|_\alpha^- = FS_{\theta_2} - \lim_{\alpha} \|x_{mn} - L_1\|_\alpha^+ = 0,$$

for each $\alpha \in [0, 1]$, since

$$0 \leq \|x_{mn} - L_1\|_\alpha^- \leq \|x_{mn} - L_1\|_\alpha^+ \leq \|x_{mn} - L_1\|_0^+$$

holds for every $m, n \in \mathbb{N}$ and for each $\alpha \in [0, 1]$.

Let I_2 be an admissible ideal of $\mathbb{N} \times \mathbb{N}$ and $(X, \|\cdot\|)$ be a fuzzy normed space. A double sequence $x = (x_{mn})$ in X is said to be I_2 -Cauchy with respect to the fuzzy norm on X if for each $\varepsilon > 0$, there exist integers $p = p(\varepsilon)$ and $q = q(\varepsilon)$ such that the set

$$\{(m, n) \in \mathbb{N} \times \mathbb{N} : \|x_{mn} - x_{pq}\|_0^+ \geq \varepsilon\}$$

belongs to I_2 .

We say that an admissible ideal $I_2 \subset 2^{\mathbb{N} \times \mathbb{N}}$ satisfies the property (AP2), if for every countable family of mutually disjoint sets $\{A_1, A_2, \dots\}$ belonging to I_2 , there exists a countable family of sets $\{B_1, B_2, \dots\}$ such that $A_j \cap B_j \in I_2^0$, i.e., $A_j \cap B_j$ is included in the finite union of rows and columns in $\mathbb{N} \times \mathbb{N}$ for each $j \in \mathbb{N}$ and $B = \bigcup_{j=1}^{\infty} B_j \in I_2$ (hence $B_j \in I_2$ for each $j \in \mathbb{N}$).

Lemma 1.1 Let $\{P_i\}_{i=1}^{\infty}$ be a countable collection of subsets of $\mathbb{N} \times \mathbb{N}$ such that $P_i \in F(I_2)$ for each i , where $F(I_2)$ is a filter associated with a strongly admissible ideal I_2 with the property (AP2). Then there exists a set $P \subset \mathbb{N} \times \mathbb{N}$ such that $P \in F(I_2)$ and the set $P \setminus P_i$ is finite for all i .

2. Main Results

In this section, we give some theorem for I_2 -convergence and I_2 -Cauchy for double sequences in fuzzy normed spaces and study some properties.

Theorem 2.1 *Let I_2 be an admissible ideal. If a double sequence (x_{mn}) in X is I_2 -convergent to L_1 , then L_1 is uniquely determined.*

Proof. Let (x_{mn}) be any double sequence and suppose that $FI_2 - \lim_{m,n \rightarrow \infty} x_{mn} = L_1$ and $FI_2 - \lim_{m,n \rightarrow \infty} x_{mn} = L_2$, where $L_1 \neq L_2$. Since $L_1 \neq L_2$, we may suppose that $L_1 > L_2$. Select $\varepsilon = \frac{L_1 - L_2}{4}$, so that the neighborhoods $(L_1 - \varepsilon, L_1 + \varepsilon)$ and $(L_2 - \varepsilon, L_2 + \varepsilon)$ of L_1 and L_2 respectively are disjoint. Since L_1 and L_2 both are I_2 -limit of the sequence (x_{mn}) , therefore, both the sets

$$A(\varepsilon) = \{(m, n) \in \mathbb{N} \times \mathbb{N} : \|x_{mn} - L_1\|_0^+ \geq \varepsilon\}$$

and

$$B(\varepsilon) = \{(m, n) \in \mathbb{N} \times \mathbb{N} : \|x_{mn} - L_2\|_0^+ \geq \varepsilon\}$$

belongs to I_2 .

This implies that the sets

$$A^c(\varepsilon) = \{(m, n) \in \mathbb{N} \times \mathbb{N} : \|x_{mn} - L_1\|_0^+ < \varepsilon\}$$

and

$$B^c(\varepsilon) = \{(m, n) \in \mathbb{N} \times \mathbb{N} : \|x_{mn} - L_2\|_0^+ < \varepsilon\}$$

belongs to $F(I_2)$. Since $F(I_2)$ is a filter on $\mathbb{N} \times \mathbb{N}$ therefore $A^c(\varepsilon) \cap B^c(\varepsilon)$ is a non empty set in $F(I_2)$. In this way we obtain a contradiction to the fact that the neighborhoods $(L_1 - \varepsilon, L_1 + \varepsilon)$ and $(L_2 - \varepsilon, L_2 + \varepsilon)$ of L_1 and L_2 respectively are disjoint. Hence we have $L_1 = L_2$. \square

Theorem 2.2 *Let I_2 be an admissible ideal, (x_{mn}) be a double sequence on X and $L_1 \in X$. Then $FP - \lim_{m,n \rightarrow \infty} x_{mn} = L_1 \Rightarrow FI_2 - \lim_{m,n \rightarrow \infty} x_{mn} = L_1$.*

Proof. Let $FP - \lim_{m,n \rightarrow \infty} x_{mn} = L_1$. For $\varepsilon > 0$ there exists a positive integer $k_0 = k_0(\varepsilon)$ such that $\|x_{mn} - L_1\|_0^+ < \varepsilon$ whenever $m, n \geq k_0$. This implies that the set

$$\begin{aligned} A(\varepsilon) &= \{(m, n) \in \mathbb{N} \times \mathbb{N} : \|x_{mn} - L_1\|_0^+ \geq \varepsilon\} \\ &\subset (\mathbb{N} \times \{1, 2, 3, \dots, k_0 - 1\}) \cup (\{1, 2, 3, \dots, k_0 - 1\} \times \mathbb{N}). \end{aligned}$$

Since I_2 is an admissible ideal, then

$$(\mathbb{N} \times \{1, 2, 3, \dots, k_0 - 1\}) \cup (\{1, 2, 3, \dots, k_0 - 1\} \times \mathbb{N}) \in I_2$$

and so $A(\varepsilon) \in I_2$. Hence, we have $FI_2 - \lim_{m,n \rightarrow \infty} x_{mn} = L_1$ \square

Theorem 2.3 *Let I_2 be an admissible ideal and $(X, \|\cdot\|)$ be a fuzzy normed space. Then a double sequence (x_{mn}) is I_2 -convergent if and only if it is I_2 -Cauchy double sequence.*

Proof. Let $x_{mn} \rightarrow L_1 (FI_2)$. Then for each $\varepsilon > 0$, we have

$$A(\varepsilon) = \{(m, n) \in \mathbb{N} \times \mathbb{N} : \|x_{mn} - L_1\|_0^+ \geq \varepsilon\} \text{ belongs to } I_2.$$

Since I_2 is an admissible ideal, there exist an $(m_0, n_0) \in \mathbb{N} \times \mathbb{N}$ such that $(m_0, n_0) \notin A(\varepsilon)$.

Let $A_1(\varepsilon) = \{(m, n) \in \mathbb{N} \times \mathbb{N} : \|x_{mn} - L_1\|_0^+ \geq 2\varepsilon\}$. Since $\|\cdot\|_0^+$ being a norm in usual sense, we get

$$\|x_{mn} - L_1\|_0^+ + \|x_{m_0 n_0} - L_1\|_0^+ \geq \|x_{mn} - x_{m_0 n_0}\|_0^+.$$

We observe that if $(m, n) \in A_1(\varepsilon)$, then $\|x_{mn} - L_1\|_0^+ + \|x_{m_0 n_0} - L_1\|_0^+ \geq 2\varepsilon$.

On the other hand since $(m_0, n_0) \notin A(\varepsilon)$, we have

$$\|x_{m_0 n_0} - L_1\|_0^+ < \varepsilon.$$

So we can conclude that $\|x_{mn} - L_1\|_0^+ \geq \varepsilon$, hence $(m, n) \in A(\varepsilon)$. This implies that $A_1(\varepsilon) \subset A(\varepsilon)$, for each $\varepsilon > 0$. This gives $A_1(\varepsilon) \in I_2$. This show that (x_{mn}) is an I_2 -Cauchy sequence.

Assume that (x_{mn}) is I_2 -Cauchy double sequence. We prove that (x_{mn}) is I_2 -convergent. To this effect, let (ε_d) be a strictly decreasing sequence of numbers converging to zero. Since (x_{mn}) is I_2 -Cauchy double sequence, there exist two strictly increasing sequences (i_d) and (j_d) of positive integers such that

$$A(\varepsilon_d) = \left\{ (m, n) \in \mathbb{N} \times \mathbb{N} : \|x_{mn} - x_{i_d j_d}\|_0^+ \geq \varepsilon_d \right\} \in I_2, (d \in \mathbb{N}).$$

This implies that

$$\emptyset \neq \left\{ (m, n) \in \mathbb{N} \times \mathbb{N} : \|x_{mn} - x_{i_d j_d}\|_0^+ < \varepsilon_d \right\} \in F(I_2), (d \in \mathbb{N}). \quad (2.1)$$

Let d and c be two positive integers such that $d \neq c$. By (2.1), both the sets

$D(\varepsilon_d) = \left\{ (m, n) \in \mathbb{N} \times \mathbb{N} : \|x_{mn} - x_{i_d j_d}\|_0^+ < \varepsilon_d \right\}$ and
 $C(\varepsilon_c) = \left\{ (m, n) \in \mathbb{N} \times \mathbb{N} : \|x_{mn} - x_{i_c j_c}\|_0^+ < \varepsilon_c \right\}$ are non empty sets in $F(I_2)$. Since $F(I_2)$ is a filter on $\mathbb{N} \times \mathbb{N}$, therefore

$$\emptyset \neq D(\varepsilon_d) \cap C(\varepsilon_c) \in F(I_2).$$

Thus for each pair d and c of positive integers with $d \neq c$, we can select a pair $(m_{dc}, n_{dc}) \in \mathbb{N} \times \mathbb{N}$ such that

$$\|x_{m_{dc} n_{dc}} - x_{i_d j_d}\|_0^+ < \varepsilon_d \text{ and } \|x_{m_{dc} n_{dc}} - x_{i_c j_c}\|_0^+ < \varepsilon_c.$$

It follows that

$$\|x_{i_d j_d} - x_{i_c j_c}\|_0^+ \leq \|x_{m_{dc} n_{dc}} - x_{i_d j_d}\|_0^+ + \|x_{m_{dc} n_{dc}} - x_{i_c j_c}\|_0^+ < \varepsilon_d + \varepsilon_c \rightarrow 0$$

as $d, c \rightarrow \infty$. This implies that $(x_{i_d j_d}), d \in \mathbb{N}$ is a Cauchy double sequence. Thus the sequence $(x_{i_d j_d})$ converges to a finite limit L_1 (say). i.e.,

$$\lim_{d, c \rightarrow \infty} x_{i_d j_c} = L_1.$$

Also, we have $\varepsilon_d \rightarrow 0$ as $d \rightarrow \infty$, so for each $\varepsilon > 0$ we can choose the positive integers d_0 such that

$$\varepsilon_{d_0} < \frac{\varepsilon}{2} \text{ and } \|x_{i_{d_0} j_{d_0}} - L_1\|_0^+ < \frac{\varepsilon}{2} \text{ for } d \geq d_0. \quad (2.2)$$

Next we prove that the set $A(\varepsilon) = \left\{ (m, n) \in \mathbb{N} \times \mathbb{N} : \|x_{mn} - L_1\|_0^+ \geq \varepsilon \right\}$ is contained in $A(\varepsilon_{d_0})$. Let $(m, n) \in A(\varepsilon)$, then we have

$$\begin{aligned} \varepsilon &\leq \|x_{mn} - L_1\|_0^+ \leq \|x_{mn} - x_{i_{d_0} j_{d_0}}\|_0^+ + \|x_{i_{d_0} j_{d_0}} - L_1\|_0^+ \\ &\leq \|x_{mn} - x_{i_{d_0} j_{d_0}}\|_0^+ + \frac{\varepsilon}{2} \end{aligned}$$

by (2.2). This implies that $\frac{\varepsilon}{2} \leq \|x_{mn} - x_{i_{d_0} j_{d_0}}\|_0^+$ and therefore by first half of (2.2) we have $\varepsilon_{d_0} \leq \|x_{mn} - x_{i_{d_0} j_{d_0}}\|_0^+$.

This implies that $(m, n) \in A(\varepsilon_{d_0})$ and therefore $A(\varepsilon)$ is contained in $A(\varepsilon_{d_0})$. Since $A(\varepsilon_{d_0})$ belongs to I_2 therefore $A(\varepsilon)$ belongs to I_2 . This proves that (x_{mn}) is I_2 -convergent to L_1 . \square

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Some Results on λ –Statistical Convergence for Double Sequences in Fuzzy Normed Spaces

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Abstract: In this paper, we have introduced λ –statistical convergence and condition of being λ –statistical Cauchy for double sequences in fuzzy normed linear spaces and we have studied some results these concepts.

1. Introduction and Preliminaries

The concept of convergence of real sequences has been extended to statistical convergence independently by Fast [7] and Schoenberg [24]. This concept was extended to the double sequences by Mursaleen and Edely [14]. The concept of ordinary convergence of a sequence of fuzzy numbers was firstly introduced by Matloka [11] and proved some basic theorems for sequences of fuzzy numbers. Mohiuddine et al. [13] studied Statistical convergence of double sequences in fuzzy normed spaces. Recently, Türkmen and Çınar [27] studied λ –statistical convergence in fuzzy normed linear spaces.

The concept of λ –statistical convergence was defined by Mursaleen[15] as follows.

Let $\lambda = (\lambda_n)$ be a non-decreasing sequence of positive real numbers tending to ∞ such that $\lambda_{n+1} \leq \lambda_n + 1$, $\lambda_1 = 1$. The set of all such sequences will be denoted by Λ . A sequence $x = (x_k)$ is said to be λ –statistically convergent or S_λ –convergent to L if for every $\varepsilon > 0$

$$\lim_{n \rightarrow \infty} \frac{1}{\lambda_n} |\{k \in I_n : |x_k - L| \geq \varepsilon\}| = 0,$$

where $I_n = [n - \lambda_n + 1, n]$. In this case, we have written $S_\lambda - \lim x = L$ or $x_k \rightarrow L(S_\lambda)$ and

$$S_\lambda = \{x : \exists L \in \mathbb{R}, S_\lambda - \lim x = L\}.$$

Now, we recall the concept of statistical convergence, double sequence, fuzzy normed spaces and some basic definitions (see [1, 5, 7–9, 12–14, 17–19, 21–23, 25–29])

Fuzzy sets are considered with respect to a nonempty base set X of elements of interest. The essential idea is that each element $x \in X$ is assigned a membership grade $u(x)$ taking values in $[0, 1]$, with $u(x) = 0$ corresponding to nonmembership, $0 < u(x) < 1$ to partial membership, and $u(x) = 1$ to full membership.

According to Zadeh a fuzzy subset of X is a nonempty subset $\{(x, u(x)) : x \in X\}$ of $X \times [0, 1]$ for some function $u : X \rightarrow [0, 1]$. The function u itself is often used for the fuzzy set.

A fuzzy set u on \mathbb{R} is called a fuzzy number if it has the following properties:

- i) u is normal, that is, there exists an $x_0 \in \mathbb{R}$ such that $u(x_0) = 1$;
- ii) u is fuzzy convex, that is, for $x, y \in \mathbb{R}$ and $0 \leq \lambda \leq 1$,

$$u(\lambda x + (1 - \lambda)y) \geq \min[u(x), u(y)];$$

- iii) u is upper semicontinuous;

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iv) $\text{suppu} = \text{cl}\{x \in \mathbb{R} : u(x) > 0\}$, or denoted by $[u]_0$, is compact.

Let $L(\mathbb{R})$ be set of all fuzzy numbers. If $u \in L(\mathbb{R})$ and $u(t) = 0$ for $t < 0$, then u is called a non-negative fuzzy number. We have written $L^*(\mathbb{R})$ by the set of all non-negative fuzzy numbers. We can say that $u \in L^*(\mathbb{R})$ if and only if $u_\alpha^- \geq 0$ for each $\alpha \in [0, 1]$. Clearly we have $\tilde{0} \in L(\mathbb{R})$. For $u \in L(\mathbb{R})$, the α level set of u is defined by

$$[u]_\alpha = \begin{cases} \{x \in \mathbb{R} : u(x) \geq \alpha\}, & \text{if } \alpha \in (0, 1] \\ \text{suppu}, & \text{if } \alpha = 0. \end{cases}$$

Some arithmetic operations for α -level sets are defined as follows: $u, v \in L(\mathbb{R})$ and $[u]_\alpha = [u_\alpha^-, u_\alpha^+]$ and $[v]_\alpha = [v_\alpha^-, v_\alpha^+]$, $\alpha \in (0, 1]$. Then

$$[u \oplus v]_\alpha = [u_\alpha^- + v_\alpha^-, u_\alpha^+ + v_\alpha^+] \quad [u \ominus v]_\alpha = [u_\alpha^- - v_\alpha^+, u_\alpha^+ - v_\alpha^-]$$

$$[u \odot v]_\alpha = [u_\alpha^- \cdot v_\alpha^-, u_\alpha^+ \cdot v_\alpha^-] \quad [\tilde{1} \oslash u]_\alpha = \left[\frac{1}{u_\alpha^+}, \frac{1}{u_\alpha^-} \right] u_\alpha^- > 0$$

For $u, v \in L(\mathbb{R})$, the supremum metric on $L(\mathbb{R})$ is defined as

$$D(u, v) = \sup_{0 \leq \alpha \leq 1} \max \{ |u_\alpha^- - v_\alpha^-|, |u_\alpha^+ - v_\alpha^+| \}.$$

It is known that D is a metric on $L(\mathbb{R})$, and $(L(\mathbb{R}), D)$ is a complete metric space. A sequence $x = (x_k)$ of fuzzy numbers is said to be convergent to the fuzzy number x_0 if for every $\varepsilon > 0$, there exists a positive integer k_0 such that $D(x_k, x_0) < \varepsilon$ for $k > k_0$. And a sequence $x = (x_k)$ of fuzzy numbers convergens to levelwise to x_0 , if and only if $\lim_{k \rightarrow \infty} [x_k]_\alpha = [x_0]_\alpha^-$ and $\lim_{k \rightarrow \infty} [x_k]_\alpha = [x_0]_\alpha^+$ where $[x_k]_\alpha = [(x_k)_\alpha^-, (x_k)_\alpha^+]$ and $[x_0]_\alpha = [(x_0)_\alpha^-, (x_0)_\alpha^+]$ for every $\alpha \in (0, 1)$.

The statistical converge of fuzzy number defined Savas is as follows;

A sequence $X = (X_k)$ of fuzzy numbers is said to be λ -statistically convergent to fuzzy numbers X_0 if every $\varepsilon > 0$

$$\lim_n \frac{1}{\lambda_n} |\{k \in I_n : d(X_k, X_0) \geq \varepsilon\}| = 0.$$

Later, many mathematicians studied statistical convergence of fuzzy numbers.

Let X be a vector space over \mathbb{R} , let $\|\cdot\| : X \rightarrow L^*(\mathbb{R})$ and the mappings $L; R$ (respectively, left norm and right norm) : $[0, 1] \times [0, 1] \rightarrow [0, 1]$ be symetric, nondecreasing in both arguments and satisfy $L(0, 0) = 0$ and $R(1, 1) = 1$.

The quadruple $(X, \|\cdot\|, L, R)$ is called fuzzy normed linear space (briefly $(X, \|\cdot\|)$ FNS) and $\|\cdot\|$ a fuzzy norm if the following axioms are satisfied

1) $\|x\| = \tilde{0}$ iff $x = \theta$,

2) $\|rx\| = |r| \odot \|x\|$ for $x \in X$, $r \in \mathbb{R}$,

3) For all $x, y \in X$

a) $\|x + y\|(s + t) \geq L(\|x\|(s), \|y\|(t))$, whenever $s \leq \|x\|_1^-$, $t \leq \|y\|_1^-$ and $s + t \leq \|x + y\|_1^-$,

b) $\|x + y\|(s + t) \leq R(\|x\|(s), \|y\|(t))$, whenever $s \geq \|x\|_1^-$, $t \geq \|y\|_1^-$ and $s + t \geq \|x + y\|_1^-$.

Let $(X, \|\cdot\|_C)$ be an ordinary normed linear space. Then a fuzzy norm $\|\cdot\|$ on X can be obtained

$$\|x\|(t) = \begin{cases} 0 & \text{if } 0 \leq t \leq a\|x\|_C \text{ or } t \geq b\|x\|_C \\ \frac{t}{(1-a)\|x\|_C} - \frac{a}{1-a} & a\|x\|_C \leq t \leq \|x\|_C \\ \frac{b}{(b-1)\|x\|_C} + \frac{t}{b-1} & \|x\|_C \leq t \leq b\|x\|_C \end{cases} \quad (1.1)$$

where $\|x\|_C$ is the ordinary norm of $x (\neq \theta)$,

$0 < a < 1$ and $1 < b < \infty$. For $x = \theta$, define $\|x\| = \tilde{0}$.

Hence, $(X, \|\cdot\|)$ is a fuzzy normed linear space. Şençimen has defined convergence in fuzzy normed spaces as follows;

Let $(X, \|\cdot\|)$ be an fuzzy normed linear space. A sequence $(x_n)_{n=1}^\infty$ in X is convergent to $x \in X$ with respect to the fuzzy norm on X and we denote by $x_n \xrightarrow{FN} x$, provided that $(D) - \lim_{n \rightarrow \infty} \|x_n - x\| = \tilde{0}$; i.e. for every $\varepsilon > 0$ there is an $N(\varepsilon) \in \mathbb{N}$ such that $D(\|x_n - x\|, \tilde{0}) < \varepsilon$ for all $n > N(\varepsilon)$. This means that for every $\varepsilon > 0$ there is an $N(\varepsilon) \in \mathbb{N}$ such that

$$\sup_{\alpha \in [0, 1]} \|x_n - x\|_\alpha^+ = \|x_n - x\|_0^+ < \varepsilon$$

for all $n \geq N(\varepsilon)$.

Let $(X, \|\cdot\|)$ be an fuzzy normed space and $\lambda \in \Lambda$. A sequence $x = (x_k)$ in X is said to be λ -statistically convergent to $L \in X$ with respect to fuzzy norm on X or $FS\lambda$ -convergent if for each $\varepsilon > 0$

$$\lim_{n \rightarrow \infty} \frac{1}{\lambda_n} |\{k \in I_n : D(\|x_k - L\|, \tilde{0}) \geq \varepsilon\}| = 0.$$

A double sequence $x = (x_{jk})$ is said to be statistically convergent to the number l if for each $\varepsilon > 0$,

$$\delta_2(\{(j, k) : j \leq m \text{ and } k \leq n, |x_{jk} - l| \geq \varepsilon\}) = 0.$$

In this case, we write $S_2 - \lim_{j,k} x_{jk} = l$.

Let $(X, \|\cdot\|)$ be a fuzzy normed space. Then a double sequence (x_{jk}) is said to be convergent to $x \in X$ with respect to the fuzzy norm on X and we write $x_{jk} \xrightarrow{FN} x$ if for every $\varepsilon > 0$ there exist a number $N = N(\varepsilon)$ such that

$$D(\|x_{jk} - x\|, \tilde{0}) < \varepsilon, \text{ for all } j, k \geq N.$$

Let $(X, \|\cdot\|)$ be an fuzzy normed space. A double sequence (x_{jk}) is said to be statistically convergent to $x \in X$ with respect to the fuzzy norm on X if for every $\varepsilon > 0$,

$$\delta_2(\{(j, k) \in \mathbb{N} \times \mathbb{N} : D(\|x_{jk} - x\|, \tilde{0}) \geq \varepsilon\}) = 0.$$

This implies that, for each $\varepsilon > 0$ the set $K(\varepsilon) = \{(j, k) \in \mathbb{N} \times \mathbb{N} : \|x_{jk} - x\|_0^+ \geq \varepsilon\}$ has naturally density zero; namely, for each $\varepsilon > 0$ $\|x_{jk} - x\|_0^+ < \varepsilon$ for almost all j, k . In this case, we write $FS_2 - \lim x_{jk} = x$ or $x_{jk} \xrightarrow{FS_2} x$. Let $\gamma = (\gamma_m)$ and $\mu = (\mu_r)$ be two non-decreasing sequences of positive real numbers each tending to ∞ and such that

$$\begin{aligned} \gamma_{m+1} &\leq \gamma_m + 1, \gamma_1 = 1 \text{ and} \\ \mu_{r+1} &\leq \mu_r + 1, \mu_1 = 1. \end{aligned}$$

Let $I_m = [m - \gamma_m + 1, m]$ and $J_r = [r - \mu_r + 1, r]$.

For any set $K \subseteq \mathbb{N} \times \mathbb{N}$, the number

$$\delta_\lambda^2(K) = \lim_{m,r \rightarrow \infty} \frac{1}{\lambda_{mr}} |\{(k, l) : (k, l) \in K \cap I_m \times J_r\}|$$

is said to be the λ -double density of K , provided the limit exists, where $\lambda_{mr} = \gamma_m \mu_r$.

We now ready to define the λ -statistical convergence for double sequences.

A double sequence $x = (x_{kl})$ in X is said to be λ -statistically convergent to $L \in X$ or S_λ^2 -convergent if for each $\varepsilon > 0$

$$P - \lim_{m,r \rightarrow \infty} \frac{1}{\lambda_{mr}} |\{(k, l) \in I_m \times J_r : |x_{kl} - L| \geq \varepsilon\}| = 0.$$

In this case, we write $x_{kl} \xrightarrow{S_\lambda^2} L$ or $x_{kl} \rightarrow L (S_\lambda^2)$ or $S_\lambda^2 - \lim x_{kl} = L$. Throughout the paper, we will denote $\lambda_{mr} = \gamma_m \mu_r$ and the collection of such sequences will be denoted by Λ_2 . Also we will get $I_m = [m - \gamma_m + 1, m]$ and $J_r = [r - \mu_r + 1, r]$

2. Main Result

In this section, we define λ -statistically convergent for double sequence and λ -statistically Cauchy for double sequences in fuzzy normed linear spaces. We also obtained some basic properties of this notion in fuzzy normed spaces.

Definition 2.1 Let $(X, \|\cdot\|)$ be a fuzzy normed space. A double sequence $x = (x_{kl})$ in X is said to be λ -statistically convergent to $L \in X$ with respect to fuzzy norm on X or FS_λ^2 -convergent if for each $\varepsilon > 0$

$$\lim_{m,r \rightarrow \infty} \frac{1}{\lambda_{mr}} |\{(k, l) \in I_m \times J_r : D(\|x_{kl} - L\|, \tilde{0}) \geq \varepsilon\}| = 0,$$

So, we have written $x_{kl} \xrightarrow{FS_\lambda^2} L$ or $x_{kl} \rightarrow L (FS_\lambda^2)$ or $FS_\lambda^2 - \lim x_{kl} = L$. This implies that for each $\varepsilon > 0$, the set

$$K(\varepsilon) = \{(k, l) \in I_m \times J_r : \|x_{kl} - L\|_0^+ \geq \varepsilon\}$$

has a natural density zero, namely, for each $\varepsilon > 0$, $\|x_{kl} - L\|_0^+ < \varepsilon$ for almost all k and l .

In this case, we have written $FS_\lambda^2 - \lim x = L$. The set of all λ -statistically convergent sequence with respect to fuzzy norm on X will be denoted by FS_λ^2 and $FS_\lambda^2 = \{x = (x_{kl}) : \exists L, FS_\lambda^2 - \lim x = L\}$. In this case, we have written throughout the paper (x_{kl}) is FS_λ^2 -convergent to $L \in X$ means that (x_{kl}) is λ -statistically convergent to $L \in X$ with respect to the fuzzy norm on X .

If $\gamma_m = m$ and $\mu_r = r$ for all m, r then the space $FS_\lambda^2(X)$ is reduced to the space $FS_2(X)$ and since $\delta_2(K) \leq \delta_\lambda^2(K)$ we have $FS_\lambda^2(X) \subset FS_2(X)$.

Lemma 2.2 Let $(X, \|\cdot\|)$ be a fuzzy normed space and $x = (x_{kl})$ be a double sequence in X . Then for each $\varepsilon > 0$, the following statements are equivalent:

- (i) $FS_\lambda^2 - \lim_{k,l \rightarrow \infty} x_{kl} = L$.
- (ii) $\delta_\lambda^2(\{(k, l) : (k, l) \in I_m \times J_r, \|x_{kl} - L\|_0^+ \geq \varepsilon\}) = 0$
- (iii) $\delta_\lambda^2(\{(k, l) : (k, l) \in I_m \times J_r, \|x_{kl} - L\|_0^+ < \varepsilon\}) = 1$
- (iii) $FS_\lambda^2 - \lim_{k,l \rightarrow \infty} \|x_{kl} - L\|_0^+ = 0$

Theorem 2.3 Let $(X, \|\cdot\|)$ be a fuzzy normed space and $\lambda \in \Lambda_2$. If a sequence (x_{kl}) is a FS_λ^2 -convergent, then FS_λ^2 -limit is unique.

Proof. Suppose that $FS_\lambda^2 - \lim x_{kl} = L_1$ and $FS_\lambda^2 - \lim x_{kl} = L_2$ and $L_2 - L_1 = 2\varepsilon > 0$. We define the following sets as $A(\varepsilon) = \{(k, l) \in I_m \times J_r : \|x_{kl} - L_1\|_0^+ \geq \frac{\varepsilon}{2}\}$ and $B(\varepsilon) = \{(k, l) \in I_m \times J_r : \|x_{kl} - L_2\|_0^+ \geq \frac{\varepsilon}{2}\}$. So that $\delta_\lambda^2(A(\varepsilon)) = 0$ and $\delta_\lambda^2(B(\varepsilon)) = 0$. It follows that, there are $k \in I_m, l \in J_r$ such that $\|x_{kl} - L_1\|_0^+ < \frac{\varepsilon}{2}$ and $\|x_{kl} - L_2\|_0^+ < \frac{\varepsilon}{2}$. Further, for these k and l we have

$$2\varepsilon = \|L_2 - L_1\|_0^+ \leq \|x_{kl} - L_2\|_0^+ + \|x_{kl} - L_1\|_0^+ < \varepsilon$$

which is a contradiction. This completes the proof. \square

The next theorem gives the algebraic characterization of λ -statistical convergence on fuzzy normed spaces.

Theorem 2.4 Let (x_{kl}) and (y_{kl}) be sequences in fuzzy normed space $(X, \|\cdot\|)$ such that $x_{kl} \xrightarrow{FS_\lambda^2} L_1$ and $y_{kl} \xrightarrow{FS_\lambda^2} L_2$ and $\lambda \in \Lambda_2$ where $L_1, L_2 \in X$. Then we have

- i) $(x_{kl} + y_{kl}) \xrightarrow{FS_\lambda^2} L_1 + L_2$,
- ii) $tx_{kl} \xrightarrow{FS_\lambda^2} tL_1$ ($t \in \mathbb{R}$),

Theorem 2.5 Let $(X, \|\cdot\|)$ be a fuzzy normed space. If a double sequence $x = (x_{kl})$ is convergent to L with respect to fuzzy norm on X then it is FS_λ^2 -convergent to L .

Proof. Let $x_{kl} \xrightarrow{FN} L$. Then for every $\varepsilon > 0$, there is a couple $(k_0, l_0) \in \mathbb{N} \times \mathbb{N}$ such that $D(\|x_{kl} - L\|, \tilde{0}) \geq \varepsilon$ for all $k \geq k_0, l \geq l_0$. Hence the set

$$\{(k, l) : k \in I_m, l \in J_r, D(\|x_{kl} - L\|, \tilde{0}) \geq \varepsilon\}$$

has natural density zero that is $FS_\lambda^2 - \lim x_{kl} = L$. \square

Definition 2.6 Let $(X, \|\cdot\|)$ be a fuzzy normed space. A sequence (x_{kl}) in X is λ -statistically Cauchy with respect to the fuzzy norm on X provided that for every $\varepsilon > 0$, there exist a positive integers t and v such that for all $k, p \geq t$ and $l, q \geq v$, such that

$$\delta_\lambda^2\{(k, l) \in I_m \times J_r : \|x_{kl} - x_{pq}\|_0^+ \geq \varepsilon\} = 0.$$

In the sequel, (x_{kl}) is FS_λ^2 -Cauchy means that (x_{kl}) is λ -statistically Cauchy with respect to the fuzzy norm on X .

Theorem 2.7 Let $(X, \|\cdot\|)$ be a fuzzy normed space and (x_{kl}) be a double sequence in X . In $(X, \|\cdot\|)$, Every FS_λ^2 -convergent sequence is also an FS_λ^2 -Cauchy with respect to the fuzzy norm on X .

Proof. Let $x_{kl} \xrightarrow{FS_\lambda^2} L$ and $\varepsilon > 0$. Then we have $\|x_{kl} - L\|_0^+ < \varepsilon/2$ for a.a. k and l . Choose a positive integers $t \leq p$ and $v \leq q$ such that $\|x_{pq} - L\|_0^+ < \varepsilon/2$. Now, $\|\cdot\|_0^+$ being a norm in the usual sense, we get

$$\begin{aligned} \|x_{kl} - x_{pq}\|_0^+ &= \|(x_{kl} - L) + (x_{pq} - L)\|_0^+ \\ &\leq \|x_{kl} - L\|_0^+ + \|x_{pq} - L\|_0^+ \\ &< \varepsilon/2 + \varepsilon/2 < \varepsilon \end{aligned}$$

for all $k, p \geq t$ and $l, q \geq v$. This shows that (x_{kl}) is FS_λ^2 -Cauchy. \square

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Almost Contraction Mappings in Cone b - Metric Spaces with Vector Coefficient over Banach Algebras

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Cone b -Metric Spaces with vector coefficient over Banach Algebras, Spectral Radius, Fixed Point Theorems. MSC: 47H04, 47H09, 47H10.

Abstract:

In this presentation we prove a fixed point theorem for almost contraction mappings in cone b -metric spaces with vector coefficient over Banach algebras

1. Introduction and Preliminaries

Throughout this presentation, we suppose that \mathcal{A} is a real Banach algebra where the multiplicative unit and the null vector will be denoted by e and θ , respectively.

Definition 1.1 (The Spectral Radius) The spectral radius of $a \in A$ is given by

$$\rho(a) := \lim_{n \rightarrow \infty} \|a^n\|^{\frac{1}{n}}.$$

Note that if $\rho(a) < 1$, then $e - a$ is invertible (see [6]) and the inverse of $e - a$ is given by

$$(e - a)^{-1} = \sum_{i=0}^{\infty} a^i. \quad (1.1)$$

Definition 1.2 (Cone) Let P be a subset of \mathcal{A} such that $\{\theta, e\} \subset P$. P is called a cone of \mathcal{A} if the following conditions hold:

- (c1) P is closed;
- (c2) $\lambda P + \mu P \subset P$ for all non-negative real numbers λ and μ ;
- (c3) $PP \subset P$ and $P \cap (-P) = \theta$.

A cone P with $\text{int}P \neq \emptyset$ is called a solid cone where $\text{int}P$ indicates the interior of P .

Definition 1.3 (Normal Cone) To each cone P of \mathcal{A} there corresponds a partial ordering \preceq on \mathcal{A} defined by $x \preceq y$ iff $y - x \in P$. By $x \prec y$ we understand that $x \preceq y$ but $x \neq y$, while $x \ll y$ stands for $y - x \in \text{int}P$. If there exists a positive real number K such that for all $x, y \in \mathcal{A}$

$$\theta \preceq x \preceq y \text{ implies } \|x\| \leq K \|y\|, \quad (1.2)$$

then a cone P is called normal. The least of K 's with the above condition is called the normal constant of P .

Definition 1.4 (Xu and Radenovic-2014, see [3]) A sequence $\{u_n\}$ in P is said to be a c -sequence if for each $c \gg \theta$ there exists $n_0 \in \mathbb{N}$ such that $u_n \ll c$ for $n \geq n_0$.

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Lemma 1.5 (Xu and Radenovic-2014, see [3]) *If $\{u_n\}$ and $\{v_n\}$ are two c -sequences in P , then $\{\alpha u_n + \beta v_n\}$ is also a c -sequence for positive real numbers α and β .*

Lemma 1.6 (Xu and Radenovic-2014, see [3]) *Let $k \in P$. If $\{u_n\}$ is a c -sequence in P , then $\{ku_n\}$ is also a c -sequence in P .*

Lemma 1.7 (Xu and Radenovic-2014, see [3]) *Let $u \in \mathcal{A}$. For each $c \gg \theta$ if $\theta \preceq u \ll c$, then $u = \theta$.*

Lemma 1.8 (Xu and Radenovic-2014, see [3]) *Let $\{u_n\}$ be a sequence in P . Then the following items are equivalent:*

- (i) $\{u_n\}$ is a c -sequence.
- (ii) For each $c \gg \theta$ there is $n_0 \in \mathbb{N}$ such that $u_n \prec c$ whenever $n \geq n_0$.
- (iii) For each $c \gg \theta$ there is $n_1 \in \mathbb{N}$ such that $u_n \preceq c$ whenever $n \geq n_1$.

Lemma 1.9 (Huang and Radenovic-2015, see [4]) *Let $h \in P$ with $\rho(h) < 1$. Then $\{u_n\}$ with $u_n = h^n$ is a c -sequence.*

Lemma 1.10 (Ozavsar-2018 (see[5]) *Let $k \in P$ such that $r(k) < 1$. Then*

$$\sum_{i=p}^n k^i \preceq k^p (e - k)^{-1} \quad (1.3)$$

for all $p \in \mathbb{N}$.

Definition 1.11 (Huang and Zhang-2007, Liu and Xu-2013, see [1, 2]) *Let \mathcal{A} be an ordered Banach algebra and $X \neq \emptyset$. A cone metric space over \mathcal{A} is given by a pair (X, d) where d is a mapping $d : X \times X \rightarrow \mathcal{A}$ satisfying*

- (1) $\theta \preceq d(x, y)$ and $d(x, y) = \theta$ if and only if $x = y$
- (2) $d(x, y) = d(y, x)$
- (3) $d(x, y) \preceq d(x, z) + d(z, y)$

for all $x, y, z \in X$ and for null vector $\theta \in \mathcal{A}$.

Notice that the class of metric spaces is contained by one of cone metric spaces over ordered Banach algebras.

Example 1.12 *Let \mathcal{A} be the usual algebra of all real valued continuous functions on $X = [0, 1]$ which also have continuous derivatives on X . If \mathcal{A} is equipped with the norm $\|f\| = \|f\|_\infty + \|f'\|_\infty$, then \mathcal{A} becomes a Banach algebra with unit $e = 1$. Moreover, $P = \{f \in \mathcal{A} | f(t) \geq 0 \text{ for all } t \in X\}$ is a nonnormal cone (see [3]). Consider a mapping $d : X \times X \rightarrow \mathcal{A}$ defined by $d(x, y)(t) = |x - y|e^t$ for all $x, y \in X$. It is obvious that (X, d) is a cone metric space on the Banach algebra \mathcal{A} .*

Definition 1.13 (Huang and Zhang-2007, see [1]) *Let $\{x_n\}$ be a sequence in a cone metric space (X, d) on \mathcal{A} . Then*

- (i) We say that $\{x_n\}$ is convergent to $x \in X$ if to each $c \gg \theta$ there corresponds a natural number n_0 such that $d(x_n, x) \ll c$ for all $n \geq n_0$. This is denoted by $\lim_{n \rightarrow \infty} x_n = x$ or $x_n \rightarrow x$ as $n \rightarrow \infty$.
- (ii) $\{x_n\}$ is called Cauchy if for each $c \gg \theta$ there is a natural number n_0 such that $d(x_m, x_n) \ll c$ for all $m, n \geq n_0$.
- (iii) A cone metric space (X, d) is called complete if every Cauchy sequence in X converges to an element x of X .

Lemma 1.14 (Xu and Radenovic-2014, see [3]) *If (X, d) is a complete cone metric space over \mathcal{A} and $\{x_n\} \subset X$ is a sequence that converges to $x \in X$, then the following assertions are true:*

- (i) $\{d(x_n, x)\}$ is a c -sequence.
- (ii) $\{d(x_n, x_{n+m})\}$ is a c -sequence for all $m \in \mathbb{N}$.

Definition 1.15 (Huang and Radenovic-2015, see [4]) *Let \mathcal{A} be a Banach algebra, $X \neq \emptyset$, $s \in \mathbb{R}$ with $s \geq 1$. If a mapping $d : X \times X \rightarrow \mathcal{A}$ holds the following*

- (i) $\theta \preceq d(x, y)$ and $d(x, y) = \theta$ if and only if $x = y$;
- (ii) $d(x, y) = d(y, x)$;

$$(iii) \ d(x,y) \preceq s[d(x,z) + d(z,y)]$$

for all $x,y,z \in X$ and for null vector $\theta \in \mathcal{A}$. Then d is said to be a cone b -metric and the pair (X,d) is said to be a **cone b -metric space** on \mathcal{A} .

Definition 1.16 (Reny and Nabwey, et al. 2017, see [7]) Let \mathcal{A} be a Banach algebra, $X \neq \emptyset$, $s \in P$ with $s \succeq e$. If a mapping $d : X \times X \rightarrow \mathcal{A}$ holds the following conditions:

- (i) $\theta \preceq d(x,y)$ and $d(x,y) = \theta$ if and only if $x = y$;
- (ii) $d(x,y) = d(y,x)$;
- (iii) $d(x,y) \preceq s[d(x,z) + d(z,y)]$

for all $x,y,z \in X$ and for the null vector $\theta \in \mathcal{A}$. Then d is said to be a **cone b -metric with vector coefficient** and the pair (X,d) is said to be a cone b -metric space with vector coefficient over \mathcal{A} .

Note that all definitions and properties in cone metric spaces over Banach algebras can be extended to cone b -metric spaces with vector coefficient over Banach algebras.

Definition 1.17 (Banach, 1922) For a usual metric space (X,d) , let $T : X \rightarrow X$ be a self mapping. T is said to be a contraction if there exists a constant $k \in (0, 1)$ such that

$$d(Tx, Ty) \preceq kd(x,y) \text{ for all } x,y \in X \quad (1.4)$$

Banach proved that a contraction mapping T has a unique fixed point x to which the sequence $\{T^n x_0\}$ converges for any x_0 in a complete metric space X .

Definition 1.18 (Berinde-2004, see [8]) Let (X,d) be a metric space and let $T : X \rightarrow X$ a mapping. If there is $k \in \mathbb{R}$ with $0 < k < 1$ and some $l \geq 0$ such that for all $x,y \in X$

$$d(Tx, Ty) \preceq kd(x,y) + ld(y, Tx), \quad (1.5)$$

then T is said to be a (k,l) -almost contraction in the usual metric spaces.

By the symmetry property of d , the condition (1.8) implies the following dual one:

$$d(Tx, Ty) \preceq kd(x,y) + ld(x, Ty), \text{ for all } x,y \in X. \quad (1.6)$$

Definition 1.19 (Liu and Xu-2013, [2]) Suppose that (X,d) is a cone metric space over a Banach algebra \mathcal{A} , and $T : X \rightarrow X$ is a self mapping. T is said to be a generalized contraction if there exists a constant vector $k \in \mathcal{A}$ with $\rho(k) \in (0, 1)$ such that

$$d(Tx, Ty) \preceq kd(x,y) \text{ for all } x,y \in X \quad (1.7)$$

where $\rho(k)$ stands for the spectral radius of k .

Note that Liu and Xu also proved that a generalized contraction in a complete cone metric space with a solid normal cone has a unique fixed point, and the corresponding sequence obtained by Picard iteration converges to this unique fixed point. Later, in 2014, Xu and Radenovic obtained their result by removing the condition of normality for cone.

Definition 1.20 (Özavşar-2018, see [5]) Let (X,d) be a cone metric space over \mathcal{A} and let $T : X \rightarrow X$ be a mapping. If there is $k \in P$ with $0 < \rho(k) < 1$ and some $l \in P$ such that for all $x,y \in X$

$$d(Tx, Ty) \preceq kd(x,y) + ld(y, Tx), \quad (1.8)$$

then we call T a generalized (k,l) -almost contraction in the setting of cone metric spaces with Banach algebras.

Note that the class of (k,l) -almost mappings given above contains those of many mappings in cone metric spaces and the usual metric spaces.

Theorem 1.21 (Özavşar-2018, see [5]) Let (X,d) be a complete cone metric space over \mathcal{A} . If $T : X \rightarrow X$ is a (k,l) -almost contraction, then T has at least one fixed point in X .

The following theorem introduces the condition for uniqueness of fixed point:

Theorem 1.22 (Özavşar-2018, see [5]) Let T be a (k,l) -almost contraction in a complete cone metric space. If T satisfies

$$d(Tx, Ty) \preceq kd(x,y) + ld(x, Tx) \text{ for all } x,y \in X, \quad (1.9)$$

then it has unique fixed point, and for any $x \in X$, the iterative sequence $\{T^n x\}$ converges to the fixed point.

Now we are ready to introduce the following theorem by following the results mentioned above:

Theorem 1.23 Let (X, d) be a complete cone b -metric space with an invertible vector coefficient $s \in P$ over \mathcal{A} and $T : X \rightarrow X$ be a self mapping such that for some $k, l \in P$,

$$d(Tx, Ty) \preceq kd(x, y) + ld(y, Tx) \text{ for all } x, y \in X. \quad (1.10)$$

If k commutes with s and $0 < \rho(sk) < 1$, then T has at least one fixed point in X .

Proof. For arbitray $x_0 \in X$, let $x_n = Tx_{n-1}$ for all $n \in \mathbb{N}$ with $n \geq 1$. By (1.10), we obtain

$$d(x_{n+1}, x_n) = d(Tx_n, Tx_{n-1}) \preceq kd(x_n, x_{n-1}), \quad (1.11)$$

which implies that

$$d(x_n, x_{n+1}) \preceq k^n d(x_0, x_1). \quad (1.12)$$

Let $m, n \in \mathbb{N}$ with $m > n$. We have from (cbm3) that

$$d(x_m, x_n) \preceq s^{m-n-1}d(x_{m-1}, x_m) + \sum_{j=n}^{m-2} s^{j-n+1}d(x_{j+1}, x_j). \quad (1.13)$$

Using the fact $s \succeq e$ together with the properties of P , we have

$$s^{m-n-1}d(x_{m-1}, x_m) \preceq s^{m-n}d(x_{m-1}, x_m), \quad (1.14)$$

implying that

$$d(x_m, x_n) \preceq \sum_{j=n}^{m-1} s^{j-n+1}d(x_{j+1}, x_j). \quad (1.15)$$

Then, by substituting (1.12) into (1.15), we have

$$d(x_m, x_n) \preceq \sum_{j=n}^{m-1} s^{j-n+1}k^j d(x_0, x_1) \quad (1.16)$$

The fact $sk = ks$ transforms (1.16) into

$$d(x_m, x_n) \preceq s^{-n+1} \left(\sum_{j=n}^{m-1} (sk)^j \right) d(x_0, x_1). \quad (1.17)$$

Since $r(sk) < 1$ and by Lemma 1.10, we have from (1.17) that

$$d(x_m, x_n) \preceq s^{-n+1}(sk)^n(e - sk)^{-1}d(x_0, x_1). \quad (1.18)$$

Since we have that $s \succeq e$ implies $s^{n-1} \succeq \dots \succeq s \succeq e$ and s is invertible, we get from (1.18) that

$$d(x_m, x_n) \preceq (sk)^n(e - sk)^{-1}d(x_0, x_1). \quad (1.19)$$

Let $u_n = (sk)^n(e - sk)^{-1}d(x_0, x_1)$. Then, using the advantage of $r(sk) < 1$ together with Lemma 1.5 and Lemma 1.9, we see that $\{u_n\}$ is a c -sequence, that is, for each $c \in \text{int}P$, there is $n_0 \in \mathbb{N}$ such that

$$d(x_n, x_m) \preceq u_n \ll c$$

whenever $n \geq n_0$. Thus, $\{x_n\}$ is a Cauchy sequence. Since X is complete cone b -metric space, $\{x_n\}$ converges to $x \in X$.

Moreover we have

$$\begin{aligned} d(x, Tx) &\preceq sd(x, x_{n+1}) + sd(x_{n+1}, Tx) \\ &= sd(x, x_{n+1}) + sd(Tx_n, Tx) \\ &\preceq sd(x, x_{n+1}) + skd(x_n, x) + sld(x, x_{n+1}) \\ &\preceq s(e + l)d(x, x_{n+1}) + skd(x_n, x). \end{aligned}$$

Let $h_n = s(e + l)d(x, x_{n+1}) + skd(x_n, x)$. Using Lemma 1.6 and Lemma 1.14 together with Lemma 1.5, we see that $\{h_n\}$ is a c -sequence, implying that for each $c \gg \theta$, there is $n_0 \in \mathbb{N}$ such that $d(x, Tx) \preceq h_n \ll c$ for $n \geq n_0$. Considering Lemma 1.7 we obtain $x = Tx$ \square

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Some Fixed Point Theorems in Cone Metric Spaces over Banach Algebras

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Keywords:

Cone Metric Spaces over Banach Algebras, Spectral Radius, Fixed Point Theorems. MSC: 47H04, 47H09, 47H10.

Abstract: In this presentation we first mention about the notion of cone metric spaces over Banach algebras. Next, we present some fixed point theorems in such spaces, which provide proper generalizations for the well known fixed point theorems

1. Introduction and Preliminaries

Throughout this presentation, we suppose that \mathcal{A} is a real Banach algebra where the multiplicative unit and the null vector will be denoted by e and θ , respectively.

Definition 1.1 (The Spectral Radius) The spectral radius of $a \in A$ is given by

$$\rho(a) := \lim_{n \rightarrow \infty} \|a^n\|^{\frac{1}{n}}.$$

Note that if $\rho(a) < 1$, then $e - a$ is invertible (see [6]) and the inverse of $e - a$ is given by

$$(e - a)^{-1} = \sum_{i=0}^{\infty} a^i. \quad (1.1)$$

Definition 1.2 (Cone) Let P be a subset of \mathcal{A} such that $\{\theta, e\} \subset P$. P is called a cone of \mathcal{A} if the following conditions hold:

- (c1) P is closed;
- (c2) $\lambda P + \mu P \subset P$ for all non-negative real numbers λ and μ ;
- (c3) $PP \subset P$ and $P \cap (-P) = \theta$.

A cone P with $\text{int}P \neq \emptyset$ is called a solid cone where $\text{int}P$ indicates the interior of P .

Definition 1.3 (Normal Cone) To each cone P of \mathcal{A} there corresponds a partial ordering \preceq on \mathcal{A} defined by $x \preceq y$ iff $y - x \in P$. By $x < y$ we understand that $x \preceq y$ but $x \neq y$, while $x \ll y$ stands for $y - x \in \text{int}P$. If there exists a positive real number K such that for all $x, y \in \mathcal{A}$

$$\theta \preceq x \preceq y \text{ implies } \|x\| \leq K \|y\|, \quad (1.2)$$

then a cone P is called normal. The least of K 's with the above condition is called the normal constant of P .

Definition 1.4 (Xu and Radenovic-2014, see [3]) A sequence $\{u_n\}$ in P is said to be a c -sequence if for each $c \gg \theta$ there exists $n_0 \in \mathbb{N}$ such that $u_n \ll c$ for $n \geq n_0$.

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Lemma 1.5 (Xu and Radenovic-2014, see [3]) *If $\{u_n\}$ and $\{v_n\}$ are two c -sequences in P , then $\{\alpha u_n + \beta v_n\}$ is also a c -sequence for positive real numbers α and β .*

Lemma 1.6 (Xu and Radenovic-2014, see [3]) *Let $k \in P$. If $\{u_n\}$ is a c -sequence in P , then $\{ku_n\}$ is also a c -sequence in P .*

Lemma 1.7 (Xu and Radenovic-2014, see [3]) *Let $u \in \mathcal{A}$. For each $c \gg \theta$ if $\theta \preceq u \ll c$, then $u = \theta$.*

Lemma 1.8 (Xu and Radenovic-2014, see [3]) *Let $\{u_n\}$ be a sequence in P . Then the following items are equivalent:*

- (i) $\{u_n\}$ is a c -sequence.
- (ii) For each $c \gg \theta$ there is $n_0 \in \mathbb{N}$ such that $u_n \prec c$ whenever $n \geq n_0$.
- (iii) For each $c \gg \theta$ there is $n_1 \in \mathbb{N}$ such that $u_n \preceq c$ whenever $n \geq n_1$.

Lemma 1.9 (Huang and Radenovic-2015, see [4]) *Let $h \in P$ with $\rho(h) < 1$. Then $\{u_n\}$ with $u_n = h^n$ is a c -sequence.*

Lemma 1.10 (Özavşar-2018 (see[5]) *Let $k \in P$ such that $r(k) < 1$. Then*

$$\sum_{i=p}^n k^i \preceq k^p (e - k)^{-1} \quad (1.3)$$

for all $p \in \mathbb{N}$.

Definition 1.11 (Huang and Zhang-2007, Liu and Xu-2013, see [1, 2]) *Let \mathcal{A} be an ordered Banach algebra and $X \neq \emptyset$. A cone metric space over \mathcal{A} is given by a pair (X, d) where d is a mapping $d : X \times X \rightarrow \mathcal{A}$ satisfying*

- (1) $\theta \preceq d(x, y)$ and $d(x, y) = \theta$ if and only if $x = y$
- (2) $d(x, y) = d(y, x)$
- (3) $d(x, y) \preceq d(x, z) + d(z, y)$

for all $x, y, z \in X$ and for null vector $\theta \in \mathcal{A}$.

Notice that the class of metric spaces is contained by one of cone metric spaces over ordered Banach algebras.

Example 1.12 *Let \mathcal{A} be the usual algebra of all real valued continuous functions on $X = [0, 1]$ which also have continuous derivatives on X . If \mathcal{A} is equipped with the norm $\|f\| = \|f\|_\infty + \|f'\|_\infty$, then \mathcal{A} becomes a Banach algebra with unit $e = 1$. Moreover, $P = \{f \in \mathcal{A} | f(t) \geq 0 \text{ for all } t \in X\}$ is a nonnormal cone (see [3]). Consider a mapping $d : X \times X \rightarrow \mathcal{A}$ defined by $d(x, y)(t) = |x - y|e^t$ for all $x, y \in X$. It is obvious that (X, d) is a cone metric space on the Banach algebra \mathcal{A} .*

Definition 1.13 (Huang and Zhang-2007, see [1]) *Let $\{x_n\}$ be a sequence in a cone metric space (X, d) on \mathcal{A} . Then*

- (i) We say that $\{x_n\}$ is convergent to $x \in X$ if to each $c \gg \theta$ there corresponds a natural number n_0 such that $d(x_n, x) \ll c$ for all $n \geq n_0$. This is denoted by $\lim_{n \rightarrow \infty} x_n = x$ or $x_n \rightarrow x$ as $n \rightarrow \infty$.
- (ii) $\{x_n\}$ is called Cauchy if for each $c \gg \theta$ there is a natural number n_0 such that $d(x_m, x_n) \ll c$ for all $m, n \geq n_0$.
- (iii) A cone metric space (X, d) is called complete if every Cauchy sequence in X converges to an element x of X .

Lemma 1.14 (Xu and Radenovic-2014, see [3]) *If (X, d) is a complete cone metric space over \mathcal{A} and $\{x_n\} \subset X$ is a sequence that converges to $x \in X$, then the following assertions are true:*

- (i) $\{d(x_n, x)\}$ is a c -sequence.
- (ii) $\{d(x_n, x_{n+m})\}$ is a c -sequence for all $m \in \mathbb{N}$.

2. Almost Contraction Mappings in the Usual Metric Spaces

Definition 2.1 (Banach, 1922) For a usual metric space (X, d) , let $T : X \mapsto X$ be a self mapping. T is said to a contraction if there exists a constant $k \in (0, 1)$ such that

$$d(Tx, Ty) \preceq kd(x, y) \text{ for all } x, y \in X \quad (2.1)$$

Banach proved that a contraction mapping T has a unique fixed point x to which the sequence $\{T^n x_0\}$ converges for any x_0 in a complete metric space X .

Definition 2.2 (Berinde-2004, see [7]) Let (X, d) be a metric space and let $T : X \rightarrow X$ a mapping. If there is $k \in \mathbb{R}$ with $0 < k < 1$ and some $l \geq 0$ such that for all $x, y \in X$

$$d(Tx, Ty) \preceq kd(x, y) + ld(y, Tx), \quad (2.2)$$

then T is said to be a (k, l) -almost contraction in the usual metric spaces.

By the symmetry property of d , the condition (3.2) implies the following dual one:

$$d(Tx, Ty) \preceq kd(x, y) + ld(x, Ty), \text{ for all } x, y \in X. \quad (2.3)$$

Berinde showed that the class of almost contractions contains those of many well known mappings in the usual metric spaces. He also proved a fixed point theorem for (k, l) -almost contractions in the usual metric spaces (X, d) .

3. Almost Contraction Mappings in Cone Metric Spaces over Banach Algebras

Definition 3.1 (Liu and Xu-2013, [2]) Suppose that (X, d) is a cone metric space over a Banach algebra \mathcal{A} , and $T : X \mapsto X$ is a self mapping. T is said to a generalized contraction if there exists a constant vector $k \in \mathcal{A}$ with $\rho(k) \in (0, 1)$ such that

$$d(Tx, Ty) \preceq kd(x, y) \text{ for all } x, y \in X \quad (3.1)$$

where $\rho(k)$ stands for the spectral radius of k .

Note that Liu and Xu also proved that a generalized contraction in a complete cone metric space with a solid normal cone has a unique fixed point, and the corresponding sequence obtained by Picard iteration converges to this unique fixed point. Later, in 2014, Xu and Radenovic obtained their result by removing the condition of normality for cone.

Definition 3.2 (Özavşar-2018, see [5]) Let (X, d) be a cone metric space over \mathcal{A} and let $T : X \rightarrow X$ be a mapping. If there is $k \in P$ with $0 < \rho(k) < 1$ and some $l \in P$ such that for all $x, y \in X$

$$d(Tx, Ty) \preceq kd(x, y) + ld(y, Tx), \quad (3.2)$$

then we call T a generalized (k, l) -almost contraction in the setting of cone metric spaces with Banach algebras.

Note that the class of (k, l) -almost mappings given above contains those of many mappings in cone metric spaces and the usual metric spaces.

Theorem 3.3 (Özavşar-2018, see [5]) Let (X, d) be a complete cone metric space over \mathcal{A} . If $T : X \rightarrow X$ is a (k, l) -almost contraction, then T has at least one fixed point in X .

The following theorem introduces the condition for uniqueness of fixed point:

Theorem 3.4 (Özavşar-2018, see [5]) Let T be a (k, l) -almost contraction in a complete cone metric space. If T satisfies

$$d(Tx, Ty) \preceq kd(x, y) + ld(x, Tx) \text{ for all } x, y \in X, \quad (3.3)$$

then it has unique fixed point, and for any $x \in X$, the iterative sequence $\{T^n x\}$ converges to the fixed point.

Example 3.5 Let $X = [0, 1] \times [0, 1]$ and consider the usual Banach algebra $\mathcal{A} = \mathbb{R}^2$ endowed with the standard norm and pointwise multiplication. For a mapping $d : X \times X \rightarrow \mathcal{A}$ defined by $d((x_1, y_1), (x_2, y_2)) = (|x_2 - x_1|, |y_2 - y_1|)$, it is obvious that (X, d) is a complete cone metric space over \mathcal{A} with solid cone $P = \{(a, b) | a \geq 0 \text{ and } b \geq 0\}$.

Example 3.6 Then a mapping $T : X \rightarrow X$ defined by

$$f(x, y) = \begin{cases} \left(\frac{x}{3}, \frac{2y}{3}\right) & \text{if } 0 \leq x \leq 1 \text{ and } 0 \leq y \leq \frac{1}{2} \\ \left(\frac{x}{3}, \frac{2y}{3} + \frac{1}{3}\right) & \text{if } 0 \leq x \leq 1 \text{ and } \frac{1}{2} < y \leq 1 \end{cases}$$

is a (k, l) -almost contraction where $k = \left(\frac{1}{3}, \frac{2}{3}\right) \in P$ with $\rho(k) < 1$ and $l = (0, 6) \succeq \theta$. The set of fixed points of T is $\{(0, 0), (0, 1)\}$.

Definition 3.7 [see [8]] Let (X, d) be a cone metric space over a Banach space E with solid cone P and let $\mathcal{N}(X)$ be a family of all nonempty subsets of X . A mapping $H : \mathcal{N}(X) \times \mathcal{N}(X) \rightarrow E$ is called an H -cone metric over a Banach space E with respect to (X, d) , if for all $A, B \in \mathcal{N}(X)$ the following conditions hold:

(H1) $H(A, B) = 0 \Rightarrow A = B$;

(H2) $H(A, B) = H(B, A)$;

(H3) For all $\varepsilon \in E$ with $\varepsilon \gg \theta$ and for all $x \in A$, there exists at least one $y \in B$ such that $d(x, y) \preceq H(A, B) + \varepsilon$;

(H4) One of the following holds:

(i) For all $\varepsilon \in E$ with $\varepsilon \gg \theta$ there is at least one $x \in A$ such that $H(A, B) \preceq d(x, y) + \varepsilon$ for all $y \in B$.

(ii) For all $\varepsilon \in E$ with $\varepsilon \gg \theta$ there is at least one $x \in B$ such that $H(A, B) \preceq d(x, y) + \varepsilon$ for all $y \in A$.

4. Multivalued Contraction Mappings in Cone b -Metric Spaces over Banach Algebras

Definition 4.1 (Huang, Radenovic-2015, see [4]) Let \mathcal{A} be a Banach algebra, $X \neq \emptyset$, $s \in \mathbb{R}$ with $s \geq 1$. If a mapping $d : X \times X \rightarrow \mathcal{A}$ holds the following

(i) $\theta \preceq d(x, y)$ and $d(x, y) = \theta$ if and only if $x = y$;

(ii) $d(x, y) = d(y, x)$;

(iii) $d(x, y) \preceq s[d(x, z) + d(z, y)]$

for all $x, y, z \in X$ and for null vector $\theta \in \mathcal{A}$. Then d is said to be a cone b -metric and the pair (X, d) is said to be a cone b -metric space on \mathcal{A} .

In [9], Özavşar introduced the Banach contraction principle for Nadler type contractions in the sense of Wardowski [8] by using the setting of cone b -metric spaces over Banach algebras as follows:

Theorem 4.2 (Özavşar-2018, see [9]) Suppose that (X, d) is a cone b -metric space with $s \in \mathbb{R}$ such that $s \geq 1$, and $T : X \rightarrow \mathcal{N}(X)$ is a set-valued mapping. If there is $k \in P$ with $r(sk) \in [0, 1)$ such that

$$H(Tx, Ty) \preceq kd(x, y) \text{ for all } x, y \in X, \tag{4.1}$$

then there is at least one $x \in X$ such that $x \in Tx$.

Note that this theorem extends the result of Nadler.

Example 4.3 Consider the Banach algebra $\mathcal{A} = \mathbb{R}^2$ endowed with the pointwise multiplication and the usual norm. Let $P = \{(x, y) \in \mathbb{R}^2 | x, y \geq 0\}$ and $X = \mathbb{R}^2$ and $p \in \mathbb{R}$ with $p > 1$. Then, using advantage of the inequality $(a + b)^p \leq 2^p(a^p + b^p)$ for all $a, b \geq 0$ and the properties of the cone P , one can show that a mapping $d : X \times X \rightarrow \mathcal{A}$ defined by $d((x_1, y_1), (x_2, y_2)) = (|x_1 - x_2|^p, |y_1 - y_2|^p)$ is a cone b -metric with $s = 2^p$ over \mathcal{A} . Let $a \otimes b$ be a closed subset of X defined by $a \otimes b := \{(x, y) \in X | 0 \leq x \leq a, 0 \leq y \leq b\}$ for $a, b \geq 0$. Now consider $\mathcal{N}(X) = \{a \otimes b | a, b \geq 0\}$. Then it is clear that a mapping $H : \mathcal{N}(X) \times \mathcal{N}(X) \rightarrow \mathcal{A}$ defined by $H(a_1 \otimes b_1, a_2 \otimes b_2) = (|a_1 - a_2|^p, |b_1 - b_2|^p)$ is H -cone b -metric with respect to (X, d) over \mathcal{A} . Let $T : X \rightarrow \mathcal{N}(X)$ given by $T(x, y) = \left| \frac{\cos x}{4} \right| \otimes \left| \frac{\cos y}{16} \right|$. Then, by using the basic properties of $|\cdot|$, one can show that

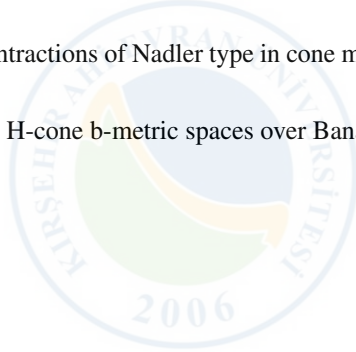
$$H(T(x_1, y_1), T(x_2, y_2)) \preceq kd((x_1, y_1), (x_2, y_2)), \tag{4.2}$$

where $k = \left(\frac{1}{s^2}, \frac{1}{s^4} \right) \in P$. Since T holds the conditions of Theorem 3.6, it has a fixed point.

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Asymptotically \mathcal{I} -Invariant Equivalence of Sequences Defined By A Modulus Function

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Abstract:

In this paper, we introduce the concepts of strongly asymptotically ideal invariant equivalence, f -asymptotically ideal invariant equivalence, strongly f -asymptotically ideal invariant equivalence and asymptotically ideal invariant statistical equivalence for sequences. Also, we investigate some relationships among them.

1. Introduction

Throughout the paper \mathbb{N} denotes the set of all natural numbers and \mathbb{R} the set of all real numbers. The concept of convergence of a real sequence has been extended to statistical convergence independently by Fast [1], Schoenberg [24] and studied by many authors. The idea of \mathcal{I} -convergence was introduced by Kostyrko et al. [2] as a generalization of statistical convergence which is based on the structure of the ideal \mathcal{I} of subset of \mathbb{N} . Several authors including Raimi [17], Schaefer [23], Mursaleen and Edely [7], Mursaleen [9], Savaş [18, 19], Nuray and Savaş [11], Pancaroğlu and Nuray [13] and some authors have studied invariant convergent sequences. The concept of strongly σ -convergence was defined by Mursaleen [8]. Savaş and Nuray [20] introduced the concepts of σ -statistical convergence and lacunary σ -statistical convergence and gave some inclusion relations. Nuray et al. [12] defined the concepts of σ -uniform density of a subset A of the set \mathbb{N} , \mathcal{I}_σ -convergence and investigated relationships between \mathcal{I}_σ -convergence and invariant convergence also \mathcal{I}_σ -convergence and $[V_\sigma]_p$ -convergence. Pancaroğlu and Nuray [13] studied Statistical lacunary invariant summability. Recently, Nuray and Ulusu [25] investigated lacunary \mathcal{I} -invariant convergence and lacunary \mathcal{I} -invariant Cauchy sequence of real numbers.

Marouf [6] presented definitions for asymptotically equivalent and asymptotic regular matrices. Patterson [14] presented asymptotically statistical equivalent sequences for nonnegative summability matrices. Patterson and Savaş [15, 22] introduced asymptotically lacunary statistically equivalent sequences and also asymptotically $\sigma\theta$ -statistical equivalent sequences. Ulusu [26, 27] studied asymptotically ideal invariant equivalence and asymptotically lacunary \mathcal{I}_σ -equivalence.

Modulus function was introduced by Nakano [10]. Maddox [5], Pehlivan [16] and many authors used a modulus function f to define some new concepts and inclusion theorems. Kumar and Sharma [3] studied lacunary equivalent sequences by ideals and modulus function.

Now, we recall the basic concepts and some definitions and notations (See [2, 4–6, 12, 14, 16]).

Let σ be a mapping of the positive integers into itself. A continuous linear functional ϕ on ℓ_∞ , the space of real bounded sequences, is said to be an invariant mean or a σ mean, if and only if,

1. $\phi(x) \geq 0$, when the sequence $x = (x_n)$ has $x_n \geq 0$ for all n ,
2. $\phi(e) = 1$, where $e = (1, 1, 1, \dots)$,

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3. $\phi(x_{\sigma(n)}) = \phi(x)$ for all $x \in \ell_\infty$.

The mappings ϕ are assumed to be one-to-one and such that $\sigma^m(n) \neq n$ for all positive integers n and m , where $\sigma^m(n)$ denotes the m th iterate of the mapping σ at n . Thus ϕ extends the limit functional on c , the space of convergent sequences, in the sense that $\phi(x) = \lim x$ for all $x \in c$. In case σ is translation mappings $\sigma(n) = n + 1$, the σ mean is often called a Banach limit and V_σ , the set of bounded sequences all of whose invariant means are equal, is the set of almost convergent sequences.

A sequence $x = (x_k)$ is said to be statistically convergent to the number L if for every $\varepsilon > 0$,

$$\lim_{n \rightarrow \infty} \frac{1}{n} \left| \left\{ k \leq n : |x_k - L| \geq \varepsilon \right\} \right| = 0,$$

where the vertical bars indicate the number of elements in the enclosed set.

A family of sets $\mathcal{I} \subseteq 2^{\mathbb{N}}$ is called an ideal if and only if

(i) $\emptyset \in \mathcal{I}$, (ii) For each $A, B \in \mathcal{I}$ we have $A \cup B \in \mathcal{I}$, (iii) For each $A \in \mathcal{I}$ and each $B \subseteq A$ we have $B \in \mathcal{I}$.

An ideal is called nontrivial if $\mathbb{N} \notin \mathcal{I}$ and nontrivial ideal is called admissible if $\{n\} \in \mathcal{I}$ for each $n \in \mathbb{N}$.

Throughout the paper we let \mathcal{I} be an admissible ideal.

Let $A \subseteq \mathbb{N}$ and

$$s_m = \min_n \left| A \cap \{ \sigma(n), \sigma^2(n), \dots, \sigma^m(n) \} \right| \quad \text{and} \quad S_m = \max_n \left| A \cap \{ \sigma(n), \sigma^2(n), \dots, \sigma^m(n) \} \right|.$$

If the limits $\underline{V}(A) = \lim_{m \rightarrow \infty} \frac{s_m}{m}$ and $\overline{V}(A) = \lim_{m \rightarrow \infty} \frac{S_m}{m}$ exist then, they are called a lower σ -uniform density and an upper σ -uniform density of the set A , respectively. If $\underline{V}(A) = \overline{V}(A)$, then $V(A) = \underline{V}(A) = \overline{V}(A)$ is called the σ -uniform density of A .

Denote by \mathcal{I}_σ the class of all $A \subseteq \mathbb{N}$ with $V(A) = 0$.

A sequence $x = (x_k)$ is said to be \mathcal{I}_σ -convergent to L if for every $\varepsilon > 0$, the set $A_\varepsilon = \{k : |x_k - L| \geq \varepsilon\}$ belongs to \mathcal{I}_σ , i.e., $V(A_\varepsilon) = 0$. It is denoted by $\mathcal{I}_\sigma - \lim x_k = L$.

The two nonnegative sequences $x = (x_k)$ and $y = (y_k)$ are said to be asymptotically equivalent if

$$\lim_k \frac{x_k}{y_k} = 1$$

(denoted by $x \sim y$).

The two nonnegative sequences $x = (x_k)$ and $y = (y_k)$ are said to be asymptotically statistical equivalent of multiple L if for every $\varepsilon > 0$,

$$\lim_n \frac{1}{n} \left| \left\{ k \leq n : \left| \frac{x_k}{y_k} - L \right| \geq \varepsilon \right\} \right| = 0$$

(denoted by $x \stackrel{S_L}{\sim} y$) and simply asymptotically statistical equivalent if $L = 1$.

The two nonnegative sequences $x = (x_k)$ and $y = (y_k)$ are said to be strongly asymptotically equivalent of multiple L with respect to the ideal \mathcal{I} if for every $\varepsilon > 0$,

$$\left\{ n \in \mathbb{N} : \frac{1}{n} \sum_{k=1}^n \left| \frac{x_k}{y_k} - L \right| \geq \varepsilon \right\} \in \mathcal{I}$$

(denoted by $x_k \stackrel{\mathcal{I}(\omega)}{\sim} y_k$) and simply strongly asymptotically equivalent with respect to the ideal \mathcal{I} , if $L = 1$.

The two nonnegative sequences $x = (x_k)$ and $y = (y_k)$ are said to be asymptotically \mathcal{I}_σ -equivalent of multiple L if for every $\varepsilon > 0$, $A_\varepsilon = \left\{ k \in \mathbb{N} : \left| \frac{x_k}{y_k} - L \right| \geq \varepsilon \right\} \in \mathcal{I}_\sigma$, i.e., $V(A_\varepsilon) = 0$. It is denoted by $x_k \stackrel{[\mathcal{I}_\sigma^L]}{\sim} y_k$.

A function $f : [0, \infty) \rightarrow [0, \infty)$ is called a modulus if

1. $f(x) = 0$ if and only if $x = 0$,
2. $f(x+y) \leq f(x) + f(y)$,

3. f is increasing,
4. f is continuous from the right at 0.

A modulus may be unbounded (for example $f(x) = x^p$, $0 < p < 1$) or bounded (for example $f(x) = \frac{x}{x+1}$).

Let f be modulus function. The two nonnegative sequences $x = (x_k)$ and $y = (y_k)$ are said to be f -asymptotically equivalent of multiple L with respect to the ideal \mathcal{I} provided that, for every $\varepsilon > 0$,

$$\left\{ k \in \mathbb{N} : f\left(\left|\frac{x_k}{y_k} - L\right|\right) \geq \varepsilon \right\} \in \mathcal{I}$$

(denoted by $x_k \overset{\mathcal{I}(f)}{\sim} y_k$) and simply f -asymptotically \mathcal{I} -equivalent if $L = 1$.

Let f be modulus function. The two nonnegative sequences $x = (x_k)$ and $y = (y_k)$ are said to be strongly f -asymptotically equivalent of multiple L with respect to the ideal \mathcal{I} provided that, for every $\varepsilon > 0$

$$\left\{ n \in \mathbb{N} : \frac{1}{n} \sum_{k=1}^n f\left(\left|\frac{x_k}{y_k} - L\right|\right) \geq \varepsilon \right\} \in \mathcal{I}$$

(denoted by $x_k \overset{\mathcal{I}(\omega_f)}{\sim} y_k$) and simply strongly f -asymptotically \mathcal{I} -equivalent if $L = 1$.

Lemma 1.1 [16] Let f be a modulus and $0 < \delta < 1$. Then, for each $x \geq \delta$ we have $f(x) \leq 2f(1)\delta^{-1}x$.

2. Main Results

Definition 2.1 The sequences $x = (x_k)$ and $y = (y_k)$ are said to be strongly asymptotically \mathcal{I} -invariant equivalent of multiple L if for every $\varepsilon > 0$,

$$\left\{ n \in \mathbb{N} : \frac{1}{n} \sum_{k=1}^n \left| \frac{x_k}{y_k} - L \right| \geq \varepsilon \right\} \in \mathcal{I}_\sigma$$

(denoted by $x_k \overset{[\mathcal{I}_\sigma^L]}{\sim} y_k$) and simply strongly asymptotically \mathcal{I} -invariant equivalent if $L = 1$.

Definition 2.2 Let f be a modulus function. The sequences $x = (x_k)$ and $y = (y_k)$ are said to be f -asymptotically \mathcal{I} -invariant equivalent of multiple L if for every $\varepsilon > 0$,

$$\left\{ k \in \mathbb{N} : f\left(\left|\frac{x_k}{y_k} - L\right|\right) \geq \varepsilon \right\} \in \mathcal{I}_\sigma$$

(denoted by $x_k \overset{\mathcal{I}_\sigma^L(f)}{\sim} y_k$) and simply f -asymptotically \mathcal{I} -invariant equivalent if $L = 1$.

Definition 2.3 Let f be a modulus function. The sequences $x = (x_k)$ and $y = (y_k)$ are said to be strongly f -asymptotically \mathcal{I} -invariant equivalent of multiple L if for every $\varepsilon > 0$,

$$\left\{ n \in \mathbb{N} : \frac{1}{n} \sum_{k=1}^n f\left(\left|\frac{x_k}{y_k} - L\right|\right) \geq \varepsilon \right\} \in \mathcal{I}_\sigma$$

(denoted by $x_k \overset{[\mathcal{I}_\sigma^L(f)]}{\sim} y_k$) and simply strongly f -asymptotically \mathcal{I} -invariant equivalent if $L = 1$.

Theorem 2.4 Let f be a modulus function. Then,

$$x_k \overset{[\mathcal{I}_\sigma^L]}{\sim} y_k \Rightarrow x_k \overset{[\mathcal{I}_\sigma^L(f)]}{\sim} y_k.$$

Proof. Let $x_k \overset{[\mathcal{I}_\sigma^L]}{\sim} y_k$ and $\varepsilon > 0$ be given. Choose $0 < \delta < 1$ such that $f(t) < \varepsilon$ for $0 \leq t \leq \delta$. Then, for

$m = 1, 2, \dots$, we can write

$$\begin{aligned} \frac{1}{n} \sum_{k=1}^n f\left(\left|\frac{x_{\sigma^k(m)}}{y_{\sigma^k(m)}} - L\right|\right) &= \frac{1}{n} \sum_{k=1}^n f\left(\left|\frac{x_{\sigma^k(m)}}{y_{\sigma^k(m)}} - L\right|\right) \\ &\quad \left|\frac{x_{\sigma^k(m)}}{y_{\sigma^k(m)}} - L\right| \leq \delta \\ &\quad + \frac{1}{n} \sum_{k=1}^n f\left(\left|\frac{x_{\sigma^k(m)}}{y_{\sigma^k(m)}} - L\right|\right) \\ &\quad \left|\frac{x_{\sigma^k(m)}}{y_{\sigma^k(m)}} - L\right| > \delta \end{aligned}$$

and so by Lemma 1.1

$$\frac{1}{n} \sum_{k=1}^n f\left(\left|\frac{x_{\sigma^k(m)}}{y_{\sigma^k(m)}} - L\right|\right) < \varepsilon + \left(\frac{2f(1)}{\delta}\right) \frac{1}{n} \sum_{k=1}^n \left|\frac{x_{\sigma^k(m)}}{y_{\sigma^k(m)}} - L\right|$$

uniformly in m . Thus, for every any $\gamma > 0$

$$\left\{n \in \mathbb{N} : \frac{1}{n} \sum_{k=1}^n f\left(\left|\frac{x_{\sigma^k(m)}}{y_{\sigma^k(m)}} - L\right|\right) \geq \gamma\right\} \subseteq \left\{n \in \mathbb{N} : \frac{1}{n} \sum_{k=1}^n \left|\frac{x_{\sigma^k(m)}}{y_{\sigma^k(m)}} - L\right| \geq \frac{(\gamma - \varepsilon)\delta}{2f(1)}\right\},$$

uniformly in m . Since $x_k \overset{[\mathcal{I}_\sigma^L]}{\sim} y_k$, it follows the later set and hence, the first set in above expression belongs to \mathcal{I}_σ . This proves that $x_k \overset{[\mathcal{I}_\sigma^L(f)]}{\sim} y_k$. \square

Definition 2.5 The sequences x_k and y_k are said to be asymptotically \mathcal{I} -invariant statistical equivalent of multiple L if for every $\varepsilon > 0$ and each $\gamma > 0$,

$$\left\{n \in \mathbb{N} : \frac{1}{n} \left| \left\{k \leq n : \left|\frac{x_{\sigma^k(m)}}{y_{\sigma^k(m)}} - L\right| \geq \varepsilon\right\} \right| \geq \gamma\right\} \in \mathcal{I}_\sigma$$

(denoted by $x_k \overset{\mathcal{I}(\mathcal{I}_\sigma)}{\sim} y_k$) and simply asymptotically \mathcal{I} -invariant statistical equivalent if $L = 1$.

Theorem 2.6 Let f be a modulus function. Then,

$$x_k \overset{[\mathcal{I}_\sigma^L(f)]}{\sim} y_k \Rightarrow x_k \overset{\mathcal{I}(\mathcal{I}_\sigma)}{\sim} y_k.$$

Proof. Assume that $x_k \overset{[\mathcal{I}_\sigma^L(f)]}{\sim} y_k$ and $\varepsilon > 0$ be given. Since for $m = 1, 2, \dots$,

$$\begin{aligned} \frac{1}{n} \sum_{k=1}^n f\left(\left|\frac{x_{\sigma^k(m)}}{y_{\sigma^k(m)}} - L\right|\right) &\geq \frac{1}{n} \sum_{k=1}^n f\left(\left|\frac{x_{\sigma^k(m)}}{y_{\sigma^k(m)}} - L\right|\right) \\ &\quad \left|\frac{x_{\sigma^k(m)}}{y_{\sigma^k(m)}} - L\right| \geq \varepsilon \\ &\geq f(\varepsilon) \cdot \frac{1}{n} \left| \left\{k \leq n : \left|\frac{x_{\sigma^k(m)}}{y_{\sigma^k(m)}} - L\right| \geq \varepsilon\right\} \right| \end{aligned}$$

it follows that for any $\gamma > 0$,

$$\left\{n \in \mathbb{N} : \frac{1}{n} \left| \left\{k \leq n : \left|\frac{x_{\sigma^k(m)}}{y_{\sigma^k(m)}} - L\right| \geq \varepsilon\right\} \right| \geq \frac{\gamma}{f(\varepsilon)}\right\} \subseteq \left\{n \in \mathbb{N} : \frac{1}{n} \sum_{k=1}^n f\left(\left|\frac{x_{\sigma^k(m)}}{y_{\sigma^k(m)}} - L\right|\right) \geq \gamma\right\},$$

uniformly in m . Since $x_k \overset{[\mathcal{I}_\sigma^L(f)]}{\sim} y_k$, so the last set belongs to \mathcal{I}_σ . But then by the definition of an ideal, the first set belongs to \mathcal{I}_σ and therefore $x_k \overset{\mathcal{I}(\mathcal{I}_\sigma)}{\sim} y_k$ \square

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Asymptotically Lacunary \mathcal{I} -Invariant Equivalence of Sequences Defined By A Modulus Function

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Abstract:

In this paper, we introduce the concepts of strongly asymptotically lacunary ideal invariant equivalence, f -asymptotically lacunary ideal invariant equivalence, strongly f -asymptotically lacunary ideal invariant equivalence and asymptotically lacunary ideal invariant statistical equivalence for sequences. Also, we investigate some relationships among them.

1. Introduction

Throughout the paper \mathbb{N} denotes the set of all natural numbers and \mathbb{R} the set of all real numbers. The concept of convergence of a real sequence has been extended to statistical convergence independently by Fast [1], Schoenberg [24] and studied by many authors. The idea of \mathcal{I} -convergence was introduced by Kostyrko et al. [2] as a generalization of statistical convergence which is based on the structure of the ideal \mathcal{I} of subset of \mathbb{N} . Several authors including Raimi [17], Schaefer [23], Mursaleen and Edely [7], Mursaleen [9], Savaş [18, 19], Nuray and Savaş [11], Pancaroğlu and Nuray [13] and some authors have studied invariant convergent sequences. The concept of strongly σ -convergence was defined by Mursaleen [8]. Savaş and Nuray [20] introduced the concepts of σ -statistical convergence and lacunary σ -statistical convergence and gave some inclusion relations. Nuray et al. [12] defined the concepts of σ -uniform density of a subset A of the set \mathbb{N} , \mathcal{I}_σ -convergence and investigated relationships between \mathcal{I}_σ -convergence and invariant convergence also \mathcal{I}_σ -convergence and $[V_\sigma]_p$ -convergence. Pancaroğlu and Nuray [13] studied Statistical lacunary invariant summability. Recently, Nuray and Ulusu [25] investigated lacunary \mathcal{I} -invariant convergence and lacunary \mathcal{I} -invariant Cauchy sequence of real numbers.

Marouf [6] presented definitions for asymptotically equivalent and asymptotic regular matrices. Patterson [14] presented asymptotically statistical equivalent sequences for nonnegative summability matrices. Patterson and Savaş [15, 22] introduced asymptotically lacunary statistically equivalent sequences and also asymptotically $\sigma\theta$ -statistical equivalent sequences. Ulusu [26, 27] studied asymptotically ideal invariant equivalence and asymptotically lacunary \mathcal{I}_σ -equivalence.

Modulus function was introduced by Nakano [10]. Maddox [5], Pehlivan [16] and many authors used a modulus function f to define some new concepts and inclusion theorems. Kumar and Sharma [3] studied lacunary equivalent sequences by ideals and modulus function.

Now, we recall the basic concepts and some definitions and notations (See [2, 4–6, 12, 14, 16]).

Let σ be a mapping of the positive integers into itself. A continuous linear functional ϕ on ℓ_∞ , the space of real bounded sequences, is said to be an invariant mean or a σ mean, if and only if,

1. $\phi(x) \geq 0$, when the sequence $x = (x_n)$ has $x_n \geq 0$ for all n ,
2. $\phi(e) = 1$, where $e = (1, 1, 1, \dots)$,

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3. $\phi(x_{\sigma(n)}) = \phi(x)$ for all $x \in \ell_\infty$.

The mappings ϕ are assumed to be one-to-one and such that $\sigma^m(n) \neq n$ for all positive integers n and m , where $\sigma^m(n)$ denotes the m th iterate of the mapping σ at n . Thus ϕ extends the limit functional on c , the space of convergent sequences, in the sense that $\phi(x) = \lim x$ for all $x \in c$. In case σ is translation mappings $\sigma(n) = n + 1$, the σ mean is often called a Banach limit and V_σ , the set of bounded sequences all of whose invariant means are equal, is the set of almost convergent sequences.

By a lacunary sequence we mean an increasing integer sequence $\theta = \{k_r\}$ such that $k_0 = 0$ and $h_r = k_r - k_{r-1} \rightarrow \infty$ as $r \rightarrow \infty$. Throughout the paper, we let θ a lacunary sequence.

The sequence $x = (x_k)$ is $S_{\sigma\theta}$ -convergent to L , if for every $\varepsilon > 0$,

$$\lim_{r \rightarrow \infty} \frac{1}{h_r} \left| \left\{ k \in I_r : |x_{\sigma^k(n)} - L| \geq \varepsilon \right\} \right| = 0, \text{ uniformly in } n=1,2,\dots$$

A family of sets $\mathcal{I} \subseteq 2^{\mathbb{N}}$ is called an ideal if and only if

(i) $\emptyset \in \mathcal{I}$, (ii) For each $A, B \in \mathcal{I}$ we have $A \cup B \in \mathcal{I}$, (iii) For each $A \in \mathcal{I}$ and each $B \subseteq A$ we have $B \in \mathcal{I}$. An ideal is called nontrivial if $\mathbb{N} \notin \mathcal{I}$ and nontrivial ideal is called admissible if $\{n\} \in \mathcal{I}$ for each $n \in \mathbb{N}$. Throughout the paper we let \mathcal{I} be an admissible ideal.

Let $A \subseteq \mathbb{N}$ and

$$s_m = \min_n \left| A \cap \{\sigma(n), \sigma^2(n), \dots, \sigma^m(n)\} \right| \text{ and } S_m = \max_n \left| A \cap \{\sigma(n), \sigma^2(n), \dots, \sigma^m(n)\} \right|.$$

If the limits $\underline{V}(A) = \lim_{m \rightarrow \infty} \frac{s_m}{m}$ and $\overline{V}(A) = \lim_{m \rightarrow \infty} \frac{S_m}{m}$ exist then, they are called a lower σ -uniform density and an upper σ -uniform density of the set A , respectively. If $\underline{V}(A) = \overline{V}(A)$, then $V(A) = \underline{V}(A) = \overline{V}(A)$ is called the σ -uniform density of A .

Denote by \mathcal{I}_σ the class of all $A \subseteq \mathbb{N}$ with $V(A) = 0$.

A sequence $x = (x_k)$ is said to be \mathcal{I}_σ -convergent to L if for every $\varepsilon > 0$, the set $A_\varepsilon = \{k : |x_k - L| \geq \varepsilon\}$ belongs to \mathcal{I}_σ , i.e., $V(A_\varepsilon) = 0$. It is denoted by $\mathcal{I}_\sigma - \lim x_k = L$.

Let $\theta = \{k_r\}$ be a lacunary sequence, $A \subseteq \mathbb{N}$ and

$$s_r = \min_n \left\{ \left| A \cap \{\sigma^m(n) : m \in I_r\} \right| \right\} \text{ and } S_r = \max_n \left\{ \left| A \cap \{\sigma^m(n) : m \in I_r\} \right| \right\}.$$

If the limits $\underline{V}_\theta(A) = \lim_{r \rightarrow \infty} \frac{s_r}{h_r}$ and $\overline{V}_\theta(A) = \lim_{r \rightarrow \infty} \frac{S_r}{h_r}$ exist then, they are called a lower lacunary σ -uniform density and an upper lacunary σ -uniform density of the set A , respectively. If $\underline{V}_\theta(A) = \overline{V}_\theta(A)$, then $V_\theta(A) = \underline{V}_\theta(A) = \overline{V}_\theta(A)$ is called the lacunary σ -uniform density of A .

Denoted by $\mathcal{I}_{\sigma\theta}$ the class of all $A \subseteq \mathbb{N}$ with $V_\theta(A) = 0$.

A sequence (x_k) is said to be lacunary \mathcal{I}_σ -convergent or $\mathcal{I}_{\sigma\theta}$ -convergent to L if for every $\varepsilon > 0$, $A_\varepsilon = \{k : |x_k - L| \geq \varepsilon\} \in \mathcal{I}_{\sigma\theta}$, i.e., $V_\theta(A_\varepsilon) = 0$. It is denoted by $\mathcal{I}_{\sigma\theta} - \lim x_k = L$.

The two nonnegative sequences $x = (x_k)$ and $y = (y_k)$ are said to be asymptotically equivalent if $\lim_{k \rightarrow \infty} \frac{x_k}{y_k} = 1$ (denoted by $x \sim y$).

The two nonnegative sequences $x = (x_k)$ and $y = (y_k)$ are strongly asymptotically lacunary invariant equivalent of multiple L if $\lim_{r \rightarrow \infty} \frac{1}{h_r} \sum_{k \in I_r} \left| \frac{x_{\sigma^k(m)}}{y_{\sigma^k(m)}} - L \right| = 0$, uniformly in m (denoted by $x \overset{N_{\sigma\theta}}{\sim} y$) and strongly simply asymptotically lacunary invariant equivalent if $L = 1$.

The two nonnegative sequences $x = (x_k)$ and $y = (y_k)$ are said to be asymptotically lacunary invariant statistical equivalent of multiple L if for every $\varepsilon > 0$,

$$\lim_{r \rightarrow \infty} \frac{1}{h_r} \left| \left\{ k \in I_r : \left| \frac{x_{\sigma^k(m)}}{y_{\sigma^k(m)}} - L \right| \geq \varepsilon \right\} \right| = 0, \text{ uniformly in } m$$

(denoted by $x \overset{S_{\sigma\theta}}{\sim} y$) and simply asymptotically lacunary invariant statistical equivalent if $L = 1$.

The two nonnegative sequences $x = (x_k)$ and $y = (y_k)$ are said to be strongly asymptotically equivalent of multiple L with respect to the ideal \mathcal{I} if for every $\varepsilon > 0$,

$$\left\{ n \in \mathbb{N} : \frac{1}{n} \sum_{k=1}^n \left| \frac{x_k}{y_k} - L \right| \geq \varepsilon \right\} \in \mathcal{I}$$

(denoted by $x_k \overset{\mathcal{I}(\omega)}{\sim} y_k$) and simply strongly asymptotically equivalent with respect to the ideal \mathcal{I} , if $L = 1$.

The two nonnegative sequences $x = (x_k)$ and $y = (y_k)$ are said to be strongly asymptotically lacunary equivalent of multiple L respect to the ideal \mathcal{I} provided that for every $\varepsilon > 0$,

$$\left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in I_r} \left| \frac{x_k}{y_k} - L \right| \geq \varepsilon \right\} \in \mathcal{I}$$

(denoted by $x_k \overset{[\mathcal{I}(N_\theta)]}{\sim} y_k$) and simply strongly asymptotically lacunary \mathcal{I} -equivalent with respect to the ideal \mathcal{I} , if $L = 1$.

The two nonnegative sequences $x = (x_k)$ and $y = (y_k)$ are said to be asymptotically lacunary statistical equivalent of multiple L with respect to the ideal \mathcal{I} provided that for every $\varepsilon > 0$ and $\gamma > 0$,

$$\left\{ r \in \mathbb{N} : \frac{1}{h_r} \left| \left\{ k \in I_r : \left| \frac{x_k}{y_k} - L \right| \geq \varepsilon \right\} \right| \geq \gamma \right\} \in \mathcal{I}$$

(denoted by $x_k \overset{\mathcal{I}(\mathcal{I}_\theta)}{\sim} y_k$) and simply asymptotically lacunary \mathcal{I} -statistical equivalent if $L = 1$.

The two nonnegative sequences $x = (x_k)$ and $y = (y_k)$ are said to be asymptotically \mathcal{I}_σ -equivalent of multiple L if for every $\varepsilon > 0$, $A_\varepsilon = \left\{ k \in \mathbb{N} : \left| \frac{x_k}{y_k} - L \right| \geq \varepsilon \right\} \in \mathcal{I}_\sigma$, i.e., $V(A_\varepsilon) = 0$. It is denoted by $x_k \overset{[\mathcal{I}_\sigma^L]}{\sim} y_k$.

The two nonnegative sequences $x = (x_k)$ and $y = (y_k)$ are said to be asymptotically $\mathcal{I}_{\sigma\theta}$ -equivalent of multiple L if for every $\varepsilon > 0$, $A_\varepsilon = \left\{ k \in I_r : \left| \frac{x_k}{y_k} - L \right| \geq \varepsilon \right\} \in \mathcal{I}_{\sigma\theta}$, i.e., $V_\theta(A_\varepsilon) = 0$. It is denoted by $x_k \overset{[\mathcal{I}_{\sigma\theta}^L]}{\sim} y_k$.

A function $f : [0, \infty) \rightarrow [0, \infty)$ is called a modulus if

1. $f(x) = 0$ if and only if $x = 0$,
2. $f(x+y) \leq f(x) + f(y)$,
3. f is increasing,
4. f is continuous from the right at 0.

A modulus may be unbounded (for example $f(x) = x^p$, $0 < p < 1$) or bounded (for example $f(x) = \frac{x}{x+1}$).

Let f be modulus function. The two nonnegative sequences $x = (x_k)$ and $y = (y_k)$ are said to be f -asymptotically equivalent of multiple L with respect to the ideal \mathcal{I} provided that, for every $\varepsilon > 0$,

$$\left\{ k \in \mathbb{N} : f \left(\left| \frac{x_k}{y_k} - L \right| \right) \geq \varepsilon \right\} \in \mathcal{I}$$

(denoted by $x_k \overset{\mathcal{I}(f)}{\sim} y_k$) and simply f -asymptotically \mathcal{I} -equivalent if $L = 1$.

Let f be modulus function. The two nonnegative sequences $x = (x_k)$ and $y = (y_k)$ are said to be strongly f -asymptotically equivalent of multiple L with respect to the ideal \mathcal{I} provided that, for every $\varepsilon > 0$

$$\left\{ n \in \mathbb{N} : \frac{1}{n} \sum_{k=1}^n f \left(\left| \frac{x_k}{y_k} - L \right| \right) \geq \varepsilon \right\} \in \mathcal{I}$$

(denoted by $x_k \overset{\mathcal{I}(\omega_f)}{\sim} y_k$) and simply strongly f -asymptotically \mathcal{I} -equivalent if $L = 1$.

Let f be a modulus function. The two nonnegative $x = (x_k)$ and $y = (y_k)$ are said to be strongly f -asymptotically lacunary equivalent of multiple L with respect to the ideal \mathcal{I} provided that for every $\varepsilon > 0$,

$$\left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in I_r} f \left(\left| \frac{x_k}{y_k} - L \right| \right) \geq \varepsilon \right\} \in \mathcal{I}$$

(denoted by $x_k \overset{[\mathcal{I}(N_\theta^f)]}{\sim} y_k$) and simply strongly f -asymptotically lacunary \mathcal{I} -equivalent if $L = 1$.

The sequences $x = (x_k)$ and $y = (y_k)$ are said to be strongly asymptotically \mathcal{I} -invariant equivalent of multiple L if for every $\varepsilon > 0$,

$$\left\{ n \in \mathbb{N} : \frac{1}{n} \sum_{k=1}^n \left| \frac{x_k}{y_k} - L \right| \geq \varepsilon \right\} \in \mathcal{I}_\sigma$$

(denoted by $x_k \overset{[\mathcal{I}_\sigma^L]}{\sim} y_k$) and simply strongly asymptotically \mathcal{I} -invariant equivalent if $L = 1$.

Let f be a modulus function. The sequences $x = (x_k)$ and $y = (y_k)$ are said to be f -asymptotically \mathcal{I} -invariant equivalent of multiple L if for every $\varepsilon > 0$,

$$\left\{ k \in \mathbb{N} : f\left(\left|\frac{x_k}{y_k} - L\right|\right) \geq \varepsilon \right\} \in \mathcal{I}_\sigma$$

(denoted by $x_k \overset{\mathcal{I}_\sigma^L(f)}{\sim} y_k$) and simply f -asymptotically \mathcal{I} -invariant equivalent if $L = 1$.

Let f be a modulus function. The sequences $x = (x_k)$ and $y = (y_k)$ are said to be strongly f -asymptotically \mathcal{I} -invariant equivalent of multiple L if for every $\varepsilon > 0$,

$$\left\{ n \in \mathbb{N} : \frac{1}{n} \sum_{k=1}^n f\left(\left|\frac{x_k}{y_k} - L\right|\right) \geq \varepsilon \right\} \in \mathcal{I}_\sigma$$

(denoted by $x_k \overset{[\mathcal{I}_\sigma^L(f)]}{\sim} y_k$) and simply strongly f -asymptotically \mathcal{I} -invariant equivalent if $L = 1$.

The sequences x_k and y_k are said to be asymptotically \mathcal{I} -invariant statistical equivalent of multiple L if for every $\varepsilon > 0$ and each $\gamma > 0$,

$$\left\{ n \in \mathbb{N} : \frac{1}{n} \left| \left\{ k \leq n : \left| \frac{x_{\sigma^k(m)}}{y_{\sigma^k(m)}} - L \right| \geq \varepsilon \right\} \right| \geq \gamma \right\} \in \mathcal{I}_\sigma$$

(denoted by $x_k \overset{\mathcal{I}(\mathcal{I}_\sigma)}{\sim} y_k$) and simply asymptotically \mathcal{I} -invariant statistical equivalent if $L = 1$.

Lemma 1.1 [16] Let f be a modulus and $0 < \delta < 1$. Then, for each $x \geq \delta$ we have $f(x) \leq 2f(1)\delta^{-1}x$.

2. Main Results

Definition 2.1 The sequences $x = (x_k)$ and $y = (y_k)$ are said to be strongly asymptotically lacunary \mathcal{I} -invariant equivalent of multiple L , if for every $\varepsilon > 0$

$$\left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in I_r} \left| \frac{x_k}{y_k} - L \right| \geq \varepsilon \right\} \in \mathcal{I}_{\sigma\theta}$$

(denoted by $x_k \overset{[\mathcal{I}_{\sigma\theta}^L]}{\sim} y_k$) and simply strongly asymptotically lacunary \mathcal{I} -invariant equivalent if $L = 1$.

Definition 2.2 Let f be a modulus function. The sequences $x = (x_k)$ and $y = (y_k)$ are said to be f -asymptotically lacunary \mathcal{I} -invariant equivalent of multiple L , if for every $\varepsilon > 0$

$$\left\{ k \in \mathbb{N} : f\left(\left|\frac{x_k}{y_k} - L\right|\right) \geq \varepsilon \right\} \in \mathcal{I}_{\sigma\theta},$$

(denoted by $x_k \overset{\mathcal{I}_{\sigma\theta}^L(f)}{\sim} y_k$) and simply f -asymptotically lacunary \mathcal{I} -invariant equivalent if $L = 1$.

Definition 2.3 Let f be a modulus function. The sequences $x = (x_k)$ and $y = (y_k)$ are said to be strongly f -asymptotically lacunary \mathcal{I} -invariant equivalent of multiple L , if for every $\varepsilon > 0$

$$\left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in I_r} f\left(\left|\frac{x_k}{y_k} - L\right|\right) \geq \varepsilon \right\} \in \mathcal{I}_{\sigma\theta}$$

(denoted by $x_k \overset{[\mathcal{I}_{\sigma\theta}^L(f)]}{\sim} y_k$) and simply strongly f -asymptotically lacunary \mathcal{I} -invariant equivalent if $L = 1$.

Theorem 2.4 Let f be a modulus function. Then,

$$x_k \overset{[\mathcal{I}_{\sigma\theta}^L]}{\sim} y_k \Rightarrow x_k \overset{[\mathcal{I}_{\sigma\theta}^L(f)]}{\sim} y_k.$$

Proof. Let $x_k \overset{[\mathcal{I}_{\sigma\theta}^L]}{\sim} y_k$ and $\varepsilon > 0$ be given. Choose $0 < \delta < 1$ such that $f(t) < \varepsilon$ for $0 \leq t \leq \delta$. Then, for

$m = 1, 2, \dots$, we can write

$$\begin{aligned} \frac{1}{h_r} \sum_{k \in I_r} f\left(\left|\frac{x_{\sigma^k(m)}}{y_{\sigma^k(m)}} - L\right|\right) &= \frac{1}{h_r} \sum_{\substack{k \in I_r \\ \left|\frac{x_{\sigma^k(m)}}{y_{\sigma^k(m)}} - L\right| \leq \delta}} f\left(\left|\frac{x_{\sigma^k(m)}}{y_{\sigma^k(m)}} - L\right|\right) \\ &\quad + \frac{1}{h_r} \sum_{\substack{k \in I_r \\ \left|\frac{x_{\sigma^k(m)}}{y_{\sigma^k(m)}} - L\right| > \delta}} f\left(\left|\frac{x_{\sigma^k(m)}}{y_{\sigma^k(m)}} - L\right|\right) \end{aligned}$$

and so by Lemma 1.1

$$\frac{1}{h_r} \sum_{k \in I_r} f\left(\left|\frac{x_{\sigma^k(m)}}{y_{\sigma^k(m)}} - L\right|\right) < \varepsilon + \left(\frac{2f(1)}{\delta}\right) \frac{1}{h_r} \sum_{k \in I_r} \left|\frac{x_{\sigma^k(m)}}{y_{\sigma^k(m)}} - L\right|$$

uniformly in m . Thus, for each any $\gamma > 0$

$$\left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in I_r} f\left(\left|\frac{x_{\sigma^k(m)}}{y_{\sigma^k(m)}} - L\right|\right) \geq \gamma \right\} \subseteq \left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in I_r} \left|\frac{x_{\sigma^k(m)}}{y_{\sigma^k(m)}} - L\right| \geq \frac{(\gamma - \varepsilon)\delta}{2f(1)} \right\},$$

uniformly in m . Since $x_k \overset{[\mathcal{I}_{\sigma\theta}^L]}{\sim} y_k$, it follows the later set and hence, the first set in above expression belongs to $\mathcal{I}_{\sigma\theta}$. This proves that $x_k \overset{[\mathcal{I}_{\sigma\theta}^L(f)]}{\sim} y_k$. \square

Definition 2.5 The sequences x_k and y_k are said to be asymptotically lacunary \mathcal{I} -invariant statistical equivalent of multiple L if for every $\varepsilon > 0$ and each $\gamma > 0$,

$$\left\{ r \in \mathbb{N} : \frac{1}{h_r} \left| \left\{ k \in I_r : \left| \frac{x_k}{y_k} - L \right| \geq \varepsilon \right\} \right| \geq \gamma \right\} \in \mathcal{I}_{\sigma\theta}$$

(denoted by $x_k \overset{\mathcal{I}(\mathcal{I}_{\sigma\theta})}{\sim} y_k$) and simply asymptotically lacunary \mathcal{I} -invariant statistical equivalent if $L = 1$.

Theorem 2.6 Let f be a modulus function. Then,

$$x_k \overset{[\mathcal{I}_{\sigma\theta}^L(f)]}{\sim} y_k \Rightarrow x_k \overset{\mathcal{I}(\mathcal{I}_{\sigma\theta})}{\sim} y_k.$$

Proof. Assume that $x_k \overset{[\mathcal{I}_{\sigma\theta}^L(f)]}{\sim} y_k$ and $\varepsilon > 0$ be given. Since for $m = 1, 2, \dots$,

$$\begin{aligned} \frac{1}{h_r} \sum_{k \in I_r} f\left(\left|\frac{x_{\sigma^k(m)}}{y_{\sigma^k(m)}} - L\right|\right) &\geq \frac{1}{h_r} \sum_{\substack{k \in I_r \\ \left|\frac{x_{\sigma^k(m)}}{y_{\sigma^k(m)}} - L\right| \geq \varepsilon}} f\left(\left|\frac{x_{\sigma^k(m)}}{y_{\sigma^k(m)}} - L\right|\right) \\ &\geq f(\varepsilon) \cdot \frac{1}{h_r} \left| \left\{ k \in I_r : \left| \frac{x_{\sigma^k(m)}}{y_{\sigma^k(m)}} - L \right| \geq \varepsilon \right\} \right| \end{aligned}$$

it follows that for any $\gamma > 0$,

$$\left\{ r \in \mathbb{N} : \frac{1}{h_r} \left| \left\{ k \in I_r : \left| \frac{x_{\sigma^k(m)}}{y_{\sigma^k(m)}} - L \right| \geq \varepsilon \right\} \right| \geq \gamma \right\} \subseteq \left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in I_r} f\left(\left|\frac{x_{\sigma^k(m)}}{y_{\sigma^k(m)}} - L\right|\right) \geq \gamma f(\varepsilon) \right\},$$

uniformly in m . Since $x_k \overset{[\mathcal{I}_{\sigma\theta}^L(f)]}{\sim} y_k$, the last set belongs to $\mathcal{I}_{\sigma\theta}$ and so by the definition of an ideal, the first set belongs to $\mathcal{I}_{\sigma\theta}$. Therefore, $x_k \overset{\mathcal{I}(\mathcal{I}_{\sigma\theta})}{\sim} y_k$. \square

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A Generalized Statistical Convergence via Ideals Defined by Folner Sequence on Amenable Semigroup

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Keywords:

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Abstract: The purpose of this study is to extend the notions of \mathcal{I} -convergence, \mathcal{I} -limit superior and \mathcal{I} -limit inferior, \mathcal{I} -cluster point and \mathcal{I} -limit point to functions defined on discrete countable amenable semigroups.

1. Introduction

Let G be a discrete countable amenable semigroup with identity in which both right and left cancelation laws hold, and $w(G)$ and $m(G)$ denote the spaces of all real valued functions and all bounded real functions f on G respectively. $m(G)$ is a Banach space with the supremum norm $\|f\|_{\infty} = \sup\{|f(g)| : g \in G\}$. Nomika [17] showed that, if G is countable amenable group, there exists a sequence $\{S_n\}$ of finite subsets of G such that (i) $G = \cup_{i=1}^{\infty} S_n$, (ii) $S_n \subset S_{n+1}$, $n = 1, 2, 3, \dots$, (iii) $\lim_{n \rightarrow \infty} \frac{|S_n g \cap S_n|}{|S_n|} = 1$, $\lim_{n \rightarrow \infty} \frac{|g S_n \cap S_n|}{|S_n|} = 1$ for all $g \in G$. Here $|A|$ denotes the number of elements in the finite set A . Any sequence of finite subsets of G satisfying (i), (ii) and (iii) is called a Folner sequence for G .

The sequence $S_n = \{0, 1, 2, \dots, n-1\}$ is a familiar Folner sequence giving rise to the classical Cesàro method of summability.

The concept of summability in amenable semigroups was introduced in [14], [15]. In [3], Douglass extended the notion of arithmetic mean to amenable semigroups and obtained a characterization for almost convergence in amenable semigroups.

In [16], the notions of convergence and statistical convergence, statistical limit point and statistical cluster point to functions on discrete countable amenable semigroups were introduced.

Fast [5] presented an interesting generalization of the usual sequential limit which he called statistical convergence for number sequences.

After studies about statistical convergence, Kostyrko, Macaj and Wilczyński defined \mathcal{I} -convergence in a metric space by using the notion of an ideal of the set of positive integers.(see [10]) Later, it was further studied by Salát, Tripathy and Ziman ([12], [13]) and many others. \mathcal{I} -convergence is a generalization of statistical convergence.

We recall the concept of asymptotic and logarithmic density of a set $A \subset \mathbb{N}$ (see [19] pp. 71, 95-96). Let $A \subset \mathbb{N}$. Put $d_n(A) = \frac{1}{n} \sum_{k=1}^n \chi_A(k)$ and $\delta_n(A) = \frac{1}{p_n} \sum_{k=1}^n \frac{\chi_A(k)}{k}$ for $n \in \mathbb{N}$, where $p_n = \sum_{k=1}^n \frac{1}{k}$. The numbers $\underline{d}(A) = \limsup_{n \rightarrow \infty} d_n(A)$ and $\overline{d}(A) = \liminf_{n \rightarrow \infty} d_n(A)$ are called the lower and upper asymptotic density of A , respectively. Similarly, the numbers $\underline{\delta}(A) = \liminf_{n \rightarrow \infty} \delta_n(A)$ and $\overline{\delta}(A) = \limsup_{n \rightarrow \infty} \delta_n(A)$ are called the lower and upper logarithmic density of A , respectively. If $\underline{d}(A) = \overline{d}(A)$ ($\underline{\delta}(A) = \overline{\delta}(A)$), then $d(A) = \underline{d}(A)$ is called the asymptotic density of A ($\delta(A) = \underline{\delta}(A)$ is called the logarithmic density of A , respectively). It is well known that for each $A \subset \mathbb{N}$, $\underline{d}(A) \leq \underline{\delta}(A) \leq \overline{\delta}(A) \leq \overline{d}(A)$.

Denote by \mathcal{I}_d , \mathcal{I}_{δ} the class of all A with $d(A) = 0$ ($\delta(A) = 0$, respectively). Then \mathcal{I}_d and \mathcal{I}_{δ} are non-trivial admissible ideals, \mathcal{I}_d -convergence coincides with the statistical convergence and \mathcal{I}_{δ} -convergence is said to be logarithmic statistical convergence.

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Recently, Das, Savas and Ghosal [2] introduced new notions, namely \mathcal{I} -statistical convergence and \mathcal{I} -lacunary statistical convergence by using ideal.

In [8], he extended the concepts of statistical limit superior and inferior (as introduced by Fridy and Orhan) to \mathcal{I} -limit superior and inferior and give some \mathcal{I} -analogue of properties of statistical limit superior and inferior for a sequence of real numbers.

The purpose of this study is to extend the notions of \mathcal{I} -convergence, \mathcal{I} -limit superior and \mathcal{I} -limit inferior, \mathcal{I} -cluster point and \mathcal{I} -limit point to functions defined on discrete countable amenable semigroups. Also, in this paper, we make a new approach to the notions of $[V, \lambda]$ -summability and λ -statistical convergence by using ideals and introduce new notions, namely, \mathcal{I} - $[V, \lambda]$ -summability and \mathcal{I} - λ -statistical convergence to functions defined on discrete countable amenable semigroups. For the particular case when the amenable semigroup is the additive positive integers, our definition and theorems yield the results of [8], [10], [14].

2. Definitions and Notations

Definition 2.1 ([16]) Let G be a discrete countable amenable semigroup with identity in which both right and left cancelation laws hold. $f \in w(G)$ is said to be convergent to s , for any Folner sequence $\{S_n\}$ for G , if for each $\varepsilon > 0$ there exists $k_0 \in \mathbb{N}$ such that $|f(g) - s| < \varepsilon$ for all $m > k_0$ and $g \in G \setminus S_m$.

Definition 2.2 ([16]) Let G be a discrete countable amenable semigroup with identity in which both right and left cancelation laws hold. $f \in w(G)$ is said to be a Cauchy sequence for any Folner sequence $\{S_n\}$ for G , if for each $\varepsilon > 0$ there exists $k_0 \in \mathbb{N}$ such that $|f(g) - f(h)| < \varepsilon$ for all $m > k_0$ and $g \in G \setminus S_m$.

Definition 2.3 ([16]) Let G be a discrete countable amenable semigroup with identity in which both right and left cancelation laws hold. $f \in w(G)$ is said to be strongly summable to s , for any Folner sequence $\{S_n\}$ for G , if

$$\lim_{n \rightarrow \infty} \frac{1}{|S_n|} \sum_{g \in S_n} |f(g) - s| = 0,$$

where $|S_n|$ denotes the cardinality of the set S_n .

The upper and lower Folner densities of a set $S \subset G$ are defined by

$$\overline{\delta}(S) = \limsup_{n \rightarrow \infty} \frac{1}{|S_n|} |\{g \in S_n : g \in S\}|$$

and

$$\underline{\delta}(S) = \liminf_{n \rightarrow \infty} \frac{1}{|S_n|} |\{g \in S_n : g \in S\}|$$

respectively $\overline{\delta}(S) = \underline{\delta}(S)$, then

$$\delta(S) = \lim_{n \rightarrow \infty} \frac{1}{|S_n|} |\{g \in S_n : g \in S\}|$$

is called Folner density of S . Take $G = \mathbb{N}$, $S_n = \{0, 1, 2, \dots, n-1\}$ and S be the set of positive integers with leading digit 1 in the decimal expansion. The set S has no Folner density with respect to the Folner sequence $\{S_n\}$, since $\underline{\delta}(S) = \frac{1}{9}$, $\overline{\delta}(S) = \frac{5}{9}$. To facilitate this idea we introduce the following notion: If f is function such that $f(g)$ satisfies property P for all g except a set of Folner density zero, we say that $f(g)$ satisfies P for "almost all g ", and abbreviate this by "a.a.g".

Definition 2.4 ([16]) Let G be a discrete countable amenable semigroup with identity in which both right and left cancelation laws hold. $f \in w(G)$ is said to be statistically convergent to s , for any Folner sequence $\{S_n\}$ for G , if for every $\varepsilon > 0$

$$\lim_{n \rightarrow \infty} \frac{1}{|S_n|} |\{g \in S_n : |f(g) - s| \geq \varepsilon\}| = 0.$$

The set of all statistically convergent functions will be denoted by $S(G)$.

Definition 2.5 ([16]) Let G be a discrete countable amenable semigroup with identity in which both right and left cancelation laws hold. $f \in w(G)$ is said to be statistical Cauchy function for any Folner sequence $\{S_n\}$ for G , if for every $\varepsilon > 0$ and $l \geq 0$, then there exists an $m \in G \setminus S_l$ such that

$$\lim_{n \rightarrow \infty} \frac{1}{|S_n|} |\{g \in S_n : |f(g) - f(m)| \geq \varepsilon\}| = 0.$$

3. Main Results

Definition 3.1 Let G be a discrete countable amenable semigroup with identity in which both right and left cancellation laws hold. $f \in w(G)$ is said to be \mathcal{I} -convergent to s for any Folner sequence $\{S_n\}$ for G , if for every $\varepsilon > 0$;

$$\{g \in S_n : |f(g) - s| \geq \varepsilon\} \in \mathcal{I};$$

i.e., $|f(g) - s| < \varepsilon$ a.a.g. The set of all \mathcal{I} -convergent sequences will be denoted by $\mathcal{I}(G)$.

In this section, we study the concepts of \mathcal{I} -limit superior and \mathcal{I} -limit inferior for a Folner sequence, give the relationship between them, and prove some basic properties of these concepts.

For any Folner sequence $\{S_n\}$ for G and for $f \in w(G)$ let B_f denote the set,

$$B_f := \{b \in \mathbb{R} : \{g \in S_n : f(g) > b\} \notin \mathcal{I}\}$$

and similarly,

$$A_f := \{a \in \mathbb{R} : \{g \in S_n : f(g) < a\} \notin \mathcal{I}\}$$

We begin with a definition.

Definition 3.2 If $f \in w(G)$, then the \mathcal{I} -limit superior of $f \in w(G)$, for any Folner sequence $\{S_n\}$ for G , is given by

$$\mathcal{I}\text{-lim sup } f : \begin{cases} \sup B_f, & \text{if } B_f \neq \emptyset, \\ -\infty, & \text{if } B_f = \emptyset, \end{cases}$$

Similarly, the \mathcal{I} -limit inferior for any Folner sequence $\{S_n\}$ for G is given by

$$\mathcal{I}\text{-lim inf } f : \begin{cases} \inf A_f, & \text{if } A_f \neq \emptyset, \\ \infty, & \text{if } A_f = \emptyset, \end{cases}$$

It is easy to see that for any $f \in w(G)$ and for any Folner sequence $\{S_n\}$ for G , $\mathcal{I}\text{-lim inf } f \leq \mathcal{I}\text{-lim sup } f$.

Definition 3.3 The function $f \in w(G)$ is said to be \mathcal{I} -bounded for any Folner sequence $\{S_n\}$ for G , if there is a number M such that

$$\{g \in S_n : |f(g)| \geq M\} \in \mathcal{I}.$$

Note that \mathcal{I} -boundedness implies that $\mathcal{I}\text{-lim sup } f$ and $\mathcal{I}\text{-lim inf } f$ are finite. The following theorem can be proved by a straightforward least upper bound argument.

Theorem 3.4 For any Folner sequence $\{S_n\}$ for G , if $\mu = \mathcal{I}\text{-lim sup } f$ is finite, then for each $\varepsilon > 0$

$$\{g \in S_n : f(g) > \mu - \varepsilon\} \notin \mathcal{I} \text{ and } \{g \in S_n : f(g) > \mu + \varepsilon\} \in \mathcal{I}. \quad (1.1)$$

Conversely, if (1.1) holds for every $\varepsilon > 0$ then $\mu = \mathcal{I}\text{-lim sup } f$.

Proof. Let $\varepsilon > 0$. Since $\mu + \varepsilon > \mu = \sup \{f : \sup B_f \notin \mathcal{I}\}$, the number $\mu + \varepsilon$ is not in $\{f : \sup B_f \notin \mathcal{I}\}$ and $\{g \in S_n : f(g) > \mu + \varepsilon\} \in \mathcal{I}$. Further $\mu - \varepsilon < \mu$ and there exists $t' \in \mathbb{R}$ such that $\mu - \varepsilon < t' < \mu$, $t' \in \{f : \sup B_f \notin \mathcal{I}\}$. Hence $\{g \in S_n : f(g) > t'\} \notin \mathcal{I}$ and also $\{g \in S_n : f(g) > \mu - \varepsilon\} \notin \mathcal{I}$. Consequently (1.1) holds.

On the other hand, suppose that the number μ fulfils (1.1) for every $\varepsilon > 0$. Then, if $\varepsilon > 0$, we have $\mu + \varepsilon \notin \{f : \sup B_f \notin \mathcal{I}\}$ and $\mathcal{I}\text{-lim sup } f \leq \mu + \varepsilon$. Since this holds for every $\varepsilon > 0$, we have $\mathcal{I}\text{-lim sup } f \leq \mu$. The first condition in (1.1) implies $\mathcal{I}\text{-lim sup } f \geq \mu - \varepsilon$ for each $\varepsilon > 0$, so we have $\mathcal{I}\text{-lim sup } f \geq \mu$. Inequalities $\mathcal{I}\text{-lim sup } f \leq \mu$ and $\mathcal{I}\text{-lim sup } f \geq \mu$ imply $\mu = \mathcal{I}\text{-lim sup } f$. \square

The dual statement for $\mathcal{I}\text{-lim sup } f$ is as follows.

Theorem 3.5 For any Folner sequence $\{S_n\}$ for G , if $\lambda = \mathcal{I}\text{-lim inf } f$ is finite, then for each $\varepsilon > 0$

$$\{g \in S_n : f(g) < \lambda + \varepsilon\} \notin \mathcal{I} \text{ and } \{g \in S_n : f(g) < \lambda - \varepsilon\} \in \mathcal{I}. \quad (2.1)$$

Conversely, if (2.1) holds for every $\varepsilon > 0$ then $\lambda = \mathcal{I}\text{-lim inf } f$.

Proof. The proof of this theorem is similar to proof of the theorem 1. \square

Theorem 3.6 For any Folner sequence $\{S_n\}$ for G ,

$$\mathcal{I}\text{-lim inf } f \leq \mathcal{I}\text{-lim sup } f.$$

Proof. First consider the case in which $\mathcal{I}\text{-limsup} f = -\infty$. Hence we have $B_f = \emptyset$, so for every b in \mathbb{R} , $\{g \in S_n : f(g) > b\} \in \mathcal{I}$ which implies that $\{g \in S_n : f(g) \leq b\} \in \mathcal{F}(\mathcal{I})$ so for every a in \mathbb{R} , $\{g \in S_n : f(g) \leq a\} \notin \mathcal{I}$. Hence $\mathcal{I}\text{-liminf} f = -\infty$.

The case in which $\mathcal{I}\text{-limsup} f = +\infty$ needs no proof, so we next assume that $\mu := \mathcal{I}\text{-limsup} f$ is finite and $\lambda := \mathcal{I}\text{-liminf} f$. Given $\varepsilon > 0$ we show that $\mu + \varepsilon \in A_f$, so that $\lambda \leq \mu + \varepsilon$. By theorem 1, $\{g \in S_n : f(g) > \mu + \varepsilon\} \in \mathcal{I}$ because $\mu = \sup B_f$. This implies $\{g \in S_n : f(g) \leq \mu + \frac{\varepsilon}{2}\} \in \mathcal{F}(\mathcal{I})$. Since

$$\left\{g \in S_n : f(g) \leq \mu + \frac{\varepsilon}{2}\right\} \subseteq \{g \in S_n : f(g) < \mu + \varepsilon\}$$

and $\mathcal{F}(\mathcal{I})$ is a filter on \mathbb{N} ,

$$\{g \in S_n : f(g) < \mu + \varepsilon\} \in \mathcal{F}(\mathcal{I}).$$

This implies

$$\{g \in S_n : f(g) < \mu + \varepsilon\} \notin \mathcal{I}.$$

Hence $\mu + \varepsilon \in A_f$. By the definition $\lambda := \mathcal{I}\text{-liminf} f$, so we conclude that $\lambda \leq \mu + \varepsilon$; and since ε is arbitrary this proves that $\lambda \leq \mu$. \square

Theorem 3.7 For any Folner sequence $\{S_n\}$ for G , \mathcal{I} -bounded function f is \mathcal{I} -convergent if and only if $\mathcal{I}\text{-limsup} f = \mathcal{I}\text{-liminf} f$.

Proof. For any Folner sequence $\{S_n\}$ for G , let $\lambda := \mathcal{I}\text{-liminf} f$ and $\mu := \mathcal{I}\text{-limsup} f$. First assume that $\mathcal{I}\text{-lim} f = s$ and $\varepsilon > 0$. Then $\{g \in S_n : |f(g) - s| \geq \varepsilon\} \in \mathcal{I}$, so that $\{g \in S_n : f(g) > s + \varepsilon\} \in \mathcal{I}$ which implies that $\mu \leq s$. We also have $\{g \in S_n : f(g) < s - \varepsilon\} \in \mathcal{I}$, which yields that $s \leq \lambda$. Therefore $\mu \leq \lambda$. Combining this with *Theorem 3* we conclude that $\mu = \lambda$.

Now assume that for any Folner sequence $\{S_n\}$ for G , $\mathcal{I}\text{-limsup} f = \mathcal{I}\text{-liminf} f$. If $\varepsilon > 0$, then (1.1) and (2.1) imply that

$$\left\{g \in S_n : f(g) > s + \frac{\varepsilon}{2}\right\} \in \mathcal{I}$$

and

$$\left\{g \in S_n : f(g) < s - \frac{\varepsilon}{2}\right\} \in \mathcal{I}.$$

Hence, for any Folner sequence $\{S_n\}$ for G , $\mathcal{I}\text{-lim} f = s$. \square

Definition 3.8 Let G be a discrete countable amenable semigroup with identity in which both right and left cancelation laws hold. $f \in w(G)$ is said to be \mathcal{I} -Cauchy function for any Folner sequence $\{S_n\}$ for G if, for each $\varepsilon > 0$ and $l \geq 0$, then there exists an $m \in G/S_l$ such that

$$\{g \in S_n : |f(g) - f(m)| \geq \varepsilon\} \in \mathcal{I}$$

i.e., $|f(g) - f(m)| < \varepsilon$ a.a.g.

Theorem 3.9 The following statements are equivalent:

- (i) $f \in w(G)$ is \mathcal{I} -convergent function
- (ii) $f \in w(G)$ is \mathcal{I} -Cauchy function.

Proof. (i) \Rightarrow (ii) To prove that (i) implies (ii) we assume that $\mathcal{I}\text{-lim} f(g) = s$. Let $\varepsilon > 0$. Then $|f(g) - s| < \frac{\varepsilon}{2}$ a.a.g and, if g_0 is chosen so that $|f(g_0) - s| < \frac{\varepsilon}{2}$ a.a.g, then we have

$$|f(g) - f(g_0)| < |f(g) - s| + |f(g_0) - s| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

a.a.g. Hence f is \mathcal{I} -Cauchy function.

(ii) \Rightarrow (i) Next $\{g \in S_n : |f(g) - f(m)| < \varepsilon\} \in \mathcal{F}(\mathcal{I})$ holds for all $\varepsilon > 0$. Then the set

$$C_\varepsilon = \{g \in S_n : f(g) \in [f(m) - \varepsilon, f(m) + \varepsilon]\} \in \mathcal{F}(\mathcal{I})$$

for all $\varepsilon > 0$. Denote $J_\varepsilon = [f(m) - \varepsilon, f(m) + \varepsilon]$.

Fix an $\varepsilon > 0$. Then $C_\varepsilon \in \mathcal{F}(\mathcal{I})$ and $C_{\frac{\varepsilon}{2}} \in \mathcal{F}(\mathcal{I})$. Hence $C_\varepsilon \cap C_{\frac{\varepsilon}{2}} \in \mathcal{F}(\mathcal{I})$. This implies

$$\begin{aligned} J &= J_\varepsilon \cap J_{\frac{\varepsilon}{2}} \neq \emptyset, \\ \{g \in S_n : f(g) \in J\} &\in \mathcal{F}(\mathcal{I}), \\ \text{diam}(J) &\leq \frac{1}{2} \text{diam}(J_\varepsilon). \end{aligned}$$

($\text{diam}(J)$ denotes the length of the interval J .) This way, by induction, we can construct the sequence of (closed) intervals $J_\varepsilon = I_0 \supseteq I_1 \supseteq \dots \supseteq I_n \supseteq \dots$ with the property $\text{diam}(I_n) \leq \frac{1}{2} \text{diam}(I_{n-1})$ ($n = 2, 3, \dots$). Then there exists a $s \in \bigcap_{n \in \mathbb{N}} I_n$ and it is routine work to verify $\mathcal{I}\text{-lim} f(g) = s$. \square

4. \mathcal{I} -Limit Points and \mathcal{I} -Cluster points

In [10] Koystkro introduced the concepts of \mathcal{I} limit point and \mathcal{I} cluster point. In this section we extend these concepts of \mathcal{I} limit point and \mathcal{I} cluster point to the functions defined on discrete countable amenable semigroups. If $f \in w(G)$ and $H \subset G$, we write $R_f(G)$ to denote the range of $f \in w(G)$. If $R_f(H)$ is a subset of $R_f(G)$ and $\lim_{n \rightarrow \infty} \frac{|H \cap S_n|}{|S_n|} = 0$ then $R_f(H)$ is called a subset of Folner density zero for any Folner sequence $\{S_n\}$ for G , or a thin subset. On the other hand $R_f(H)$ is a nonthin subset of $R_f(G)$ if $\lim_{n \rightarrow \infty} \frac{|H \cap S_n|}{|S_n|} \neq 0$.

Definition 4.1 The number s is a \mathcal{I} limit point for an $f \in w(G)$, for any Folner sequence $\{S_n\}$ for G , provided that there is nonthin subset of $R_f(G)$ that f \mathcal{I} -converges to s in it.

Definition 4.2 The number c is a \mathcal{I} cluster point for an $f \in w(G)$, for any Folner sequence $\{S_n\}$ for G , provided that for each $\varepsilon > 0$ the set $\{g \in S_n : |f(g) - c| < \varepsilon\} \notin \mathcal{I}$.

For $f \in w(G)$, let $L_f(G)$, $\mathcal{I}(\Lambda_f(G))$, $\mathcal{I}(\Gamma_f(G))$ denote the sets of all ordinary limit points, \mathcal{I} limit points and \mathcal{I} cluster points of f , respectively. It is clear that $\mathcal{I}(\Lambda_f(G)) \subseteq \mathcal{I}(\Gamma_f(G)) \subseteq L_f(G)$.

Theorem 4.3 Let $f \in w(G)$ be \mathcal{I} -bounded for any Folner sequence $\{S_n\}$ for G and let $\mathcal{I}(\Gamma_f(G))$ be the set of all \mathcal{I} cluster points of f , for any Folner sequence $\{S_n\}$ for G . Then

$$\mathcal{I}\text{-lim sup } f = \max \mathcal{I}(\Gamma_f(G)).$$

Proof. Put $\mathcal{I}\text{-lim sup } f = \mu$. Suppose $\mu' > \mu$. First we show that μ' is not in $\mathcal{I}(\Gamma_f(G))$. We have

$$\mu = \sup S, S = \{t : \{g \in S_n : f(g) > t\} \notin \mathcal{I}\}. \quad (7.1)$$

Choose $\varepsilon > 0$ such that $\mu < \mu' - \varepsilon < \mu'$. Then $\mu' - \varepsilon \notin S$ and

$$\{g \in S_n : f(g) > \mu' - \varepsilon\} \in \mathcal{I}.$$

It follows from the definition of \mathcal{I} cluster point for an $f \in w(G)$ that $\mu' \notin \mathcal{I}(\Gamma_f(G))$.

We show $\mu \in \mathcal{I}(\Gamma_f(G))$. Let $\eta > 0$. It follows from (7.1) that there is a $t_0 \in \mathbb{R}$ such that $\mu - \eta < t_0 \leq \mu$, $t_0 \in S$. Hence

$$\{g \in S_n : f(g) > t_0\} \notin \mathcal{I}. \quad (7.2)$$

Simultaneously, since $\mu + \frac{\eta}{2} \notin S$, we have

$$\{g \in S_n : f(g) > \mu + \frac{\eta}{2}\} \in \mathcal{I}. \quad (7.2')$$

It follows from (7.2) and (7.2') $\{g \in S_n : f(g) \in (\mu - \eta, \mu + \eta)\} \notin \mathcal{I}$ and $\mu \in \mathcal{I}(\Gamma_f(G))$. \square

Remark 4.4 It can be shown for a \mathcal{I} -bounded sequence $\{S_n\}$ for G the equality

$$\mathcal{I}\text{-lim inf } f = \min \mathcal{I}(\Gamma_f(G)).$$

Example 4.5 Take $G = \mathbb{Z}$, $H = \{0, \pm 1, \pm 3, \pm 5, \pm 7, \dots\}$, $S_n = [-n, n]$ and define f as follows:

$$f(g) = \begin{cases} 0, & \text{if } g \in H, \\ 1, & \text{if } g \notin G \setminus H. \end{cases}$$

Then $L_f(G) = \{0, 1\}$ and $\mathcal{I}(\Lambda_f(G)) = \{0\}$.

5. Relationship between \mathcal{I}_d and \mathcal{I}_δ -Convergence and Cesàro summability

Recall that the Folner sequence $\{S_n\}$ for G is said to be strongly $(C, 1)$ -summable to s if and only if $\lim_{n \rightarrow \infty} \frac{1}{|S_n|} \sum_{g \in S_n} |f(g) - s| = 0$.

If the Folner sequence $\{S_n\}$ for G is bounded, then $\mathcal{I}_d\text{-lim } f = s$ implies $(C, 1)\text{-lim } f(g) = s$. The converse is obviously not true. (e.g. $\{S_n\} = \{0, 1, 0, 1, \dots\}$). However $f \in m(G)$ is bounded, the \mathcal{I}_d -convergence to some number is equivalent to strongly Cesàro-summability to the same number. But, for \mathcal{I}_δ -convergence the situation is different.

Proposition 5.1 Let $f \in w(G)$ be \mathcal{I} -bounded for any Folner sequence $\{S_n\}$ for G such that $\mathcal{I}_\delta\text{-lim } f = 0$ and $(C, 1)\text{-lim } f$ does not exist.

Proof. Put $S = \bigcup_{n=2}^{\infty} S_n$, where $S_n = \{n^{n^2} + 1, n^{n^2} + 2, \dots, n^{n^2+1}\}$ for $n \in \mathbb{N}$, $n \geq 2$. If $S(k) = d_k(S)$ for $k \in \mathbb{N}$, then

$$\bar{d}(S) \geq \limsup_{n \rightarrow \infty} \frac{S(n^{n^2+1})}{n^{n^2+1}} \geq \limsup_{n \rightarrow \infty} \frac{n^{n^2+1} - n^{n^2}}{n^{n^2+1}} = 1.$$

Hence $\bar{d}(S) = 1$. Simultaneously $\sum_{k=1}^n \frac{1}{k} = \ln n + \gamma + O\left(\frac{1}{n}\right)$, where γ is an Euler constant, we have $\sum_{j \in S_n} \frac{1}{j} = \ln n + O\left(\frac{1}{n^2}\right)$ for all $n \in \mathbb{N}$, $n \geq 2$. From this by a simple calculation we get

$$\bar{\delta}(S) \leq \lim_{n \rightarrow \infty} \frac{\sum_{k=1}^n \ln k + O(1)}{\frac{n^{n^2+1}}{\sum_{j=1}^n \frac{1}{j}}} \leq \lim_{n \rightarrow \infty} \frac{n \ln n + O(1)}{(n^2 + 1) \ln n + O(1)} = 0.$$

So we have $\delta(S) = 0$ and consequently $\underline{d}(S) = 0$. So $d(S)$ does not exist.

Define f as follows:

$$f(g) = \begin{cases} 0, & \text{if } g \in G \setminus S, \\ 1, & \text{if } g \in S. \end{cases}$$

Since $\delta(S) = 0$ we have \mathcal{I}_{δ} -lim $f = 0$. But $(C, 1)$ -lim $f(g)$ does not exist. \square

6. Conclusion

The convergence of folner sequences on amenable semigroups has been recently studied by several authors. In this study, we extend concepts of statistical limit superior and inferior (as introduced by Nuray and Rhoades) to \mathcal{I} -limit superior and inferior and give some \mathcal{I} -analogue of properties of statistical limit superior and inferior for folner sequences on amenable semigroup. We investigate some properties of the new concepts. So, we have extended some well-known results.

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Lacunary \mathcal{I}_2 -Invariant Convergence of Double Sequences of Functions on Amenable Semigroups

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Abstract: In this paper, the concept of lacunary uniform density of any subset A of the set $\mathbb{N} \times \mathbb{N}$ is defined. Associate with this, the concept of lacunary \mathcal{I}_2 -invariant convergence for double Folner sequences $\{S_{kj}\}$ for G was given. Also, we examine relationships between this new type convergence concept and the concepts of lacunary invariant convergence and p -strongly lacunary invariant convergence of double Folner sequences. Finally, introducing lacunary \mathcal{I}_2^* -invariant convergence and lacunary \mathcal{I}_2 -invariant Cauchy concepts for double Folner sequences, we give the relationships among these concepts and relationships with lacunary \mathcal{I}_2 -invariant convergence concept.

1. Introduction

The idea of the statistical convergence was given by Zygmund [36] in the first edition of his monograph published in Warsaw in 1935. The concept of statistical convergence was introduced by Fast [31] and then reintroduced by Schoenberg [29] independently. Over the years, statistical convergence has been developed in ([7], [30], [32], [33], [34], [13]) and turned out very useful to resolve many convergence problems arising in Analysis.

A number sequence $x = (x_k)$ is said to be statistically convergent to the number l if for every $\varepsilon > 0$, $\lim_{n \rightarrow \infty} \frac{1}{n} |\{k \leq n : |x_k - l| \geq \varepsilon\}| = 0$. In this case we write $st - \lim_{k \rightarrow \infty} x_k = l$. Statistical convergence is a natural generalization of ordinary convergence. If $\lim_{k \rightarrow \infty} x_k = l$, then $st - \lim_{k \rightarrow \infty} x_k = l$. The converse does not hold in general.

Mursaleen and Edely [13] extended the above idea from single to double sequences of scalars and established relations between statistical convergence and strongly Cesàro summable double sequences.

Lacunary statistical convergence was defined by Fridy and Orhan [35]. Çakan and Altay [34] presented multidimensional analogues of the results presented by Fridy and Orhan [35].

By a lacunary sequence we mean an increasing integer sequence $\theta = \{k_r\}$ such that $k_0 = 0$ and $h_r = k_r - k_{r-1} \rightarrow \infty$ as $r \rightarrow \infty$. Throughout this paper the intervals determined by θ will be denoted by $I_r = (k_{r-1}, k_r]$.

A sequence $x = (x_k)$ is said to be lacunary statistically convergent to the number L if for every $\varepsilon > 0$, $\lim_{r \rightarrow \infty} \frac{1}{h_r} |\{k \in I_r : |x_k - L| \geq \varepsilon\}| = 0$. In this case we write $S_\theta - \lim_{k \rightarrow \infty} x_k = L$ or $x_k \rightarrow L(S_\theta)$.

The idea of \mathcal{I} -convergence was introduced by Kostyrko et al. [6] as a generalization of statistical convergence which is based on the structure of the ideal \mathcal{I} of subset of the set of natural numbers. Das et al. [25], [8] introduced the concept of \mathcal{I} -convergence of double sequences in a metric space and studied some properties of this convergence.

The notion of lacunary ideal convergence of real sequences was introduced in [20]. Das et al. [24] introduced new notions, namely \mathcal{I} -statistical convergence and \mathcal{I} -lacunary statistical convergence by using ideal. Belen et al. [1] introduced the notion of ideal statistical convergence of double sequences, which is a new generalization of the notions of statistical convergence and usual convergence. Kumar et al. [39] introduced \mathcal{I} -lacunary statistical convergence of double sequences. More investigation and applications on this notion can be found in [21].

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Several authors have studied invariant convergent sequences (see, [11],[12], [22], [23], [38]). Savaş and Nuray [26] introduced the concepts of σ -statistically convergence and lacunary σ -statistically convergence and gave some inclusion relations. Pancaroğlu and Nuray [40] defined the concept lacunary invariant summability and p -strongly lacunary invariant summability. The concept of lacunary strongly σ -convergence was introduced by Savaş [22].

In [16], the concepts of σ -uniform density of subsets A of the set \mathbb{N} of positive integers and corresponding \mathcal{I}_σ -convergence were introduced. Also, inclusion relations between \mathcal{I}_σ -convergence and invariant convergence also \mathcal{I}_σ -convergence and $[V_\sigma]_p$ were given [16]. Recently, the concept of lacunary σ -uniform density of the set $A \subset \mathbb{N}$, lacunary \mathcal{I}_σ -convergence, lacunary \mathcal{I}_σ^* -convergence, lacunary \mathcal{I}_σ -Cauchy, lacunary \mathcal{I}_σ^* -Cauchy sequences of real numbers were defined by Ulusu and Nuray [14] and similar concepts can be seen in [16]. Ulusu et al. [15] defined the lacunary \mathcal{I}_2 -invariant convergence for double sequence.

Let σ be a one-to-one mapping of the set of positive integers into itself such that $\sigma^m(n) = (\sigma^{m-1}(n))$, $m = 1, 2, 3, \dots$. A continuous linear functional Φ on l_∞ , the space of real bounded sequences, is said to be an invariant mean or a σ mean, if and only if,

- (1) $\Phi(x) \geq 0$, for all sequences $x = (x_n)$ with $x_n \geq 0$ for all n ;
- (2) $\Phi(e) = 1$, where $e = (1, 1, 1, \dots)$;
- (3) $\Phi(x_{\sigma(n)}) = \Phi(x)$ for all $x \in l_\infty$.

The mapping Φ are assumed to be one-to-one such that $\sigma^m(n) \neq n$ for all positive integers n and m , where $\sigma^m(n)$ denotes the m th iterate of the mapping σ at n . Thus, Φ extends the limit functional on c , the space of convergent sequences, in the sense that $\Phi(x) = \lim x$, for all $x \in c$. In case σ is translation mapping $\sigma(n) = n + 1$, the σ mean is often called a Banach limit and V_σ , the set of bounded sequences all of whose invariant means are equal, is the set of almost convergent sequences.

A sequence $x = (x_k)$ is σ -statistically convergent to L if for every $\varepsilon > 0$, $\lim_{m \rightarrow \infty} \frac{1}{m} \left| \left\{ k \leq m : \left| x_{\sigma^k(n)} - L \right| \geq \varepsilon \right\} \right| = 0$, uniformly in n .

In this case, we write $S_\sigma - \lim x = L$ or $x_k \rightarrow L(S_\sigma)$.

Nuray et al. [16] introduced the concepts of σ -uniform density and \mathcal{I}_σ -convergence.

A sequence $x = (x_k)$ is \mathcal{I}_σ convergent to the number L if for every $\varepsilon > 0$, $A_\varepsilon = \{k : |x_k - L| \geq \varepsilon\} \in \mathcal{I}_\sigma$, that is $V(A_\varepsilon) = 0$. In this case, we write $\mathcal{I}_\sigma - \lim x = L$.

Let G be a discrete countable amenable semigroup with identity in which both right and left cancelation laws hold, and $w(G)$ and $m(G)$ denote the spaces of all real valued functions and all bounded real functions f on G respectively. $m(G)$ is a Banach space with the supremum norm $\|f\|_\infty = \sup\{|f(g)| : g \in G\}$. Nomika [28] showed that, if G is countable amenable group, there exists a sequence $\{S_n\}$ of finite subsets of G such that (i) $G = \cup_{i=1}^\infty S_n$, (ii) $S_n \subset S_{n+1}$, $n = 1, 2, 3, \dots$, (iii) $\lim_{n \rightarrow \infty} \frac{|S_n g \cap S_n|}{|S_n|} = 1$, $\lim_{n \rightarrow \infty} \frac{|g S_n \cap S_n|}{|S_n|} = 1$ for all $g \in G$. Here $|A|$ denotes the number of elements in the finite set A . Any sequence of finite subsets of G satisfying (i), (ii) and (iii) is called a Folner sequence for G .

The sequence $S_n = \{0, 1, 2, \dots, n-1\}$ is a familiar Folner sequence giving rise to the classical Cesàro method of summability.

The concept of summability in amenable semigroups was introduced in [9], [10]. In [4], Douglas extended the notion of arithmetic mean to amenable semigroups and obtained a characterization for almost convergence in amenable semigroups.

In [27], the notions of convergence and statistical convergence, statistical limit point and statistical cluster point to functions on discrete countable amenable semigroups were introduced.

Let G be a discrete countable amenable semigroup with identity in which both right and left cancelation laws hold. $f \in w(G)$ is said to be \mathcal{I} -convergent to s for any Folner sequence $\{S_n\}$ for G , if for every $\varepsilon > 0$; $\{g \in S_n : |f(g) - s| \geq \varepsilon\} \in \mathcal{I}$; i.e., $|f(g) - s| < \varepsilon$ a.a.g. The set of all \mathcal{I} -convergent sequences will be denoted by $\mathcal{I}(G)$.

Let G be a discrete countable amenable semigroup with identity in which both right and left cancelation laws hold. The function $f \in w(G)$ is said to be \mathcal{I} -invariant convergent to s for any Folner sequence $\{S_n\}$ for G if for every $\varepsilon > 0$;

$$\{g \in S_n : |f(g) - s| \geq \varepsilon\}$$

belongs to \mathcal{I}_σ ; i.e., $V(A_\varepsilon) = 0$. The set of all \mathcal{I} -invariant convergent sequences will be denoted by $\mathcal{I}_\sigma(G)$.

Let G be a discrete countable amenable semigroup with identity in which both right and left cancelation laws hold. The function $f \in w(G)$ is said to be invariant convergent to s for any Folner sequence $\{S_n\}$ for G if

$$\lim_{n \rightarrow \infty} \frac{1}{|S_n|} \sum_{1 \leq k \leq |S_n| \& g \in S_n} f(g \sigma^k(m)) = s, \text{ uniformly in } m.$$

In this case, we write $f \rightarrow s(V_\sigma)$.

A double sequence $x = (x_{mn})$ of real numbers is said to be convergent to $L \in \mathbb{R}$ in Pringsheim's sense (shortly, p -convergent to $L \in \mathbb{R}$), if for any $\varepsilon > 0$, there exists $N_\varepsilon \in \mathbb{N}$ such that $|x_{mn} - L| < \varepsilon$, whenever $m, n > N_\varepsilon$. In this case, we write

$$\lim_{m,n \rightarrow \infty} x_{mn} = L.$$

We recall that a subset K of $\mathbb{N} \times \mathbb{N}$ is said to have natural density $d(K)$ if

$$d(K) = \lim_{m,n \rightarrow \infty} \frac{K(m,n)}{m.n},$$

where $K(m,n) = |\{(j,k) \in \mathbb{N} \times \mathbb{N} : j \leq m, k \leq n\}|$.

A nontrivial ideal \mathcal{I}_2 of $\mathbb{N} \times \mathbb{N}$ is called strongly admissible if $\{i\} \times \mathbb{N}$ and $\mathbb{N} \times \{i\}$ belong to \mathcal{I}_2 for each $i \in \mathbb{N}$. It is evident that a strongly admissible ideal is admissible also.

Throughout the paper we take \mathcal{I}_2 as a strongly admissible ideal in $\mathbb{N} \times \mathbb{N}$.

$$\mathcal{I}_2^0 = \left\{ A \subset 2^{\mathbb{N} \times \mathbb{N}} : (\exists m(A) \in \mathbb{N}) (i, j \geq m(A) \implies (i, j) \notin A) \right\}.$$

Then \mathcal{I}_2^0 is a strongly admissible ideal if and only if $\mathcal{I}_2^0 \subset \mathcal{I}_2$.

Let (X, ρ) be a metric space. A double sequence $x = (x_{mn})$ in X is said to be \mathcal{I}_2 -convergent to $L \in X$, if for any $\varepsilon > 0$ we have

$$A(\varepsilon) = \{(m,n) \in \mathbb{N} \times \mathbb{N} : \rho(x_{mn}, L) \geq \varepsilon\} \in \mathcal{I}_2.$$

In this case, we say that x is \mathcal{I}_2 -convergent and we write

$$\mathcal{I}_2 - \lim_{m,n \rightarrow \infty} x_{mn} = L.$$

An admissible ideal $\mathcal{I}_2 \subset 2^{\mathbb{N} \times \mathbb{N}}$ satisfies the property (AP2) if for every countable family of mutually disjoint sets $\{E_1, E_2, \dots\}$ belonging to \mathcal{I}_2 , there exists a countable family of sets $\{F_1, F_2, \dots\}$ such that $E_j \Delta F_j \in \mathcal{I}_2^0$, i.e., $E_j \Delta F_j$ is included in the finite union of rows and columns in $\mathbb{N} \times \mathbb{N}$ for each $j \in \mathbb{N}$ and $F = \bigcup_{j=1}^{\infty} F_j \in \mathcal{I}_2$ (hence $F_j \in \mathcal{I}_2$ for each $j \in \mathbb{N}$).

A double sequence $\theta = \{(k_r, j_u)\}$ is called double lacunary sequence if there exist two increasing sequences of integers (k_r) and (j_u) such that

$$k_0 = 0, h_r = k_r - k_{r-1} \rightarrow \infty \text{ and } j_0 = 0, \bar{h}_u = j_u - j_{u-1} \rightarrow \infty, r, u \rightarrow \infty.$$

We will use the following notation $k_{us} := k_{u!s}$, $h_{us} := h_u \bar{h}_s$ and θ_{us} is determined by

$$I_{us} := \{(k, j) : k_{r-1} < k \leq k_r \text{ and } j_{u-1} < j \leq j_u\},$$

$$q_u := \frac{k_r}{k_{r-1}}, \bar{q}_u := \frac{j_u}{j_{u-1}}$$

Throughout the paper, by $\theta = \{(k_r, j_u)\}$ we will denote a double lacunary sequence of positive real numbers, respectively, unless otherwise stated.

2. Main results

Definition 2.1 $f \in w(G)$ is said to be lacunary invariant convergent to s , for any double Følner sequence $\{S_{kj}\}$ for G , if

$$\lim_{r,u \rightarrow \infty} \frac{1}{h_{ru}} \sum_{k,j \in I_{ru} \& g \in S_{kj}} f(g^{\sigma^k(m), \sigma^j(n)}) = s,$$

uniformly in n, m and it is denoted by $S_{kj} \rightarrow s (V_2^{\sigma\theta}(G))$.

Definition 2.2 Let $\theta = \{(k_r, j_u)\}$ be a double lacunary sequence, $A \subset \mathbb{N} \times \mathbb{N}$ and

$$P_{ru} := \min_{m,n} \left| \left\{ A \cap \left\{ (\sigma^k(m), \sigma^j(n)) : (k, j) \in I_{ru} \right\} \right\} \right|$$

and

$$P_{ru} := \max_{m,n} \left| \left\{ A \cap \left\{ (\sigma^k(m), \sigma^j(n)) : (k, j) \in I_{ru} \right\} \right\} \right|$$

If the following limit exist

$$\underline{V}_2^\theta(A) := \lim_{r,u \rightarrow \infty} \frac{P_{ru}}{P_{ru}}, \overline{V}_2^\theta(A) := \lim_{r,u \rightarrow \infty} \frac{P_{ru}}{p_{ru}},$$

then they are called a lower lacunary σ -uniform density and an upper lacunary σ -uniform density of the set A , respectively. If $V_2^\theta(A) = \overline{V}_2^\theta(A)$, then $V_2^\theta(A) = \underline{V}_2^\theta(A) = \overline{V}_2^\theta(A)$ is called the lacunary σ -uniform density of A . Denote by $\mathcal{I}_2^{\sigma\theta}$ the class of all $A \subseteq \mathbb{N} \times \mathbb{N}$ with $V_2^\theta(A) = 0$.

Throughout the paper we take $\mathcal{I}_2^{\sigma\theta}$ as a strongly admissible ideal in $\mathbb{N} \times \mathbb{N}$.

Definition 2.3 The function $f \in w(G)$ is said to be \mathcal{I}_2 -bounded for any Folner sequence $\{S_{k,j}\}$ for G , if there is a number M such that

$$\{(k, j) \in I_{ru} \& g \in S_{k,j} : |f(g)| \geq M\} \in \mathcal{I}_2.$$

Definition 2.4 $f \in w(G)$ is said to be lacunary \mathcal{I}_2 -invariant convergent, or $\mathcal{I}_2^{\sigma\theta}$ -convergent to s , for any double Folner sequence $\{S_{k,j}\}$ for G , if for every $\varepsilon > 0$, the set

$$A(\varepsilon) = \{(k, j) \in I_{ru} \& g \in S_{k,j} : |f(g_{\sigma^k(m), \sigma^j(n)}) - s| \geq \varepsilon\}$$

belongs to $\mathcal{I}_2^{\sigma\theta}$; i.e., $V_2^\theta A(\varepsilon) = 0$, uniformly in m, n . In this case, we write

$$\mathcal{I}_2^{\sigma\theta}\text{-}\lim f(g_{\sigma^k(m), \sigma^j(n)}) = s \text{ or } f(g_{\sigma^k(m), \sigma^j(n)}) \rightarrow s(\mathcal{I}_2^{\sigma\theta}), \text{ where } g \in S_{k,j}.$$

Theorem 2.5 Suppose that $\{S_{k,j}\}$ is a bounded double Folner sequence. If $\{S_{k,j}\}$ is lacunary \mathcal{I}_2 -invariant convergent to s , then $\{S_{k,j}\}$ is lacunary invariant convergent to s .

Proof. Let $\theta = \{(k_r, j_u)\}$ be a double lacunary sequence, $m, n \in \mathbb{N}$ be arbitrary, $\varepsilon > 0$. Now we calculate

$$t(k, j, r, u) := \left| \frac{1}{h_{ru}} \sum_{k, j \in I_{ru} \& g \in S_{k,j}} f(g_{\sigma^k(m), \sigma^j(n)}) - s \right|.$$

We have

$$t(k, j, r, u) \leq t^{(1)}(k, j, r, u) + t^{(2)}(k, j, r, u)$$

where

$$t^{(1)}(m, n, f) := \frac{1}{h_{ru}} \sum_{\substack{k, j \in I_{ru} \& g \in S_{k,j} \\ |f(g_{\sigma^k(m), \sigma^j(n)}) - s| \geq \varepsilon}} |f(g_{\sigma^k(m), \sigma^j(n)}) - s|$$

and

$$t^{(2)}(m, n, x) = \frac{1}{h_{ru}} \sum_{\substack{k, j \in I_{ru} \& g \in S_{k,j} \\ |f(g_{\sigma^k(m), \sigma^j(n)}) - s| < \varepsilon}} |f(g_{\sigma^k(m), \sigma^j(n)}) - s|.$$

Therefore, we have $t^{(2)}(m, n, f) < \varepsilon$, for every $m, n = 1, 2, \dots$. The boundedness of double folner sequence is implies that there exist $M > 0$ such that

$$|f(g_{\sigma^k(m), \sigma^j(n)}) - s| \leq M, \quad (j = 1, 2, \dots; m = 1, 2, \dots),$$

then this implies that

$$\begin{aligned} t^{(1)}(m, n, f) &\leq \frac{M}{h_{ru}} \left| \left\{ k, j \in I_{ru} \& g \in S_{k,j} : |f(g_{\sigma^k(m), \sigma^j(n)}) - s| \geq \varepsilon \right\} \right| \\ &\leq M \cdot \frac{\max_m |\{1 < j < |S_n| : f_k(|f(g_{\sigma^k(m), \sigma^j(n)}) - s|) \geq \varepsilon\}|}{h_{ru}} \\ &= M \cdot \frac{S_{ru}}{h_{ru}}. \end{aligned}$$

Hence, $\{S_{k,j}\}$ is lacunary invariant summable to s . □

But converse of the theorem does not hold. For example, $\{S_{k,j}\}$ is the double sequence defined by the following;

$$S_{k,j} := \begin{cases} 1, & \text{if } k_{r-1} < k < k_{r-1} + \lceil \sqrt{h_r} \rceil, \\ & j_{r-1} < j < j_{r-1} + \lceil \sqrt{h_r} \rceil, \text{ and } k+j \text{ is an even integer.} \\ 0, & \text{if } k_{r-1} < k < k_{r-1} + \lceil \sqrt{h_r} \rceil, \\ & j_{r-1} < j < j_{r-1} + \lceil \sqrt{h_r} \rceil, \text{ and } k+j \text{ is an old integer.} \end{cases}$$

where $\sigma(m) = m + 1$ and $\sigma(n) = n + 1$, this folner sequence is lacunary invariant convergent to $\frac{1}{2}$, but it is not lacunary \mathcal{I}_2 -invariant convergent.

Definition 2.6 $f \in w(G)$ is said to be strongly lacunary invariant convergent to s , for any double Folner sequence $\{S_{kj}\}$ for G , if

$$\lim_{r,u \rightarrow \infty} \frac{1}{h_{ru}} \sum_{k,j \in I_{ru} \& g \in S_{kj}} |f(g_{\sigma^k(m), \sigma^j(n)}) - s| = 0,$$

uniformly in n, m and it is denoted by $S_{kj} \rightarrow s ([V_2^{\sigma\theta}(G)])$.

Definition 2.7 $f \in w(G)$ is said to be p -strongly lacunary invariant convergent ($0 < p < \infty$) to s , for any double Folner sequence $\{S_{kj}\}$ for G , if

$$\lim_{r,u \rightarrow \infty} \frac{1}{h_{ru}} \sum_{k,j \in I_{ru} \& g \in S_{kj}} |f(g_{\sigma^k(m), \sigma^j(n)}) - s|^p = 0,$$

uniformly in n, m and it is denoted by $S_{kj} \rightarrow s ([V_2^{\sigma\theta}(G)]_p)$.

Theorem 2.8 If a double sequence $\{S_{kj}\}$ for G is p -strongly lacunary invariant convergent to s , then $\{S_{kj}\}$ is lacunary \mathcal{I}_2 -invariant convergent to s .

Proof. Let $S_{kj} \rightarrow s ([V_2^{\sigma\theta}(G)]_p)$, $0 < p < \infty$. Suppose $\varepsilon > 0$. Then for every double lacunary sequence $\theta = \{(k_r, j_u)\}$ and for every $m, n \in \mathbb{N}$, we have

$$\begin{aligned} & \sum_{k,j \in I_{ru} \& g \in S_{kj}} |f(g_{\sigma^k(m), \sigma^j(n)}) - s|^p \\ & \geq \sum_{\substack{k,j \in I_{ru} \& g \in S_{kj} \\ |f(g_{\sigma^k(m), \sigma^j(n)}) - s| \geq \varepsilon}} |f(g_{\sigma^k(m), \sigma^j(n)}) - s|^p \\ & \geq \varepsilon^p \cdot \left| \left\{ k, j \in I_{ru} \& g \in S_{kj} : |f(g_{\sigma^k(m), \sigma^j(n)}) - s| \geq \varepsilon \right\} \right| \\ & \geq \varepsilon^p \cdot \max_{m,n} \left| \left\{ k, j \in I_{ru} \& g \in S_{kj} : |f(g_{\sigma^k(m), \sigma^j(n)}) - s| \geq \varepsilon \right\} \right| \end{aligned}$$

and

$$\begin{aligned} & \frac{1}{h_{ru}} \sum_{k,j \in I_{ru} \& g \in S_{kj}} |f(g_{\sigma^k(m), \sigma^j(n)}) - s|^p \\ & \geq \varepsilon^p \cdot \frac{\max_{m,n} \left| \left\{ k, j \in I_{ru} \& g \in S_{kj} : |f(g_{\sigma^k(m), \sigma^j(n)}) - s| \geq \varepsilon \right\} \right|}{h_{ru}} = \varepsilon^p \cdot \frac{S_{ru}}{h_{ru}} \end{aligned}$$

This implies $\lim_{r,u \rightarrow \infty} \frac{S_{ru}}{h_{ru}} = 0$ and so $\{S_{kj}\}$ is $\mathcal{I}_2^{\sigma\theta}$ -convergent to s . \square

Theorem 2.9 If a double sequence $\{S_{kj}\} \in I_\infty^2$ and $\{S_{kj}\}$ is lacunary \mathcal{I}_2 -invariant convergent to s , then $\{S_{kj}\}$ is p -strongly lacunary invariant convergent to s ($0 < p < \infty$).

Proof. Now suppose that $\{S_{kj}\} \in I_\infty^2$ and $\{S_{kj}\}$ is lacunary \mathcal{I}_2 -invariant convergent to s . Let $0 < p < \infty$ and $\varepsilon > 0$. By assumption, we have $V_2^\theta(A_\varepsilon) = 0$. The boundedness of $\{S_{kj}\}$ implies that there exist $M > 0$ such that

$$|f(g_{\sigma^k(m), \sigma^j(n)}) - s| \leq M, \quad (k, j) \in I_{ru}; m, n = 1, 2, \dots$$

Observe that for every $m, n \in \mathbb{N}$ we have that

$$\begin{aligned}
& \frac{1}{h_{ru}} \sum_{k, j \in I_{ru} \& g \in S_{kj}} \left| f \left(g_{\sigma^k(m), \sigma^j(n)} \right) - s \right|^p \\
&= \frac{1}{h_{ru}} \sum_{k, j \in I_{ru} \& g \in S_{kj} \& \left| f \left(g_{\sigma^k(m)} \right) - s \right| \geq \varepsilon} \left| f \left(g_{\sigma^k(m), \sigma^j(n)} \right) - s \right|^p \\
&+ \frac{1}{h_{ru}} \sum_{k, j \in I_{ru} \& g \in S_{kj} \& \left| f \left(g_{\sigma^k(m)} \right) - s \right| < \varepsilon} \left| f \left(g_{\sigma^k(m), \sigma^j(n)} \right) - s \right|^p \\
&\leq M \cdot \frac{\max_m \left| \left\{ 1 \leq j \leq |S_n| : \left| f \left(g_{\sigma^k(m), \sigma^j(n)} \right) - s \right| \geq \varepsilon \right\} \right|}{|S_n|} + \varepsilon^p \\
&\leq M \cdot \frac{S_{ru}}{h_{ru}} + \varepsilon^p.
\end{aligned}$$

Hence, we obtain

$$\lim_{r, u \rightarrow \infty} \frac{1}{h_{ru}} \sum_{k, j \in I_{ru} \& g \in S_{kj}} \left| f \left(g_{\sigma^k(m), \sigma^j(n)} \right) - s \right| = 0, \text{ uniformly in } m, n.$$

This completes the proof of the theorem. \square

Definition 2.10 A double Folner sequence $\{S_{kj}\}$ is lacunary \mathcal{I}_2^* -invariant convergent or $\mathcal{I}_2^{\sigma\theta}$ -convergent to s if and only if there exists a set $M_2 \in \mathcal{F}(\mathcal{I}_2^{\sigma\theta})$ ($H = \mathbb{N} \times \mathbb{N} \setminus M_2 \in \mathcal{I}_2^{\sigma\theta}$) such that

$$\lim_{k, j \rightarrow \infty} S_{kj} = s, (k, j) \in M_2.$$

In this case, we write $\mathcal{I}_2^{\sigma\theta}$ - $\lim_{k, j \rightarrow \infty} S_{kj} = s$ or $S_{kj} \rightarrow s$ ($\mathcal{I}_2^{\sigma\theta}$).

Theorem 2.11 If a double sequence $\{S_{kj}\}$ is lacunary \mathcal{I}_2^* -invariant convergent to s , then this sequence is lacunary \mathcal{I}_2 -invariant convergent to s .

Proof. Since $\mathcal{I}_2^{\sigma\theta}$ - $\lim_{k, j \rightarrow \infty} S_{kj} = s$ there exists a set $M_2 \in \mathcal{F}(\mathcal{I}_2^{\sigma\theta})$ ($H = \mathbb{N} \times \mathbb{N} \setminus M_2 \in \mathcal{I}_2^{\sigma\theta}$) such that

$$\lim_{k, j \rightarrow \infty} S_{kj} = s, (k, j) \in M_2.$$

Given $\varepsilon > 0$. By (1), there exists $k_0, j_0 \in \mathbb{N}$ such that $\left| f \left(g_{\sigma^k(m), \sigma^j(n)} \right) - s \right| < \varepsilon$, for all $(k, j) \in M_2$ and $k > k_0, j > j_0$. Hence for every $\varepsilon > 0$, we have

$$\begin{aligned}
T(\varepsilon) &= \left\{ (k, j) \in \mathbb{N} \times \mathbb{N} : \left| f \left(g_{\sigma^k(m), \sigma^j(n)} \right) - s \right| > \varepsilon \right\} \\
&\subset H \cup (M_2 \cap ((\{1, 2, \dots, (k_0 - 1)\} \times \mathbb{N}) \cup (\mathbb{N} \times \{1, 2, \dots, (j_0 - 1)\}))).
\end{aligned}$$

Since $\mathcal{I}_2^{\sigma\theta} \subset 2^{\mathbb{N} \times \mathbb{N}}$ is strongly admissible ideal,

$$H \cup (M_2 \cap ((\{1, 2, \dots, (k_0 - 1)\} \times \mathbb{N}) \cup (\mathbb{N} \times \{1, 2, \dots, (j_0 - 1)\}))) \in \mathcal{I}_2^{\sigma\theta},$$

so we have $T(\varepsilon) \in \mathcal{I}_2^{\sigma\theta}$ that is $V_2^\theta(T(\varepsilon)) = 0$. Hence, $\mathcal{I}_2^{\sigma\theta}$ - $\lim_{k, j \rightarrow \infty} S_{kj} = s$. \square

The converse of the Theorem 4 holds if $\mathcal{I}_2^{\sigma\theta}$ has property (AP2).

Theorem 2.12 Let $\mathcal{I}_2^{\sigma\theta}$ has property (AP2). If a double sequence $\{S_{kj}\}$ is lacunary \mathcal{I}_2 -invariant convergent to s , then this sequence is lacunary \mathcal{I}_2^* -invariant convergent to s .

Finally, we define the concepts of lacunary \mathcal{I}_2 -invariant Cauchy and lacunary \mathcal{I}_2^* -invariant Cauchy sequences.

Definition 2.13 A double Folner sequence $\{S_{kj}\}$ is said to be lacunary \mathcal{I}_2 -invariant Cauchy sequence or $\mathcal{I}_2^{\sigma\theta}$ -Cauchy sequence, if for every $\varepsilon > 0$, there exist numbers $s = s(\varepsilon), t = t(\varepsilon) \in \mathbb{N}$ such that

$$A(\varepsilon) = \left\{ (k, j), (s, t) \in I_{ru} : \left| f(S_{kj}) - f(S_{st}) \right| \geq \varepsilon \right\} \in \mathcal{I}_2^{\sigma\theta},$$

that is, $V_2^\theta(A(\varepsilon)) = 0$.

Definition 2.14 A double Folner sequence $\{S_{kj}\}$ is said to be lacunary \mathcal{I}_2^* -invariant Cauchy sequence or $\mathcal{I}_2^{\sigma\theta}$ -Cauchy sequence, if there exists a set $M_2 \in \mathcal{F}(\mathcal{I}_2^{\sigma\theta})$, ($H = \mathbb{N} \times \mathbb{N} \setminus M_2 \in \mathcal{I}_2^{\sigma\theta}$) such that for every $(k, j), (s, t) \in M_2$

$$\lim_{k,j,s,t \rightarrow \infty} |f(S_{kj}) - f(S_{st})| = 0.$$

The proof of the theorems are similar to the proof of Theorems in ([17], [18], [19]) so we omit them

Theorem 2.15 If a double Folner sequence $\{S_{kj}\}$ is $\mathcal{I}_2^{\sigma\theta}$ -convergent, then $\{S_{kj}\}$ is $\mathcal{I}_2^{\sigma\theta}$ -Cauchy sequence.

Theorem 2.16 If a double Folner sequence $\{S_{kj}\}$ is $\mathcal{I}_2^{\sigma\theta}$ -Cauchy sequence, then $\mathcal{I}_2^{\sigma\theta}$ -Cauchy sequence.

Theorem 2.17 Let $\mathcal{I}_2^{\sigma\theta}$ has property (AP_2) . If a double sequence $\{S_{kj}\}$ is $\mathcal{I}_2^{\sigma\theta}$ -Cauchy sequence, then $\{S_{kj}\}$ is $\mathcal{I}_2^{\sigma\theta}$ -Cauchy sequence.

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4th INTERNATIONAL CONFERENCE ON ANALYSIS AND ITS APPLICATIONS

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On the New Solutions of (3+1)-Dimensional Potential-YTSF Equation by $\tan(\frac{F(\xi)}{2})$ -Expansion Method

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The $\tan(\frac{F(\xi)}{2})$ -expansion method, the (3+1)-dimensional potential-YTSF equation, trigonometric function solutions, hyperbolic function solutions, exponential function solutions, rational function solutions.

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Abstract: In this paper, we employ the $\tan(\frac{F(\xi)}{2})$ -expansion method to explore the solution structure of (3+1)-dimensional potential-YTSF equation. We obtain new solitary wave solutions in form of trigonometric function, hyperbolic function, exponential function and rational function solutions. We also plot two- and three-dimensional graphics for some of the obtained solutions. In this study, all the computations are performed with the aid of Mathematica 9 and consequently a comprehensive conclusion is submitted.

4th INTERNATIONAL CONFERENCE ON ANALYSIS AND ITS APPLICATIONS

1. Introduction

In the field of nonlinear science such as biology, chemistry physics and mathematical physics, etc. nonlinear partial differential equations (NLPDEs) are used frequently to explain models like fluid dynamics, plasma physics, optic fibers, chemical kinematics, etc. Owing to the significant role of NLPDEs in many phenomena, the investigation of solutions of these equations has become more attractive for researchers. In order to obtain solutions to the equations in this category, various methods are developed and applied such as (G'/G) -expansion method [1], the extended (G'/G) method [2], generalized (G'/G) method [3], Kudryashov method [4], tanh function method [5], jacobi elliptic function method [6] and many other analytical methods are used in [7–14, 16].

In this study, we apply the $\tan(\frac{F(\xi)}{2})$ -expansion method [15] to the (3+1)-dimensional potential-YTSF equation, given by [17]:

$$-4u_{xt} + u_{xxxz} + 4u_x u_{xz} + 2u_{xx} u_z + 3u_{yy} = 0. \quad (1.1)$$

Several of approaches have been formulated by many researchers to investigate the solution structures of equation (1.1), such as exp-function method [18], the kudryashov method [4], the extended (G'/G) -expansion method [19], the homoclinic test technique [20].

The remaining part of this study if formed as follows: the description of $\tan(\cdot)$ -expansion method is given in section 2, the application of this method to the (3+1)-dimensional potential-YTSF equation is presented in section 3, in section 4 two- and three-dimensional plots of some obtained solutions are shown and finally the conclusion is presented in section 5.

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2. Description of the Method

Consider the following NLPDE

$$H(u, u^2, u_x, u_{xx}, u_t, u_{tt}, u_{xt}, \dots) = 0, \quad (2.1)$$

where H is a function of $u(x, t)$ and its partial derivatives in which the highest order derivatives and the nonlinear terms are connected. Performing the travelling wave transformation $u(x, t) = U(\xi)$, $\xi = x - kt$, Eq. (2.1) reduces to the following ordinary differential equation (ODE):

$$Q(U', U'', U''', \dots) = 0, \quad (2.2)$$

where Q is a function in $U(\xi)$ and its derivatives with respect to ξ and k is the wave velocity. The solution of the Eq. (2.2) is given as follows:

$$U(\xi) = \sum_{i=0}^m A_i \left(p + \tan\left(\frac{F(\xi)}{2}\right) \right)^i + \sum_{i=1}^m B_i \left(p + \tan\left(\frac{F(\xi)}{2}\right) \right)^{-i}, \quad A_m \neq 0, B_m \neq 0, \quad (2.3)$$

where m is a positive integer that can be determined by applying the balancing technique to the highest order derivative and the highest nonlinear term in Eq. (2.2). The coefficients $A_i (0 \leq i \leq m)$, $B_i (1 \leq i \leq m)$ are constants to be determined and $F = F(\xi)$ supplies the first order nonlinear ordinary differential equation (NODE), given by [15]:

$$F'(\xi) = a \sin(F(\xi)) + b \cos(F(\xi)) + c. \quad (2.4)$$

Substituting Eq. (2.3), it's derivative along with Eq. (2.4) into Eq. (2.1) and simplifying, produces algebraic system of equations for $\tan\left(\frac{F(\xi)}{2}\right)^i$, $\cot\left(\frac{F(\xi)}{2}\right)^i$. Then, all the coefficients of $\tan\left(\frac{F(\xi)}{2}\right)^i$, $\cot\left(\frac{F(\xi)}{2}\right)^i$ have to vanish. After collecting this separated algebraic system equations, we calculate $k, p, A_0, A_1, B_1, \dots, A_m, B_m$. For the solutions of Eq. (2.4), see [15].

3. Implementation of the Method

Consider the (3+1)-dimensional potential-YTSE equation given in Eq. (1.1), section (1).

Using the travelling wave transformation $u = U(\xi)$, $\xi = x + y + z - kt$, Eq. (1.1) reduces to the following NODE:

$$(4k+3)U' + 3(U')^2 + U''' = 0, \quad (3.1)$$

by applying the balancing technique on Eq. (3.1) by considering the highest derivative U''' and the highest power nonlinear term $(U')^2$, we obtain $m = 1$.

Using $m = 1$ together with Eq. (2.3), we choose the solution to Eq. (3.1) as follows:

$$U(\xi) = A_0 + A_1 \left(p + \tan\left(\frac{F(\xi)}{2}\right) \right) + B_1 \left(p + \tan\left(\frac{F(\xi)}{2}\right) \right)^{-1}, \quad (3.2)$$

substituting Eq. (3.2) into Eq. (3.1) and simplifying together with Eq. (2.4), we obtain a system of algebraic equations in terms of A_1, B_1, p, k, a, b, c . We solve the system of the algebraic equation for the coefficients A_1, B_1, p, k, a, b, c , and we obtain a different cases of solution to the coefficients. We obtain a number of families of soliton solutions under each case by considering solutions to Eq. (2.4) given by [15].

Case-1: $A_1 = 0, B_1 = b + c - 2ap - bp^2 + cp^2, k = \frac{1}{4}(-3 - a^2 - b^2 + c^2)$, has the following families of solutions:

Solution-1: $a^2 + b^2 - c^2 < 0$ and $b - c \neq 0$

$$u(x, y, z, t) = -\frac{(b-c)(2ap + b(p^2 - 1) - c(1 + p^2))}{a + p(b-c) - \sqrt{c^2 - a^2 - b^2} \tan[\beta_1]}, \quad (3.3)$$

where $\beta_1 = \frac{1}{8}\sqrt{c^2 - a^2 - b^2}((3 + a^2 + b^2 - c^2)t + 4(x + y + z))$

Solution-2: $a^2 + b^2 - c^2 > 0, b \neq 0$ and $c = 0$

$$u(x, y, z, t) = \frac{b(b - 2ap - bp^2)}{a + bp + \sqrt{a^2 + p^2} \tanh \left[\frac{1}{8} \sqrt{a^2 + b^2} ((3 + a^2 + b^2)t + 4(x + y + z)) \right]}, \quad (3.4)$$

Solution-3: $a^2 + b^2 - c^2 < 0$, $c \neq 0$ and $b = 0$

$$u(x, y, z, t) = \frac{c(c - 2ap + cp^2)}{cp - a + \sqrt{c^2 - a^2} \tan \left[\frac{1}{8} \sqrt{c^2 - a^2} ((3 + a^2 - c^2)t + 4(x + y + z)) \right]}, \quad (3.5)$$

Solution-4: $a^2 + b^2 = c^2$

$$u(x, y, z, t) = \frac{b - 2ap - bp^2 + \sqrt{a^2 + b^2} (1 + p^2)}{p - \frac{(b + \sqrt{a^2 + b^2}) \left(2 + a \left(\frac{3t}{4} + x + y + z \right) \right)}{a^2 \left(\frac{3t}{4} + x + y + z \right)}}, \quad (3.6)$$

Solution-5: $c = a$

$$u(x, y, z, t) = \frac{b + a(p - 1)^2 - bp^2}{\frac{1 - (a + b)e^{\frac{1}{4}b((3 + b^2)t + 4(x + y + z))}}{-1 + (a - b)e^{\frac{1}{4}b((3 + b^2)t + 4(x + y + z))}} + p}, \quad (3.7)$$

Solution-6: $a = c$

$$u(x, y, z, t) = -\frac{(p - 1)(b + c + bp - cp)}{\frac{1 + (b + c)e^{\frac{1}{4}b((3 + b^2)t + 4(x + y + z))}}{-1 + (b - c)e^{\frac{1}{4}b((3 + b^2)t + 4(x + y + z))}} + p}, \quad (3.8)$$

Solution-7: $c = -a$

$$u(x, y, z, t) = \frac{a - b + p(a + b)}{-1 + \frac{2b}{(1 + p) \left(a + b - e^{\frac{1}{4}b((3 + b^2)t + 4(x + y + z))} \right)}}, \quad (3.9)$$

Solution-8: $b = -c$

$$u(x, y, z, t) = 2p \left(c - \frac{a}{p + (a - cp)e^{\frac{1}{4}a((3 + a^2)t + 4(x + y + z))}} \right), \quad (3.10)$$

Solution-9: $b = 0$ and $a = c$

$$u(x, y, z, t) = \frac{c^2(p - 1)^2(3t + 4(x + y + z))}{c(p - 1)(3t + 4(x + y + z)) - 8}, \quad (3.11)$$

Solution-10: $a = 0$ and $b = c$

$$u(x, y, z, t) = \frac{2c}{p + c \left(\frac{3t}{4} + x + y + z \right)}, \quad (3.12)$$

Solution-11: $a = 0$ and $b = -c$

$$u(x, y, z, t) = \frac{2cp^2}{p - \frac{1}{c \left(\frac{3t}{4} + x + y + z \right)}}, \quad (3.13)$$

Solution-12: $a = 0$ and $b = 0$

$$u(x, y, z, t) = \frac{c(1 + p^2)}{p + \tan \left[\frac{1}{2}c \left(-\frac{1}{4}(c^2 - 3)t + x + y + z \right) \right]}. \quad (3.14)$$

Case-2: $A_1 = b - c$, $B_1 = 0$, $k = \frac{1}{4}(-3 - a^2 - b^2 + c^2)$,
has the following families of solutions:

Solution-1: $a^2 + b^2 - c^2 < 0$ and $b - c \neq 0$

$$u(x, y, z, t) = a + p(b - c) - \sqrt{c^2 - a^2 - b^2} \tan[\beta_2], \quad (3.15)$$

where $\beta_2 = \frac{1}{8} \sqrt{c^2 - a^2 - b^2} ((3 + a^2 + b^2 - c^2)t + 4(x + y + z))$

Solution-2: $a^2 + b^2 - c^2 > 0$ and $b - c \neq 0$

$$u(x, y, z, t) = a + p(b - c) + \sqrt{a^2 + b^2 - c^2} \tanh[\beta_3], \quad (3.16)$$

where $\beta_3 = \frac{1}{8} \sqrt{a^2 + b^2 - c^2} ((3 + a^2 + b^2 - c^2)t + 4(x + y + z))$

Solution-3: $a^2 + b^2 - c^2 > 0$, $b \neq 0$ and $c = 0$

$$u(x, y, z, t) = a + bp + \sqrt{a^2 + b^2} \tanh\left[\frac{1}{8} \sqrt{a^2 + b^2} ((3 + a^2 + b^2)t + 4(x + y + z))\right], \quad (3.17)$$

Solution-4: $a^2 + b^2 - c^2 < 0$, $c \neq 0$ and $b = 0$

$$u(x, y, z, t) = a - cp - \sqrt{c^2 - a^2} \tan\left[\frac{1}{8} \sqrt{c^2 - a^2} ((3 + a^2 - c^2)t + 4(x + y + z))\right], \quad (3.18)$$

Solution-5: $a^2 + b^2 = c^2$

$$u(x, y, z, t) = (b - \sqrt{a^2 + b^2}) \left(p - \frac{(b + \sqrt{a^2 + b^2}) \left(2 + a \left(\frac{3t}{4} + x + y + z \right) \right)}{a^2 \left(\frac{3t}{4} + x + y + z \right)} \right), \quad (3.19)$$

Solution-6: $c = a$

$$u(x, y, z, t) = a - ap + b \left(1 - \frac{2}{1 + (b - a)e^{\frac{1}{4}b((3+b^2)t+4(x+y+z))}} + p \right), \quad (3.20)$$

Solution-7: $a = c$

$$u(x, y, z, t) = c - cp + b \left(1 + \frac{2}{-1 + (b - c)e^{\frac{1}{4}b((3+b^2)t+4(x+y+z))}} + p \right), \quad (3.21)$$

Solution-8: $c = -a$

$$u(x, y, z, t) = (a + b) \left(1 - \frac{2b}{a + b - e^{\frac{1}{4}b((3+b^2)t+4(x+y+z))}} + p \right), \quad (3.22)$$

Solution-9: $b = -c$

$$u(x, y, z, t) = 2 \left(a + \frac{a}{-1 + ce^{\frac{1}{4}a((3+a^2)t+4(x+y+z))}} - cp \right), \quad (3.23)$$

Solution-10: $b = 0$ and $a = c$

$$u(x, y, z, t) = c - cp + \frac{8}{3t + 4(x + y + z)}, \quad (3.24)$$

Solution-11: $a = 0$ and $b = -c$

$$u(x, y, z, t) = -2cp + \frac{2}{\frac{3t}{4} + x + y + z}, \quad (3.25)$$

Solution-12: $a = 0$ and $b = 0$

$$u(x, y, z, t) = -c \left(p + \tan \left[\frac{1}{2} c \left(-\frac{1}{4} (c^2 - 3)t + x + y + z \right) \right] \right), \quad (3.26)$$

Case-3: $A_1 = b - c$, $B_1 = b + c + bp^2 - cp^2$, $a = cp - bp$,
 $k = \frac{1}{4} (-3 - b^2 - 3bB_1 + 3cB_1 + c^2 - b^2p^2 + 2bcp^2 - c^2p^2)$,

has the following families of solutions:

Solution-1: $a^2 + b^2 - c^2 < 0$ and $b - c \neq 0$

$$u(x, y, z, t) = \sqrt{-(b-c)(b+c+p^2(b-c))} \left(-1 + \cot^2[\beta_3] \right) \tan[\beta_3], \quad (3.27)$$

where $\beta_3 = \frac{1}{8} \sqrt{-(b-c)(b+c+p^2(b-c))} ((3+4b^2-4c^2+4p^2(b-c)^2)t + 4(x+y+z))$

Solution-2: $a^2 + b^2 - c^2 > 0$ and $b - c \neq 0$

$$u(x, y, z, t) = \sqrt{b^2 - c^2 + (bp - cp)^2} \left(1 + \coth^2[\beta_4] \right) \tanh[\beta_4], \quad (3.28)$$

where $\beta_4 = \frac{1}{8} \sqrt{b^2 - c^2 + (bp - cp)^2} ((3+4b^2-4c^2+4p^2(b-c)^2)t + 4(x+y+z))$

Solution-3: $a^2 + b^2 - c^2 > 0$, $b \neq 0$ and $c = 0$

$$u(x, y, z, t) = 2b \sqrt{1+p^2} \coth \left[\frac{1}{4} b \sqrt{1+p^2} ((3+4b^2(1+p^2))t + 4(x+y+z)) \right], \quad (3.29)$$

Solution-4: $a^2 + b^2 - c^2 < 0$, $c \neq 0$ and $b = 0$

$$u(x, y, z, t) = 2c \sqrt{1-p^2} \cot \left[\frac{1}{4} c \sqrt{1-p^2} ((3+4c^2(p^2-1))t + 4(x+y+z)) \right], \quad (3.30)$$

Solution-5: $a = c$

$$u(x, y, z, t) = 2b + \frac{1}{-\frac{1}{4b} + \frac{be^{\frac{1}{2}b((3+4b^2)t+4(x+y+z))}}{4(p-1)^2}}, \quad (3.31)$$

Solution-6: $c = -a$

$$u(x, y, z, t) = -\frac{2b(b^2 + (1+p)^2 e^{\frac{1}{2}b((3+4b^2)t+4(x+y+z))}}{b^2 - (1+p)^2 e^{\frac{1}{2}b((3+4b^2)t+4(x+y+z))}}, \quad (3.32)$$

Solution-7: $b = 0$ and $a = c$

$$u(x, y, z, t) = \frac{8}{3t + 4(x+y+z)}, \quad (3.33)$$

Solution-8: $a = 0$ and $b = c$

$$u(x, y, z, t) = \frac{2c}{p + c \left(\frac{3t}{4} + x + y + z \right)}, \quad (3.34)$$

Case-4: $A_1 = 0$, $B_1 = b + c + bp^2 - cp^2$, $a = cp - bp$,
 $k = \frac{1}{8} (-6 + b^2 - 3bB_1 + 3cB_1 - c^2 + b^2p^2 - 2bcp^2 + c^2p^2)$,
has the following families of solutions:

Solution-1: $a^2 + b^2 - c^2 < 0$ and $b - c \neq 0$

$$u(x, y, z, t) = \sqrt{-(b-c)(b+c+p^2(b-c))} \cot[\beta_5], \quad (3.35)$$

where $\beta_5 = \frac{1}{8} \sqrt{-(b-c)(b+c+p^2(b-c))} ((3+b^2-c^2+p^2(b-c)^2)t + 4(x+y+z))$

Solution-2: $a^2 + b^2 - c^2 > 0$ and $b - c \neq 0$

$$u(x, y, z, t) = \sqrt{b^2 - c^2 + (bp - cp)^2} \coth[\beta_6], \quad (3.36)$$

where $\beta_6 = \frac{1}{8} \sqrt{b^2 - c^2 + (bp - cp)^2} ((3+b^2-c^2+p^2(b-c)^2)t + 4(x+y+z))$

Solution-3: $a^2 + b^2 - c^2 > 0$, $b \neq 0$ and $c = 0$

$$u(x, y, z, t) = b\sqrt{1+p^2} \coth \left[\frac{1}{8} b\sqrt{1+p^2} ((3+b^2(1+p^2))t + 4(x+y+z)) \right], \quad (3.37)$$

Solution-4: $a^2 + b^2 - c^2 < 0$, $c \neq 0$ and $b = 0$

$$u(x, y, z, t) = c\sqrt{1-p^2} \cot \left[\frac{1}{8} c\sqrt{1-p^2} ((3+c^2(p^2-1))t + 4(x+y+z)) \right], \quad (3.38)$$

Solution-5: $c = a$

$$u(x, y, z, t) = b - \frac{2b(p-1)}{-1 + be^{\frac{1}{4}b((3+b^2)t+4(x+y+z))} + p}, \quad (3.39)$$

Solution-6: $a = c$

$$u(x, y, z, t) = b + \frac{2b(p-1)}{1 + be^{\frac{1}{4}b((3+b^2)t+4(x+y+z))} - p}, \quad (3.40)$$

Solution-7: $c = -a$

$$u(x, y, z, t) = b - \frac{2b^2}{b + (1+p)e^{\frac{1}{4}b((3+b^2)t+4(x+y+z))}}, \quad (3.41)$$

Solution-8: $b = -c$

$$u(x, y, z, t) = 2cp \left(1 - \frac{2}{1 + ce^{\frac{1}{2}cp((3+4c^2p^2)t+4(x+y+z))}} \right), \quad (3.42)$$

4. Graphics

In this section, we parade the two- and three-dimensional plots of some obtained solutions.

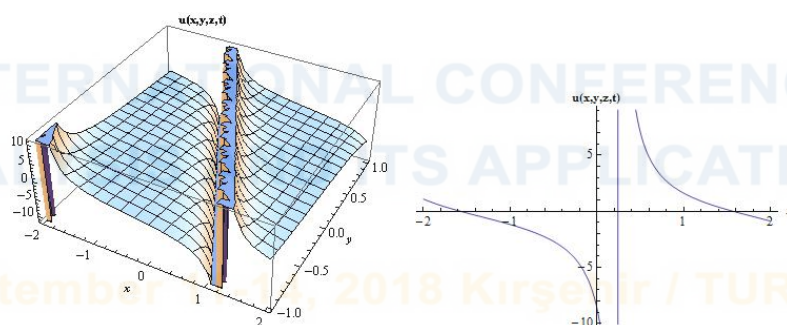


Fig. 1. The 3D and 2D surfaces of Eq. (3.3) by considering the values $a = 1$, $b = 2$, $c = 3$, $p = 0.5$, $t = 0.001$, $z = 0.002$, $-2 < x < 2$, $-1 < y < 1$ and $y = 0.003$ for the 2D graphic.

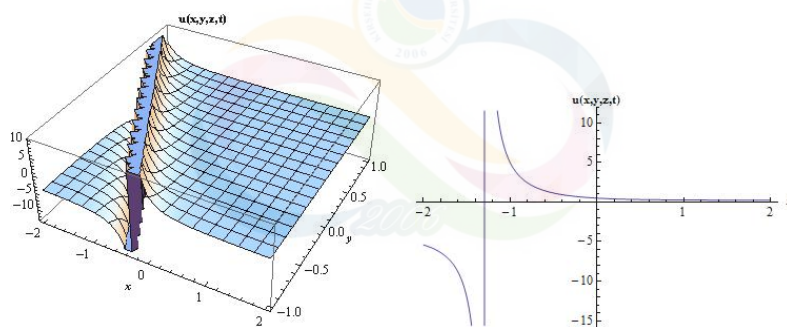


Fig. 2. The 3D and 2D surfaces of Eq. (3.5) by considering the values $a = 1$, $b = 2$, $p = 0.5$, $t = 0.001$, $z = 0.002$, $-2 < x < 2$, $-1 < y < 1$ and $y = 0.003$ for the 2D graphic.

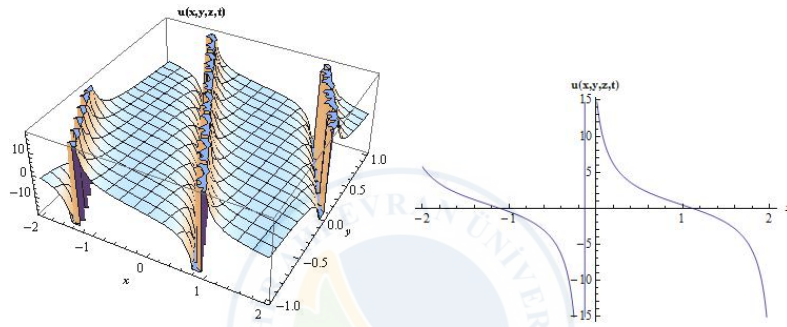


Fig. 3. The 3D and 2D surfaces of Eq. (3.6) by considering the values $a = 1, b = 2, p = 0.5, t = 0.001, z = 0.002, -2 < x < 2, -1 < y < 1$ and $y = 0.003$ for the 2D graphic.

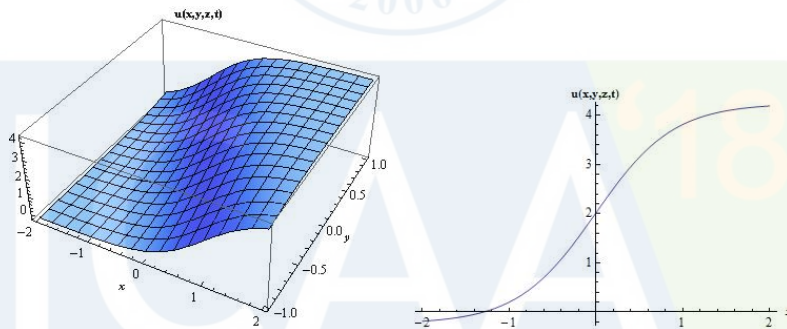


Fig. 4. The 3D and 2D surfaces of Eq. (3.17) by considering the values $a = 1, b = 2, p = 0.5, t = 0.001, z = 0.002, -2 < x < 2, -1 < y < 1$ and $y = 0.003$ for the 2D graphic.

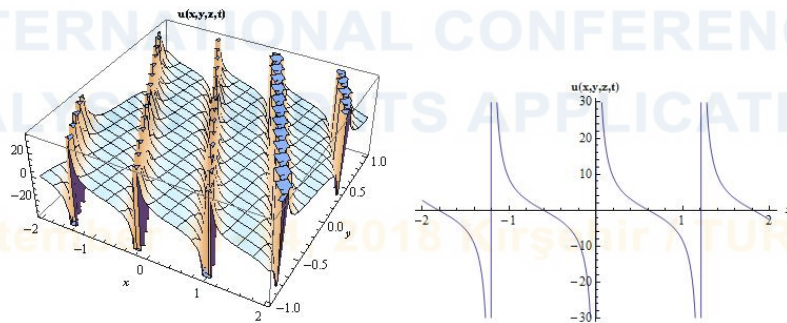


Fig. 5. The 3D and 2D surfaces of Eq. (3.30) by considering the values $c = 3, t = 0.001, z = 0.002, -2 < x < 2, -1 < y < 1$ and $y = 0.003$ for the 2D graphic.

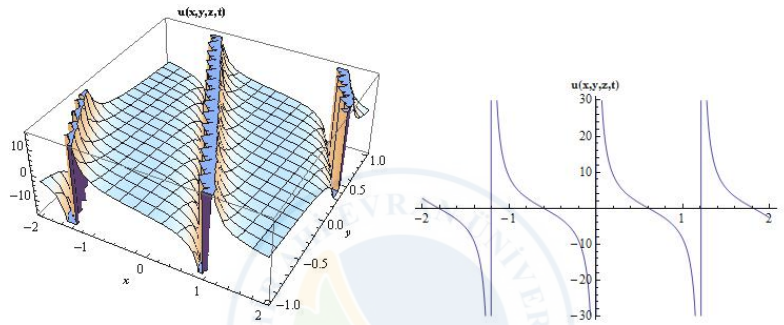


Fig. 6. The 3D and 2D surfaces of Eq. (3.38) by considering the values $c = 3$, $t = 0.001$, $z = 0.002$, $-2 < x < 2$, $-1 < y < 1$ and $y = 0.003$ for the 2D graphic.

5. Conclusion

In this article, we successfully applied the powerful $\tan(\frac{F(\xi)}{2})$ -expansion method to the (3+1)-dimensional potential-YTSF equation. We are able to construct a number of new solitary wave solutions with trigonometric, hyperbolic, exponential and rational function structures. We carried out all the computation in this paper with help of Wolfram Mathematica 9, we checked all the obtained solutions and they indeed satisfied equation (1.1), and we also plot the two- and three-dimensional graphics of some obtained solutions using the same computer program. When we compare our obtained results with the results obtained in [17], we observed that our results are newly constructed solutions with the different solution structures. We can finally say that the $\tan(\frac{F(\xi)}{2})$ -expansion method is easy and highly computerized method that can be applied to different class of complicated NLPDEs.

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Chebyshev Polynomial Coefficient Results on A Subclass of Analytic and Bi-Univalent Functions Involving Quasi-Subordination

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Chebyshev polynomials,
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MSC: 30C45

Abstract: In this study, we use the Chebyshev polynomial expansion to construct a new subclass of bi-univalent functions involving quasi-subordination.

1. Introduction and Definitions

Let \mathbb{E} be the unit disc

$$\{z \in \mathbb{C} : |z| < 1\},$$

and let \mathcal{A} be the class of functions analytic in \mathbb{E} , satisfying the conditions

$$f(0) = 0, \quad f'(0) = 1.$$

Then each $f \in \mathcal{A}$ has the Taylor expansion

$$f(z) = z + \sum_{m=2}^{\infty} a_m z^m. \quad (1.1)$$

The class of this kind of functions is represented by \mathcal{S} .

Let the functions f, g be analytic in \mathbb{E} . If there exists a Schwarz function ϖ , which is analytic in \mathbb{E} under the conditions

$$\varpi(0) = 0, \quad |\varpi(z)| < 1,$$

such that

$$f(z) = g(\varpi(z)) \quad (z \in \mathbb{E}),$$

then the function f is subordinate to g in \mathbb{E} , and indite $f(z) \prec g(z)$ ($z \in \mathbb{E}$).

Furthermore, for two analytic functions f and g , the function f is said to be quasi-subordinate to g in \mathbb{E} and written as

$$f(z) \prec_q g(z) \quad (z \in \mathbb{E})$$

if there exists an analytic function φ ($|\varphi(z)| \leq 1$) such that $\frac{f(z)}{\varphi(z)}$ analytic in \mathbb{E} and

$$\frac{f(z)}{\varphi(z)} \prec g(z) \quad (z \in \mathbb{E})$$

that is, there exists a Schwarz function $\varpi(z)$ such that $f(z) = \varphi(z)g(\varpi(z))$. Also, one observes that if $\varphi(z) = 1$, then $f(z) = g(\varpi(z))$ so that $f(z) \prec g(z)$ in \mathbb{E} . Also notice that if $\varpi(z) = z$, then $f(z) = \varphi(z)g(z)$ and it is said that is majorized by g and written $f(z) \ll g(z)$ in \mathbb{E} . Hence it is obvious that quasi-subordination

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is a generalization of subordination as well as majorization (see, e.g. [14], [13], [12] for works related to quasi-subordination).

The Koebe One-Quarter Theorem (see [6]) ensures that the image of \mathbb{E} under every $f \in \mathcal{S}$ contains a disc of radius $\frac{1}{4}$. So, every $f \in \mathcal{S}$ has an inverse f^{-1} which satisfies

$$f^{-1}(f(z)) = z \quad (z \in \mathbb{E})$$

and

$$f(f^{-1}(w)) = w \quad \left(|w| < r_0(f), r_0(f) \geq \frac{1}{4} \right),$$

where

$$g(w) = f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3) w^3 - (5a_2^3 - 5a_2 a_3 + a_4) w^4 + \dots$$

A function $f \in \mathcal{A}$ is said to be bi-univalent in \mathbb{E} if both f and f^{-1} are univalent in \mathbb{E} . The class consisting of bi-univalent functions are denoted by σ . A concept of bi-univalent analytic functions is due to Lewin [11]. Ever since then, the first two coefficients $|a_2|$ and $|a_3|$ of the bi-univalent functions was used by many researchers, see for example, [1], [3], [4], [8], [16], [17], [18]. The coefficient estimate problem for Taylor-Maclaurin coefficients $|a_n|$ for $n \geq 4$ is presumably still an open problem. Anyway, using Faber polynomial expansions, several authors obtained coefficient estimates $|a_n|$ for classes bi-univalent functions, see for example [2], [9], [10].

One of the important tools in numerical analysis, from both theoretical and practical points of view, is Chebyshev polynomials. There are four kinds of Chebyshev polynomials. The majority of research papers dealing with specific orthogonal polynomials of Chebyshev family, contain mainly results of Chebyshev polynomials of first and second kinds $T_m(z)$ and $U_m(z)$ and their numerous uses in different applications, see for example, Doha [7] and Mason [15].

The Chebyshev polynomials of the second kinds are well known. In the case of a real variable x on $(-1, 1)$, they are defined by

$$U_m(x) = \frac{\sin(m+1)\theta}{\sin \theta},$$

where the subscript m denotes the polynomial degree and where $x = \cos \theta$.

Now, we consider the function that is the generating function of a Chebyshev polynomial:

$$\begin{aligned} H(z, t) &= \frac{1}{1 - 2tz + z^2} \quad t \in \left(\frac{1}{2}, 1\right] \\ &= 1 + \sum_{m=1}^{\infty} \frac{\sin(m+1)\theta}{\sin \theta} z^m \quad (z \in \mathbb{E}). \end{aligned}$$

If we choose $t = \cos \theta$, $\theta \in \left(-\frac{\pi}{3}, \frac{\pi}{3}\right)$, then

$$H(z, t) = 1 + 2\cos \theta z + (3\cos^2 \theta - \sin^2 \theta) z^2 + \dots \quad (z \in \mathbb{E}).$$

Following see, we can write

$$H(z, t) = 1 + U_1(t)z + U_2(t)z^2 + \dots \quad \left(t \in \left(\frac{1}{2}, 1\right], z \in \mathbb{E} \right), \quad (1.2)$$

where $U_{m-1} = \frac{\sin(m \arccost)}{\sqrt{1-t^2}}$ ($m \in \mathbb{N}$) are the Chebyshev polynomials of the second kind. Also it is known that

$$U_m(t) = 2tU_{m-1}(t) - U_{m-2}(t),$$

and

$$U_1(t) = 2t, U_2(t) = 4t^2 - 1, U_3(t) = 8t^3 - 4t, \dots \quad (1.3)$$

Now, we establish a new subclass of analytic and bi-univalent functions based on quasi-subordination.

Definition 1.1 A function $f \in \sigma$ is said to be in the class

$$W_{\sigma}(\beta, t, \gamma) \quad \left(0 \leq \beta \leq 1, t \in \left(\frac{1}{2}, 1\right], \gamma \in \mathbb{C} \setminus \{0\} \right)$$

if the following quasi-subordinations are satisfied:

$$\frac{1}{\gamma} \left[\left\{ \beta f'(z) + (1-\beta) \frac{zf'(z)}{f(z)} \right\} - 1 \right] \prec_q (H(z,t) - 1) \quad (z \in \mathbb{E}) \quad (1.4)$$

and

$$\frac{1}{\gamma} \left[\left\{ \beta g'(w) + (1-\beta) \frac{wg'(w)}{g(w)} \right\} - 1 \right] \prec_q (H(w,t) - 1) \quad (w \in \mathbb{E}), \quad (1.5)$$

where $g = f^{-1}$.

It is clear from the definition that $f \in W_\sigma(\beta, t, \gamma)$ if and only if there exists a function φ ($|\varphi(z)| \leq 1$) such that

$$\frac{(1-\gamma) \frac{f(z)}{z} + \gamma(D_q f)(z)}{\varphi(z)} \prec (H(z,t) - 1) \quad (z \in \mathbb{E})$$

and

$$\frac{(1-\gamma) \frac{g(w)}{w} + \gamma(D_q g)(w)}{\varphi(w)} \prec (H(w,t) - 1) \quad (w \in \mathbb{E}),$$

where the function $H(z,t)$ is given by (1.2), and that the function φ , analytic in \mathbb{E} , is of the form:

$$\varphi(z) = d_0 + d_1 z + d_2 z^2 + \dots \quad (|\varphi(z)| \leq 1, z \in \mathbb{E}). \quad (1.6)$$

Motivated by the earlier work of Dziok et al. [5], we study the Chebyshev polynomial expansions to provide estimates for the initial coefficients of a newly-constructed subclass of bi-univalent functions.

2. Coefficient bounds for the function class $W_\sigma(\beta, t, \gamma)$

Theorem 2.1 Let f given by (1.1) be in the class $W_\sigma(\beta, t, \gamma)$. Then

$$|a_2| \leq \frac{2|\gamma d_0| t \sqrt{2t}}{\sqrt{4(1+2\beta)\gamma t^2 - (1+\beta)^2(4t^2-1)}}$$

and

$$|a_3| \leq \frac{4|\gamma d_0|^2 t^2}{(1+\beta)^2} + \frac{2t|\gamma|(|d_0| + |d_1|)}{2+\beta}.$$

Proof. Let $f \in W_\sigma(\beta, t, \gamma)$. The inequalities (1.4) and (1.5) imply the existence of two Schwarz functions

$$u(z) = \sum_{n=1}^{\infty} u_n z^n$$

and

$$v(w) = \sum_{n=1}^{\infty} v_n w^n,$$

with

$$|u_k| \leq 1, |v_k| \leq 1 \quad (\forall k \in \mathbb{N}) \quad (2.1)$$

so that

$$\begin{aligned} \frac{1}{\gamma} \left[\left\{ \beta f'(z) + (1-\beta) \frac{zf'(z)}{f(z)} \right\} - 1 \right] &= \varphi(z) (H(u(z), t) - 1) \\ \frac{1}{\gamma} \left[\left\{ \beta g'(w) + (1-\beta) \frac{wg'(w)}{g(w)} \right\} - 1 \right] &= \varphi(w) (H(v(w), t) - 1), \end{aligned}$$

or equivalently

$$\begin{aligned} \frac{1}{\gamma} \left[\left\{ \beta f'(z) + (1-\beta) \frac{zf'(z)}{f(z)} \right\} - 1 \right] &= U_1(t) u_1 d_0 z \\ &+ [(U_1(t) u_2 + U_2(t) u_1^2) d_0 + U_1(t) u_1 d_1] z^2 + \dots \end{aligned} \quad (2.2)$$

and

$$\frac{1}{\gamma} \left[\left\{ \beta g'(w) + (1 - \beta) \frac{wg'(w)}{g(w)} \right\} - 1 \right] = U_1(t)v_1d_0w + [(U_1(t)v_2 + U_2(t)v_1^2)d_0 + U_1(t)v_1d_1]w^2 + \dots \quad (2.3)$$

It follows from (2.2) and (2.3) that

$$\frac{1 + \beta}{\gamma} a_2 = U_1(t)u_1d_0, \quad (2.4)$$

$$\frac{(2 + \beta)a_3 - (1 - \beta)a_2^2}{\gamma} = U_1(t)u_1d_1 + (U_2(t)u_1^2 + U_1(t)u_2)d_0, \quad (2.5)$$

$$-\frac{1 + \beta}{\gamma} a_2 = U_1(t)v_1d_0, \quad (2.6)$$

$$\frac{3(1 + \beta)a_2^2 - (2 + \beta)a_3}{\gamma} = U_1(t)v_1d_1 + (U_2(t)v_1^2 + U_1(t)v_2)d_0. \quad (2.7)$$

From the equations (2.4) and (2.6), we can easily see that

$$u_1 = -v_1, \quad (2.8)$$

$$\frac{2(1 + \beta)^2}{\gamma^2} a_2^2 = U_1^2(t)(u_1^2 + v_1^2)d_0^2. \quad (2.9)$$

If we add (2.5) to (2.7), we get

$$\frac{2(1 + 2\beta)}{\gamma} a_2^2 = U_1(t)(u_2 + v_2)d_0 + U_2(t)(u_1^2 + v_1^2)d_0 + U_1(t)(u_1 + v_1)d_1. \quad (2.10)$$

Using (2.9) in equality (2.10),

$$\left[\frac{2(1 + 2\beta)}{\gamma} - \frac{2U_2(t)(1 + \beta)^2}{U_1^2(t)\gamma^2 d_0} \right] a_2^2 = U_1(t)(u_2 + v_2)d_0. \quad (2.11)$$

From (1.3), (2.11) and (2.1), we get

$$|a_2| \leq \frac{2|\gamma d_0|t\sqrt{2t}}{\sqrt{4(1 + 2\beta)\gamma^2 t^2 - (1 + \beta)^2(4t^2 - 1)}}.$$

Next, if we subtract (2.7) from (2.5), we obtain

$$\frac{2(2 + \beta)}{\gamma} (a_3 - a_2^2) = U_1(t)(u_2 - v_2)d_0 + U_2(t)(u_1^2 - v_1^2)d_0 + U_1(t)(u_1 - v_1)d_1. \quad (2.12)$$

Then, in view of (2.8) and (2.9), also (2.12)

$$a_3 = \frac{U_1^2(t)\gamma^2 d_0^2}{2(1 + \beta)^2} (u_1^2 + v_1^2) + \frac{2U_1(t)\gamma d_1 u_1 + U_1(t)\gamma d_0 (u_2 - v_2)}{2(2 + \beta)}.$$

Notice that from (1.3) and (2.1)

$$|a_3| \leq \frac{4|\gamma d_0|^2 t^2}{(1 + \beta)^2} + \frac{2t|\gamma|(|d_0| + |d_1|)}{2 + \beta}.$$

□

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On A Subclass of Bi-Univalent Functions with The Faber Polynomial Expansion

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Abstract:

In this study, we use the Faber polynomial expansions to obtain upper bounds for $|a_n|$ ($n > 3$) coefficients of functions belong to a subclass of bi-univalent functions involving the Jackson q -derivative operator in the open unit disc

$$\mathbb{E} = \{z \in \mathbb{C} : |z| < 1\}.$$

1. Introduction, Definitions and Notations

Let \mathcal{A} denote the class of functions of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad (1.1)$$

which are analytic in the open unit disc \mathbb{E} . By \mathcal{S} we denote the subclass of \mathcal{A} consisting of functions the form (1.1) which are also univalent in \mathbb{E} . Further, \mathcal{P} be the class of functions consisting of φ , such that

$$\varphi(z) = 1 + \sum_{n=1}^{\infty} \varphi_n z^n,$$

which are regular in the open unit disc \mathbb{E} and satisfy the condition $\Re(\varphi(z)) > 0$ in \mathbb{E} .

In order to introduce the principles of subordination and quasi-subordination, we let f and g be two analytic functions in \mathbb{E} . We say that f is subordinate to the function g , written as:

$$f(z) \prec g(z) \quad (z \in \mathbb{E}),$$

if there exists a Schwarz function $\varpi(z) = \sum_{n=1}^{\infty} c_n z^n$, analytic in \mathbb{E} , with

$$\varpi(0) = 0 \quad \text{and} \quad |\varpi(z)| < 1 \quad (z \in \mathbb{E})$$

such that

$$f(z) = g(\varpi(z)) \quad (z \in \mathbb{E}).$$

For the Schwarz function $\varpi(z)$, we note that $|c_n| < 1$ (see Duren [11]).

Furthermore, for two analytic functions f and g , the function f is said to be quasi-subordinate to the function g in \mathbb{E} , written as:

$$f(z) \prec_q g(z) \quad (z \in \mathbb{E})$$

if there exists an analytic function φ ($|\varphi(z)| \leq 1$) such that the function $\frac{f(z)}{\varphi(z)}$ analytic in \mathbb{E} and

$$\frac{f(z)}{\varphi(z)} \prec g(z) \quad (z \in \mathbb{E})$$

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that is, there exists the above-mentioned Schwarz function $\varpi(z)$ such that

$$f(z) = \varphi(z)g(\varpi(z)).$$

One observes that, in the special case when $\varphi(z) = 1$, the quasi-subordination coincides with the usual subordination. Also notice that if $\varpi(z) = z$, then $f(z) = \varphi(z)g(z)$ and it is said that f is majorized by g and written $f(z) \ll g(z)$ in \mathbb{E} . Hence it is obvious that quasi-subordination is a generalization of subordination as well as majorization (see [22]).

The Koebe One-Quarter Theorem [11] states that the image of \mathbb{E} under every function f in the normalized univalent function class \mathcal{S} contains a disc of radius $\frac{1}{4}$. Thus, clearly, every such univalent function has an inverse f^{-1} which satisfies the following condition:

$$f^{-1}(f(z)) = z \quad (z \in \mathbb{E})$$

and

$$f(f^{-1}(w)) = w \quad \left(|w| < r_0(f); r_0(f) \geq \frac{1}{4} \right),$$

where

$$g(w) = f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3) w^3 - (5a_2^3 - 5a_2 a_3 + a_4) w^4 + \dots$$

A function $f \in \mathcal{A}$ is said to be bi-univalent in \mathbb{E} if both f and f^{-1} are univalent in \mathbb{E} . Let σ denote the class of bi-univalent functions defined in the unit disc \mathbb{E} . For a brief history of functions in the class, see [26] (see also [10], [9], [18], [20]). Recently, Srivastava *et al.* [26], Altınkaya and Yalçın [6] made an effort to introduce various subclasses of the bi-univalent function class σ and found non-sharp coefficient estimates on the initial coefficients $|a_2|$ and $|a_3|$ (see also [27], [28]). But the coefficient problem for each one of the following Taylor–Maclaurin coefficients

$$|a_n|, \quad n \in \mathbb{N} \setminus \{1, 2\}; \quad \mathbb{N} = \{1, 2, 3, \dots\}$$

is still an open problem. In the literature, there are only a few works determining the general coefficient bounds $|a_n|$ for the analytic bi-univalent functions ([7], [15], [17]).

The Faber polynomials introduced by Faber [12] play an important role in various areas of mathematical sciences, especially in Geometric Function Theory. Grunsky [14] succeeded in establishing a set of conditions for a given function which are necessary and in their totality sufficient for the univalence of this function, and in these conditions the coefficients of the Faber polynomials play an important role. Schiffer [23] gave a differential equations for univalent functions solving certain extremum problems with respect to coefficients of such functions; in this differential equation appears again a polynomial which is just the derivative of a Faber polynomial (see, for details, [24]). Using the Faber polynomial expansion of functions $f \in \mathcal{A}$ of the form (1.1), the coefficients of its inverse map $g = f^{-1}$ may be expressed as follows (see [3]):

$$g(w) = f^{-1}(w) = w + \sum_{n=2}^{\infty} \frac{1}{n} K_{n-1}^{-n}(a_2, a_3, \dots) w^n,$$

where

$$\begin{aligned} K_{n-1}^{-n} &= \frac{(-n)!}{(-2n+1)!(n-1)!} a_2^{n-1} + \frac{(-n)!}{[2(-n+1)]!(n-3)!} a_2^{n-3} a_3 \\ &+ \frac{(-n)!}{(-2n+3)!(n-4)!} a_2^{n-4} a_4 \\ &+ \frac{(-n)!}{[2(-n+2)]!(n-5)!} a_2^{n-5} [a_5 + (-n+2)a_3^2] \\ &+ \frac{(-n)!}{(-2n+5)!(n-6)!} a_2^{n-6} [a_6 + (-2n+5)a_3 a_4] \\ &+ \sum_{j \geq 7} a_2^{n-j} V_j, \end{aligned} \tag{1.2}$$

such that V_j ($7 \leq j \leq n$) is a homogeneous polynomial in the variables a_2, a_3, \dots, a_n (see, for details, [4]). In particular, the first three terms of K_{n-1}^{-n} are given below:

$$\begin{aligned}\frac{1}{2}K_1^{-2} &= -a_2, \\ \frac{1}{3}K_2^{-3} &= 2a_2^2 - a_3, \\ \frac{1}{4}K_3^{-4} &= -(5a_2^3 - 5a_2a_3 + a_4).\end{aligned}\tag{1.3}$$

In general, an expansion of K_n^p is given by (see [3])

$$K_n^p = pa_n + \frac{p(p-1)}{2}E_n^2 + \frac{p!}{(p-3)!3!}E_n^3 + \dots + \frac{p!}{(p-n)!n!}E_n^n \quad (p \in \mathbb{Z}),\tag{1.4}$$

where

$$\mathbb{Z} = \{0, \pm 1, \pm 2, \dots\} \quad \text{and} \quad E_n^p = E_n^p(a_2, a_3, \dots)$$

and, alternatively, by (see [1] and [2])

$$E_n^m(a_1, a_2, \dots, a_n) = \sum_{m=1}^{\infty} \frac{m!(a_1)^{\mu_1} \dots (a_n)^{\mu_n}}{\mu_1! \dots \mu_n!},\tag{1.5}$$

while $a_1 = 1$, and the sum is taken over all nonnegative integers μ_1, \dots, μ_n satisfying the following conditions:

$$\begin{aligned}\mu_1 + \mu_2 + \dots + \mu_n &= m, \\ \mu_1 + 2\mu_2 + \dots + n\mu_n &= n.\end{aligned}\tag{1.6}$$

Evidently,

$$E_n^n(a_1, a_2, \dots, a_n) = a_1^n.$$

In the field of Geometric Functions Theory, various subclasses of analytic functions have been studied from different viewpoints. The fractional q -calculus is the important tools that are used to investigate subclasses of analytic functions. Historically speaking, a firm footing of the usage of the the q -calculus in the context of Geometric Function Theory was actually provided and the basic (or q -) hypergeometric functions were first used in Geometric Function Theory in a book chapter by Srivastava (see, for details, [25]). In fact, the theory of univalent functions can be described by using the theory of the q -calculus. Moreover, in recent years, such q -calculus operators as the fractional q -integral and fractional q -derivative operators were used to construct several subclasses of analytic functions (see, for example, [5], [8] and [21]).

For the convenience, we provide some basic definitions and concept details of q -calculus which are used in this paper. We suppose throughout the paper that $0 < q < 1$. We recall the definitions of fractional q -calculus operators of complex valued function f . We shall follow the notation and terminology in [13].

Definition 1.1 (See [16]) The q -derivative of a function f is defined on a subset of \mathbb{C} is given by

$$(D_q f)(z) = \frac{f(z) - f(qz)}{(1-q)z} \quad (z \neq 0),\tag{1.7}$$

and $(D_q f)(0) = f'(0)$ provided $f'(0)$ exists.

Note that

$$\lim_{q \rightarrow 1^-} (D_q f)(z) = \lim_{q \rightarrow 1^-} \frac{f(z) - f(qz)}{(1-q)z} = \frac{df(z)}{dz}$$

if f is differentiable. From (1.7), we deduce that

$$(D_q f)(z) = 1 + \sum_{n=2}^{\infty} [n]_q a_n z^{n-1},\tag{1.8}$$

where the symbol $[n]_q$ denotes the so-called the twin-basic number is a natural generalization of the q -number, that is

$$[n]_q = \frac{1 - q^n}{1 - q} \quad (q \neq 1).$$

The object of this paper is to introduce a new subclass of bi-univalent functions defined by using the Jackson q -derivative operator and use the Faber polynomial expansion techniques to derive bounds for the general Taylor-Maclaurin coefficients $|a_n|$ for the functions in this class. We also obtain estimates for the initial coefficients $|a_2|$ and $|a_3|$ of these functions.

2. The general Taylor-Maclaurin coefficients $|a_n|$

We begin this section by introducing the function class $T_\sigma(q; \gamma)$ by means of the following definition.

Definition 2.1 Let the function $\Psi \in \mathcal{P}$ be univalent in \mathbb{E} and let $\Psi(\mathbb{E})$ be symmetrical about the real axis with

$$\Psi'(0) > 0.$$

We say that a function $f \in \sigma$ is in the class

$$T_\sigma(q; \gamma) \quad (\gamma \geq 1)$$

if the following quasi-subordinations hold true:

$$(1 - \gamma) \frac{f(z)}{z} + \gamma(D_q f)(z) \prec_q \Psi(z) \quad (z \in \mathbb{E}) \quad (2.1)$$

and

$$(1 - \gamma) \frac{g(w)}{w} + \gamma(D_q g)(w) \prec_q \Psi(w) \quad (w \in \mathbb{E}), \quad (2.2)$$

where $g = f^{-1}$.

It is clear from Definition 2.1 that $f \in T_\sigma(q; \gamma)$ if and only if there exists a function h ($|h(z)| \leq 1$) such that

$$\frac{(1 - \gamma) \frac{f(z)}{z} + \gamma(D_q f)(z)}{h(z)} \prec (\Psi(z)) \quad (z \in \mathbb{E})$$

and

$$\frac{(1 - \gamma) \frac{g(w)}{w} + \gamma(D_q g)(w)}{h(w)} \prec (\Psi(w)) \quad (w \in \mathbb{E}).$$

Throughout this paper, we suppose that the function $\Psi \in \mathcal{P}$ is of the form:

$$\Psi(z) = 1 + B_1 z + B_2 z^2 + \dots \quad (B_1 > 0, z \in \mathbb{E}).$$

and that the function h , analytic in \mathbb{E} , is of the form:

$$h(z) = H_0 + H_1 z + H_2 z^2 + \dots \quad (|h(z)| \leq 1, z \in \mathbb{E}).$$

Our main result is given by Theorem 2.2 below.

Theorem 2.2 Let f given by (1.1) be in the class $T_\sigma(q; \gamma)$. If $a_m = 0$ for $2 \leq m \leq n-1$, then

$$|a_n| \leq \frac{B_1 + |H_{n-1}|}{1 + \left([n]_q - 1\right) \gamma} \quad (n > 3).$$

Proof. For analytic functions f of the form (1.1), we have

$$(1 - \gamma) \frac{f(z)}{z} + \gamma(D_q f)(z) = \sum_{n=2}^{\infty} \left[1 + \left([n]_q - 1\right) \gamma\right] a_n z^{n-1} \quad (2.3)$$

and

$$\begin{aligned} (1 - \gamma) \frac{g(w)}{w} + \gamma(D_q g)(w) &= \sum_{n=1}^{\infty} \left[1 + \left([n]_q - 1\right) \gamma\right] b_n w^{n-1} \\ &= \sum_{n=1}^{\infty} \left[1 + \left([n]_q - 1\right) \gamma\right] \\ &\quad \times \frac{1}{n} K_{n-1}^{-n} (a_2, a_3, \dots, a_n) w^{n-1}. \end{aligned} \quad (2.4)$$

On the other hand, the inequalities (2.1) and (2.2) imply the existence of two Schwarz functions $u(z) = \sum_{n=1}^{\infty} c_n z^n$

and $v(w) = \sum_{n=1}^{\infty} d_n w^n$ so that

$$(1 - \gamma) \frac{f(z)}{z} + \gamma(D_q f)(z) = h(z) \Psi(u(z)) \quad (2.5)$$

and

$$(1 - \gamma) \frac{g(w)}{w} + \gamma(D_q g)(w) = h(z)\Psi(v(w)). \quad (2.6)$$

Thus, from (2.3) and (2.5) yields

$$\left[1 + \left([n]_q - 1\right) \gamma\right] a_n = H_{n-1} + \sum_{t=1}^{n-1} \sum_{k=1}^t B_k E_n^k(c_1, c_2, \dots, c_n) H_{n-(t+1)} \quad (H_0 = 1). \quad (2.7)$$

Similarly, by using (2.4) and (2.6), we find that

$$\left[1 + \left([n]_q - 1\right) \gamma\right] b_n = H_{n-1} + \sum_{t=1}^{n-1} \sum_{k=1}^t B_k E_n^k(d_1, d_2, \dots, d_n) H_{n-(t+1)}. \quad (2.8)$$

We note that, for $a_m = 0$ ($2 \leq m \leq n - 1$), we have

$$b_n = -a_n$$

and so

$$\begin{aligned} \left[1 + \left([n]_q - 1\right) \gamma\right] a_n &= B_1 c_{n-1} + H_{n-1}, \\ -\left[1 + \left([n]_q - 1\right) \gamma\right] a_n &= B_1 d_{n-1} + H_{n-1}. \end{aligned}$$

Now taking the absolute values of either of the above two equations and using the facts that $|c_{n-1}| \leq 1$ and $|d_{n-1}| \leq 1$, we obtain

$$|a_n| = \frac{|B_1 c_{n-1} + H_{n-1}|}{\left|1 + \left([n]_q - 1\right) \gamma\right|} = \frac{|B_1 d_{n-1} + H_{n-1}|}{\left|1 + \left([n]_q - 1\right) \gamma\right|} \leq \frac{B_1 + |H_{n-1}|}{1 + \left([n]_q - 1\right) \gamma}, \quad (2.9)$$

which evidently completes the proof of Theorem 2.2. \square

Corollary 2.3 If we take $h(z) = 1$ and $\Psi(z) = \left(\frac{1+z}{1-z}\right)^\xi$ ($0 < \xi \leq 1$) which gives $B_1 = 2\xi$, in Theorem 2.2, then we obtain

$$|a_n| \leq \frac{2\xi}{1 + \left([n]_q - 1\right) \gamma} \quad (n > 3).$$

Corollary 2.4 If we take $h(z) = 1$ and $\Psi(z) = \frac{1+(1-2\xi)z}{1-z}$ ($0 < \xi \leq 1$) which gives $B_1 = 2(1 - \xi)$, in Theorem 2.2, then we obtain

$$|a_n| \leq \frac{2(1 - \xi)}{1 + \left([n]_q - 1\right) \gamma} \quad (n > 3).$$

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On The Bounds of General Subclasses of Analytic and Bi-Univalent Functions Associated with Subordination

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Abstract:

In this study, we define several new subclasses of bi-univalent functions involving a differential operator in the open unit disc

$$\mathbb{E} = \{z \in \mathbb{C} : |z| < 1\}.$$

Moreover, we obtain estimates on the coefficients for functions belonging to these classes.

1. Introduction

Let \mathcal{A} be the class of functions analytic in \mathbb{E} , satisfying the conditions

$$f(0) = 0, \quad f'(0) = 1.$$

Then each $f \in \mathcal{A}$ has the Taylor expansion

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \quad (1.1)$$

The class of this kind of functions is represented by \mathcal{S} . Further, let \mathcal{P} be the class of functions $\Phi(z) = 1 + \sum_{n=1}^{\infty} \Phi_n z^n$ that are analytic in E and satisfy the condition $\Re(\Phi(z)) > 0$ in \mathbb{E} . By the Caratheodory's lemma (e.g., see [10]) we have $|\Phi_n| \leq 2$.

Let the functions f, g be analytic in \mathbb{E} . If there exists a Schwarz function ϖ , which is analytic in \mathbb{E} under the conditions

$$\varpi(0) = 0, \quad |\varpi(z)| < 1,$$

such that

$$f(z) = g(\varpi(z)) \quad (z \in \mathbb{E}),$$

then, the function f is subordinate to g in E , and indite $f(z) \prec g(z)$ ($z \in \mathbb{E}$).

The Koebe One-Quarter Theorem (see [10]) ensures that the image of \mathbb{E} under every $f \in \mathcal{S}$ contains a disc of radius $\frac{1}{4}$. So, every $f \in \mathcal{S}$ has an inverse f^{-1} which satisfies

$$f^{-1}(f(z)) = z \quad (z \in \mathbb{E})$$

and

$$f(f^{-1}(w)) = w \left(|w| < r_0(f), \quad r_0(f) \geq \frac{1}{4} \right),$$

where

$$g(w) = f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3) w^3 - (5a_2^3 - 5a_2 a_3 + a_4) w^4 + \dots$$

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A function $f \in \mathcal{A}$ is said to be bi-univalent in \mathbb{E} if both f and f^{-1} are univalent in \mathbb{E} . The class consisting of bi-univalent functions are denoted by σ . A concept of bi-univalent analytic functions is due to Lewin [14]. Ever since then, the first two coefficients $|a_2|$ and $|a_3|$ of the bi-univalent functions was used by many researchers, see for example, [5], [7], [8], [11], [16], [17], [18]. The coefficient estimate problem for Taylor-Maclaurin coefficients $|a_n|$ for $n \geq 4$ is presumably still an open problem. Anyway, using Faber polynomial expansions, several authors obtained coefficient estimates $|a_n|$ for classes bi-univalent functions, see for example [6], [12], [13].

The Faber polynomials introduced by Faber play an important role in various areas of mathematical sciences, especially in Geometric Function Theory. By using the Faber polynomial expansion of functions $f \in \mathcal{A}$ of the form (1.1), the coefficients of its inverse map $g = f^{-1}$ may be expressed as follows (see Airault and Bouali [3]):

$$g(w) = f^{-1}(w) = w + \sum_{n=2}^{\infty} \frac{1}{n} K_{n-1}^{-n}(a_2, a_3, \dots) w^n,$$

where

$$\begin{aligned} K_{n-1}^{-n} &= \frac{(-n)!}{(-2n+1)!(n-1)!} a_2^{n-1} + \frac{(-n)!}{[2(-n+1)]!(n-3)!} a_2^{n-3} a_3 \\ &+ \frac{(-n)!}{(-2n+3)!(n-4)!} a_2^{n-4} a_4 \\ &+ \frac{(-n)!}{[2(-n+2)]!(n-5)!} a_2^{n-5} [a_5 + (-n+2)a_3^2] \\ &+ \frac{(-n)!}{(-2n+5)!(n-6)!} a_2^{n-6} [a_6 + (-2n+5)a_3 a_4] \\ &+ \sum_{j \geq 7} a_2^{n-j} V_j, \end{aligned} \quad (1.2)$$

such that V_j ($7 \leq j \leq n$) is a homogeneous polynomial in the variables a_2, a_3, \dots, a_n (see, for details, Airault and Ren [4]). In particular, the first three terms of K_{n-1}^{-n} are given below:

$$\begin{aligned} \frac{1}{2} K_1^{-2} &= -a_2, \\ \frac{1}{3} K_2^{-3} &= 2a_2^2 - a_3, \\ \frac{1}{4} K_3^{-4} &= -(5a_2^3 - 5a_2 a_3 + a_4). \end{aligned} \quad (1.3)$$

In general, an expansion of K_n^p is given by (see Airault and Bouali [3])

$$K_n^p = p a_n + \frac{p(p-1)}{2} E_n^2 + \frac{p!}{(p-3)!3!} E_n^3 + \dots + \frac{p!}{(p-n)!n!} E_n^n \quad (p \in \mathbb{Z}), \quad (1.4)$$

where

$$\mathbb{Z} = \{0, \pm 1, \pm 2, \dots\} \quad \text{and} \quad E_n^p = E_n^p(a_2, a_3, \dots)$$

and, alternatively, by (see Airault [1] and [2])

$$E_n^m(a_1, a_2, \dots, a_n) = \sum_{m=1}^{\infty} \frac{m! (a_1)^{\mu_1} \dots (a_n)^{\mu_n}}{\mu_1! \dots \mu_n!}, \quad (1.5)$$

while $a_1 = 1$, and the sum is taken over all nonnegative integers μ_1, \dots, μ_n satisfying the following conditions:

$$\begin{aligned} \mu_1 + \mu_2 + \dots + \mu_n &= m, \\ \mu_1 + 2\mu_2 + \dots + n\mu_n &= n. \end{aligned} \quad (1.6)$$

Evidently,

$$E_n^n(a_1, a_2, \dots, a_n) = a_1^n.$$

Now, we establish a new subclass of analytic and bi-univalent functions based on subordination.

Definition 1.1 A function $f \in \sigma$ is said to be in the class

$$M_{\sigma}(\mu, \beta, \Phi) \quad (\mu \geq 0, \beta \geq 1; z, w \in \mathbb{E})$$

if the following subordinations are satisfied:

$$(1 - \beta) \left(\frac{f(z)}{z} \right)^{\mu} + \beta f'(z) \left(\frac{f(z)}{z} \right)^{\mu-1} \prec \Phi(z) \quad (1.7)$$

and

$$(1 - \beta) \left(\frac{g(w)}{w} \right)^{\mu} + \beta g'(w) \left(\frac{g(w)}{w} \right)^{\mu-1} \prec \Phi(w), \quad (1.8)$$

where $g = f^{-1}$.

2. Coefficient bounds for the function class $M_{\sigma}(\mu, \beta, \Phi)$

Theorem 2.1 Let f given by (1.1) be in the class $M_{\sigma}(\mu, \beta, \Phi)$. Then

$$|a_2| \leq \min \left\{ \frac{2}{\mu + \beta}, \sqrt{\frac{8}{(\mu + 2\beta)(\beta + 1)}} \right\}$$

and

$$|a_3| \leq \min \left\{ \frac{4}{(\mu + \beta)^2} + \frac{2}{\mu + 2\beta}, \frac{8}{(\mu + 2\beta)(\mu + 1)} + \frac{2}{\mu + \beta\lambda} \right\}.$$

Proof. Let $f \in M_{\sigma}(\mu, \beta, \Phi)$. The inequalities (1.7) and (1.8) imply the existence of two positive real part functions

$$\varpi(z) = 1 + \sum_{n=1}^{\infty} t_n z^n$$

and

$$\vartheta(w) = 1 + \sum_{n=1}^{\infty} s_n w^n,$$

where $\Re(\varpi(z)) > 0$ and $\Re(\vartheta(w)) > 0$ in \mathcal{D} so that

$$(1 - \beta) \left(\frac{f(z)}{z} \right)^{\mu} + \beta f'(z) \left(\frac{f(z)}{z} \right)^{\mu-1} = \Phi(\varpi(z)), \quad (2.1)$$

$$(1 - \beta) \left(\frac{g(w)}{w} \right)^{\mu} + \beta g'(w) \left(\frac{g(w)}{w} \right)^{\mu-1} = \Phi(\vartheta(w)). \quad (2.2)$$

It follows from (2.1) and (2.2) that

$$(\mu + \beta)a_2 = \Phi_1 t_1, \quad (2.3)$$

$$(\mu + 2\beta) \left[\frac{\mu - 1}{2} a_2^2 + a_3 \right] = \Phi_1 t_2 + \Phi_2 t_1^2 \quad (2.4)$$

and

$$-(\mu + \beta)a_2 = \Phi_1 s_1, \quad (2.5)$$

$$(\mu + 2\beta) \left[\frac{\mu + 3}{2} a_2^2 - a_3 \right] = \Phi_1 s_2 + \Phi_2 s_1^2. \quad (2.6)$$

From (2.3) or (2.5) we obtain

$$|a_2| \leq \frac{|\Phi_1 t_1|}{\mu + \beta} = \frac{|\Phi_1 s_1|}{\mu + \beta} \leq \frac{2}{\mu + \beta}. \quad (2.7)$$

Adding (2.4) to (2.6) implies

$$(\mu + 2\beta)(\mu + 1)a_2^2 = \Phi_1(t_2 + s_2) + \Phi_2(t_1^2 + s_1^2)$$

or, equivalently

$$|a_2| \leq \sqrt{\frac{8}{(\mu + 2\beta)(\mu + 1)}}. \quad (2.8)$$

Next, in order to find the bound on the coefficient $|a_3|$, we subtract (2.6) from (2.4). We thus get

$$2(\mu + 2\beta)(a_3 - a_2^2) = \Phi_1(t_2 - s_2) + \Phi_2(t_1^2 - s_1^2) \quad (2.9)$$

or

$$\begin{aligned} |a_3| &\leq |a_2|^2 + \frac{|\Phi_1(t_2 - s_2)|}{2(\mu + 2\beta)} \\ &= |a_2|^2 + \frac{2}{\mu + 2\beta} \end{aligned} \quad (2.10)$$

Upon substituting the value of a_2^2 from (2.7) and (2.8) into (2.10), it follows that

$$|a_3| \leq \frac{4}{(\mu + \beta)^2} + \frac{2}{\mu + 2\beta}$$

and

$$|a_3| \leq \frac{8}{(\mu + 2\beta)(\mu + 1)} + \frac{2}{\mu + 2\beta}.$$

□

Theorem 2.2 Let $f \in M_{\sigma}(\mu, \beta, \Phi)$. If $a_m = 0$ with $2 \leq m \leq n-1$, then

$$|a_n| \leq \frac{2}{\mu + (n-1)\beta} \quad (n \geq 4).$$

Proof. By using the Faber polynomial expansion of functions $f \in \mathcal{A}$ of the form (1.1) and its inverse map $g = f^{-1}$, we can write

$$(1 - \beta) \left(\frac{f(z)}{z} \right)^{\mu} + \beta f'(z) \left(\frac{f(z)}{z} \right)^{\mu-1} = 1 + \sum_{n=2}^{\infty} G_{n-1}(a_2, a_3, \dots, a_n) a_n z^{n-1} \quad (2.11)$$

and

$$(1 - \beta) \left(\frac{g(w)}{w} \right)^{\mu} + \beta g'(w) \left(\frac{g(w)}{w} \right)^{\mu-1} = 1 + \sum_{n=1}^{\infty} G_{n-1}(A_2, A_3, \dots, A_n) a_n w^{n-1},$$

where

$$G_1 = (\mu + \beta)a_2, \quad (2.12)$$

$$G_2 = (\mu + 2\beta) \left[\frac{\mu-1}{2} a_2^2 + a_3 \right],$$

$$G_3 = (\mu + 3\beta) \left[\frac{(\mu-1)(\mu-2)}{3!} a_2^3 + (\mu-1)a_2 a_3 + a_4 \right]$$

and, in general (see [9])

$$\begin{aligned} G_{n-1} &= ([\mu + (n-1)\beta] \times [(\mu-1)!]) \\ &\times \left[\sum_{i_1+2i_2+\dots+(n-1)i_{n-1}=n-1} \frac{a_2^{i_1} a_3^{i_2} \dots a_n^{i_{n-1}}}{i_1! i_2! \dots i_{n-1}! [\mu - (i_1 + i_2 + \dots + i_{n-1})]!} \right]. \end{aligned}$$

Next, by using the Faber polynomial expansion of functions $\varpi, \vartheta \in \mathcal{P}$, we also obtain

$$\Phi(\varpi(z)) = 1 + \sum_{n=1}^{\infty} \sum_{k=1}^n \Phi_k F_n^k(t_1, t_2, \dots, t_n) z^n$$

and

$$\Phi(\vartheta(w)) = 1 + \sum_{n=1}^{\infty} \sum_{k=1}^n \Phi_k F_n^k(s_1, s_2, \dots, s_n) w^n.$$

Comparing the corresponding coefficients yields

$$[\mu + (n-1)\beta] a_n = \sum_{k=1}^{n-1} \Phi_k F_{n-1}^k(t_1, t_2, \dots, t_{n-1}) \quad n \geq 2$$

and

$$[\mu + (n-1)\beta]A_n = \sum_{k=1}^{n-1} \Phi_k F_{n-1}^k(s_1, s_2, \dots, s_{n-1}) \quad (n \geq 2).$$

Note that for $a_m = 0$, $2 \leq m \leq n-1$, we have $A_n = -a_n$ and so

$$\begin{aligned} [\mu + (n-1)\beta]a_n &= \Phi_1 t_{n-1} \\ -[\mu + (n-1)\beta]a_n &= \Phi_1 s_{n-1} \end{aligned}$$

Now taking the absolute values of either of the above two equations and using the facts that $|\Phi_1| \leq 2$, $|t_{n-1}| \leq 1$ and $|s_{n-1}| \leq 1$, we obtain

$$|a_n| \leq \frac{|\Phi_1 t_{n-1}|}{\mu + (n-1)\beta} = \frac{|\Phi_1 s_{n-1}|}{\mu + (n-1)\beta} \leq \frac{2}{\mu + (n-1)\beta}. \quad (2.13)$$

□

The next theorem restricts our attention to the Fekete-Szegő inequalities over the class $M_\sigma(\mu, \beta, \Phi)$ for a real parameter.

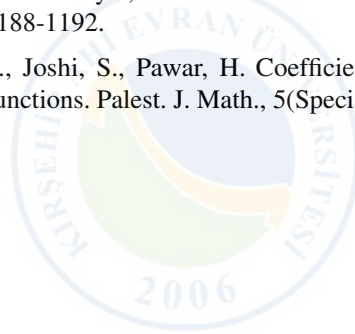
Theorem 2.3 (See [15]) Suppose that the function f of the form (1.1) belongs to the class $M_\sigma(\mu, \beta, \Phi)$. Then, for a real number η

$$|a_3 - \eta a_2^2| \leq \begin{cases} \frac{\Phi_1}{\mu+2\beta}; & |\eta - 1| \leq \frac{\mu+1}{2} \left| 1 + \frac{2(\Phi_2 - \Phi_1)(\mu+\beta)^2}{(\mu+2\beta)(\mu+1)\Phi_1^2} \right| \\ \frac{2\Phi_1^3|\eta-1|}{|(\mu+2\beta)(\mu+1)\Phi_1^2 + 2(\Phi_2 - \Phi_1)(\mu+\beta)^2|}; & |\eta - 1| \geq \frac{\mu+1}{2} \left| 1 + \frac{2(\Phi_2 - \Phi_1)(\mu+\beta)^2}{(\mu+2\beta)(\mu+1)\Phi_1^2} \right| \end{cases}.$$

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Certain Convex Harmonic Functions Defined by Subordination

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Abstract:

We investigate certain subclasses of convex harmonic univalent functions defined by subordination. We obtain coefficient bounds, distortion theorems, extreme points, convolution and convex combinations for these classes of functions. Also relevant connections of the results presented here with diverse known results are briefly denoted.

1. Introduction and Preliminaries

A continuous complex valued function $f = u + iv$ defined in a simply connected complex domain $D \subset \mathbb{C}$ is said to be harmonic in D if both u and v are real harmonic in D . Consider the functions U and V analytic in D so that $u = \Re(U)$ and $v = \Im(V)$. Then the harmonic function f can be expressed by

$$f(z) = h(z) + \overline{g(z)}, \quad z \in D,$$

where $h = (U + V)/2$ and $g = (U - V)/2$. We call h the analytic part and g co-analytic part of f . If g is identically zero then f reduces to the analytic case. A necessary and sufficient condition for f to be locally univalent and sense-preserving in D is that $|g'(z)| < |h'(z)|$, $z \in D$ (see Clunie and Sheil-Small [2]).

Let H denote the class of functions $f = h + \overline{g}$ which are harmonic sense-preserving, and univalent in the open unit disk $\mathbb{E} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$ with $f(0) = f_z(0) - 1 = 0$. Thus, any function $f \in H$ can be written in the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n + \sum_{n=1}^{\infty} \overline{b_n} z^n, \quad |b_1| < 1. \quad (1.1)$$

Also, let \overline{H} denote the subclass of H consisting of functions $f = h + \overline{g}$ so that the functions h and g take the form

$$h(z) = z - \sum_{n=2}^{\infty} |a_n| z^n \text{ and } g(z) = - \sum_{n=1}^{\infty} |b_n| z^n, \quad |b_1| < 1. \quad (1.2)$$

Recently, Kim et al. [7], studied a family of complex valued harmonic convex univalent functions related to uniformly convex analytic functions, denoted by $HCV(k, \alpha)$, $0 \leq k < \infty$, so that $f = h + \overline{g} \in HCV(k, \alpha)$ if and only if

$$\Re \left\{ 1 + (1 + ke^{i\theta}) \frac{z^2 h''(z) + 2z g'(z) + \overline{z^2 g''(z)}}{z h'(z) - \overline{z g'(z)}} \right\} \geq \alpha,$$

$\theta \in \mathbb{R}$, $0 \leq \alpha < 1$. When $\alpha = k = 0$ and $k = 0$, this class is denoted by HC and $HC(\alpha)$, respectively. These classes have been studied by Silverman [9], Avcı and Zlotkiewicz [1], Öztürk and Yalçın [8], Jahangiri [6], Yalçın [10], Yalçın and Öztürk [11].

We say that a function $f \in H$ is subordinate to a function $F \in H$, and write $f \prec F$, if there exists a complex valued function w which maps \mathbb{E} into oneself with $w(0) = 0$, such that $f(z) = F(w(z))$ ($z \in \mathbb{E}$).

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Furthermore, if the function F is univalent in \mathbb{E} , then we have the following equivalence:

$$f(z) \prec F(z) \Leftrightarrow f(0) = F(0) \text{ and } f(\mathbb{E}) \subset F(\mathbb{E}).$$

Denote by $HC(k, A, B)$ the subclass of H consisting of functions f of the form (1.1) that satisfy the condition

$$1 + (1 + ke^{i\theta}) \frac{z^2 h''(z) + \overline{2zg'(z) + z^2 g''(z)}}{zh'(z) - zg'(z)} \prec \frac{1 + Az}{1 + Bz}, \quad (1.3)$$

where $-B \leq A < B \leq 1$, $0 \leq k < \infty$ and $\theta \in \mathbb{R}$.

Finally, we let $\overline{HC}(k, A, B) \equiv HC(k, A, B) \cap \overline{H}$.

By suitably specializing the parameters, the classes $HC(k, A, B)$ reduces to the various subclasses of harmonic univalent functions. Such as,

- (i) $HC(0, A, B) = K_H(A, B)$ ([5]),
- (ii) $HC(k, 2\alpha - 1, 1) = HCV(k, \alpha)$, $0 \leq \alpha < 1$ ([7]),
- (iii) $HC(1, 2\gamma - 1, 1) = RS_H(1, \gamma)$, $0 \leq \gamma < 1$ ([12]),
- (iv) $HC(0, 2\alpha - 1, 1) = HC(\alpha)$, $0 \leq \alpha < 1$ ([1], [8], [6]),
- (v) $HC(0, -1, 1) = HC$ ([9]).

Making use of the techniques and methodology used by Dziok (see [3], [4]), Dziok et al. [5], in this paper, we find necessary and sufficient conditions, distortion bounds, compactness and extreme points for the above defined class $\overline{HC}(k, A, B)$.

2. Main Results

For functions f_1 and $f_2 \in H$ of the form

$$f_m(z) = z + \sum_{n=2}^{\infty} a_{m,n} z^n + \sum_{n=1}^{\infty} \overline{b_{m,n} z^n}, \quad (z \in \mathbb{E}, m = 1, 2),$$

we define the Hadamard product of f_1 and f_2 by

$$(f_1 * f_2)(z) = z + \sum_{n=2}^{\infty} a_{1,n} a_{2,n} z^n + \sum_{n=1}^{\infty} \overline{b_{1,n} b_{2,n} z^n} \quad (z \in \mathbb{E}).$$

First we state and prove the necessary and sufficient conditions for harmonic functions in $HC(k, A, B)$.

Theorem 2.1 Let $f \in H$. Then $f \in HC(k, A, B)$ if and only if

$$f(z) * \phi(z; \zeta) \neq 0, \quad (\zeta \in \mathbb{C}, |\zeta| = 1, z \in \mathbb{E} \setminus \{0\}),$$

where

$$\begin{aligned} \phi(z; \zeta) &= \frac{(B-A)\zeta z + 2(1 + ke^{i\theta})z^2 + [A + (1 + 2ke^{i\theta})B]\zeta z^2}{(1-z)^3} \\ &+ \frac{2(1 + ke^{i\theta})\bar{z} + [A + (1 + 2ke^{i\theta})B]\zeta \bar{z} + (B-A)\zeta \bar{z}^2}{(1-\bar{z})^3}. \end{aligned}$$

Proof. Let $f \in H$ be of the form (1.1). Then $f \in HC(k, A, B)$ if and only if it satisfies (1.3) or equivalently

$$1 + (1 + ke^{i\theta}) \frac{z^2 h''(z) + \overline{2zg'(z) + z^2 g''(z)}}{zh'(z) - zg'(z)} \neq \frac{1 + A\zeta}{1 + B\zeta}, \quad (2.1)$$

where $\zeta \in \mathbb{C}$, $|\zeta| = 1$ and $z \in \mathbb{E} \setminus \{0\}$. Since

$$h(z) = h(z) * \frac{z}{1-z}, \quad g(z) = g(z) * \frac{z}{1-z},$$

$$zh'(z) = h(z) * \frac{z}{(1-z)^2}, \quad zg'(z) = g(z) * \frac{z}{(1-z)^2}$$

and

$$z^2 h''(z) = h(z) * \frac{2z^2}{(1-z)^3}, \quad z^2 g''(z) = g(z) * \frac{2z^2}{(1-z)^3},$$

the inequality (2.1) yields

$$\begin{aligned}
 & (1+B\zeta) \left[zh'(z) + (1+ke^{i\theta})(z^2h''(z) + \overline{z^2g''(z)}) + (1+2ke^{i\theta})\overline{zg'(z)} \right] \\
 & - (1+A\zeta) \left[zh'(z) - \overline{zg'(z)} \right] \\
 = & h(z) * \left\{ (1+B\zeta) \left[\frac{z}{(1-z)^2} + (1+ke^{i\theta}) \frac{2z^2}{(1-z)^3} \right] - (1+A\zeta) \frac{z}{(1-z)^2} \right\} \\
 & + \overline{g(z)} * \left\{ (1+B\zeta) \left[(1+2ke^{i\theta}) \frac{\bar{z}}{(1-\bar{z})^2} + (1+ke^{i\theta}) \frac{2\bar{z}^2}{(1-\bar{z})^3} \right] \right. \\
 & \quad \left. + (1+A\zeta) \frac{\bar{z}}{(1-\bar{z})^2} \right\} \\
 = & h(z) * \frac{(B-A)\zeta z + 2(1+ke^{i\theta})z^2 + [A + (1+2ke^{i\theta})B]\zeta z^2}{(1-z)^3} \\
 & + \overline{g(z)} * \frac{2(1+ke^{i\theta})\bar{z} + [A + (1+2ke^{i\theta})B]\zeta\bar{z} + (B-A)\zeta\bar{z}^2}{(1-\bar{z})^3} \\
 = & f(z) * \phi(z; \zeta) \neq 0.
 \end{aligned}$$

□

Here we state a result due to Silverman [9], which we will use throughout this paper.

Theorem 2.2 Let f be of the form (1.1). If

$$\sum_{n=2}^{\infty} n^2 |a_n| + \sum_{n=1}^{\infty} n^2 |b_n| \leq 1, \tag{2.2}$$

then f is harmonic, sense preserving, univalent in \mathbb{E} , and $f \in HC$. Condition (2.2) is also necessary if $f \in HC \cap \overline{H}$.

Now we state and prove a sufficient coefficient bound for the class $HC(k, A, B)$.

Theorem 2.3 Let f be of the form (1.1). If $0 \leq k < \infty$, $-B \leq A < B \leq 1$, and

$$\sum_{n=2}^{\infty} \Phi_n |a_n| + \sum_{n=1}^{\infty} \Psi_n |b_n| \leq B - A, \tag{2.3}$$

where

$$\Phi_n = n[(1+k)(Bn+n-1) - A - Bk] \tag{2.4}$$

and

$$\Psi_n = n[(1+k)(Bn+n+1) + A + Bk], \tag{2.5}$$

then f is harmonic, sense preserving, univalent in \mathbb{E} , and $f \in HC(k, A, B)$.

Proof. Since $n(B-A) \leq (1+k)(Bn+n-1) - A - Bk$ and $n(B-A) \leq (1+k)(Bn+n+1) + A + Bk$ for $0 \leq k < \infty$, and $-B \leq A < B \leq 1$, it follows from Theorem 2.2 that $f \in HC$ and hence f is sense preserving and convex univalent in \mathbb{E} . Now, we only need to show that if (1.3) holds then $f \in HC(k, A, B)$. By definition of subordination, $f \in HC(k, A, B)$ if and only if there exists a complex valued function w ; $w(0) = 0$, $|w(z)| < 1$ ($z \in \mathbb{E}$) such that

$$1 + (1+ke^{i\theta}) \frac{z^2h''(z) + \overline{2zg'(z)} + \overline{z^2g''(z)}}{zh'(z) - \overline{zg'(z)}} = \frac{1+Aw(z)}{1+Bw(z)}$$

or equivalently

$$\left| \frac{(1+ke^{i\theta})(z^2h''(z) + \overline{2zg'(z)}) + 2(1+ke^{i\theta})\overline{zg'(z)}}{(B-A)zh'(z) + B(1+ke^{i\theta})(z^2h''(z) + \overline{z^2g''(z)}) + [A+B(1+2ke^{i\theta})]\overline{zg'(z)}} \right| < 1. \tag{2.6}$$

Substituting for $z^2 h''(z)$, $zh'(z)$, $z^2 g''(z)$ and $zg'(z)$ in (2.6), we obtain

$$\begin{aligned}
 & \left| (1 + ke^{i\theta}) \left(z^2 h''(z) + \overline{z^2 g''(z)} + 2z \overline{g'(z)} \right) \right| \\
 & - \left| (B - A)zh'(z) + B(1 + ke^{i\theta})(z^2 h''(z) + \overline{z^2 g''(z)}) \right. \\
 & \quad \left. + [A + B(1 + 2ke^{i\theta})]\overline{zg'(z)} \right| \\
 & = \left| \sum_{n=2}^{\infty} n(n-1)(1 + ke^{i\theta})a_n z^n + \sum_{n=1}^{\infty} n(n+1)(1 + ke^{i\theta})\overline{b_n z^n} \right| \\
 & - \left| (B - A)z + \sum_{n=2}^{\infty} n [Bn - A + Bke^{i\theta}(n-1)] a_n z^n \right. \\
 & \quad \left. + \sum_{n=1}^{\infty} n [Bn + A + Bke^{i\theta}(n+1)] \overline{b_n z^n} \right| \\
 & \leq \sum_{n=2}^{\infty} n[(1+k)(Bn+n-1) - A - Bk] |a_n| |z|^n \\
 & \quad + \sum_{n=1}^{\infty} n[(1+k)(Bn+n+1) + A + Bk] |b_n| |z|^n - (B - A) |z| \\
 & \leq |z| \left\{ \sum_{n=2}^{\infty} n[(1+k)(Bn+n-1) - A - Bk] |a_n| \right. \\
 & \quad \left. + \sum_{n=1}^{\infty} n[(1+k)(Bn+n+1) + A + Bk] |b_n| - (B - A) \right\} \\
 & < 0,
 \end{aligned}$$

by (2.3). \square

The harmonic functions

$$f(z) = z + \sum_{n=2}^{\infty} \frac{B-A}{\Phi_n} x_n z^n + \sum_{n=1}^{\infty} \frac{B-A}{\Psi_n} y_n \bar{z}^n, \quad (2.7)$$

where $\sum_{n=2}^{\infty} |x_n| + \sum_{n=1}^{\infty} |y_n| = 1$, show that the coefficient bound given by in Theorem 2.3 is sharp. Since

$$\sum_{n=2}^{\infty} \Phi_n |a_n| + \sum_{n=1}^{\infty} \Psi_n |b_n| = (B-A) \sum_{n=2}^{\infty} |x_n| + (B-A) \sum_{n=1}^{\infty} |y_n| = B-A,$$

the functions of the form (2.7) are in $HC(k, A, B)$.

Next we show that the bound (2.3) is also necessary for $\overline{HC}(k, A, B)$.

Theorem 2.4 Let $f = h + \bar{g}$ with h and g of the form (1.2). Then $f \in \overline{HC}(k, A, B)$ if and only if the condition (2.3) holds.

Proof. In view of Theorem 2.3, we only need to show that $f \notin \overline{HC}(k, A, B)$ if condition (2.3) does not hold. We note that a necessary and sufficient condition for $f = h + \bar{g}$ given by (1.2) to be in $\overline{HC}(k, A, B)$ is that the coefficient condition (2.3) to be satisfied. Equivalently, we must have

$$\left| \frac{- \sum_{n=2}^{\infty} n(n-1)(1 + ke^{i\theta}) |a_n| z^n - \sum_{n=1}^{\infty} n(n+1)(1 + ke^{i\theta}) |b_n| \bar{z}^n}{(B - A)z - \sum_{n=2}^{\infty} n [Bn - A + Bke^{i\theta}(n-1)] |a_n| z^n - \sum_{n=1}^{\infty} n [Bn + A + Bke^{i\theta}(n+1)] |b_n| \bar{z}^n} \right| < 1.$$

For $z = r < 1$ we obtain

$$\frac{\sum_{n=2}^{\infty} n(n-1)(1+k)|a_n|r^{n-1} + \sum_{n=1}^{\infty} n(n+1)(1+k)|b_n|r^{n-1}}{(B-A) - \sum_{n=2}^{\infty} n[Bn-A+Bk(n-1)]|a_n|r^{n-1} - \sum_{n=1}^{\infty} n[Bn+A+Bk(n+1)]|b_n|r^{n-1}} < 1. \quad (2.8)$$

If condition (2.3) does not hold then condition (2.8) does not hold for r sufficiently close to 1. Thus there exists $z_0 = r_0$ in $(0, 1)$ for which the quotient (2.8) is greater than 1. This contradicts the required condition for $f \in \overline{HC}(k, A, B)$ and so the proof is completed. \square

Theorem 2.5 Let $f \in \overline{HC}(k, A, B)$. Then for $|z| = r < 1$, we have

$$|f(z)| \leq (1 + |b_1|)r + \frac{1}{2} \left(\frac{B-A}{(1+k)(2B+1)-A-Bk} - \frac{(1+k)(B+2)+A+Bk}{(1+k)(2B+1)-A-Bk} |b_1| \right) r^2$$

and

$$|f(z)| \geq (1 - |b_1|)r - \frac{1}{2} \left(\frac{B-A}{(1+k)(2B+1)-A-Bk} - \frac{(1+k)(B+2)+A+Bk}{(1+k)(2B+1)-A-Bk} |b_1| \right) r^2.$$

Proof. We only prove the right hand inequality. The proof for the left hand inequality is similar and will be omitted. Let $f \in \overline{HC}(k, A, B)$. Taking the absolute value of f we have

$$\begin{aligned} |f(z)| &\leq (1 + |b_1|)r + \sum_{n=2}^{\infty} (|a_n| + |b_n|) r^n \\ &\leq (1 + |b_1|)r + \frac{B-A}{2[(1+k)(2B+1)-A-Bk]} \\ &\quad \times \sum_{n=2}^{\infty} (\Phi_n |a_n| + \Psi_n |b_n|) r^n \\ &\leq (1 + |b_1|)r + \frac{\{B-A - [(1+k)(B+2)+A+Bk]|b_1|\}}{2[(1+k)(2B+1)-A-Bk]} r^2. \end{aligned}$$

□

The following covering result follows from the left hand inequality in Theorem 2.5.

Corollary 2.6 Let $f = h + \bar{g}$ with h and g of the form (1.2). If $f \in \overline{HC}(k, A, B)$ then

$$\left\{ w : |w| < \frac{2(1+k)(B+1) + (B-A)(1-3|b_1|)}{2[(1+k)(2B+1)-A-Bk]} \right\} \subset f(\mathbb{E}).$$

Theorem 2.7 Set

$$h_1(z) = z, \quad h_n(z) = z - \frac{B-A}{\Phi_n} z^n, \quad (n = 2, 3, \dots)$$

and

$$g_n(z) = z - \frac{B-A}{\Psi_n} \bar{z}^n, \quad (n = 1, 2, \dots).$$

Then $f \in \overline{HC}(k, A, B)$ if and only if it can be expressed as

$$f(z) = \sum_{n=1}^{\infty} (x_n h_n(z) + y_n g_n(z)),$$

where $x_n \geq 0$, $y_n \geq 0$ and $\sum_{n=1}^{\infty} (x_n + y_n) = 1$. In particular, the extreme points of $\overline{HC}(k, A, B)$ are $\{h_n\}$ and $\{g_n\}$.

Proof. Suppose

$$\begin{aligned} f(z) &= \sum_{n=1}^{\infty} (x_n h_n(z) + y_n g_n(z)) \\ &= \sum_{n=1}^{\infty} (x_n + y_n)z - \sum_{n=2}^{\infty} \frac{B-A}{\Phi_n} x_n z^n - \sum_{n=1}^{\infty} \frac{B-A}{\Psi_n} y_n \bar{z}^n. \end{aligned}$$

Then

$$\begin{aligned} \sum_{n=2}^{\infty} \Phi_n |a_n| + \sum_{n=1}^{\infty} \Psi_n |b_n| &= (B-A) \sum_{n=2}^{\infty} x_n + (B-A) \sum_{n=1}^{\infty} y_n \\ &= (B-A)(1 - x_1) \leq B-A \end{aligned}$$

and so $f \in \overline{HC}(k, A, B)$. Conversely, if $f \in \overline{HC}(k, A, B)$, then

$$|a_n| \leq \frac{B-A}{\Phi_n} \text{ and } |b_n| \leq \frac{B-A}{\Psi_n}.$$

Set

$$x_n = \frac{\Phi_n}{B-A} |a_n| \text{ (} n = 2, 3, \dots \text{)} \text{ and } y_n = \frac{\Psi_n}{B-A} |b_n| \text{ (} n = 1, 2, \dots \text{)}.$$

Then note by Theorem 2.4, $0 \leq x_n \leq 1$ ($n = 2, 3, \dots$) and $0 \leq y_n \leq 1$ ($n = 1, 2, \dots$). We define

$$x_1 = 1 - \sum_{n=2}^{\infty} x_n - \sum_{n=1}^{\infty} y_n$$

and note that by Theorem 2.4, $x_1 \geq 0$. Consequently, we obtain $f(z) = \sum_{n=1}^{\infty} (x_n h_n(z) + y_n g_n(z))$ as required. \square

Now we show that $\overline{HC}(k, A, B)$ is closed under convex combinations of its members.

Theorem 2.8 *The class $\overline{HC}(k, A, B)$ is closed under convex combination.*

Proof. For $i = 1, 2, 3, \dots$ let $f_i \in \overline{HC}(k, A, B)$, where f_i is given by

$$f_i(z) = z - \sum_{n=2}^{\infty} |a_{ni}| z^n - \sum_{n=1}^{\infty} |b_{ni}| \bar{z}^n.$$

Then by (2.3),

$$\sum_{n=1}^{\infty} (\Phi_n |a_{ni}| + \Psi_n |b_{ni}|) \leq 2(B-A). \tag{2.9}$$

For $\sum_{i=1}^{\infty} t_i = 1$, $0 \leq t_i \leq 1$, the convex combination of f_i may be written as

$$\sum_{i=1}^{\infty} t_i f_i(z) = z - \sum_{n=2}^{\infty} \left(\sum_{i=1}^{\infty} t_i |a_{ni}| \right) z^n - \sum_{n=1}^{\infty} \left(\sum_{i=1}^{\infty} t_i |b_{ni}| \right) \bar{z}^n.$$

Then by (2.6),

$$\begin{aligned} \sum_{n=1}^{\infty} \left(\Phi_n \sum_{i=1}^{\infty} t_i |a_{ni}| + \Psi_n \sum_{i=1}^{\infty} t_i |b_{ni}| \right) &= \sum_{i=1}^{\infty} t_i \left(\sum_{n=1}^{\infty} [\Phi_n |a_{ni}| + \Psi_n |b_{ni}|] \right) \\ &\leq 2(B-A) \sum_{i=1}^{\infty} t_i = 2(B-A). \end{aligned}$$

This is the condition required by (2.3) and so $\sum_{i=1}^{\infty} t_i f_i(z) \in \overline{HC}(k, A, B)$. \square

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Certain Subclasses of Harmonic Univalent Functions Associated with A Multiplier Linear Operator

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Abstract:

In the paper, we introduce new subclasses of functions defined by multiplier differential operator and give coefficient bounds for these subclasses. Also, we obtain necessary and sufficient convolution conditions, distortion bounds and extreme points for these subclasses of functions.

1. Introduction and Preliminaries

Let $U_r = \{z \in \mathbb{C} : |z| < r\}$ be open disk of radius r of complex plane and let $U = U_1$ be the open unit disk. We denote by A the class of analytic functions on U .

A harmonic mapping f of the simply connected domain D is a complex-valued function of the form $f = h + \bar{g}$, where h and g analytic and $h(0) = h'(0) - 1 = 0$, $g(0) = 0$. We call h and g analytic and co-analytic part of f , respectively. The Jacobian of f is given by $J_{f(z)} = |f_z(z)|^2 - |f_{\bar{z}}(z)|^2 = |h'(z)|^2 - |g'(z)|^2$. A result of Lewy [13] states that f is locally univalent if and only if its Jacobian is never zero, and is sense-preserving if the Jacobian is positive.

By SH we denote the class of complex-valued, sense-preserving univalent harmonic mappings that are normalized in U . Then for $f = h + \bar{g} \in SH$, we may express the analytic functions h and g as

$$h(z) = z + \sum_{k=2}^{\infty} a_k z^k, \quad g(z) = \sum_{k=1}^{\infty} b_k z^k; \quad |b_1| < 1. \quad (1.1)$$

Note that SH reduces to the class S of normalized analytic univalent sense-preserving functions in U , if the co-analytic part of f is identically zero.

The subclass SH^0 of SH consists of all functions in SH which have the additional property $g'(0) = b_1 = 0$. Also, notice that $S \subset SH^0 \subset SH$.

In 1984 Clunie and Sheil-Small [6] investigated the class SH as well as its geometric subclasses and obtained some coefficient bounds. Since then, there have been several related papers on SH and its subclasses such as Avcı and Złotkiewicz [3], Silverman [15], Jahangiri [11], Silverman and Silvia [16], Yaşar and Yalçın [17], Yalçın [19] studied the harmonic univalent functions.

For $f \in A$, the differential operator $D_{\alpha, \mu}^n(\lambda, w)$ was introduced by Bucur et al. [4], where $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $\mu, \lambda, w \geq 0$, $0 \leq \alpha \leq \mu w^\lambda$.

Next, for $f \in SH$ of the form (1.1) Altinkaya and Yalçın [1] defined the modified differential operator

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$D_{\alpha,\mu}^n(\lambda, w) : SH \rightarrow SH$ by

$$\begin{aligned} D_{\alpha,\mu}^0(\lambda, w)f(z) &= f(z), \\ D_{\alpha,\mu}^1(\lambda, w)f(z) &= (\alpha - \mu w^\lambda)f(z) + (\mu w^\lambda - \alpha + 1)(zf_z(z) - \bar{z}f_{\bar{z}}(z)), \\ &\vdots \\ D_{\alpha,\mu}^n(\lambda, w)f(z) &= D_{\alpha,\mu}^1(\lambda, w)(D_{\alpha,\mu}^{n-1}(\lambda, w)f(z)). \end{aligned} \quad (1.2)$$

If f is given by (1.1), then from (1.2), we see that

$$D_{\alpha,\mu}^n(\lambda, w)f(z) = D_{\alpha,\mu}^n(\lambda, w)h(z) + (-1)^n \overline{D_{\alpha,\mu}^n(\lambda, w)g(z)},$$

where

$$D_{\alpha,\mu}^n(\lambda, w)h(z) = z + \sum_{k=2}^{\infty} [(k-1)(\mu w^\lambda - \alpha) + k]^n a_k z^k$$

and

$$D_{\alpha,\mu}^n(\lambda, w)g(z) = \sum_{k=1}^{\infty} [(k+1)(\mu w^\lambda - \alpha) + k]^n b_k z^k,$$

where $\mu, \lambda, w \geq 0$, $0 \leq \alpha \leq \mu w^\lambda$ (Altinkaya and Yalçın [1]).

We remark that when $f \in A$, for $\alpha = \mu = 0$ we get Sălăgean differential operator [14], for $\alpha = \lambda = w = 1$ we get the operator introduced by Al-Oboudi [2], for $\lambda = w = 1$ we obtain the operator introduced by Darus and Ibrahim [7], if $\alpha = 1$ we get the operator introduced by Darus and Faisal [8].

Also, we notice that when $f \in SH$, for $\alpha = \mu = 0$ we obtain modified Sălăgean differential operator introduced by Jahangiri et al. [12], and if $\alpha = \lambda = w = 1$ we get the operator introduced by Yaşar and Yalçın [18].

The Hadamard product of functions f_1 and f_2 of the form

$$f_t(z) = z + \sum_{k=2}^{\infty} a_{t,k} z^k + \sum_{k=1}^{\infty} \overline{b_{t,k}} z^k \quad (z \in U, t = \{1, 2\})$$

is defined by

$$(f_1 * f_2)(z) = z + \sum_{k=2}^{\infty} a_{1,k} a_{2,k} z^k + \sum_{k=1}^{\infty} \overline{b_{1,k} b_{2,k}} z^k.$$

Also if f is given by (1.1), then we have

$$\begin{aligned} D_{\alpha,\mu}^n(\lambda, w)f(z) &= f(z) * \underbrace{(\psi_1(z) + \overline{\psi_2(z)}) * \dots * (\psi_1(z) + \overline{\psi_2(z)})}_{n \text{ times}}, \\ &= h(z) * \underbrace{\psi_1(z) * \dots * \psi_1(z)}_{n \text{ times}} + \overline{g(z) * \underbrace{\psi_2(z) * \dots * \psi_2(z)}_{n \text{ times}}}, \end{aligned}$$

where

$$\psi_1(z) = \frac{z + (\mu w^\lambda - \alpha)z^2}{(1-z)^2}, \quad \psi_2(z) = \frac{(\mu w^\lambda - \alpha)z^2 + (2\alpha - 2\mu w^\lambda - 1)z}{(1-z)^2}.$$

With a view to recalling the principle of subordination between analytic functions, let the functions f and g be analytic in U . Given functions $f, g \in A$, f is subordinate to g if there exists a complex-valued function ϕ which maps U into itself with $\phi(0) = 0$ such that $f(z) = g(\phi(z))$, $z \in U$. We denote this subordination by $f(z) \prec g(z)$, $z \in U$. In particular, if the function g is univalent in U , the above subordination is equivalent to $f(0) = g(0)$, $f(U) \subset g(U)$.

Denote by

$$SH_{\alpha,\mu,w}^{n,\lambda}(A, B) \quad \left(\mu, \lambda, w \geq 0, 0 \leq \alpha \leq \mu w^\lambda, -B \leq A < B \leq 1 \right)$$

the subclass of SH consisting of functions f of the form (1.1) that satisfy the condition

$$\frac{D_{\alpha,\mu}^{n+1}(\lambda, w)f(z)}{D_{\alpha,\mu}^n(\lambda, w)f(z)} \prec \frac{1 + Az}{1 + Bz}, \quad (1.3)$$

where $D_{\alpha,\mu}^n(\lambda, w)f(z)$ is defined by (1.2).

By suitably specializing the parameters, the class $SH_{\alpha,\mu,w}^{n,\lambda}(A, B)$ reduces to the various subclasses of harmonic univalent functions. Such as,

- $SH_{\alpha,\mu,w}^{n,\lambda}(2\beta - 1, 1) = SH(\lambda, w, n, \alpha, \beta)$ (Altinkaya and Yalçın [1]),
- $SH_{1,\mu,1}^{n,1}(2\beta - 1, 1) = SH(\mu, n, \beta)$; $0 \leq \beta < 1$, $\mu \geq 1$ (Yaşar and Yalçın [18]),
- $SH_{0,0,w}^{n,\lambda}(2\beta - 1, 1) = H(n, \beta)$; $0 \leq \beta < 1$, (Jahangiri et al. [12]),
- $SH_{0,0,w}^{0,\lambda}(-1, 1) = SH^*(0)$ (Avcı and Zlotkiewicz [3], Silverman [15], Silverman and Silvia [16]),
- $SH_{0,0,w}^{0,\lambda}(2\beta - 1, 1) = SH^*(\beta)$; $0 \leq \beta < 1$, (Jahangiri [11]),
- $SH_{0,0,w}^{1,\lambda}(-1, 1) = KH(0)$ (Avcı and Zlotkiewicz [3], Silverman [15], Silverman and Silvia [16]),
- $SH_{0,0,w}^{1,\lambda}(2\beta - 1, 1) = KH(\beta)$; $0 \leq \beta < 1$, (Jahangiri [11]),
- $SH_{0,0,w}^{n,\lambda}(A, B) = H_n(A, B)$ (Dziok et al. [10]),
- $SH_{0,0,w}^{0,\lambda}(A, B) = SH^*(A, B)$ (Dziok [9]),
- $SH_{0,\mu,1}^{n,1}(A, B) = SH(\mu, n, A, B)$ (Çakmak et al. [5]).

Making use of the techniques and methodology used by Dziok [9], Dziok et al. [10], in this paper we find necessary and sufficient conditions, distortion bounds, compactness and extreme points for the above defined class $SH_{\alpha,\mu,w}^{n,\lambda}(A, B)$.

2. Main Results

First, we provide a necessary and sufficient convolution condition for the harmonic functions in $SH_{\alpha,\mu,w}^{n,\lambda}(A, B)$.

Theorem 2.1 For $z \in U \setminus \{0\}$, let $f \in SH$. Then $f \in SH_{\alpha,\mu,w}^{n,\lambda}(A, B)$ if and only if

$$D_{\alpha,\mu}^n(\lambda, w)f(z) * \varphi(z; \zeta) \neq 0 \quad (\zeta \in \mathbb{C}, |\zeta| = 1),$$

where

$$\varphi(z; \zeta) = z \frac{(B-A)\zeta + [1 + \mu w^\lambda - \alpha + (A+B(\mu w^\lambda - \alpha))\zeta]z}{(1-z)^2} + \bar{z} \frac{2(\alpha - \mu w^\lambda - 1) + [2B(\alpha - \mu w^\lambda) - (B+A)]\zeta + [\mu w^\lambda - \alpha + 1 + (A+B(\mu w^\lambda - 1))\zeta]\bar{z}}{(1-\bar{z})^2}.$$

Proof. Let $f \in SH$. Then $f \in SH_{\alpha,\mu,w}^{n,\lambda}(A, B)$ if and only if the condition (1.3) holds or equivalently

$$\frac{D_{\alpha,\mu}^{n+1}(\lambda, w)f(z)}{D_{\alpha,\mu}^n(\lambda, w)f(z)} \neq \frac{1+A\zeta}{1+B\bar{\zeta}} \quad (\zeta \in \mathbb{C}, |\zeta| = 1). \quad (2.1)$$

Now for

$$D_{\alpha,\mu}^n(\lambda, w)f(z) = D_{\alpha,\mu}^n(\lambda, w)f(z) * \left(\frac{z}{1-z} + \frac{\bar{z}}{1-\bar{z}} \right)$$

and

$$D_{\alpha,\mu}^{n+1}(\lambda, w)f(z) = D_{\alpha,\mu}^n(\lambda, w)f(z) * \left(\psi_1(z) + \overline{\psi_2(z)} \right),$$

the inequality (2.1) yields

$$\begin{aligned} & (1+B\zeta)D_{\alpha,\mu}^{n+1}(\lambda, w)f(z) - (1+A\zeta)D_{\alpha,\mu}^n(\lambda, w)f(z) \\ &= D_{\alpha,\mu}^n(\lambda, w)f(z) * \left\{ \frac{(1+B\zeta)[z+(\mu w^\lambda - \alpha)z^2]}{(1-z)^2} + \frac{(1+B\zeta)[(2\alpha - 2\mu w^\lambda - 1)\bar{z} + (\mu w^\lambda - \alpha)\bar{z}^2]}{(1-\bar{z})^2} \right\} \\ & - D_{\alpha,\mu}^n(\lambda, w)f(z) * \left\{ \frac{(1+A\zeta)z}{1-z} + \frac{(1+A\zeta)\bar{z}}{1-\bar{z}} \right\} \\ &= D_{\alpha,\mu}^n(\lambda, w)f(z) * \left\{ \frac{(B-A)\zeta z + [1 + \mu w^\lambda - \alpha + (A+B(\mu w^\lambda - \alpha))\zeta]z^2}{(1-z)^2} \right. \\ & \left. + \frac{[2(\alpha - \mu w^\lambda - 1) + (2B(\alpha - \mu w^\lambda) - (B+A))\zeta]\bar{z} + [\mu w^\lambda - \alpha + 1 + (A+B(\mu w^\lambda - \alpha))\zeta]\bar{z}^2}{(1-\bar{z})^2} \right\} \\ &= D_{\alpha,\mu}^n(\lambda, w)f(z) * \varphi(z; \zeta) \neq 0. \end{aligned}$$

□

A sufficient coefficient for the functions in $SH_{\alpha,\mu,w}^{n,\lambda}(A,B)$ is provided in the following.

Theorem 2.2 Let $f = h + \bar{g}$ be so that h and g are given by (1.1). Then $f \in SH_{\alpha,\mu,w}^{n,\lambda}(A,B)$, if

$$\sum_{k=2}^{\infty} D_k |a_k| + \sum_{k=1}^{\infty} E_k |b_k| \leq B - A, \quad (2.2)$$

where

$$D_k = [(k-1)(\mu w^\lambda - \alpha) + k]^n \{ (k-1)(\mu w^\lambda - \alpha)(1+B) + k(1+B) - (1+A) \} \quad (2.3)$$

and

$$E_k = [(k+1)(\mu w^\lambda - \alpha) + k]^n \{ (k+1)(\mu w^\lambda - \alpha)(1+B) + k(1+B) + (1+A) \}. \quad (2.4)$$

Proof. It is easy to see that the theorem is true for $f(z) = z$. So, we assume that $a_k \neq 0$ or $b_k \neq 0$ for $k \geq 2$. Since $D_k \geq k(B-A)$ and $E_k \geq k(B-A)$ by (2.2), we obtain

$$\begin{aligned} |h'(z)| - |g'(z)| &\geq 1 - \sum_{k=2}^{\infty} k |a_k| |z|^{k-1} - \sum_{k=1}^{\infty} k |b_k| |z|^{k-1} \\ &\geq 1 - |z| \left(\sum_{k=2}^{\infty} k |a_k| + \sum_{k=1}^{\infty} k |b_k| \right) \\ &\geq 1 - \frac{|z|}{B-A} \left(\sum_{k=2}^{\infty} D_k |a_k| + \sum_{k=1}^{\infty} E_k |b_k| \right) \\ &\geq 1 - |z| > 0. \end{aligned}$$

Therefore f is sense preserving and locally univalent in U . For the univalence condition, consider $z_1, z_2 \in U$ so that $z_1 \neq z_2$. Then

$$\left| \frac{z_1^k - z_2^k}{z_1 - z_2} \right| = \left| \sum_{m=1}^k z_1^{m-1} z_2^{k-m} \right| \leq \sum_{m=1}^k |z_1^{m-1}| |z_2^{k-m}| < k, \quad k \geq 2.$$

Hence

$$\begin{aligned} \left| \frac{f(z_1) - f(z_2)}{h(z_1) - h(z_2)} \right| &\geq 1 - \left| \frac{g(z_1) - g(z_2)}{h(z_1) - h(z_2)} \right| = 1 - \left| \frac{\sum_{k=1}^{\infty} b_k (z_1^k - z_2^k)}{(z_1 - z_2) + \sum_{k=2}^{\infty} a_k (z_1^k - z_2^k)} \right| \\ &> 1 - \frac{\sum_{k=1}^{\infty} k |b_k|}{1 - \sum_{k=2}^{\infty} k |a_k|} \geq 1 - \frac{\sum_{k=1}^{\infty} \frac{E_k}{B-A} |b_k|}{1 - \sum_{k=2}^{\infty} \frac{D_k}{B-A} |a_k|} \geq 0, \end{aligned}$$

which proves univalence.

On the other hand, $f \in SH_{\alpha,\mu,w}^{n,\lambda}(A,B)$ if and only if there exists a complex valued function ϕ ; $\phi(0) = 0$, $|\phi(z)| < 1$ ($z \in U$) such that

$$\frac{D_{\alpha,\mu}^{n+1}(\lambda, w)f(z)}{D_{\alpha,\mu}^n(\lambda, w)f(z)} = \frac{1 + A\phi(z)}{1 + B\phi(z)}$$

or equivalently

$$\left| \frac{D_{\alpha,\mu}^{n+1}(\lambda, w)f(z) - D_{\alpha,\mu}^n(\lambda, w)f(z)}{BD_{\alpha,\mu}^{n+1}(\lambda, w)f(z) - AD_{\alpha,\mu}^n(\lambda, w)f(z)} \right| < 1, \quad (z \in U). \quad (2.5)$$

The above inequality (2.5) holds, since for $|z| = r$ ($0 < r < 1$) we obtain

$$\begin{aligned}
 & \left| D_{\alpha,\mu}^{n+1}(\lambda, w)f(z) - D_{\alpha,\mu}^n(\lambda, w)f(z) \right| - \left| BD_{\alpha,\mu}^{n+1}(\lambda, w)f(z) - AD_{\alpha,\mu}^n(\lambda, w)f(z) \right| \\
 &= \left| \sum_{k=2}^{\infty} [(k-1)(\mu w^\lambda - \alpha) + k]^n (k-1)(\mu w^\lambda - \alpha + 1) a_k z^k \right. \\
 & \quad \left. + (-1)^{n+1} \sum_{k=1}^{\infty} [(k+1)(\mu w^\lambda - \alpha) + k]^n (k+1)(\mu w^\lambda - \alpha + 1) \overline{b_k z^k} \right| \\
 & \quad - \left| (B-A)z + \sum_{k=2}^{\infty} [(k-1)(\mu w^\lambda - \alpha) + k]^n [B(k-1)(\mu w^\lambda - \alpha) + Bk - A] a_k z^k \right. \\
 & \quad \left. + (-1)^{n+1} \sum_{k=1}^{\infty} [(k+1)(\mu w^\lambda - \alpha) + k]^n [B(k+1)(\mu w^\lambda - \alpha) + Bk + A] \overline{b_k z^k} \right| \\
 & \leq \sum_{k=2}^{\infty} [(k-1)(\mu w^\lambda - \alpha) + k]^n \{ (k-1)(\mu w^\lambda - \alpha)(1+B) + k(1+B) - (1+A) \} |a_k| |z|^k \\
 & \quad + \sum_{k=1}^{\infty} [(k+1)(\mu w^\lambda - \alpha) + k]^n \{ (k+1)(\mu w^\lambda - \alpha)(1+B) + k(1+B) + (1+A) \} |b_k| |z|^k \\
 & \quad - (B-A) |z| \\
 & < |z| \left\{ \sum_{k=2}^{\infty} D_k |a_k| + \sum_{k=1}^{\infty} E_k |b_k| - (B-A) \right\} \leq 0,
 \end{aligned}$$

therefore $f \in SH_{\alpha,\mu,w}^{n,\lambda}(A, B)$, and so the proof is completed. \square

Next we show that the condition (2.2) is also necessary for the functions $f \in SH$ to be in the class $SHT_{\alpha,\mu,w}^{n,\lambda}(A, B) = T^n \cap SH_{\alpha,\mu,w}^{n,\lambda}(A, B)$ where T^n is the class of functions $f = h + \bar{g} \in SH$ so that

$$f = h + \bar{g} = z - \sum_{k=2}^{\infty} |a_k| z^k + (-1)^n \sum_{k=1}^{\infty} |b_k| \bar{z}^k \quad (z \in U). \quad (2.6)$$

Theorem 2.3 Let $f = h + \bar{g}$ be defined by (2.6). Then $f \in SHT_{\alpha,\mu,w}^{n,\lambda}(A, B)$ if and only if the condition (2.2) holds.

Proof. The ‘if’ part follows from Theorem 2.2. For the ‘only-if’ part, assume that $f \in SHT_{\alpha,\mu,w}^{n,\lambda}(A, B)$, then by (2.5) we have

$$\left| \frac{\sum_{k=2}^{\infty} [(k-1)(\mu w^\lambda - \alpha) + k]^n (k-1)(\mu w^\lambda - \alpha + 1) |a_k| z^k + \sum_{k=1}^{\infty} [(k+1)(\mu w^\lambda - \alpha) + k]^n (k+1)(\mu w^\lambda - \alpha + 1) |b_k| \bar{z}^k}{(B-A)z - \sum_{k=2}^{\infty} [(k-1)(\mu w^\lambda - \alpha) + k]^n [B(k-1)(\mu w^\lambda - \alpha) + Bk - A] |a_k| z^k - \sum_{k=1}^{\infty} [(k+1)(\mu w^\lambda - \alpha) + k]^n [B(k+1)(\mu w^\lambda - \alpha) + Bk + A] |b_k| \bar{z}^k} \right| < 1.$$

For $z = r < 1$ we obtain

$$\frac{\left\{ \sum_{k=2}^{\infty} [(k-1)(\mu w^\lambda - \alpha) + k]^n (k-1)(\mu w^\lambda - \alpha + 1) |a_k| + \sum_{k=1}^{\infty} [(k+1)(\mu w^\lambda - \alpha) + k]^n (k+1)(\mu w^\lambda - \alpha + 1) |b_k| \right\} r^{k-1}}{B-A - \left\{ \sum_{k=2}^{\infty} [(k-1)(\mu w^\lambda - \alpha) + k]^n [B(k-1)(\mu w^\lambda - \alpha) + Bk - A] |a_k| + \sum_{k=1}^{\infty} [(k+1)(\mu w^\lambda - \alpha) + k]^n [B(k+1)(\mu w^\lambda - \alpha) + Bk + A] |b_k| \right\} r^{k-1}} < 1.$$

Thus, for D_k and E_k as defined by (2.3) and (2.4), we have

$$\sum_{k=2}^{\infty} D_k |a_k| r^{k-1} + \sum_{k=1}^{\infty} E_k |b_k| r^{k-1} < B - A \quad (0 \leq r < 1). \quad (2.7)$$

Let $\{\sigma_k\}$ be the sequence of partial sums of the series

$$\sum_{k=2}^{\infty} D_k |a_k| + \sum_{k=1}^{\infty} E_k |b_k|.$$

Then $\{\sigma_k\}$ is a nondecreasing sequence and by (2.7) it is bounded above by $B - A$. Thus, it is convergent and

$$\sum_{k=2}^{\infty} D_k |a_k| + \sum_{k=1}^{\infty} E_k |b_k| = \lim_{k \rightarrow \infty} \sigma_k \leq B - A.$$

This gives the condition (2.2). \square

In the following we show that the class of functions of the form (2.6) is convex and compact.

Theorem 2.4 *The class $SHT_{\alpha,\mu,w}^{n,\lambda}(A,B)$ is a convex and compact subset of SH .*

Proof. Let $f_t \in SHT_{\alpha,\mu,w}^{n,\lambda}(A,B)$, where

$$f_t(z) = z - \sum_{k=2}^{\infty} |a_{t,k}| z^k + (-1)^n \sum_{k=1}^{\infty} |b_{t,k}| \bar{z}^k \quad (z \in U, t \in \mathbb{N}). \quad (2.8)$$

Then $0 \leq \eta \leq 1$, let $f_1, f_2 \in SHT_{\alpha,\mu,w}^{n,\lambda}(A,B)$ be defined by (2.8). Then

$$\begin{aligned} \kappa(z) &= \eta f_1(z) + (1-\eta) f_2(z) \\ &= z - \sum_{k=2}^{\infty} (\eta |a_{1,k}| + (1-\eta) |a_{2,k}|) z^k \\ &\quad + (-1)^n \sum_{k=1}^{\infty} (\eta |b_{1,k}| + (1-\eta) |b_{2,k}|) \bar{z}^k \end{aligned}$$

and

$$\begin{aligned} &\sum_{k=2}^{\infty} D_k [\eta |a_{1,k}| + (1-\eta) |a_{2,k}|] + \sum_{k=1}^{\infty} E_k [\eta |b_{1,k}| + (1-\eta) |b_{2,k}|] \\ &= \eta \left\{ \sum_{k=2}^{\infty} D_k |a_{1,k}| + \sum_{k=1}^{\infty} E_k |b_{1,k}| \right\} + (1-\eta) \left\{ \sum_{k=2}^{\infty} D_k |a_{2,k}| + \sum_{k=1}^{\infty} E_k |b_{2,k}| \right\} \\ &\leq \eta(B-A) + (1-\eta)(B-A) = B-A. \end{aligned}$$

Thus, the function $\kappa = \eta f_1 + (1-\eta) f_2$ belongs to the class $SHT_{\alpha,\mu,w}^{n,\lambda}(A,B)$. This means that the class $SHT_{\alpha,\mu,w}^{n,\lambda}(A,B)$ is convex.

However, for $f_t \in SHT_{\alpha,\mu,w}^{n,\lambda}(A,B)$, $t \in \mathbb{N}$ and $|z| \leq r$ ($0 < r < 1$), we get

$$\begin{aligned} |f_t(z)| &\leq r + \sum_{k=2}^{\infty} |a_{t,k}| r^k + \sum_{k=1}^{\infty} |b_{t,k}| r^k \\ &\leq r + \sum_{k=2}^{\infty} D_k |a_{t,k}| r^k + \sum_{k=1}^{\infty} E_k |b_{t,k}| r^k \\ &\leq r + (B-A)r^2. \end{aligned}$$

Therefore, $SHT_{\alpha,\mu,w}^{n,\lambda}(A,B)$ is locally uniformly bounded. Let

$$f_t(z) = z - \sum_{k=2}^{\infty} |a_{t,k}| z^k + (-1)^n \sum_{k=1}^{\infty} |b_{t,k}| \bar{z}^k \quad (z \in U, t \in \mathbb{N})$$

and let $f = h + \bar{g}$ be so that h and g are given by (1.1). Using Theorem 2.3 we obtain

$$\sum_{k=2}^{\infty} D_k |a_{t,k}| + \sum_{k=1}^{\infty} E_k |b_{t,k}| \leq (B-A). \quad (2.9)$$

If we assume that $f_t \rightarrow f$, then we conclude that $|a_{t,k}| \rightarrow |a_k|$ and $|b_{t,k}| \rightarrow |b_k|$ as $k \rightarrow \infty$ ($t \in \mathbb{N}$). Let $\{\sigma_k\}$ be the sequence of partial sums of the series $\sum_{k=2}^{\infty} D_k |a_{t,k}| + \sum_{k=1}^{\infty} E_k |b_{t,k}|$. Then $\{\sigma_k\}$ is a nondecreasing sequence and by (2.9) it is bounded above by $B-A$. Thus, it is convergent and

$$\sum_{k=2}^{\infty} D_k |a_{t,k}| + \sum_{k=1}^{\infty} E_k |b_{t,k}| = \lim_{k \rightarrow \infty} \sigma_k \leq B-A.$$

Therefore $f \in SHT_{\alpha,\mu,w}^{n,\lambda}(A,B)$ and therefore the class $SHT_{\alpha,\mu,w}^{n,\lambda}(A,B)$ is closed. In consequence, the class $SHT_{\alpha,\mu,w}^{n,\lambda}(A,B)$ is compact subset of SH , which completes the proof. \square

We continue with the following theorem.

Theorem 2.5 Extreme points of the class $SHT_{\alpha,\mu,w}^{n,\lambda}(A,B)$ are the functions f of the form (1.1) where $h = h_k$ and $g = g_k$ are of the form

$$h_1(z) = z, \quad h_k(z) = z - \frac{B-A}{D_k} z^k, \quad (2.10)$$

$$g_k(z) = (-1)^n \frac{B-A}{E_k} \bar{z}^k \quad (z \in U, k \geq 2).$$

Proof. Let $g_k = \eta f_1 + (1 - \eta) f_2$ where $0 < \eta < 1$ and $f_1, f_2 \in SHT_{\alpha,\mu,w}^{n,\lambda}(A,B)$ are functions of the form

$$f_t(z) = z - \sum_{k=2}^{\infty} |a_{t,k}| z^k + (-1)^n \sum_{k=2}^{\infty} |b_{t,k}| \bar{z}^k \quad (z \in U, t \in \{1, 2\}).$$

Then, by (2.10), we have

$$|b_{1,k}| = |b_{2,k}| = \frac{B-A}{E_k}$$

and therefore $a_{1,t} = a_{2,t} = 0$ for $t \in \{2, 3, \dots\}$ and $b_{1,t} = b_{2,t} = 0$ for $t \in \{2, 3, \dots\} \setminus \{k\}$. It follows that $g_k(z) = f_1(z) = f_2(z)$ and g_k are in the class of extreme points of the function class $SHT_{\alpha,\mu,w}^{n,\lambda}(A,B)$. Similarly, we can verify that the functions $h_k(z)$ are the extreme points of the class $SHT_{\alpha,\mu,w}^{n,\lambda}(A,B)$. Now, suppose that a function f of the form (1.1) is in the family of extreme points of the class $SHT_{\alpha,\mu,w}^{n,\lambda}(A,B)$ and f is not of the form (2.10). Then there exists $m \in \{2, 3, \dots\}$ such that

$$0 < |a_m| < \frac{B-A}{[(m-1)(\mu w^\lambda - \alpha) + m]^n \{(m-1)(\mu w^\lambda - \alpha)(1+B) + m(1+B) - (1+A)\}}$$

or

$$0 < |b_m| < \frac{B-A}{[(m+1)(\mu w^\lambda - \alpha) + m]^n \{(m+1)(\mu w^\lambda - \alpha)(1+B) + m(1+B) + (1+A)\}}.$$

If

$$0 < |a_m| < \frac{B-A}{[(m-1)(\mu w^\lambda - \alpha) + m]^n \{(m-1)(\mu w^\lambda - \alpha)(1+B) + m(1+B) - (1+A)\}},$$

then putting

$$\eta = \frac{|a_m|[(m-1)(\mu w^\lambda - \alpha) + m]^n \{(m-1)(\mu w^\lambda - \alpha)(1+B) + m(1+B) - (1+A)\}}{B-A}$$

and

$$\Phi = \frac{f - \eta h_m}{1 - \eta},$$

we have $0 < \eta < 1$, $h_m \neq \Phi$.

Therefore, f is not in the family of extreme points of the class $SHT_{\alpha,\mu,w}^{n,\lambda}(A,B)$. Similarly, if

$$0 < |b_m| < \frac{B-A}{[(m+1)(\mu w^\lambda - \alpha) + m]^n \{(m+1)(\mu w^\lambda - \alpha)(1+B) + m(1+B) + (1+A)\}},$$

then putting

$$\eta = \frac{|b_m|[(m+1)(\mu w^\lambda - \alpha) + m]^n \{(m+1)(\mu w^\lambda - \alpha)(1+B) + m(1+B) + (1+A)\}}{B-A}$$

and

$$\Phi = \frac{f - \eta g_m}{1 - \eta},$$

we have $0 < \eta < 1$, $g_m \neq \Phi$.

It follows that f is not in the family of extreme points of the class $SHT_{\alpha,\mu,w}^{n,\lambda}(A,B)$ and so the proof is completed. \square

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Results on A Class of Harmonic Univalent Functions Involving A New Differential Operator

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Abstract:

In this paper, a new class of complex-valued harmonic univalent functions defined by using a new differential operator is introduced. We investigate coefficient bounds, distortion inequalities, extreme points and inclusion results for this class.

1. Introduction and Preliminaries

Harmonic functions are famous for their use in the study of minimal surfaces and also play important roles in a variety of problems in applied mathematics (e.g. see Choquet [4], Dorff [5], Duren [6]). A continuous function $f = u + iv$ is a complex valued harmonic function in a complex domain \mathbb{C} if both u and v are real harmonic in \mathbb{C} . In any simply connected domain $D \subset \mathbb{C}$ we can write $f = h + \bar{g}$, where h and g are analytic in D . We call h the analytic part and g the co-analytic part of f . A necessary and sufficient condition for f to be locally univalent and sense-preserving in D is that $|h'(z)| > |g'(z)|$ in D (see [3]).

Denote by SH the class of functions $f = h + \bar{g}$ that are harmonic univalent and sense-preserving in the unit disk

$$U = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$$

for which $f(0) = f_z(0) - 1 = 0$. Then for $f = h + \bar{g} \in SH$, we may express the analytic functions h and g as

$$h(z) = z + \sum_{k=2}^{\infty} a_k z^k, \quad g(z) = \sum_{k=1}^{\infty} b_k z^k. \quad (1.1)$$

Therefore

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k + \overline{\sum_{k=1}^{\infty} b_k z^k}, \quad |b_1| < 1.$$

Note that SH reduces to the class S of normalized analytic univalent functions in U if the co-analytic part of f is identically zero.

In 1984 Clunie and Sheil-Small [3] investigated the class SH as well as its geometric subclasses and obtained some coefficient bounds. Since then, there has been several related papers on SH and its subclasses such as Avcı and Zlotkiewicz [1], Silverman [10], Silverman and Silvia [11], Jahangiri [7] studied the harmonic univalent functions.

The differential operator $D_{\alpha, \mu}^n(\lambda, w)$ ($n \in \mathbb{N}_0$) was introduced by Bucur et al. [2]. For $f = h + \bar{g}$ given by (1.1), we define the following differential operator:

$$D_{\alpha, \mu}^n(\lambda, w)f(z) = D_{\alpha, \mu}^n(\lambda, w)h(z) + (-1)^n \overline{D_{\alpha, \mu}^n(\lambda, w)g(z)},$$

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where

$$D_{\alpha,\mu}^n(\lambda,w)h(z) = z + \sum_{k=2}^{\infty} [(k-1)(\mu w^\lambda - \alpha) + k]^n a_k z^k$$

and

$$D_{\alpha,\mu}^n(\lambda,w)g(z) = \sum_{k=1}^{\infty} [(k+1)(\mu w^\lambda - \alpha) + k]^n b_k z^k,$$

where $\mu, \lambda, w \geq 0, 0 \leq \alpha \leq \mu w^\lambda$, with $D_{\alpha,\mu}^n(\lambda,w)f(0) = 0$.

Motivated by the differential operator $D_{\alpha,\mu}^n(\lambda,w)$, we define generalization of the differential operator for a function $f = h + \bar{g}$ given by (1.1).

$$\begin{aligned} D_{\alpha,\mu}^0(\lambda,w)f(z) &= D^0 f(z) = h(z) + \overline{g(z)}, \\ D_{\alpha,\mu}^1(\lambda,w)f(z) &= (\alpha - \mu w^\lambda)(h(z) + \overline{g(z)}) + (\mu w^\lambda - \alpha + 1)(zh'(z) - \overline{zg'(z)}), \\ &\vdots \\ D_{\alpha,\mu}^n(\lambda,w)f(z) &= D(D_{\alpha,\mu}^{n-1}(\lambda,w)f(z)). \end{aligned} \quad (1.2)$$

If f is given by (1.1), then from (1.2), we see that

$$D_{\alpha,\mu}^n(\lambda,w)f(z) = z + \sum_{k=2}^{\infty} [(k-1)(\mu w^\lambda - \alpha) + k]^n a_k z^k + (-1)^n \sum_{k=1}^{\infty} [(k+1)(\mu w^\lambda - \alpha) + k]^n \overline{b_k z^k}. \quad (1.3)$$

When $w = \alpha = 0$, we get modified Salagean differential operator [9].

Denote by $SH(\lambda, w, n, \alpha, \beta)$ the subclass of SH consisting of functions f of the form (1.1) that satisfy the condition

$$\Re \left(\frac{D_{\alpha,\mu}^{n+1}(\lambda,w)f(z)}{D_{\alpha,\mu}^n(\lambda,w)f(z)} \right) \geq \beta \quad (0 \leq \beta < 1), \quad (1.4)$$

where $D_{\alpha,\mu}^n(\lambda,w)f(z)$ is defined by (1.3).

We let the subclass $\overline{SH}(\lambda, w, n, \alpha, \beta)$ consisting of harmonic functions $f_n = h + \bar{g}_n$ in SH so that h and g_n are of the form

$$h(z) = z - \sum_{k=2}^{\infty} a_k z^k, \quad g_n(z) = (-1)^n \sum_{k=1}^{\infty} b_k z^k, \quad a_k, b_k \geq 0. \quad (1.5)$$

By suitably specializing the parameters, the classes $SH(\lambda, w, n, \alpha, \beta)$ reduces to the various subclasses of harmonic univalent functions. Such as,

- (i) $SH(0, 0, 0, 0, 0) = SH^*(0)$ (Avcı [1], Silverman [10], Silverman and Silvia [11]),
- (ii) $SH(0, 0, 0, 0, \beta) = SH^*(\beta)$ (Jahangiri [7]),
 $SH(0, 0, 0, 0, \beta) = \overline{SH}(1, 0, \beta)$ (Yalçın [12]),
- (iii) $SH(0, 0, 1, 0, 0) = KH(0)$ (Avcı [1], Silverman [10], Silverman and Silvia [11]),
- (iv) $SH(0, 0, 1, 0, \beta) = KH(\beta)$ (Jahangiri [7]),
 $SH(0, 0, 1, 0, \beta) = \overline{SH}(2, 1, \beta)$ (Yalçın [12]),
- (v) $SH(0, 0, n, 0, \beta) = H(n, \beta)$ (Jahangiri et al. [8]),
 $SH(0, 0, n, 0, \beta) = \overline{SH}(n+1, n, \beta)$ (Yalçın [12]),

The object of the present paper is to give sufficient condition for functions $f = h + \bar{g}$ where h and g are given by (1.1) to be in the class $SH(\lambda, w, n, \alpha, \beta)$; and it is shown that this coefficient condition is also necessary for functions belonging to the subclass $\overline{SH}(\lambda, w, n, \alpha, \beta)$. Also, we obtain coefficient bounds, distortion inequalities, extreme points and inclusion results for this class.

2. Coefficient Bounds

Theorem 2.1 Let $f = h + \bar{g}$ be so that h and g are given by (1.1). Furthermore, let

$$\sum_{k=2}^{\infty} (k - \beta) [(k-1)(\mu w^\lambda - \alpha) + k]^n |a_k| + \sum_{k=1}^{\infty} (k + \beta) [(k+1)(\mu w^\lambda - \alpha) + k]^n |b_k| \leq 1 - \beta, \quad (2.1)$$

where $\mu, \lambda, w \geq 0, 0 \leq \alpha \leq \mu w^\lambda, n \in \mathbb{N}_0, 0 \leq \beta < 1$. Then f is sense-preserving, harmonic univalent in U and $f \in SH(\lambda, w, n, \alpha, \beta)$.

Proof. If $z_1 \neq z_2$,

$$\begin{aligned} \left| \frac{f(z_1) - f(z_2)}{h(z_1) - h(z_2)} \right| &\geq 1 - \left| \frac{g(z_1) - g(z_2)}{h(z_1) - h(z_2)} \right| = 1 - \left| \frac{\sum_{k=1}^{\infty} b_k (z_1^k - z_2^k)}{(z_1 - z_2) + \sum_{k=2}^{\infty} a_k (z_1^k - z_2^k)} \right| \\ &> 1 - \frac{\sum_{k=1}^{\infty} k |b_k|}{1 - \sum_{k=2}^{\infty} k |a_k|} \\ &\geq 1 - \frac{\sum_{k=1}^{\infty} \frac{(k+\beta)[(k+1)(\mu w^\lambda - \alpha) + k]^n}{1-\beta} |b_k|}{1 - \sum_{k=2}^{\infty} \frac{(k-\beta)[(k-1)(\mu w^\lambda - \alpha) + k]^n}{1-\beta} |a_k|} \\ &\geq 0, \end{aligned}$$

which proves univalence. Note that f is sense preserving in U . This is because

$$\begin{aligned} |h'(z)| &\geq 1 - \sum_{k=2}^{\infty} k |a_k| |z|^{k-1} > 1 - \sum_{k=2}^{\infty} \frac{(k-\beta)[(k-1)(\mu w^\lambda - \alpha) + k]^n}{1-\beta} |a_k| \\ &\geq \sum_{k=1}^{\infty} \frac{(k+\beta)[(k+1)(\mu w^\lambda - \alpha) + k]^n}{1-\beta} |b_k| > \sum_{k=1}^{\infty} k |b_k| |z|^{k-1} \\ &\geq |g'(z)|. \end{aligned}$$

Using the fact that $\Re(w) \geq \beta$ if and only if $|1 - \beta + w| \geq |1 + \beta - w|$, it suffices to show that

$$\left| (1-\beta)D_{\alpha,\mu}^n(\lambda, w) + D_{\alpha,\mu}^{n+1}(\lambda, w)f(z) \right| - \left| (1+\beta)D_{\alpha,\mu}^n(\lambda, w) - D_{\alpha,\mu}^{n+1}(\lambda, w) \right| \geq 0. \quad (2.2)$$

Substituting for $D_{\alpha,\mu}^{n+1}(\lambda, w)f(z)$ and $D_{\alpha,\mu}^n(\lambda, w)f(z)$ in (2.2), we obtain

$$\begin{aligned} &\left| (1-\beta)D_{\alpha,\mu}^n(\lambda, w) + D_{\alpha,\mu}^{n+1}(\lambda, w)f(z) \right| - \left| (1+\beta)D_{\alpha,\mu}^n(\lambda, w)f(z) - D_{\alpha,\mu}^{n+1}(\lambda, w)f(z) \right| \\ &\geq 2(1-\beta)|z| - \sum_{k=2}^{\infty} \left[(k+1-\beta) + (k-1)(\mu w^\lambda - \alpha) \right] \left[(k-1)(\mu w^\lambda - \alpha) + k \right]^n |a_k| |z|^k \\ &\quad - \sum_{k=1}^{\infty} \left[(k-1+\beta) + (k-1)(\mu w^\lambda - \alpha) \right] \left[(k+1)(\mu w^\lambda - \alpha) + k \right]^n |b_k| |z|^k \\ &\quad - \sum_{k=2}^{\infty} \left[(k-1-\beta) + (k-1)(\mu w^\lambda - \alpha) \right] \left[(k-1)(\mu w^\lambda - \alpha) + k \right]^n |a_k| |z|^k \\ &\quad - \sum_{k=1}^{\infty} \left[(k+1+\beta) + (k-1)(\mu w^\lambda - \alpha) \right] \left[(k+1)(\mu w^\lambda - \alpha) + k \right]^n |b_k| |z|^k \\ &\geq 2(1-\beta)|z| \left(1 - \sum_{k=2}^{\infty} \frac{(k-\beta)[(k-1)(\mu w^\lambda - \alpha) + k]^n}{1-\beta} |a_k| \right. \\ &\quad \left. - \sum_{k=1}^{\infty} \frac{(k+\beta)[(k+1)(\mu w^\lambda - \alpha) + k]^n}{1-\beta} |b_k| \right). \end{aligned}$$

This last expression is non-negative by (2.1), and so the proof is completed. \square

Theorem 2.2 Let $f_n = h + \bar{g}_n$ be given by (1.5). Then $f_n \in \overline{SH}(\lambda, w, n, \alpha, \beta)$ if and only if

$$\sum_{k=2}^{\infty} (k-\beta) \left[(k-1)(\mu w^\lambda - \alpha) + k \right]^n |a_k| + \sum_{k=1}^{\infty} (k+\beta) \left[(k+1)(\mu w^\lambda - \alpha) + k \right]^n |b_k| \leq 1 - \beta, \quad (2.3)$$

where $\mu, \lambda, w \geq 0, 0 \leq \alpha \leq \mu w^\lambda, n \in \mathbb{N}_0, 0 \leq \beta < 1$.

Proof. The "if" part follows from Theorem 2.1 upon noting that $\overline{SH}(\lambda, w, n, \alpha, \beta) \subset SH(\lambda, w, n, \alpha, \beta)$. For the "only if" part, we show that $f \notin \overline{SH}(\lambda, w, n, \alpha, \beta)$ if the condition (2.3) does not hold. Note that a necessary and sufficient condition for $f_n = h + \overline{g}_n$ given by (1.5), to be in $\overline{SH}(\lambda, w, n, \alpha, \beta)$ is that the condition (1.4) to be satisfied. This is equivalent to

$$\Re \left\{ \frac{(1-\beta)z - \sum_{k=2}^{\infty} (k-\beta) [(k-1)(\mu w^\lambda - \alpha) + k]^n a_k z^k}{z - \sum_{k=2}^{\infty} [(k-1)(\mu w^\lambda - \alpha) + k]^n a_k z^k + \sum_{k=1}^{\infty} [(k+1)(\mu w^\lambda - \alpha) + k]^n b_k \bar{z}^k} - \frac{-\sum_{k=1}^{\infty} (k+\beta) [(k+1)(\mu w^\lambda - \alpha) + k]^n b_k \bar{z}^k}{z - \sum_{k=2}^{\infty} [(k-1)(\mu w^\lambda - \alpha) + k]^n a_k z^k + \sum_{k=1}^{\infty} [(k+1)(\mu w^\lambda - \alpha) + k]^n b_k \bar{z}^k} \right\} \geq 0. \quad (2.4)$$

The above condition must hold for all values of z , $|z| = r < 1$. Upon choosing the values of z on the positive real axis where $0 \leq z = r < 1$ we must have

$$\frac{(1-\beta) - \sum_{k=2}^{\infty} (k-\beta) [(k-1)(\mu w^\lambda - \alpha) + k]^n a_k r^{k-1}}{1 - \sum_{k=2}^{\infty} [(k-1)(\mu w^\lambda - \alpha) + k]^n a_k r^{k-1} + \sum_{k=1}^{\infty} [(k+1)(\mu w^\lambda - \alpha) + k]^n b_k r^{k-1}} - \frac{-\sum_{k=1}^{\infty} (k+\beta) [(k+1)(\mu w^\lambda - \alpha) + k]^n b_k r^{k-1}}{1 - \sum_{k=2}^{\infty} [(k-1)(\mu w^\lambda - \alpha) + k]^n a_k r^{k-1} + \sum_{k=1}^{\infty} [(k+1)(\mu w^\lambda - \alpha) + k]^n b_k r^{k-1}} \geq 0. \quad (2.5)$$

If the condition (2.3) does not hold, then the numerator in (2.5) is negative for r sufficiently close to 1. Hence there exist $z_0 = r_0$ in $(0, 1)$ for which the quotient in (2.5) is negative. This contradicts the required condition for $f_n \in \overline{SH}(\lambda, w, n, \alpha, \beta)$ and so the proof is complete. \square

3. Distortion Inequalities and Extreme Points

Theorem 3.1 Let $f_n \in \overline{SH}(\lambda, w, n, \alpha, \beta)$. Then for $|z| = r < 1$ we have

$$|f_n(z)| \leq (1+b_1)r + \left(\frac{(1-\beta)}{(2-\beta)[\mu w^\lambda - \alpha + 2]^n} - \frac{(1+\beta)[2(\mu w^\lambda - \alpha) + 1]^n}{(2-\beta)[\mu w^\lambda - \alpha + 2]^n} b_1 \right) r^2$$

and

$$|f_n(z)| \geq (1-b_1)r - \left(\frac{(1-\beta)}{(2-\beta)[\mu w^\lambda - \alpha + 2]^n} - \frac{(1+\beta)[2(\mu w^\lambda - \alpha) + 1]^n}{(2-\beta)[\mu w^\lambda - \alpha + 2]^n} b_1 \right) r^2.$$

Proof. We only prove the right hand inequality. The proof for the left hand inequality is similar and will be

omitted. Let $f_n \in \overline{SH}(\lambda, w, n, \alpha, \beta)$. Taking the absolute value of f_n we have

$$\begin{aligned}
 |f_n(z)| &\leq (1+b_1)r + \sum_{k=2}^{\infty} (a_k + b_k)r^k \\
 &\leq (1+b_1)r + \sum_{k=2}^{\infty} (a_k + b_k)r^2 \\
 &= (1+b_1)r + \frac{(1-\beta)r^2}{(2-\beta)[\mu w^\lambda - \alpha + 2]^n} \sum_{k=2}^{\infty} \frac{(2-\beta)[\mu w^\lambda - \alpha + 2]^n}{(1-\beta)} [a_k + b_k] \\
 &\leq (1+b_1)r + \frac{(1-\beta)r^2}{(2-\beta)[\mu w^\lambda - \alpha + 2]^n} \\
 &\quad \times \sum_{k=2}^{\infty} \left(\frac{(k-\beta)[(k-1)(\mu w^\lambda - \alpha) + k]^n}{1-\beta} a_k \right. \\
 &\quad \left. + \frac{(k+\beta)[(k+1)(\mu w^\lambda - \alpha) + k]^n}{1-\beta} b_k \right) \\
 &\leq (1+b_1)r + \frac{(1-\beta)}{(2-\beta)[\mu w^\lambda - \alpha + 2]^n} \left(1 - \frac{(1+\beta)[2(\mu w^\lambda - \alpha) + 1]^n}{1-\beta} b_1 \right) r^2 \\
 &\leq (1+b_1)r + \left(\frac{(1-\beta)}{(2-\beta)[\mu w^\lambda - \alpha + 2]^n} - \frac{(1+\beta)[2(\mu w^\lambda - \alpha) + 1]^n}{(2-\beta)[\mu w^\lambda - \alpha + 2]^n} b_1 \right) r^2.
 \end{aligned}$$

The following covering result follows from the left hand inequality in Theorem 3.1. □

Corollary 3.2 Let f_n of the form (1.5) be so that $f_n \in \overline{SH}(\lambda, w, n, \alpha, \beta)$. Then

$$\left\{ w : |w| < \frac{(2-\beta)[\mu w^\lambda - \alpha + 2]^{n-1+\beta}}{(2-\beta)[\mu w^\lambda - \alpha + 2]^n} - \frac{(2-\beta)[\mu w^\lambda - \alpha + 2]^{n-(1+\beta)[2(\mu w^\lambda - \alpha) + 1]^n}}{(2-\beta)[\mu w^\lambda - \alpha + 2]^n} b_1 \right\} \subset f_n(U).$$

Theorem 3.3 Let f_n be given by (1.5). Then $f_n \in \overline{SH}(\lambda, w, n, \alpha, \beta)$ if and only if

$$f_n(z) = \sum_{k=1}^{\infty} (X_k h_k(z) + Y_k g_{n_k}(z)),$$

where

$$h_1(z) = z, \quad h_k(z) = z - \frac{1-\beta}{(k-\beta)[(k-1)(\mu w^\lambda - \alpha) + k]^n} z^k; \quad (k \geq 2),$$

$$g_{n_k}(z) = z + (-1)^n \frac{1-\beta}{(k+\beta)[(k+1)(\mu w^\lambda - \alpha) + k]^n} z^k; \quad (k \geq 1),$$

$$\sum_{k=1}^{\infty} (X_k + Y_k) = 1, X_k \geq 0, Y_k \geq 0.$$

In particular, the extreme points of $\overline{SH}(\lambda, w, n, \alpha, \beta)$ are $\{h_k\}$ and $\{g_{n_k}\}$.

Proof. For functions f_n of the form (1.5) we may write

$$\begin{aligned}
 f_n(z) &= \sum_{k=1}^{\infty} (X_k h_k(z) + Y_k g_{n_k}(z)) \\
 &= \sum_{k=1}^{\infty} (X_k + Y_k) z - \sum_{k=2}^{\infty} \frac{1-\beta}{(k-\beta)[(k-1)(\mu w^\lambda - \alpha) + k]^n} X_k z^k \\
 &\quad + (-1)^n \sum_{k=1}^{\infty} \frac{1-\beta}{(k+\beta)[(k+1)(\mu w^\lambda - \alpha) + k]^n} Y_k z^k.
 \end{aligned}$$

Then

$$\begin{aligned} & \sum_{k=2}^{\infty} \frac{(k-\beta) [(k-1)(\mu w^\lambda - \alpha) + k]^n}{1-\beta} \left(\frac{1-\beta}{(k-\beta) [(k-1)(\mu w^\lambda - \alpha) + k]^n} X_k \right) \\ & + \sum_{k=1}^{\infty} \frac{(k+\beta) [(k+1)(\mu w^\lambda - \alpha) + k]^n}{1-\beta} \left(\frac{1-\beta}{(k+\beta) [(k+1)(\mu w^\lambda - \alpha) + k]^n} Y_k \right) \\ & = \sum_{k=2}^{\infty} X_k + \sum_{k=1}^{\infty} Y_k = 1 - X_1 \leq 1, \text{ and so } f_n \in \overline{SH}(\lambda, w, n, \alpha, \beta). \end{aligned}$$

Conversely, if $f_n \in \overline{SH}(\lambda, w, n, \alpha, \beta)$, then

$$a_k \leq \frac{1-\beta}{(k-\beta) [(k-1)(\mu w^\lambda - \alpha) + k]^n}$$

and

$$b_k \leq \frac{1-\beta}{(k+\beta) [(k+1)(\mu w^\lambda - \alpha) + k]^n}.$$

Setting

$$X_k = \frac{(k-\beta) [(k-1)(\mu w^\lambda - \alpha) + k]^n}{1-\beta} a_k; \quad (k \geq 2),$$

$$Y_k = \frac{(k+\beta) [(k+1)(\mu w^\lambda - \alpha) + k]^n}{1-\beta} b_k; \quad (k \geq 1)$$

and

$$X_1 = 1 - \left(\sum_{k=2}^{\infty} X_k + \sum_{k=1}^{\infty} Y_k \right),$$

where $X_1 \geq 0$. Then

$$f_n(z) = X_1 z + \sum_{k=2}^{\infty} X_k h_k(z) + \sum_{k=1}^{\infty} Y_k g_{n_k}(z)$$

as required. \square

4. Inclusion Results

Theorem 4.1 *The class $\overline{SH}(\lambda, w, n, \alpha, \beta)$ is closed under convex combinations.*

Proof. Let $f_{n_i} \in \overline{SH}(\lambda, w, n, \alpha, \beta)$ for $i = 1, 2, \dots$, where f_{n_i} is given by

$$f_{n_i}(z) = z - \sum_{k=2}^{\infty} a_{k_i} z^k + (-1)^n \sum_{k=1}^{\infty} b_{k_i} \bar{z}^k.$$

Then by (2.3),

$$\sum_{k=2}^{\infty} \frac{(k-\beta) [(k-1)(\mu w^\lambda - \alpha) + k]^n}{1-\beta} a_{k_i} + \sum_{k=1}^{\infty} \frac{(k+\beta) [(k+1)(\mu w^\lambda - \alpha) + k]^n}{1-\beta} b_{k_i} \leq 1. \quad (4.1)$$

For $\sum_{i=1}^{\infty} t_i = 1$, $0 \leq t_i \leq 1$, the convex combination of f_{n_i} may be written as

$$\sum_{i=1}^{\infty} t_i f_{n_i}(z) = z - \sum_{k=2}^{\infty} \left(\sum_{i=1}^{\infty} t_i a_{k_i} \right) z^k + (-1)^n \sum_{k=1}^{\infty} \left(\sum_{i=1}^{\infty} t_i b_{k_i} \right) \bar{z}^k.$$

Then by (4.1),

$$\begin{aligned}
 & \sum_{k=2}^{\infty} \frac{(k-\beta) [(k-1)(\mu w^\lambda - \alpha) + k]^n}{1-\beta} \left(\sum_{i=1}^{\infty} t_i a_{k_i} \right) \\
 & + \sum_{k=1}^{\infty} \frac{(k+\beta) [(k+1)(\mu w^\lambda - \alpha) + k]^n}{1-\beta} \left(\sum_{i=1}^{\infty} t_i b_{k_i} \right) \\
 = & \sum_{i=1}^{\infty} t_i \left(\sum_{k=2}^{\infty} \frac{(k-\beta) [(k-1)(\mu w^\lambda - \alpha) + k]^n}{1-\beta} a_{k_i} \right. \\
 & \left. + \sum_{k=1}^{\infty} \frac{(k+\beta) [(k+1)(\mu w^\lambda - \alpha) + k]^n}{1-\beta} b_{k_i} \right) \\
 \leq & \sum_{i=1}^{\infty} t_i = 1.
 \end{aligned}$$

This is the condition required by (2.3) and so $\sum_{i=1}^{\infty} t_i f_{n_i}(z) \in \overline{SH}(\lambda, w, n, \alpha, \beta)$. \square

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Statistical Convergence of Minima and Minimizers of Sequences of Functions

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Abstract: In this paper, we show that, under some statistical level boundedness assumptions, statistical epi-convergence of a sequence (f_n) to a function f implies the statistical convergence of the minimum values of (f_n) to the minimum value of f . Furthermore, in case (f_n) and f have a unique minimum point, we shall prove that the sequence of the minimizers of (f_n) statistically converges to the minimizer of f .

1. Introduction

In the late of 1960's, epi-convergence is first studied by Wijsman [15, 16] where it is called infimal convergence. After Wijsman's initial contributions, it is studied by Mosco [10] on variational inequalities, by Joly [6] on topological structures compatible with epi-convergence, by Salinetti and Wets [12] on equisemicontinuous families of convex functions, by Attouch [2] on the relationship between the epi-convergence of convex functions and the graphical convergence of their subgradient mappings, and by McLinden and Bergstrom [9] on the preservation of epi-convergence under various operations performed on convex functions. Furthermore, Dal Maso [8] called it Γ -convergence. The term epi-convergence is used by Wets [14] in 1980 for the first time. Epi-convergence is needed to solve some mathematical problems including stochastic optimization, variational problems and partial differential equations.

In this part fundamental definitions and theorems will be given. First of all, let (X, d) be a metric space and $f, (f_n)$ are functions defined on X with $n \in \mathbb{N}$. If it is not mentioned explicitly the symbol d stands for the metric on X .

Let $K \subseteq \mathbb{N}$ and if the limit $\delta(K) = \lim_{n \rightarrow \infty} \frac{1}{n} |\{k \leq n : k \in K\}|$ exists then it is called asymptotic density of K where $|\{k \leq n : k \in K\}|$ denotes the number of elements of K not exceeding n (see [1, 11]).

If $\delta(K_1) = \delta(K_2) = 1$, then $\delta(K_1 \cap K_2) = \delta(K_1 \cup K_2) = 1$.

If $\delta(K_1) = \delta(K_2) = 0$, then $\delta(K_1 \cap K_2) = \delta(K_1 \cup K_2) = 0$.

Statistical convergence of a sequence of scalars was introduced by Fast [3]. Let $x = (x_k)$ be a sequence of real or complex numbers. If for all $\varepsilon > 0$, there exists L such that,

$$\lim_{n \rightarrow \infty} \frac{1}{n} |\{k \leq n : |x_k - L| \geq \varepsilon\}| = 0,$$

then the sequence (x_k) is statistically convergent to L .

The concepts of statistical limit superior and statistical limit inferior were introduced by Fridy and Orhan [4]. Let k be a positive integer and x be a real number sequence. Define the sets B_x and A_x as

$$B_x := \{b \in \mathbb{R} : \delta(\{n : x_n > b\}) \neq 0\}, \quad A_x := \{a \in \mathbb{R} : \delta(\{n : x_n < a\}) \neq 0\}.$$

Then statistical limit superior and statistical limit inferior of x is given by

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$$st\text{-}\limsup x := \begin{cases} \sup B_x & \text{if } B_x \neq \emptyset, \\ -\infty & \text{if } B_x = \emptyset. \end{cases}$$

$$st\text{-}\liminf x := \begin{cases} \inf A_x & \text{if } A_x \neq \emptyset, \\ +\infty & \text{if } A_x = \emptyset. \end{cases}$$

Lemma 1.1 [4] If $\beta = st\text{-}\limsup x$ is finite, then for every $\varepsilon > 0$,

$$\delta(\{k \in \mathbb{N} : x_k > \beta - \varepsilon\}) \neq 0 \text{ and } \delta(\{k \in \mathbb{N} : x_k > \beta + \varepsilon\}) = 0 \quad (1.1)$$

Conversely, if (1.1) holds for every $\varepsilon > 0$ then $\beta = st\text{-}\limsup x$.

The dual statement for $st\text{-}\liminf x$ is as follows:

Lemma 1.2 [4] If $\alpha = st\text{-}\liminf x$ is finite, then for every $\varepsilon > 0$,

$$\delta(\{k \in \mathbb{N} : x_k < \alpha + \varepsilon\}) \neq 0 \text{ and } \delta(\{k \in \mathbb{N} : x_k < \alpha - \varepsilon\}) = 0 \quad (1.2)$$

Conversely, if (1.2) holds for every $\varepsilon > 0$ then $\alpha = st\text{-}\liminf x$.

A point $\xi \in X$ is called a statistical limit point of a sequence $x = (x_k)$ if there is a set $K = k_1 < k_2 < k_3 < \dots$ with $\delta(K) \neq 0$ such that $x_{k_n} \rightarrow \xi$ as $n \rightarrow \infty$. The set of all statistical limit points of a sequence x will be denoted by Λ_x .

A point $\xi \in X$ is called a statistical cluster point of $x = (x_k)$ if for any $\varepsilon > 0$,

$$\delta(\{k \in \mathbb{N} : d(x_k, \xi) < \varepsilon\}) \neq 0.$$

The set of all statistical cluster points of x will be denoted by Γ_x .

Let L_x denote the set of all limit points ξ (accumulation points) of the sequence x ; i.e. $\xi \in L_x$ if there exists an infinite set $K = k_1 < k_2 < k_3 < \dots$ such that $x_{k_n} \rightarrow \xi$ as $n \rightarrow \infty$.

Obviously we have $\Lambda_x \subseteq \Gamma_x \subseteq L_x$.

In our study we will be interested much more on sequence of functions. Statistical convergence on sequence of functions is defined by Gökhan and Güngör [5].

Following definitions are statistical inner and outer limits on the concept of set convergence which is fundamental to define statistical epi-limit using sets. In this paper, we deal with Painlevé-Kuratowski [7] convergence and actually its statistical version will be studied here which is defined by Sever and Talo [13]. In set convergence, following collections of subsets of \mathbb{N} play an important role for defining statistical inner and outer limits on sequence of sets.

$$\mathcal{S} := \{N \subset \mathbb{N} : \delta(N) = 1\},$$

$$\mathcal{S}^\# := \{N \subset \mathbb{N} : \delta(N) \neq 0\}.$$

Definition 1.3 [13] Let (X, d) be a metric space. The statistical inner limit and statistical outer limit of a sequence (A_n) of closed subsets of X are defined as follows:

$$st\text{-}\liminf_n A_n := \{x \mid \forall V \in \mathcal{N}(x), \exists N \in \mathcal{S}, \forall n \in N : A_n \cap V \neq \emptyset\},$$

$$st\text{-}\limsup_n A_n := \{x \mid \forall V \in \mathcal{N}(x), \exists N \in \mathcal{S}^\#, \forall n \in N : A_n \cap V \neq \emptyset\}.$$

Proposition 1.4 [13] Let (X, d) be a metric space and (A_n) be a sequence of closed subsets of X . Then

$$st\text{-}\liminf_n A_n = \{x \mid \exists N \in \mathcal{S}, \forall n \in N, \exists y_n \in A_n : \lim_n y_n = x\}.$$

Proposition 1.5 [13] Let (X, d) be a metric space and (A_n) be a sequence of closed subsets of X . Then

$$st\text{-}\limsup_n A_n = \{x \mid \exists N \in \mathcal{S}^\#, \forall n \in N, \exists y_n \in A_n : x \in \Gamma_{y_n}\}.$$

Let f be a function defined on X , the epigraph of f is the set $epif := \{(x, \alpha) \in X \times \mathbb{R} \mid \alpha \geq f(x)\}$ and its level set is defined by $lev_{\leq \alpha} f := \{x \in X \mid f(x) \leq \alpha\}$.

Definition 1.6 Let (X, d) be a metric space and (f_n) a sequence of lower semicontinuous functions defined from X to $\overline{\mathbb{R}}$. The lower statistical epi-limit, $e_{st}\text{-}\liminf_n f_n$ is defined by the help of the sequence of sets:

$$epi(e_{st}\text{-}\liminf_n f_n) := st\text{-}\limsup_n(epi f_n). \quad (1.3)$$

Similarly, the upper statistical epi-limit $e_{st}\text{-}\limsup_n f_n$ is defined:

$$epi(e_{st}\text{-}\limsup_n f_n) := st\text{-}\liminf_n(epi f_n). \quad (1.4)$$

When these two functions are equal, we get statistical epi-limit function:

$$f = st\text{-}\lim_n f_n := e_{st}\text{-}\limsup_n f_n = e_{st}\text{-}\liminf_n f_n.$$

Definition 1.7 Let (X, d) be a metric space and (f_n) a sequence of lower semicontinuous functions from X into $\overline{\mathbb{R}}$, for every $x \in X$, lower and upper statistical epi-limit functions are defined by

$$\left(e_{st}\text{-}\liminf_n f_n\right)(x) := \sup_{V \in \mathcal{N}(x)} st\text{-}\liminf_n \inf_{y \in V} f_n(y)$$

$$\left(e_{st}\text{-}\limsup_n f_n\right)(x) := \sup_{V \in \mathcal{N}(x)} st\text{-}\limsup_n \inf_{y \in V} f_n(y)$$

If there exists a function $f : X \rightarrow \overline{\mathbb{R}}$ such that $e_{st}\text{-}\liminf_n f_n = e_{st}\text{-}\limsup_n f_n = f$, then we write $f = e_{st}\text{-}\lim_n f_n$ and we say that (f_n) is e_{st} -convergent to f on X .

Definition 1.8 [8] For every function $f : X \rightarrow \overline{\mathbb{R}}$ the lower semicontinuous envelope $sc^- f$ of f is defined for every $x \in X$ by

$$(sc^- f)(x) = \sup_{g \in \mathcal{G}(f)} g(x),$$

where $\mathcal{G}(f)$ is the set of all lower semicontinuous functions g on X such that $g(y) \leq f(y)$ for every $y \in X$.

Proposition 1.9 [8] Let $f : X \rightarrow \overline{\mathbb{R}}$ be a function. Then

$$(sc^- f)(x) = \sup_{V \in \mathcal{N}(x)} \inf_{y \in V} f(y)$$

for every $x \in X$ where $\mathcal{N}(x)$ is the neighbourhood of x .

2. Main Result

In this part, we deal with infimum values of statistical lower and upper epi-limits in open and compact sets. We also define statistical equi-coerciveness which is a necessary condition for statistical convergence of infimum value of (f_n) to infimum value of f .

Definition 2.1 Let X be a metric space and (f_n) be a sequence of functions defined on $X \rightarrow \overline{\mathbb{R}}$. The functions F^l and F^u are defined as

$$F^l = e_{st}\text{-}\liminf_n f_n, \quad F^u = e_{st}\text{-}\limsup_n f_n.$$

Theorem 2.2 Let (X, d) be a metric space and (f_n) a sequence of lower semicontinuous functions defined from X to $\overline{\mathbb{R}}$. For any open subset U of X , the following inequalities are valid.

$$\inf_{x \in U} F^l(x) \geq st\text{-}\liminf_n \inf_{x \in U} f_n(x)$$

$$\inf_{x \in U} F^u(x) \geq st\text{-}\limsup_n \inf_{x \in U} f_n(x).$$

Proof. We shall prove the first one, the other one being analogous. For every $x \in U$ we have $U \in \mathcal{N}(x)$ and

$$F^l(x) = \sup_{U \in \mathcal{N}(x)} st\text{-}\liminf_n \inf_{y \in U} f_n(y) \geq st\text{-}\liminf_n \inf_{y \in U} f_n(y)$$

by definition of statistical lower epi-limit. The inequality is valid for every $x \in U$, hence

$$\inf_{x \in U} F^l(x) \geq st\text{-}\liminf_n \inf_{y \in U} f_n(y).$$

□

Theorem 2.3 Let (X, d) be a metric space and (f_n) a sequence of lower semicontinuous functions defined from X to \mathbb{R} and let K be a compact subset of X . Then the following inequality is valid.

$$\min_{x \in K} F^l(x) \leq st\text{-}\liminf_n \inf_{x \in K} f_n(x).$$

Proof. We know that the function F^l is lower semicontinuous on X and it has a minimum in a compact set. Assume that

$$st\text{-}\liminf_n \inf_{x \in K} f_n(x) \leq \alpha$$

for an arbitrary $\alpha \in \mathbb{R}$. Hence for all $\varepsilon > 0$,

$$\delta(\{n \in \mathbb{N} : \inf_{x \in K} f_n(x) < \alpha + \varepsilon\}) \neq 0.$$

It means that there exists $M \in \mathcal{S}^\#$ such that for all $m \in M$

$$\inf_{x \in K} f_m(x) < \alpha + \varepsilon$$

Now we match every $\inf_{x \in K} f_m(x)$ with $sc^- f_m(x_m)$ points and it can be written as

$$sc^- f_m(x_m) < \alpha + \varepsilon$$

for all $m \in M$. Since K is a compact set and $x_m \in K$, there exists cluster points of $(x_m)_{m \in M}$ in K . For any $y \in \Gamma_{(x_m)}$ and for all $U \in N(y)$ we have $\delta(\{m \in M : x_m \in U\}) \neq 0$. Let us call this set as $M' = \{m \in M : x_m \in U\}$. For all $m \in M'$

$$\inf_{x \in U} f_m(x) \leq sc^- f_m(x_m) = \inf_{x \in K} f_m(x) < \alpha + \varepsilon.$$

Then, for all $\varepsilon > 0$ we have

$$st\text{-}\liminf_n \inf_{x \in U} f_n(x) \leq \alpha.$$

By taking supremum over all $U \in N(y)$ we obtain

$$F^l(y) = e_{st}\text{-}\liminf_n f_n(y) \leq \alpha.$$

Since $y \in K$, we have also $\min_{x \in K} F^l(x) \leq F^l(y) \leq \alpha$. Finally, we obtain

$$\min_{x \in K} F^l(x) \leq \alpha$$

which gives $\min_{x \in K} F^l(x) \leq st\text{-}\liminf_n \inf_{x \in K} f_n(x)$ and we are done. \square

Theorem 2.4 Suppose that there exists a countably compact subset K of X such that $\inf_{x \in X} f_n(x) = \inf_{x \in K} f_n(x)$ for every $n \in N$ with $N \in \mathcal{S}^\#$, then $e_{st}\text{-}\liminf_n f_n$ attains its minimum on X and

$$\min_{x \in X} F^l(x) = st\text{-}\liminf_n \inf_{x \in X} f_n(x).$$

Proof. First of all, by our choice of index set, we have

$$st\text{-}\liminf_m \inf_{x \in X} f_m(x) = st\text{-}\liminf_m \inf_{x \in K} f_m(x). \quad (2.1)$$

By Theorem (2.2), if we replace U with X , we get

$$\inf_{x \in X} F^l(x) \geq st\text{-}\liminf_n \inf_{x \in X} f_n(x)$$

By Theorem (2.3) and equality (2.1) we obtain

$$\inf_{x \in X} F^l(x) \leq \min_{x \in K} F^l(x) \leq st\text{-}\liminf_n \inf_{x \in X} f_n(x) \leq st\text{-}\liminf_m \inf_{x \in X} f_m(x) = st\text{-}\liminf_m \inf_{x \in K} f_m(x).$$

Hence,

$$\min_{x \in X} F^l(x) = st\text{-}\liminf_n \inf_{x \in X} f_n(x).$$

\square

Definition 2.5 The sequence (f_n) is statistically equi-coercive on X , if for every $t \in \mathbb{R}$ there exists a closed compact subset K_t of X and $N \in \mathcal{S}$ such that $\{f_n \leq t\} \subseteq K_t$ for every $n \in N$.

Theorem 2.6 Suppose that (f_n) is statistically equi-coercive in X . Then F^l and F^u are coercive and

$$\min_{x \in X} F^l(x) = st\text{-}\liminf_n \inf_{x \in X} f_n(x).$$

Proof. It is clear that F^l and F^u are coercive and lower semicontinuous and hence they attain their minimum on X . If we apply $U = X$, by Theorem 2.2 we get the inequality

$$\min_{x \in X} F^l(x) \geq st\text{-}\liminf_n \inf_{x \in X} f_n(x).$$

Hence, it is enough to show

$$\min_{x \in X} F^l(x) \leq st\text{-}\liminf_n \inf_{x \in X} f_n(x).$$

Assume that the right hand side of the inequality is less than $+\infty$ and $st\text{-}\liminf_n \inf_{x \in X} f_n(x) = \alpha$. It means

$$\delta(\{n \in \mathbb{N} : \alpha - \varepsilon < \inf_{x \in X} f_n(x) < \alpha + \varepsilon\}) \neq 0$$

for all $\varepsilon > 0$. Let us call this set as $M = \{n \in \mathbb{N} : \alpha - \varepsilon < \inf_{x \in X} f_n(x) < \alpha + \varepsilon\}$. Since the sequence (f_n) is statistically equi-coercive, for every $t \in \mathbb{R}$ there exists a compact subset K of X such that $\{f_m \leq t\} \subseteq K$ for all $m \in M$. Moreover, for all $m \in M$ we have

$$\inf_{x \in X} f_m(x) = \inf_{x \in K} f_m(x).$$

Let us define $G^l = e_{st}\text{-}\liminf_m f_m$ and apply Theorem 2.4 we obtain

$$\min_{x \in X} G^l(x) = st\text{-}\liminf_m \inf_{x \in X} f_m(x) = st\text{-}\liminf_n \inf_{x \in X} f_n(x).$$

It is obvious that $F^l \leq G^l$. Hence we have $\min_{x \in X} F^l(x) \leq st\text{-}\liminf_n \inf_{x \in X} f_n(x)$ and we are done. \square

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On the σ -Stable Families

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Abstract: Given a positive number σ , the set of all complex numbers with real parts less than $-\sigma$, is called σ -shifted Hurwitz stability region. For a linear system if all roots of the characteristic polynomial belong to a shifted region, the system is said to be σ -stable. This property is very important in the investigation of performance and stability problems. In this report we consider σ -stability problem for uncertain linear systems. For an interval family, we find the largest value of σ for which the interval family is stable. We establish sufficient conditions for segment stability which is important for the application of the Edge theorem.

1. Introduction

Consider n th order polynomial

$$a(s) = a_1 + a_2s + \dots + a_n s^{n-1} + a_{n+1}s^n \quad (1.1)$$

with $a_{n+1} \neq 0$. If all roots of (1.1) satisfy the condition $\text{Re}(s) < -\sigma$, then $a(s)$ is called σ -stable, where $\sigma > 0$. The case $\sigma = 0$ is known as Hurwitz stability. A necessary condition for Hurwitz stability of (1.1) is the positivity or negativity of all coefficients a_1, a_2, \dots, a_{n+1} . Therefore, without loss of generality we will assume that $a_i > 0, i = 1, 2, \dots, n+1$.

Define the following subset from \mathbb{R}_+^{n+1} :

$$\mathcal{D}_\sigma = \{a = (a_1, a_2, \dots, a_{n+1})^T \in \mathbb{R}_+^{n+1} : \text{The polynomial } a(s) \text{ is } \sigma\text{-stable}\}.$$

Here \mathbb{R}_+^{n+1} stands for the set

$$\{(x_1, x_2, \dots, x_{n+1})^T \in \mathbb{R}^{n+1} : x_i > 0, i = 1, 2, \dots, n+1\}.$$

By the continuity property of roots of (1.1) (see [3]) the set \mathcal{D}_σ is open.

In control theory, the performance of a linear time invariant system is dependent on the location of the closed loop roots. The decay rate is determined by the roots that are closest to the imaginary axis. In [1], σ -stability problem for a polynomial polytope is considered, a necessary and sufficient condition for robust σ -stability is obtained in terms of the Hurwitz matrices. In [2], σ -stabilization problem of an unstable plant by PID controller is considered. Here a constructive determination of the set of stabilizing controllers in the parameter space is given.

2. σ -Stability Test

Introduce the new variable $t = s + \sigma$. Then $s = t - \sigma$,

$$\begin{aligned} \text{Re}(s) &= \text{Re}(t) - \sigma, \\ \text{Re}(s) < -\sigma &\Leftrightarrow \text{Re}(t) < 0. \end{aligned}$$

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Define a new polynomial $p(t)$ in the variable t :

$$p(t) = a_{n+1}(t - \sigma)^n + a_n(t - \sigma)^{n-1} + \cdots + a_2(t - \sigma) + a_1.$$

Then

$$a(s) \text{ is } \sigma\text{-stable} \Leftrightarrow p(t) \text{ is Hurwitz stable.}$$

Simple calculation gives the following.

$$p(t) = b_1 + b_2t + \cdots + b_nt^{n-1} + b_{n+1}t^n, \quad (2.1)$$

where $b = Ma$ and

$$b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_{n+1} \end{bmatrix}, \quad a = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_{n+1} \end{bmatrix},$$

$$M = \begin{bmatrix} 1 & -\sigma & \sigma^2 & -\sigma^3 & \cdots & \cdot & (-1)^n \sigma^n \\ 0 & 1 & -C_2^1 \sigma & C_3^2 \sigma^2 & \cdots & \cdot & (-1)^{n-1} C_n^{n-1} \sigma^{n-1} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & -C_n^1 \sigma \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 \end{bmatrix}.$$

The matrix M is $(n+1) \times (n+1)$ dimensional and upper triangular.

For example, if $n = 3$ then

$$M = \begin{bmatrix} 1 & -\sigma & \sigma^2 & -\sigma^3 \\ 0 & 1 & -2\sigma & 3\sigma^2 \\ 0 & 0 & 1 & -3\sigma \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Proposition 2.1 *The polynomial $a(s)$ is σ -stable if and only if the polynomial $p(t)$ (2.1) is Hurwitz stable.*

Given Hurwitz stable vector $b \in \mathbb{R}^{n+1}$, the σ -stable vector $a \in \mathbb{R}^{n+1}$ can be calculated as $a = M^{-1}b$. From the construction it follows that the inverse matrix M^{-1} can be obtained from M after replacing σ by $-\sigma$.

For example, if $n = 3$ then

$$M^{-1} = \begin{bmatrix} 1 & \sigma & \sigma^2 & \sigma^3 \\ 0 & 1 & 2\sigma & 3\sigma^2 \\ 0 & 0 & 1 & 3\sigma \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

3. Maximal Stability Boundary for Interval Polynomial Family

Assume that the coefficients of the polynomial (1.1) vary in some intervals, that is assume that

$$a_i \in [a_i^-, a_i^+] \quad (i = 1, 2, \dots, n+1).$$

The obtained family \mathcal{P} is called an interval family and is denoted by

$$\begin{aligned} \mathcal{P} &= \sum_{i=0}^n [a_{i+1}^-, a_{i+1}^+] s^i \\ &= [a_1^-, a_1^+] + [a_2^-, a_2^+] s + \cdots + [a_{n+1}^-, a_{n+1}^+] s^n. \end{aligned} \quad (3.1)$$

By the well-known Kharitonov theorem [3], robust Hurwitz stability of (3.1) is equivalent to the stability of the four Kharitonov polynomials.

The following example shows that in the σ -stable case stability of the four Kharitonov polynomials is not sufficient for robust stability of the interval family (3.1).

Example 3.1 Consider the interval polynomial family with $\sigma = 0.25$

$$\mathcal{P} = [0.3, 0.5] + [1.5, 1.8]s + [2.1, 2.5]s^2 + [0.2, 0.3]s^3.$$

The corresponding four Kharitonov polynomials and its roots are

		roots:
$K_1(s)$	$= 0.3 + 1.5s + 2.5s^2 + 0.3s^3,$	$-7.7, \quad -0.316 \pm 0.172i$
$K_2(s)$	$= 0.5 + 1.8s + 2.1s^2 + 0.2s^3,$	$-9.588, \quad -0.455 \pm 0.23i$
$K_3(s)$	$= 0.5 + 1.5s + 2.1s^2 + 0.3s^3,$	$-6.241, \quad -0.379 \pm 0.351i$
$K_4(s)$	$= 0.3 + 1.8s + 2.5s^2 + 0.2s^3$	$-0.255, \quad -0.5, \quad -11.744.$

The polynomial $p(s) = 0.3 + 1.8s + 2.1s^2 + 0.3s^3$ is a member of the interval family and its roots are -0.22267 , -0.74439 and -0.603293 which is not σ -stable.

Theorem 3.2 Given interval family (3.1), let $a^1(s), a^2(s), \dots, a^m(s)$ be extreme polynomials. Let α be a real number defined as

$$\alpha = \max \{ \operatorname{Re}(s) : s \text{ is a root of one of the polynomials } a^i(s) \text{ (} i = 1, 2, \dots, m \text{)} \}.$$

Then α is the σ -stability boundary of the interval family (3.1). That is, \mathcal{P} is robust σ -stable for all $\sigma < -\alpha$ and \mathcal{P} is not robust σ -stable for $\sigma = -\alpha$.

Proof. By the known result, the interval family (3.1) is robust σ -stable if and only if all extreme polynomials are σ -stable [4].

If $\sigma < -\alpha$, then any root of any polynomial $a^i(s)$ satisfies the condition $\operatorname{Re}(s) \leq \alpha < -\sigma$. Therefore all extreme polynomials are stable and consequently \mathcal{P} is robust stable.

If $\sigma = -\alpha$, then the family \mathcal{P} has a member $a(s)$ with the root s satisfying the condition $\operatorname{Re}(s) = \alpha = -\sigma$ which implies that $a(s)$ is not σ -stable. \square

Example 3.3 Consider the interval polynomial family described by

$$\mathcal{P} = [0.2, 0.6] + [2.1, 2.2]s + [8, 8.1]s^2 + [2.1, 2.5]s^3 + [2.9, 3.1]s^4 + 0.2s^5.$$

The family has 32 extreme polynomials:

$$\begin{aligned} a^1(s) &= 0.2 + 2.1s + 8s^2 + 2.1s^3 + 2.9s^4 + 0.2s^5, \\ &\vdots \\ a^{32}(s) &= 0.6 + 2.2s + 8.1s^2 + 2.5s^3 + 3.1s^4 + 0.2s^5. \end{aligned}$$

Calculations give $\alpha = -0.119849$ and the family is robust σ -stable for $\sigma < 0.119849$.

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In this section, we consider σ -stability of a polynomial segment with stable end points. Stability problem of a given segment is important due to Edge theorem [5], which states that a polynomial polytope with σ -stable edges is robust σ -stable.

The following example shows that stability of the end points does not imply the stability of the whole segment.

Example 4.1 Let $\sigma = 1$. Consider the polynomial segment joining the two σ -stable polynomials

$$a(s) = 17.57 + 38s + 31s^2 + 10s^3 \text{ and } c(s) = 21.57 + 42s + 32s^2 + 10s^3.$$

$$\begin{aligned} \text{Roots of } a(s) &: -1.095, \quad -1.002 \pm 0.774i, \\ \text{roots of } c(s) &: -1.196, \quad -1.001 \pm 0.894i. \end{aligned}$$

For $\lambda = 0.5$, the polynomial

$$\frac{1}{2}a(s) + \frac{1}{2}c(s)$$

has roots: $-1.152, \quad -0.998 \pm 0.836i$ and is not σ -stable.

Let

$$a(s) = a_1 + a_2s + \dots + a_{n+1}s^n, \quad c(s) = c_1 + c_2s + \dots + c_{n+1}s^n$$

be two σ -stable polynomials, $a = (a_1, a_2, \dots, a_{n+1})^T$, $c = (c_1, c_2, \dots, c_{n+1})^T$.

In the below, we give sufficient conditions for segment stability.

Theorem 4.2 Let m^1, m^2, \dots, m^{n+1} be $(n+1)$ -dimensional row vectors of the matrix M . Assume that

$$\begin{aligned} \langle m^1, a-c \rangle = \langle m^3, a-c \rangle = \dots = \langle m^{n+1}, a-c \rangle = 0 & \quad (\text{if } n \text{ is even}), \\ \langle m^1, a-c \rangle = \langle m^3, a-c \rangle = \dots = \langle m^n, a-c \rangle = 0 & \quad (\text{if } n \text{ is odd}). \end{aligned}$$

Then the segment $[a(s), c(s)] = \{(1-\lambda)a(s) + \lambda c(s) : \lambda \in [0, 1]\}$ is σ -stable.

Proof. We set $b = Ma$, $d = Mc$. Then $b(s)$ and $d(s)$ are Hurwitz stable. Given any $\lambda \in (0, 1)$,

$$\begin{aligned} (1-\lambda)a + \lambda c &= (1-\lambda)M^{-1}b + \lambda M^{-1}d \\ &= M^{-1}[(1-\lambda)b + \lambda d]. \end{aligned}$$

By conditions of the Theorem 2,

$$\langle m^1, a \rangle = \langle m^1, c \rangle, \langle m^3, a \rangle = \langle m^3, c \rangle, \dots$$

which imply that even parts of $b(s)$ and $d(s)$ are the same.

Therefore by the known theorem $(1-\lambda)b(s) + \lambda d(s)$ is Hurwitz stable [6]. By the above construction $(1-\lambda)a(s) + \lambda c(s)$ is σ -stable. \square

Similarly, from the analogous result for odd parts follows

Theorem 4.3 Assume that

$$\begin{aligned} \langle m^2, a-c \rangle = \langle m^4, a-c \rangle = \dots = \langle m^n, a-c \rangle = 0 & \quad (\text{if } n \text{ is even}), \\ \langle m^2, a-c \rangle = \langle m^4, a-c \rangle = \dots = \langle m^{n+1}, a-c \rangle = 0 & \quad (\text{if } n \text{ is odd}). \end{aligned}$$

Then the segment $[a(s), c(s)] = \{(1-\lambda)a(s) + \lambda c(s) : \lambda \in [0, 1]\}$ is σ -stable.

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On Some Asymptotically Equivalence Types for Double Sequences and Relations among Them

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Abstract:

In this study, we give definitions of asymptotically lacunary invariant equivalence, strongly asymptotically lacunary invariant equivalence and asymptotically lacunary ideal invariant equivalence for double sequences. We also examine the existence of some relations among these new equivalence definitions.

1. Introduction and Background

Throughout the paper \mathbb{N} denotes the set of natural numbers.

Many authors have studied on the concepts of invariant mean and invariant convergence (see, [10, 11, 13, 14, 19, 20, 25]).

Let σ be a mapping of the positive integers into themselves. A continuous linear functional ϕ on ℓ_∞ , the space of real bounded sequences, is said to be an invariant mean or a σ -mean if it satisfies following conditions:

1. $\phi(x) \geq 0$, when the sequence $x = (x_n)$ has $x_n \geq 0$ for all n ,
2. $\phi(e) = 1$, where $e = (1, 1, 1, \dots)$ and
3. $\phi(x_{\sigma(n)}) = \phi(x_n)$ for all $x \in \ell_\infty$.

The mappings σ are assumed to be one-to-one and such that $\sigma^m(n) \neq n$ for all positive integers n and m , where $\sigma^m(n)$ denotes the m th iterate of the mapping σ at n . Thus, ϕ extends the limit functional on c , the space of convergent sequences, in the sense that $\phi(x) = \lim x$ for all $x \in c$.

In the case σ is translation mappings $\sigma(n) = n + 1$, the σ -mean is often called a Banach limit.

The concept of lacunary strong σ -convergence was introduced by Savaş [21] and then Pancaroğlu and Nuray [15] defined the concept of lacunary invariant summability.

The idea of \mathcal{I} -convergence was introduced by Kostyrko et al. [6] which is based on the structure of the ideal \mathcal{I} of subset of the set \mathbb{N} . For more detail, see [7].

A family of sets $\mathcal{I} \subseteq 2^{\mathbb{N}}$ is called an ideal if and only if (i) $\emptyset \in \mathcal{I}$, (ii) For each $A, B \in \mathcal{I}$ we have $A \cup B \in \mathcal{I}$, (iii) For each $A \in \mathcal{I}$ and each $B \subseteq A$ we have $B \in \mathcal{I}$.

An ideal is called non-trivial if $\mathbb{N} \notin \mathcal{I}$ and non-trivial ideal is called admissible if $\{n\} \in \mathcal{I}$ for each $n \in \mathbb{N}$.

Recently, the concept of $\sigma\theta$ -uniform density of any subset A of the set \mathbb{N} and corresponding the concept of $\mathcal{I}_{\sigma\theta}$ -convergence for real sequences were introduced by Ulusu and Nuray [28].

Several convergence concepts for double sequences and some properties of these concepts which are noted following can be seen in [1, 2, 8, 12, 16, 18, 23].

A double sequence $x = (x_{kj})$ is said to be bounded if there exists an $M > 0$ such that $|x_{kj}| < M$ for all k and j , i.e., if $\sup_{k,j} |x_{kj}| < \infty$.

The set of all bounded double sequences will be denoted by ℓ_∞^2 .

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A non-trivial ideal \mathcal{I}_2 of $\mathbb{N} \times \mathbb{N}$ is called strongly admissible ideal if $\{i\} \times \mathbb{N}$ and $\mathbb{N} \times \{i\}$ belong to \mathcal{I}_2 for each $i \in \mathbb{N}$.

It is evident that a strongly admissible ideal is admissible also.

Let (X, ρ) be a metric space and \mathcal{I}_2 be a strongly admissible ideal in $\mathbb{N} \times \mathbb{N}$. A double sequence $x = (x_{mn})$ in X is said to be \mathcal{I}_2 -convergent to $L \in X$ if for every $\varepsilon > 0$,

$$A(\varepsilon) = \{(m, n) \in \mathbb{N} \times \mathbb{N} : \rho(x_{mn}, L) \geq \varepsilon\} \in \mathcal{I}_2.$$

It is denoted by $\mathcal{I}_2 - \lim_{m, n \rightarrow \infty} x_{mn} = L$.

The double sequence $\theta_2 = \{(k_r, j_u)\}$ is called double lacunary sequence if there exist two increasing sequence of integers such that

$$k_0 = 0, h_r = k_r - k_{r-1} \rightarrow \infty \text{ and } j_0 = 0, \bar{h}_u = j_u - j_{u-1} \rightarrow \infty \text{ as } r, u \rightarrow \infty.$$

We use the following notations in the sequel:

$$k_{ru} = k_r j_u, h_{ru} = h_r \bar{h}_u, I_{ru} = \{(k, j) : k_{r-1} < k \leq k_r \text{ and } j_{u-1} < j \leq j_u\}.$$

Recently, the definitions of some invariant convergence for double sequences were presented in a study by Ulusu et al. [27] as below:

Let $\theta_2 = \{(k_r, j_u)\}$ be a double lacunary sequence. A double sequence $x = (x_{kj})$ is said to be lacunary invariant convergent to L if

$$\lim_{r, u \rightarrow \infty} \frac{1}{h_{ru}} \sum_{k, j \in I_{ru}} x_{\sigma^k(m), \sigma^j(n)} = L,$$

uniformly in $m, n = 1, 2, \dots$ and it is denoted by $x_{kj} \rightarrow L(V_2^{\sigma\theta})$.

A double sequence $x = (x_{kj})$ is said to be strongly lacunary invariant convergent to L if

$$\lim_{r, u \rightarrow \infty} \frac{1}{h_{ru}} \sum_{k, j \in I_{ru}} |x_{\sigma^k(m), \sigma^j(n)} - L| = 0,$$

uniformly in m, n and it is denoted by $x_{kj} \rightarrow L([V_2^{\sigma\theta}])$.

Let $\theta_2 = \{(k_r, j_u)\}$ be a double lacunary sequence, $A \subseteq \mathbb{N} \times \mathbb{N}$ and

$$s_{ru} = \min_{m, n} \left| A \cap \{(\sigma^k(m), \sigma^j(n)) : (k, j) \in I_{ru}\} \right|,$$

$$S_{ru} = \max_{m, n} \left| A \cap \{(\sigma^k(m), \sigma^j(n)) : (k, j) \in I_{ru}\} \right|.$$

If the following limits exist

$$\underline{V}_2^\theta(A) = \lim_{r, u \rightarrow \infty} \frac{s_{ru}}{h_{ru}} \text{ and } \overline{V}_2^\theta(A) = \lim_{r, u \rightarrow \infty} \frac{S_{ru}}{h_{ru}},$$

then they are called a lower lacunary σ -uniform density and an upper lacunary σ -uniform density of the set A , respectively. If $\underline{V}_2^\theta(A) = \overline{V}_2^\theta(A)$, then $V_2^\theta(A) = \underline{V}_2^\theta(A) = \overline{V}_2^\theta(A)$ is called the lacunary σ -uniform density of A .

Denoted by $\mathcal{I}_2^{\sigma\theta}$ the class of all $A \subseteq \mathbb{N} \times \mathbb{N}$ with $V_2^\theta(A) = 0$.

A double sequence $x = (x_{kj})$ is said to be lacunary \mathcal{I}_2 -invariant convergent or $\mathcal{I}_2^{\sigma\theta}$ -convergent to L if for every $\varepsilon > 0$

$$A_\varepsilon = \{(k, j) \in I_{ru} : |x_{kj} - L| \geq \varepsilon\} \in \mathcal{I}_2^{\sigma\theta},$$

i.e., $V_2^\theta(A_\varepsilon) = 0$. It is denoted by $\mathcal{I}_2^{\sigma\theta} - \lim x_{kj} = L$ or $x_{kj} \rightarrow L(\mathcal{I}_2^{\sigma\theta})$.

Marouf [9] presented definitions for asymptotically equivalent sequences and asymptotic regular matrices. Then, the concept of asymptotically equivalence has been developed by many researchers (see, [3, 5, 17, 22, 24]).

Two nonnegative sequences $x = (x_k)$ and $y = (y_k)$ are said to be asymptotically equivalent if

$$\lim_k \frac{x_k}{y_k} = 1.$$

It is denoted by $x \sim y$.

Hazarika and Kumar [4] presented some asymptotically equivalence definitions for double sequences as follows: Two nonnegative double sequences $x = (x_{kl})$ and $y = (y_{kl})$ are said to be P -asymptotically equivalent if

$$P - \lim_{k,l} \frac{x_{kl}}{y_{kl}} = 1,$$

denoted by $x \sim^P y$.

Two nonnegative double sequences $x = (x_{kl})$ and $y = (y_{kl})$ are said to be asymptotically \mathcal{I}_2 -equivalent of multiple L if for every $\varepsilon > 0$

$$\left\{ (k, l) \in \mathbb{N} \times \mathbb{N} : \left| \frac{x_{kl}}{y_{kl}} - L \right| \geq \varepsilon \right\} \in \mathcal{I}_2,$$

denoted by $x \sim^{\mathcal{I}_2} y$ and simply asymptotically \mathcal{I}_2 -equivalent if $L = 1$.

Recently, Ulusu [26] by defining the concept of lacunary \mathcal{I}_σ -asymptotically equivalence and the concepts of lacunary σ -asymptotically equivalence for real sequences, studied some relationships among these concepts.

2. Main Results

In this study, we give definitions of asymptotically lacunary invariant equivalence, strongly asymptotically lacunary invariant equivalence and asymptotically lacunary ideal invariant equivalence for double sequences. We also examine the existence of some relations among these new equivalence definitions.

Definition 2.1 Two nonnegative double sequence $x = (x_{kj})$ and $y = (y_{kj})$ are said to be asymptotically lacunary σ_2 -equivalent of multiple L if

$$\lim_{r,u \rightarrow \infty} \frac{1}{h_{ru}} \sum_{k,j \in I_{ru}} \frac{x_{\sigma^k(m), \sigma^j(n)}}{y_{\sigma^k(m), \sigma^j(n)}} = L,$$

uniformly in m and n . In this case, we write $x \sim_{N_{2(L)}^{\sigma\theta}} y$ and simply asymptotically lacunary σ_2 -equivalent if $L = 1$.

Definition 2.2 Two nonnegative double sequences $x = (x_{kj})$ and $y = (y_{kj})$ are said to be asymptotically lacunary \mathcal{I}_2 -invariant equivalent of multiple L if for every $\varepsilon > 0$

$$A_\varepsilon^\sim := \left\{ (k, j) \in I_{ru} : \left| \frac{x_{kj}}{y_{kj}} - L \right| \geq \varepsilon \right\} \in \mathcal{I}_2^{\sigma\theta},$$

i.e., $V_2^\theta(A_\varepsilon^\sim) = 0$. In this case, we write $x \sim_{\mathcal{I}_2^{\sigma\theta}} y$ and simply asymptotically lacunary \mathcal{I}_2 -invariant equivalent if $L = 1$.

The set of all asymptotically lacunary \mathcal{I}_2 -invariant equivalent of multiple L sequences will be denoted by $\mathcal{I}_{2(L)}^{\sigma\theta}$.

Theorem 2.3 Suppose that $x = (x_{kj}), y = (y_{kj}) \in \ell_\infty^\theta$. If x and y are asymptotically lacunary \mathcal{I}_2 -invariant equivalent of multiple L , then these sequences are asymptotically lacunary σ_2 -equivalent of multiple L .

Proof. Let $m, n \in \mathbb{N}$ be arbitrary and $\varepsilon > 0$. Now, we calculate

$$t(\theta_2, m, n) := \left| \frac{1}{h_{ru}} \sum_{k,j \in I_{ru}} \frac{x_{\sigma^k(m), \sigma^j(n)}}{y_{\sigma^k(m), \sigma^j(n)}} - L \right|.$$

We have

$$t(\theta_2, m, n) \leq t_1(\theta_2, m, n) + t_2(\theta_2, m, n),$$

where

$$t_1(\theta_2, m, n) := \frac{1}{h_{ru}} \sum_{\substack{k,j \in I_{ru} \\ \left| \frac{x_{\sigma^k(m), \sigma^j(n)}}{y_{\sigma^k(m), \sigma^j(n)}} - L \right| \geq \varepsilon}} \left| \frac{x_{\sigma^k(m), \sigma^j(n)}}{y_{\sigma^k(m), \sigma^j(n)}} - L \right|$$

and

$$t_2(\theta_2, m, n) := \frac{1}{h_{ru}} \sum_{\substack{k,j \in I_{ru} \\ \left| \frac{x_{\sigma^k(m), \sigma^j(n)}}{y_{\sigma^k(m), \sigma^j(n)}} - L \right| < \varepsilon}} \left| \frac{x_{\sigma^k(m), \sigma^j(n)}}{y_{\sigma^k(m), \sigma^j(n)}} - L \right|.$$

We get $t_2(\theta_2, m, n) < \varepsilon$, for every $m, n = 1, 2, \dots$. The boundedness of x and y implies that there exists a $M > 0$ such that

$$\left| \frac{x_{\sigma^k(m), \sigma^j(n)}}{y_{\sigma^k(m), \sigma^j(n)}} - L \right| \leq M,$$

for all $k, j \in I_{ru}$ and for every m, n . Then, this implies that

$$\begin{aligned} t_1(\theta_2, m, n) &\leq \frac{M}{h_{ru}} \left| \left\{ (k, j) \in I_{ru} : \left| \frac{x_{\sigma^k(m), \sigma^j(n)}}{y_{\sigma^k(m), \sigma^j(n)}} - L \right| \geq \varepsilon \right\} \right| \\ &\leq M \frac{\max_{m, n} \left| \left\{ (k, j) \in I_{ru} : \left| \frac{x_{\sigma^k(m), \sigma^j(n)}}{y_{\sigma^k(m), \sigma^j(n)}} - L \right| \geq \varepsilon \right\} \right|}{h_{ru}} = M \frac{S_{ru}}{h_{ru}}, \end{aligned}$$

hence $x \stackrel{N_{2(L)}^{\sigma\theta}}{\sim} y$. □

The converse of Theorem 2.3 does not hold. For example, $x = (x_{kj})$ and $y = (y_{kj})$ are the sequences defined by following;

$$x_{kj} := \begin{cases} 2, & \text{if } k_{r-1} < k < k_{r-1} + [\sqrt{h_r}], \text{ and } k+j \text{ is an even integer.} \\ & j_{r-1} < j < j_{r-1} + [\sqrt{h_u}], \\ 0, & \text{if } k_{r-1} < k < k_{r-1} + [\sqrt{h_r}], \text{ and } k+j \text{ is an odd integer.} \\ & j_{r-1} < j < j_{r-1} + [\sqrt{h_u}], \end{cases}$$

$$y_{kj} := 1.$$

When $\sigma(m) = m + 1$ and $\sigma(n) = n + 1$, this sequences are asymptotically lacunary σ_2 -equivalent but they are not asymptotically lacunary \mathcal{I}_2 -invariant equivalent.

Definition 2.4 Two nonnegative double sequence $x = (x_{kj})$ and $y = (y_{kj})$ are said to be strongly asymptotically lacunary σ_2 -equivalent of multiple L if

$$\lim_{r, u \rightarrow \infty} \frac{1}{h_{ru}} \sum_{k, j \in I_{ru}} \left| \frac{x_{\sigma^k(m), \sigma^j(n)}}{y_{\sigma^k(m), \sigma^j(n)}} - L \right| = 0,$$

uniformly in m and n . In this case, we write $x \stackrel{[N_{2(L)}^{\sigma\theta}]}{\sim} y$ and simply strongly asymptotically lacunary σ_2 -equivalent if $L = 1$.

The set of all strongly asymptotically lacunary invariant equivalent of multiple L sequences will be denoted by $[\mathfrak{N}_{2(L)}^{\sigma\theta}]$.

Theorem 2.5 If double sequences $x = (x_{kj})$ and $y = (y_{kj})$ are strongly asymptotically lacunary σ_2 -equivalent of multiple L , then these sequences are asymptotically lacunary \mathcal{I}_2 -invariant equivalent of multiple L .

Proof. Let $x \stackrel{[N_{2(L)}^{\sigma\theta}]}{\sim} y$ and given $\varepsilon > 0$. Then, for every $m, n \in \mathbb{N}$ we have

$$\begin{aligned} \sum_{k, j \in I_{ru}} \left| \frac{x_{\sigma^k(m), \sigma^j(n)}}{y_{\sigma^k(m), \sigma^j(n)}} - L \right| &\geq \sum_{k, j \in I_{ru}} \left| \frac{x_{\sigma^k(m), \sigma^j(n)}}{y_{\sigma^k(m), \sigma^j(n)}} - L \right| \\ &\geq \sum_{\left| \frac{x_{\sigma^k(m), \sigma^j(n)}}{y_{\sigma^k(m), \sigma^j(n)}} - L \right| \geq \varepsilon} \left| \frac{x_{\sigma^k(m), \sigma^j(n)}}{y_{\sigma^k(m), \sigma^j(n)}} - L \right| \\ &\geq \varepsilon \cdot \left| \left\{ (k, j) \in I_{ru} : \left| \frac{x_{\sigma^k(m), \sigma^j(n)}}{y_{\sigma^k(m), \sigma^j(n)}} - L \right| \geq \varepsilon \right\} \right| \\ &\geq \varepsilon \cdot \max_{m, n} \left| \left\{ (k, j) \in I_{ru} : \left| \frac{x_{\sigma^k(m), \sigma^j(n)}}{y_{\sigma^k(m), \sigma^j(n)}} - L \right| \geq \varepsilon \right\} \right| \end{aligned}$$

and so

$$\begin{aligned} \frac{1}{h_{ru}} \sum_{k, j \in I_{ru}} \left| \frac{x_{\sigma^k(m), \sigma^j(n)}}{y_{\sigma^k(m), \sigma^j(n)}} - L \right| &\geq \varepsilon \cdot \frac{\max_{m, n} \left| \left\{ (k, j) \in I_{ru} : \left| \frac{x_{\sigma^k(m), \sigma^j(n)}}{y_{\sigma^k(m), \sigma^j(n)}} - L \right| \geq \varepsilon \right\} \right|}{h_{ru}} \\ &= \varepsilon \cdot \frac{S_{ru}}{h_{ru}} \end{aligned}$$

This implies that $\lim_{r,u \rightarrow \infty} \frac{S_{ru}}{h_{ru}} = 0$ and so $x \overset{\mathcal{I}^{\sigma\theta}}{\sim} y$. □

Theorem 2.6 Suppose that $x = (x_{kj}), y = (y_{kj}) \in \ell_{\infty}^2$. If double sequences x and y are asymptotically lacunary \mathcal{I}_2 -invariant equivalent of multiple L , then these sequences strongly asymptotically lacunary σ_2 -equivalent of multiple L .

Proof. Suppose that $x, y \in \ell_{\infty}^2$ and $x \overset{\mathcal{I}^{\sigma\theta}}{\sim} y$. Let $\varepsilon > 0$. By assumption, we have $V_2^{\theta}(A_{\varepsilon}^{\sim}) = 0$. The boundedness of x and y implies that there exists an $M > 0$ such that

$$\left| \frac{x_{\sigma^k(m), \sigma^j(n)}}{y_{\sigma^k(m), \sigma^j(n)}} - L \right| \leq M$$

for all $k, j \in I_{ru}$ and for every $m, n \in \mathbb{N}$ we have

$$\begin{aligned} \frac{1}{h_{ru}} \sum_{k,j \in I_{ru}} \left| \frac{x_{\sigma^k(m), \sigma^j(n)}}{y_{\sigma^k(m), \sigma^j(n)}} - L \right| &= \frac{1}{h_{ru}} \sum_{k,j \in I_{ru}} \left| \frac{x_{\sigma^k(m), \sigma^j(n)}}{y_{\sigma^k(m), \sigma^j(n)}} - L \right| \\ &\quad \left| \frac{x_{\sigma^k(m), \sigma^j(n)}}{y_{\sigma^k(m), \sigma^j(n)}} - L \right| \geq \varepsilon \\ &\quad + \frac{1}{h_{ru}} \sum_{k,j \in I_{ru}} \left| \frac{x_{\sigma^k(m), \sigma^j(n)}}{y_{\sigma^k(m), \sigma^j(n)}} - L \right| \\ &\quad \left| \frac{x_{\sigma^k(m), \sigma^j(n)}}{y_{\sigma^k(m), \sigma^j(n)}} - L \right| < \varepsilon \\ &\leq M \frac{\max_{m,n} \left| \left\{ (k, j) \in I_{ru} : \left| \frac{x_{\sigma^k(m), \sigma^j(n)}}{y_{\sigma^k(m), \sigma^j(n)}} - L \right| \geq \varepsilon \right\} \right|}{h_{ru}} + \varepsilon \\ &\leq M \frac{S_{ru}}{h_{ru}} + \varepsilon. \end{aligned}$$

Hence, we obtain

$$\lim_{r,u \rightarrow \infty} \frac{1}{h_{ru}} \sum_{k,j \in I_{ru}} \left| \frac{x_{\sigma^k(m), \sigma^j(n)}}{y_{\sigma^k(m), \sigma^j(n)}} - L \right| = 0,$$

uniformly in m and n . □

Theorem 2.7

$$\mathcal{I}_{2(L)}^{\sigma\theta} \cap \ell_{\infty}^2 = [\mathcal{I}_{2(L)}^{\sigma\theta}] \cap \ell_{\infty}^2.$$

Proof. This is an immediate consequence of Theorem 2.5 and Theorem 2.6. □

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Statistical Lacunary Invariant Summability of Double Sequences

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Abstract:

In this study, we give definitions of lacunary σ -summability, strongly p -lacunary σ -summability and statistical lacunary σ -convergence for double sequences. We also examine the existence of some relations among the definitions of statistical lacunary σ -convergence, lacunary invariant statistical convergence and strongly p -lacunary σ -summability.

1. Introduction and Background

The concept of statistical convergence was first introduced by Fast [2] and since then it has been studied by Šalát [15], Fridy [3] and many others, too.

A sequence $x = (x_k)$ is said to be statistically convergent to L if for every $\varepsilon > 0$

$$\lim_{n \rightarrow \infty} \frac{1}{n} \left| \{k \leq n : |x_k - L| \geq \varepsilon\} \right| = 0,$$

where the vertical bars indicate the number of elements in the enclosed set.

Let σ be a mapping of the positive integers into themselves. A continuous linear functional ϕ on ℓ_∞ , the space of real bounded sequences, is said to be an invariant mean or a σ -mean if it satisfies following conditions:

1. $\phi(x) \geq 0$, when the sequence $x = (x_n)$ has $x_n \geq 0$ for all n ,
2. $\phi(e) = 1$, where $e = (1, 1, 1, \dots)$ and
3. $\phi(x_{\sigma(n)}) = \phi(x_n)$ for all $x \in \ell_\infty$.

The mappings σ are assumed to be one-to-one and such that $\sigma^m(n) \neq n$ for all positive integers n and m , where $\sigma^m(n)$ denotes the m th iterate of the mapping σ at n . Thus, ϕ extends the limit functional on c , the space of convergent sequences, in the sense that $\phi(x) = \lim x$ for all $x \in c$.

In the case σ is translation mappings $\sigma(n) = n + 1$, the σ -mean is often called a Banach limit.

Many authors have studied on the concepts of invariant mean and invariant convergence (see, [5, 6, 8, 10, 14, 16, 20]).

By a lacunary sequence we mean an increasing integer sequence $\theta = \{k_r\}$ such that $k_0 = 0$ and $h_r = k_r - k_{r-1} \rightarrow \infty$ as $r \rightarrow \infty$. The intervals determined by θ is denoted by $I_r = (k_{r-1}, k_r]$ (see, [9]).

The space of lacunary strong σ -convergent sequences L_θ was defined by Savaş [17] as below:

$$L_\theta = \left\{ x = (x_k) : \lim_{r \rightarrow \infty} \frac{1}{h_r} \sum_{k \in I_r} |x_{\sigma^k(m)} - L| = 0, \text{ uniformly in } m \right\}.$$

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Savaş and Nuray [18] introduced the concept of lacunary σ -statistically convergent sequence as follows:

Let $\theta = \{k_r\}$ be a lacunary sequence. A sequence $x = (x_k)$ is said to be $S_{\sigma\theta}$ -convergent to L if for every $\varepsilon > 0$

$$\lim_{r \rightarrow \infty} \frac{1}{h_r} \left| \{k \in I_r : |x_{\sigma^k(m)} - L| \geq \varepsilon\} \right| = 0,$$

uniformly in m . It is denoted by $x_k \rightarrow L(S_{\sigma\theta})$.

The concept of lacunary invariant summability and the space $[V_{\theta\sigma}]_q$ were defined by Pancaroğlu and Nuray [11] as below:

Let $\theta = \{k_r\}$ be a lacunary sequence. A sequence $x = (x_k)$ is said to be lacunary invariant summable to L if

$$\lim_{r \rightarrow \infty} \frac{1}{h_r} \sum_{k \in I_r} x_{\sigma^k(m)} = L,$$

uniformly in m .

Let $0 < q < \infty$. A sequence $x = (x_k)$ is said to be strongly lacunary q -invariant convergent to L if

$$\lim_{r \rightarrow \infty} \frac{1}{h_r} \sum_{k \in I_r} |x_{\sigma^k(m)} - L|^q = 0,$$

uniformly in m . It is denoted by $x_k \rightarrow L([V_{\theta\sigma}]_q)$

The concepts of convergence for double sequences have been studied by many authors (see, [1, 4, 12, 13, 21]).

A double sequence $x = (x_{kj})$ is said to be convergent to L in Pringsheim's sense if for every $\varepsilon > 0$, there exists $N_\varepsilon \in \mathbb{N}$ such that $|x_{kj} - L| < \varepsilon$, whenever $k, j \geq N_\varepsilon$.

A double sequence $x = (x_{kj})$ is said to be bounded if there exists an $M > 0$ such that $|x_{kj}| < M$ for all k and j , i.e., if $\sup_{k,j} |x_{kj}| < \infty$.

The set of all bounded double sequences will be denoted by ℓ_∞^2 .

Mursaleen and Edely [7] introduced the concept of statistically convergence for double sequences as follows:

A double sequence $x = (x_{kj})$ is said to be statistically convergent to L if for every $\varepsilon > 0$

$$\lim_{m,n \rightarrow \infty} \frac{1}{mn} \left| \{(k, j), k \leq m \text{ and } j \leq n : |x_{kj} - L| \geq \varepsilon\} \right| = 0.$$

The double sequence $\theta_2 = \{k_r, j_u\}$ is called double lacunary sequence if there exist two increasing sequence of integers such that

$$k_0 = 0, h_r = k_r - k_{r-1} \rightarrow \infty \text{ and } j_0 = 0, \bar{h}_u = j_u - j_{u-1} \rightarrow \infty \text{ as } r, u \rightarrow \infty.$$

We use the following notations in the sequel: 2018 Kirşehir / TURKEY

$$k_{ru} = k_r j_u, h_{ru} = h_r \bar{h}_u, I_{ru} = \{(k, j) : k_{r-1} < k \leq k_r \text{ and } j_{u-1} < j \leq j_u\}.$$

Using the double lacunary sequence concept, the concept of lacunary σ -statistically convergence for double sequences and similar concepts were defined by Savaş and Patterson [19] as below:

Let $\theta_2 = \{k_r, j_u\}$ be a double lacunary sequence. A double sequence $x = (x_{kj})$ is said to be lacunary invariant statistically convergent to L if for every $\varepsilon > 0$

$$\lim_{r,u \rightarrow \infty} \frac{1}{h_{ru}} \left| \{(k, j) \in I_{ru} : |x_{\sigma^k(m), \sigma^j(n)} - L| \geq \varepsilon\} \right| = 0,$$

uniformly in m and n . It is denoted by $x_{kj} \rightarrow L(S_2^{\sigma\theta})$.

A double sequence $x = (x_{kj})$ is said to be strongly lacunary invariant convergent to L if

$$\lim_{r,u \rightarrow \infty} \frac{1}{h_{ru}} \sum_{k,j \in I_{ru}} |x_{\sigma^k(m), \sigma^j(n)} - L| = 0,$$

uniformly in m and n . It is denoted by $x_{kj} \rightarrow L([V_2^{\sigma\theta}])$.

2. Statistical Lacunary Invariant Summability of Double Sequences

In this study, we give definitions of lacunary σ -summability, strongly p -lacunary σ -summability and statistical lacunary σ -convergence for double sequences. We also examine the existence of some relations among the definitions of statistical lacunary σ -convergence, lacunary invariant statistical convergence and strongly p -lacunary σ -summability.

Definition 2.1 Let $\theta_2 = \{k_r, j_u\}$ be a double lacunary sequence. A double sequence $x = (x_{kj})$ is said to be statistical lacunary σ -convergent to L if for every $\varepsilon > 0$

$$\lim_{v,w \rightarrow \infty} \frac{1}{vw} \left| \left\{ (k, j), k \leq v \text{ and } j \leq w : \left| \frac{1}{h_{ru}} \sum_{k,j \in I_{ru}} x_{\sigma^k(m), \sigma^j(n)} - L \right| \geq \varepsilon \right\} \right| = 0,$$

uniformly in m and n . In this case, we write $x_{kj} \rightarrow L(S_2^{\theta\sigma})$.

In other words, a double sequence $x = (x_{kj})$ is statistical lacunary σ -convergent to L if and only if the sequence

$$\left(\frac{1}{h_{ru}} \sum_{k,j \in I_{ru}} x_{\sigma^k(m), \sigma^j(n)} \right)$$

is statistical convergent to L .

Theorem 2.2 Assume that $x = (x_{kj}) \in \ell_{\infty}^2$. If x is lacunary invariant statistical convergent to L , then this sequence is statistical lacunary σ -convergent to L .

Proof. Let $x = (x_{kj})$ be a bounded double sequence and lacunary invariant statistical convergent to L . Let take a set $A(\varepsilon)$ as follows:

$$A(\varepsilon) = \{k_{r-1} \leq k \leq k_r, j_{u-1} \leq j \leq j_u : |x_{\sigma^k(m), \sigma^j(n)} - L| \geq \varepsilon\},$$

for each $m \geq 1$ and $n \geq 1$. Then we have

$$\begin{aligned} \left| \frac{1}{h_{ru}} \sum_{(k,j) \in I_{ru}} x_{\sigma^k(m), \sigma^j(n)} - L \right| &= \left| \frac{1}{h_{ru}} \sum_{(k,j) \in I_{ru}} (x_{\sigma^k(m), \sigma^j(n)} - L) \right| \\ &\leq \left| \frac{1}{h_{ru}} \sum_{(k,j) \in A(\varepsilon)} (x_{\sigma^k(m), \sigma^j(n)} - L) \right| \\ &\leq \left(\sup_{k,j} |x_{\sigma^k(m), \sigma^j(n)} - L| \right) \frac{1}{h_{ru}} |A(\varepsilon)| \rightarrow 0 \end{aligned}$$

as $r, u \rightarrow \infty$, which implies

$$\left| \frac{1}{h_{ru}} \sum_{(k,j) \in I_{ru}} x_{\sigma^k(m), \sigma^j(n)} - L \right| \rightarrow 0,$$

for all m and n . That is, $x = (x_{kj})$ is statistical lacunary σ -convergent to L . \square

Definition 2.3 Let $\theta_2 = \{k_r, j_u\}$ be a double lacunary sequence. A double sequence $x = (x_{kj})$ is said to be lacunary σ -summable to L if

$$\lim_{r,u \rightarrow \infty} \frac{1}{h_{ru}} \sum_{k,j \in I_{ru}} x_{\sigma^k(m), \sigma^j(n)} = L,$$

uniformly in m and n . In this case, we write $x_{kj} \rightarrow L(V_2^{\sigma\theta})$.

Definition 2.4 Let $\theta_2 = \{k_r, j_u\}$ be a double lacunary sequence and $0 < p < \infty$. A double sequence $x = (x_{kj})$ is said to be strongly p -lacunary σ -summable to L if

$$\lim_{r,u \rightarrow \infty} \frac{1}{h_{ru}} \sum_{k,j \in I_{ru}} |x_{\sigma^k(m), \sigma^j(n)} - L|^p = 0,$$

uniformly in m and n . In this case, we write $x_{kj} \rightarrow L([V_2^{\sigma\theta}]_p)$.

Theorem 2.5 If a double sequence $x = (x_{kj})$ is strongly p -lacunary σ -summable to L , then this sequence is lacunary invariant statistical convergent to L .

Proof. Let $x = (x_{kj})$ is strongly p -lacunary σ -summable to L . Then, for each $m \geq 1$ and $n \geq 1$

$$\begin{aligned} \frac{1}{h_{ru}} \sum_{(k,j) \in I_{ru}} |x_{\sigma^k(m), \sigma^j(n)} - L|^p &= \frac{1}{h_{ru}} \sum_{\substack{(k,j) \in I_{ru} \\ |x_{\sigma^k(m), \sigma^j(n)} - L| \geq \varepsilon}} |x_{\sigma^k(m), \sigma^j(n)} - L|^p \\ &+ \frac{1}{h_{ru}} \sum_{\substack{(k,j) \in I_{ru} \\ |x_{\sigma^k(m), \sigma^j(n)} - L| < \varepsilon}} |x_{\sigma^k(m), \sigma^j(n)} - L|^p, \end{aligned}$$

therefore we have

$$\begin{aligned} \frac{1}{h_{ru}} \sum_{(k,j) \in I_{ru}} |x_{\sigma^k(m), \sigma^j(n)} - L|^p &\geq \frac{1}{h_{ru}} \sum_{\substack{(k,j) \in I_{ru} \\ |x_{\sigma^k(m), \sigma^j(n)} - L| \geq \varepsilon}} |x_{\sigma^k(m), \sigma^j(n)} - L|^p \\ &\geq \frac{1}{h_{ru}} \varepsilon^p \cdot |A(\varepsilon)|. \end{aligned}$$

So if limit is taken as $r, u \rightarrow \infty$, we have

$$\varepsilon^p \cdot \lim_{r, u \rightarrow \infty} \frac{1}{h_{ru}} \left| \{(k, j) \in I_{ru} : |x_{\sigma^k(m), \sigma^j(n)} - L| \geq \varepsilon\} \right| \leq \lim_{r, u \rightarrow \infty} \frac{1}{h_{ru}} \sum_{(k,j) \in I_{ru}} |x_{\sigma^k(m), \sigma^j(n)} - L|^p \rightarrow 0.$$

That is, $x = (x_{kj})$ is lacunary invariant statistical convergent to L . \square

Theorem 2.6 Assume that $x = (x_{kj}) \in \ell_\infty^2$. If x is lacunary invariant statistical convergent to L , then this sequence is strongly p -lacunary σ -summable to L .

Proof. Suppose that $x = (x_{kj})$ is a bounded double sequence and lacunary invariant statistical convergent to L . Since x is bounded, there exists $M > 0$ such that

$$|x_{\sigma^k(m), \sigma^j(n)} - L| \leq M$$

uniformly in m and n . Now that $x = (x_{kj})$ is lacunary invariant statistical convergent to L , for every $\varepsilon > 0$ we have

$$\lim_{r, u \rightarrow \infty} \frac{1}{h_{ru}} \left| \{(k, j) \in I_{ru} : |x_{\sigma^k(m), \sigma^j(n)} - L| \geq \varepsilon\} \right| = 0$$

uniformly in m and n . Also, we can write

$$\begin{aligned} \frac{1}{h_{ru}} \sum_{(k,j) \in I_{ru}} |x_{\sigma^k(m), \sigma^j(n)} - L|^p &= \frac{1}{h_{ru}} \sum_{\substack{(k,j) \in I_{ru} \\ (k,j) \in A(\varepsilon)}} |x_{\sigma^k(m), \sigma^j(n)} - L|^p \\ &+ \frac{1}{h_{ru}} \sum_{\substack{(k,j) \in I_{ru} \\ (k,j) \notin A(\varepsilon)}} |x_{\sigma^k(m), \sigma^j(n)} - L|^p \\ &= t^{(1)}(r, u) + t^{(2)}(r, u) \end{aligned}$$

where

$$t^{(1)}(r, u) = \frac{1}{h_{ru}} \sum_{\substack{(k,j) \in I_{ru} \\ (k,j) \in A(\varepsilon)}} |x_{\sigma^k(m), \sigma^j(n)} - L|^p$$

and

$$t^{(2)}(r, u) = \frac{1}{h_{ru}} \sum_{\substack{(k,j) \in I_{ru} \\ (k,j) \notin A(\varepsilon)}} |x_{\sigma^k(m), \sigma^j(n)} - L|^p.$$

Now if $(k, j) \notin A(\varepsilon)$, then $t^{(2)}(r, u) < \varepsilon^p$. If $(k, j) \in A(\varepsilon)$, then

$$t^{(1)}(r, u) \leq \left(\sup_{k,j} |x_{\sigma^k(m), \sigma^j(n)} - L| \right) \frac{|A(\varepsilon)|}{h_{ru}} \leq M \frac{|A(\varepsilon)|}{h_{ru}} \rightarrow 0.$$

Thus

$$\frac{1}{h_{ru}} \sum_{(k,j) \in I_{ru}} |x_{\sigma^k(m), \sigma^j(n)} - L|^p \rightarrow 0,$$

uniformly in m and n . \square

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On Existence and Uniqueness of Some Class Nonlinear Eigenvalue Problem

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Abstract: We will investigate on existence and uniqueness of some class nonlinear eigenvalue problem

1. Introduction

In this paper, we derive a new boundedness and compactness result for the Hardy operator in variable exponent Lebesgue spaces (VELS) $L^{p(\cdot)}(0, l)$. A maximally weak condition is assumed on the exponent function. The last time, such a study was carry out in [1,2,3,4,5,6,7,8,9]. For a study the Dirichlet problem of some class nonlinear eigenvalue problem with nonstandard growth condition the obtained results is applied. Such equations arise in the studies of the so called Winslow effect physical phenomena [11] in the smart materials. In this connection, we mention recent studies for the multidimensional cases with application of Ambrosetti-Rabinoviches Mountain pass theorem approaches (see, e.g. in [1,10, 12]).

Theorem 1.1 Let $q, p(0, l) \rightarrow (1, \infty)$ be measurable functions with $q(x) \geq p(x)$ on $(0, l)$. Assume p be monotony increasing and the function $x^{-1/(p'(x)+\delta)}$ is almost decreasing on $(0, l)$. Then operator H boundedly acts the space $L^p(0, l)$ into $L^{q(\cdot), -1/p' - 1/q(\cdot)}(0, l)$. Moreover, the norm of mapping depends on p^-, p^+, δ, β .

Theorem 1.2 Let $q, p(0, l) \rightarrow (1, \infty)$ be measurable functions such that $\infty > q^+ \geq q(x) \geq p(x) \geq p^- > 1$ for all $x \in (0, l)$. Assume that p be monotony increasing and $x^{-1/p'+\varepsilon}$ is almost decreasing. Then the identity operator maps boundedly the space $W_{p(\cdot)}^1(0, l)$ into $L^{q(\cdot), -1/p' - 1/q(\cdot)}(0, l)$. Moreover, the norm of mapping is estimated by a constant depending on $p^-, p^+, q, \varepsilon, \beta$.

Notice, Theorem 1.2 states the inequality

$$\|yx^{-1/p' - 1/q(\cdot)}\|_{L^{q(\cdot), -1/p' - 1/q(\cdot)}(0, l)} \leq \|y'\|_{L^{p(\cdot)}(0, l)} \quad (1.1)$$

for any absolutely continues function $y : (0, l) \rightarrow \mathbb{R}$ with $y(0) = 0$.

In the given assertions, $L^{p, \alpha}(0, l)$ denotes the space of measurable functions with finite norm $\|yx^\alpha\|_{L^{p(\cdot)}(0, l)}$, while $W_{p(\cdot), \alpha}^1(0, l)$ stands the space of absolutely continuous functions y with $y(0) = 0$ and finite norm

$$\|y\|_{W_{p(\cdot)}^1} = \|y'\|_{L^{p(\cdot)}}.$$

We say, the function $\alpha : (0, l) \rightarrow (0, \infty)$ is almost increasing (decreasing) if there exists a constant $C > 0$ such that for any $0 < t_1 < t_2 < l$ it holds $\alpha(t_1) \leq C\alpha(t_2)$ ($\alpha(t_1) \geq C\alpha(t_2)$) We need the following assertion

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Lemma 1.3 Let $p(x)$ be increasing for $x \in (0, l)$. Let $t \in A_n(x) = (2^{-n-1}x, 2^{-n}x]$. Then it holds

$$t^{-1/(p'(t))} \leq Ct^{-1/(p_{x,n}^-)'} \quad (1.2)$$

where $p_{x,n} = \inf_{t \in A_n(x)} p(t)$

Proof. Let $y \in A_n(x)$ be a point with $t^{-1/(p'(y))} \leq 2t^{-1/(p_{x,n}^-)'}$. Let $y < t$ and both lie in $A_n(x)$. Then using almost decreasing of $x^{-1/p'+\varepsilon}$ it follows that

$$t^{-1/(p'(t))+\varepsilon} \leq cy^{-1/(p'(y))+\varepsilon}$$

Using $t, y \in A_n(x)$, $(p_{x,n}^-)' > 1$ it follows

$$t^{-1/(p'(t))} \leq 2^\varepsilon Cy^{-1/(p'(y))} \leq 2^{2+\varepsilon} Ct^{-1/(p_{x,n}^-)'}$$

Now let $y > t$, then using increasing of p , $1/p'$ also will be increasing. Since $1/(p'(t)) < 1/(p'(y))$, it follows that

$$(1/t)^{1/(p'(t))} \leq C(1/t)^{1/(p'(y))} \leq 2Ct^{-1/(p_{x,n}^-)'},$$

where $C = l^{1/(p^-)' + 1/(p^+)}$

The Lemma 1.3 has been proved.

Proof of Theorem 1.1. Let $f : (0, l) \rightarrow (0, \infty)$ be a positive measurable function. It holds the identity

$$Hf(x) = \sum_{n=1}^{\infty} \int_{2^{-n-1}x}^{2^{-n}x} f(t) dt \quad (1.3)$$

Assume $\|f\|_p = 1$. Using the triangle property of $p(\cdot)$ - norms

$$\|x^\alpha Hf\|_{q(\cdot)} \leq \sum_{n=1}^{\infty} \|x^\alpha \int_{A_n(x)} f(t) dt\|_{q(\cdot)}, \quad (1.4)$$

with $\alpha(x) = -1/p'(x) - 1/q(x)$ (recall $A_n(x) = (2^{-n-1}x, 2^{-n}x]$) Derive estimation for every summand in (4). In this purpose get estimation for the proper modular

$$I_{q(\cdot)}(x^{\alpha(\cdot)} \int_{A_n(x)} f(t) dt) = \int_0^l (x^{\alpha(\cdot)} \int_{A_n(x)} f(t) dt)^{q(x)} dx$$

Applying the assumption on p (decreasing of $x^{-1/p'+\varepsilon}$ and using the expression for $q(x) = 1/(-\alpha - 1/(p'(x)))$) we have

$$\begin{aligned} I_q(x^{-1/p'-1/q} \int_{A_n(x)} f(t) dt) &= \int_0^l (x^{-1/p'+\varepsilon} \int_{A_n(x)} f(t) dt)^{q(x)} dx \setminus x^{1+\varepsilon q(x)} \\ &\leq C^{q^+} 2^{-n\varepsilon q^+} \int_0^l dx \setminus x \left(\int_{A_n(x)} f(t) t^{-1/(p'(t))} dt \right)^{q(x)} \end{aligned} \quad (1.5)$$

Notice, we have used that $x^{-1/p'(x)+\varepsilon} \leq Ct^{-1/p'(t)+\varepsilon}$ for any $0 < x < l$ and that $2^{-n-1}x < t \leq 2^{-n}x$ by using the almost decreasing of $x^{-1/p'(x)+\varepsilon}$.

Therefore, from (4) using Holders inequality, it follows

$$\begin{aligned} &I_q(x^{\alpha(\cdot)} \int_{A_n(x)} f(t) dt) \\ &\leq C^{q^+} 2^{-n\varepsilon q^+} \int_0^l dx/x \left(\int_{A_n(x)} (f(t))^{p_{x,n}^-} dt \right)^{q(x)/(p_{x,n}^-)} \left(\int_{A_n(x)} t^{-(p_{x,n}^-)'/(p'(t))} dt \right)^{q(x)/(p_{x,n}^-)} \end{aligned} \quad (1.6)$$

Applying this Lemma 1.3 and estimate (2) it follows from (6) that

$$I_q(x^{\alpha(\cdot)}) \int_{A_n(x)} f(t) dt \leq \int_0^l dx/x \left(\int_{A_n(x)} (f(t))^{p_{x,n}^-} dt \right)^{q(x)/(p_{x,n}^-)} (Cl n 2)^{q^+ \setminus p^-} 2^{-n\epsilon q^-} C^{q^+}$$

Since

$$\int_{A_n(x)} (f(t))^{p_{x,n}^-} dt \leq \int_{A_n(x)} (f(t))^{p(t)} dt + \int_{A_n(x)} dt \leq 1 + 2^{-n} x \leq 1 + 2^{-n} l \leq l + 1.$$

it follows

$$\begin{aligned} & I_q(x^\alpha \int_{A_n(x)} f(t) dt) \\ & \leq (Cl n 2)^{q^+} 2^{-n\epsilon q^-} \int_0^l dx/x (1/(l+1)) \int_{A_n(x)} (f(t))^{p_{x,n}^-} dt^{(q(x))/(p_{x,n}^-)} (l+1)^{q^+} \\ & \leq (Cl n 2(l+1))^{q^+} \int_0^l (1/(l+1)) \int_{A_n(x)} [(f(t))^{p(t)} + 1] dt^{(p(x))/(p_{x,n}^-)} dx/x \\ & \leq 2^{-n\epsilon q^-} C^{q^+} (Cl n 2)^{q^+} (l+1)^{q^+ - 1} \int_0^l dx/x \left(\int_{A_n(x)} [(f(t))^{p(t)} + 1] dt \right). \end{aligned}$$

Hence,

$$\begin{aligned} I_q(x^{\alpha(\cdot)}) \int_{A_n(x)} (t) dt & \leq C_3 2^{-n\epsilon q^-} C^{q^+} \int_0^l \left(\int_{A_n(x)} [(f(t))^{p(t)} + 1] dt \right) dx/x \\ & \leq C_3 \int_0^{2^{-n} l} [(f(t))^{p(t)} + 1] \int_{2^{n_t}}^{2^{n+1} t} dx/x = C^{q^+} C_3 2^{-n\epsilon q^-} \ln 2 \int_0^{2^{-n} l} [(f(t))^{p(t)} + 1] dt \end{aligned}$$

$$\leq C^{q^+} C_3 2^{-n\epsilon q^-} \ln 2 (1 + 2^{-l}) C_4 2^{-n\epsilon q^-}$$

Therefore, it has been proved that

$$I_q(x^{-1/p' - 1/q} \int_{A_n(x)} f(t) dt) \leq C_4 2^{-n\epsilon q^-}$$

which implies

$$\|x^{-1/p' - 1/q} \int_{A_n(x)} f(t) dt\|_{(q(\cdot); (0, l))} \leq C_4^{1/q^+} 2^{-n\epsilon q^- / q^+} \quad (1.7)$$

Inserting (7) in (4), we get

$$\|x^{-1/p' - 1/q} Hf\|_{(q(\cdot); (0, l))} \leq C_4^{1/q^+} \sum_{n=1}^{\infty} 2^{-n\epsilon q^- / q^+} = C_5$$

The Theorem 1.1 has been proved.

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Obtaining The Finite Difference Approximation of Transmission Conditions of Deformation Problem for Multilayered Materials

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Abstract: In this study, we use finite element method to obtain the numerical solution of the plane deformation problem for multilayered materials. The mathematical model of the problem is expressed by the system of Lamé equations. Some differences of the mechanical properties of the materials composed the layers make it impossible to solve such problems with classical finite difference methods. In this work, to ensure continuity at the common boundary between the layers, we obtain the numerical expressions of the transmission conditions by using the finite element method. The relation between the numerical expressions obtained by using finite element method and finite differences method is shown.

1. Introduction

The contact problem related to the deformation of a rigid punch was considered by many authors [1]-[3]. The paper [1] is devoted to the analysis of the infinitesimal deformations of a linear elastic anisotropic layer by using Stroh formalism method. The work [4] deals with the contact problem of a stiff spherical indenter with a composite plate by dint of the commercial software and the problem are simulated by a 2-D axisymmetric model. The results numerically obtained in [5] show independence of the indentation response of an orthotropic laminate from the material, the author demonstrate dependence of the thickness of the multilayered material. In the paper [6] plane and axisymmetric contact problems for a three-layered elastic half-space are considered. In the present paper, we give an analysis and numerical solution of the boundary value problem for the Lamé system, modeling the contact problem for a multilayered material. By using the biquadratic basic functions, the transmission conditions are obtained on the boundaries of interlayer by the Finite Element Method and the interlayer stresses are analyzed.

2. Problem Formulation

The mathematical model of the contact problem related to the deformation of a rigid punch with a frictional pressure of a finite dimensional elastic material is expressed by the boundary value problem for the Lamé equation as follow [7]:

$$(\lambda + \mu)\text{grad}(\text{div}u(x)) + \mu\nabla^2u(x) = F(x), \quad x \in \Omega \quad (1)$$

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$$\begin{cases} u_2(x_1, 0) \leq -\alpha + \varphi(x_1), & \sigma_{22}(u(x_1, x_2)) \leq 0, \\ \sigma_{22}(u(x_1, x_2)) [u_2(x_1, 0) + \alpha - \varphi(x_1)] = 0, & (x_1, x_2) \in \Gamma_0; \\ \sigma_{11}(u(x_1, x_2)) = 0, & (x_1, x_2) \in \Gamma_\sigma; \\ u_1(x_1, x_2) = 0, & u_2(x_1, x_2) = 0, & (x_1, x_2) \in \Gamma_u; \\ u_1(0, x_2) = 0, & (0, x_2) \in \Gamma_1; \\ \sigma_{12}(u(x_1, x_2)) = 0, & (x_1, x_2) \in \partial\Omega. \end{cases} \quad (2)$$

Here $\Omega := \{(x_1, x_2) \in \mathbb{R}^2 : 0 < x_1 < l_{x_1}, -l_{x_2} < x_2 < 0, l_{x_1} > 0, l_{x_2} > 0\}$ is the region occupied by the cross-section of the material under the influence of the punch and $\Gamma_0, \Gamma_\sigma, \Gamma_u, \Gamma_1 \subset \partial\Omega$ are the relevant parts of the boundary of the region Ω (Fig. 1). Namely, $\Gamma_\sigma = \{(l_{x_1}, x_2) : -l_{x_2} < x_2 < 0\}$, $\Gamma_u = \{(x_1, -l_{x_2}) : 0 < x_1 < l_{x_1}\}$, $\Gamma_0 = \{(x_1, 0) : 0 < x_1 < l_{x_1}\}$, $\Gamma_1 = \{(0, x_2) : -l_{x_2} < x_2 < 0\}$. Since the condition on the upper boundary Γ_0 is given by inequality, the contact region of the punch $\Gamma_c = \{(x_1, x_2) \in \Gamma_0 : u_2 = -\alpha + \varphi(x_1)\}$ is not certain and the problem is nonlinear.

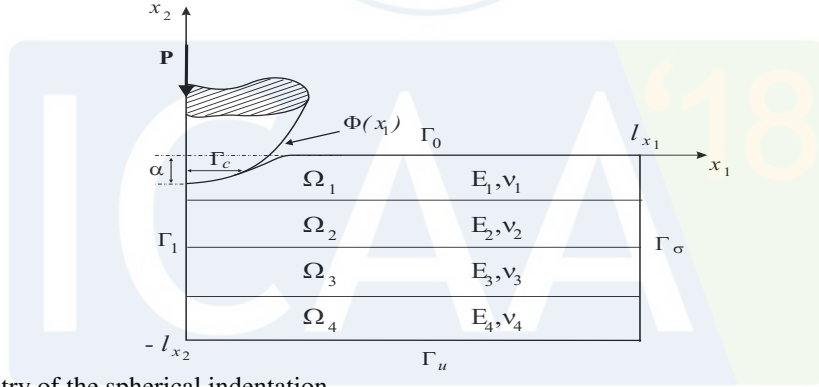


Fig. 1. Geometry of the spherical indentation

The solution of the problem (1)-(2) minimizes by the following functional

$$J(u) = (Au, u) - 0.5b(v), \quad u \in V,$$

on the set

$$V = \{u \in H^1(\Omega) : u_1(0, x_2) = 0, (0, x_2) \in \Gamma_1; u_1(x_1, -l_{x_2}) = u_2(x_1, -l_{x_2}) = 0, \\ (x_1, -l_{x_2}) \in \Gamma_u; u_2(x_1, 0) \leq -\alpha + \varphi(x_1), (x_1, 0) \in \Gamma_0\}$$

in the Sobolov space $H^1(\Omega) := W_2^1(\Omega) \times W_2^1(\Omega)$.

Here the bilinear and the linear parts of above functional have the following form

$$(Au, v) = \sum_k \int \int_{\Omega_k} \left\{ \left[(\lambda_k + 2\mu_k) \frac{\partial u_1}{\partial x_1} + \lambda_k \frac{\partial u_2}{\partial x_2} \right] \frac{\partial v_1}{\partial x_1} + \right. \\ \left. \mu_k \left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1} \right) \left(\frac{\partial v_1}{\partial x_2} + \frac{\partial v_2}{\partial x_1} \right) + \left[\lambda_k \frac{\partial u_1}{\partial x_1} + (\lambda_k + 2\mu_k) \frac{\partial u_2}{\partial x_2} \right] \frac{\partial v_2}{\partial x_2} \right\} dx_1 dx_2, \quad (3)$$

$$b(v) = \sum_k \int \int_{\Omega_k} [F_1 v_1 + F_2 v_2] dx_1 dx_2, \quad (4)$$

respectively.

3. Finite-Element Formulation

Let us use here the biquadratic basic functions $\xi_{ij}(x_1, x_2)$ to analyze the problem (1)-(2). Here $\xi_{ij}(x_{1,pq}, x_{2,pq}) = \begin{cases} 1 & (i, j) = (p, q), \\ 0 & (i, j) \neq (p, q) \end{cases}$ and the compact support of the basic function is $\bar{\Omega}_{ij} = \overline{e_{i-1j-1} \cup e_{i-1j} \cup e_{ij-1} \cup e_{ij}}$. A numerical solution (u^h, v^h) has the following form:

$$u^h(x_1, x_2) = \sum_{(ij)} u_{ij} \xi_{ij}(x_1, x_2), \quad v^h(x_1, x_2) = \sum_{(ij)} v_{ij} \xi_{ij}(x_1, x_2).$$

The local stiffness matrix (LSM) of the finite element e_{ij} is constructed as follows

$$\mathcal{L}_{ij} = \begin{bmatrix} \mathcal{L}_{11}^{(ij)} & \mathcal{L}_{12}^{(ij)} \\ \mathcal{L}_{21}^{(ij)} & \mathcal{L}_{22}^{(ij)} \end{bmatrix}.$$

Elements of the LSM are calculated by the formulas

$$\begin{aligned} [\mathcal{L}_{11}^{(ij)}] &= \left[\iint_{e_{ij}} \left\{ (\lambda + 2\mu) \frac{\partial \xi_{ij}}{\partial x_1} \frac{\partial \xi_{kl}}{\partial x_1} + \mu \frac{\partial \xi_{ij}}{\partial x_2} \frac{\partial \xi_{kl}}{\partial x_2} \right\} dx_1 dx_2 \right], \\ [\mathcal{L}_{12}^{(ij)}] &= \left[\iint_{e_{ij}} \left\{ \lambda \frac{\partial \xi_{ij}}{\partial x_2} \frac{\partial \xi_{kl}}{\partial x_1} + \mu \frac{\partial \xi_{ij}}{\partial x_1} \frac{\partial \xi_{kl}}{\partial x_2} \right\} dx_1 dx_2 \right], \\ [\mathcal{L}_{21}^{(ij)}] &= \left[\iint_{e_{ij}} \left\{ \lambda \frac{\partial \xi_{ij}}{\partial x_1} \frac{\partial \xi_{kl}}{\partial x_2} + \mu \frac{\partial \xi_{ij}}{\partial x_2} \frac{\partial \xi_{kl}}{\partial x_1} \right\} dx_1 dx_2 \right], \\ [\mathcal{L}_{22}^{(ij)}] &= \left[\iint_{e_{ij}} \left\{ (\lambda + 2\mu) \frac{\partial \xi_{ij}}{\partial x_2} \frac{\partial \xi_{kl}}{\partial x_2} + \mu \frac{\partial \xi_{ij}}{\partial x_1} \frac{\partial \xi_{kl}}{\partial x_1} \right\} dx_1 dx_2 \right], \quad k, l = \overline{1, 9}. \end{aligned}$$

So, using well-known finite-element technology we calculate the LSM $\mathcal{L}_{ij} = \{(l_{pq})\}$, $p, q = \overline{1, 18}$ for the elements e_{ij} . We can define unknown vectors corresponding to $e_{i-1j-1}, e_{i-1j}, e_{ij-1}, e_{ij}$ neighboring with point (i, j) (Figure 2) as follows:

$$\begin{aligned} \omega_{i-1j-1} &= (u_{i-1j-1}, u_{i-1j-\frac{1}{2}}, u_{i-1j}, u_{i-\frac{1}{2}j-1}, u_{i-\frac{1}{2}j-\frac{1}{2}}, u_{i-\frac{1}{2}j}, u_{ij-1}, u_{ij-\frac{1}{2}}, u_{ij}, \\ &\quad v_{i-1j-1}, v_{i-1j-\frac{1}{2}}, v_{i-1j}, v_{i-\frac{1}{2}j-1}, v_{i-\frac{1}{2}j-\frac{1}{2}}, v_{i-\frac{1}{2}j}, v_{ij-1}, v_{ij-\frac{1}{2}}, v_{ij}), \\ \omega_{i-1j} &= (u_{i-1j}, u_{i-1j+\frac{1}{2}}, u_{i-1j+1}, u_{i-\frac{1}{2}j}, u_{i-\frac{1}{2}j+\frac{1}{2}}, u_{i-\frac{1}{2}j+1}, u_{ij}, u_{ij+\frac{1}{2}}, u_{ij+1}, \\ &\quad v_{i-1j}, v_{i-1j+\frac{1}{2}}, v_{i-1j+1}, v_{i-\frac{1}{2}j}, v_{i-\frac{1}{2}j+\frac{1}{2}}, v_{i-\frac{1}{2}j+1}, v_{ij}, v_{ij+\frac{1}{2}}, v_{ij+1}), \\ \omega_{ij-1} &= (u_{ij-1}, u_{ij-\frac{1}{2}}, u_{ij}, u_{i+\frac{1}{2}j-1}, u_{i+\frac{1}{2}j-\frac{1}{2}}, u_{i+\frac{1}{2}j}, u_{i+1j-1}, u_{i+1j-\frac{1}{2}}, u_{i+1j}, \\ &\quad v_{ij-1}, v_{ij-\frac{1}{2}}, v_{ij}, v_{i+\frac{1}{2}j-1}, v_{i+\frac{1}{2}j-\frac{1}{2}}, v_{i+\frac{1}{2}j}, v_{i+1j-1}, v_{i+1j-\frac{1}{2}}, v_{i+1j}), \\ \omega_{ij} &= (u_{ij}, u_{ij+\frac{1}{2}}, u_{ij+1}, u_{i+\frac{1}{2}j}, u_{i+\frac{1}{2}j+\frac{1}{2}}, u_{i+\frac{1}{2}j+1}, u_{i+1j}, u_{i+1j+\frac{1}{2}}, u_{i+1j+1}, \\ &\quad v_{ij}, v_{ij+\frac{1}{2}}, v_{ij+1}, v_{i+\frac{1}{2}j}, v_{i+\frac{1}{2}j+\frac{1}{2}}, v_{i+\frac{1}{2}j+1}, v_{i+1j}, v_{i+1j+\frac{1}{2}}, v_{i+1j+1}). \end{aligned}$$

The nodal points of all finite elements are numerated from down to up and from left to right. The finite-element e_{ij} has its index (ij) corresponding to the lower-left vertice.

In this context, to derive the discrete analogue of equilibrium equation, as well as contact and interlaminar stresses, the following technique is suggested.

In order to obtain the equation for the central point of the finite element we have to multiply the displacement vector corresponding to this finite element with ninth (tenth) line of the LSM.

In order to obtain the discrete form for the Lamé system (1) on the nodal points of mesh $(x_{1,ij}, x_{2,ij}) \in \Omega_k$ we use four finite elements neighbouring with this point (Figure 2). So, using the components of the local stiffness matrix and above four vectors we can write their contribution to the discrete form of first (second) equilibrium equation in the form

$$\begin{aligned} & \text{(seventeenth (eighteenth) line of } \mathcal{L}_{i-1j-1}) \times \omega_{i-1j-1}^T + \\ & \text{(thirteenth (fourteenth) line of } \mathcal{L}_{i-1j}) \times \omega_{i-1j}^T + \\ & \text{(fifth (sixth) line of } \mathcal{L}_{ij-1}) \times \omega_{ij-1}^T + \\ & \text{(first (second) line of } \mathcal{L}_{ij}) \times \omega_{ij}^T. \end{aligned} \quad (5)$$

After non difficult transformations, we can write the discrete form for the system (1) on k -th layer Ω_k by using finite differences notations:

$$\begin{cases} -h_i \tau_j [(\lambda_k + 2\mu_k) u_{\overline{x_1}x_1} + \mu_k u_{\overline{x_2}x_2} + \frac{\lambda_k + \mu_k}{2} (v_{x_1x_2} + v_{\overline{x_1}\overline{x_2}})] = F_{1,ij}^h, \\ -h_i \tau_j [\mu_k v_{\overline{x_1}x_1} + (\lambda_k + 2\mu_k) v_{\overline{x_2}x_2} + \frac{\lambda_k + \mu_k}{2} (u_{x_1x_2} + u_{\overline{x_1}\overline{x_2}})] = F_{2,ij}^h, \end{cases} \quad (6)$$

where $F_{1,ij}^h$ and $F_{2,ij}^h$ are values of components of internal forces F on the nodal point $(x_{1,ij}, x_{2,ij})$. by using the notations of finite differences

Let us denote interlayer stress on the common border by $\sigma_{N,ij}^k$ ($\sigma_{T,ij}^k$), $\sigma_{N,ij}^{k+1}$ ($\sigma_{T,ij}^{k+1}$). In order to obtain the approximating expression of $\sigma_{N,ij}^k$ ($\sigma_{T,ij}^k$) we have to multiply $\mathcal{L}_{i-1,j}$ by $\omega_{i-1,j}$ and \mathcal{L}_{ij} by ω_{ij} , respectively. Then we have to sum up the results of that multiplying. Now let us approximate $\sigma_{N,ij}^{k+1}$ ($\sigma_{T,ij}^{k+1}$) on the upper boundary of lower layer. In order to do that we have to multiply line eighteenth (seventeenth) of $\mathcal{L}_{i-1,j-1}$ by $\omega_{i-1,j-1}$ and line sixth (fifth) of \mathcal{L}_{ij-1} by ω_{ij-1} , respectively. Then we have to sum up the results of that multiplying.

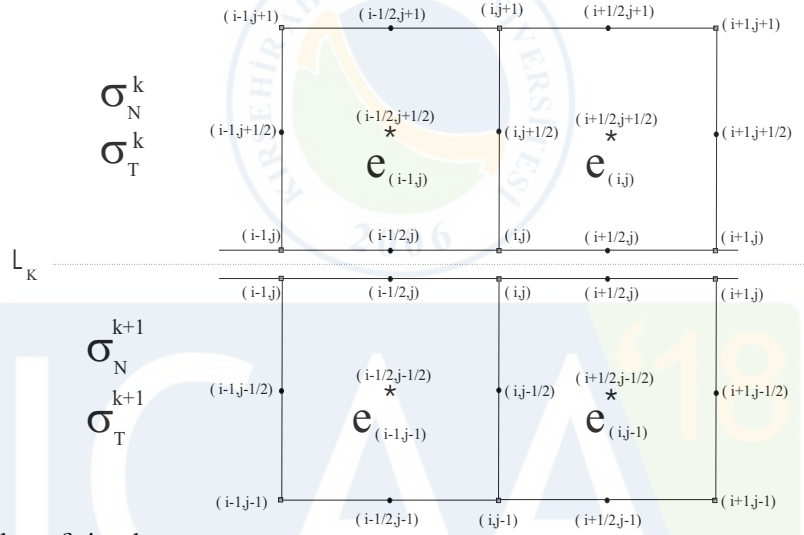


Fig. 2. The interlayer finite elements

The discrete analogues of normal ($\sigma_N^{h,(k)}$) and tangential ($\sigma_T^{h,(k)}$) components of stresses on k-th layer Ω_k have the following form:

$$\sigma_N^{h,(k)} = -\lambda_k \frac{u_{\bar{x}_1} + \check{u}_{x_1}}{2} - (\lambda_k + 2\mu_k)v_{\bar{x}_2} - \frac{\tau}{2}\mu_k[u_{x_1x_2} + v_{\bar{x}_1x_1}], \quad (7)$$

$$\sigma_T^{h,(k)} = -\mu_k(u_{x_2} + \frac{v_{\bar{x}_1} + \hat{v}_{x_1}}{2}) - \frac{\tau}{2}[(\lambda_k + 2\mu_k)u_{\bar{x}_1x_1} + \lambda_k v_{x_1x_2}]. \quad (8)$$

Analogously, we can obtain $\sigma_N^{h,(k+1)}$ and $\sigma_T^{h,(k+1)}$ as follows

$$\sigma_N^{h,(k+1)} = \lambda_{k+1} \frac{u_{x_1} + \check{u}_{\bar{x}_1}}{2} + (\lambda_{k+1} + 2\mu_{k+1})v_{\bar{x}_2} - \frac{\tau}{2}\mu_{k+1}(u_{\bar{x}_1x_2} + v_{\bar{x}_1x_1}), \quad (9)$$

$$\sigma_T^{h,(k+1)} = \mu_{k+1}(u_{\bar{x}_2} + \frac{v_{x_1} + \check{v}_{\bar{x}_1}}{2}) - \frac{\tau}{2}[(\lambda_{k+1} + 2\mu_{k+1})u_{\bar{x}_1x_1} + \lambda_{k+1}v_{\bar{x}_1x_2}]. \quad (10)$$

Using (7)-(10), it is not difficult to show that the following transmission conditions

$$\sigma_T^{h,(k)} - \sigma_T^{h,(k+1)} = 0, \quad \sigma_N^{h,(k)} - \sigma_N^{h,(k+1)} = 0 \quad (11)$$

are satisfied. In order to determinate contact domain a_c we have to calculate σ_N on the upper side of the body. We calculate this value the same way as for the upper boundary of lower layer.

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