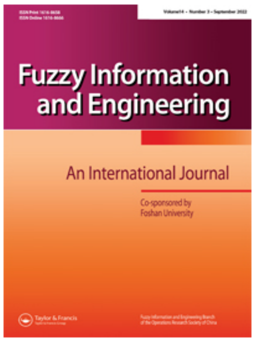

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Fibonacci Ideal Convergence on Intuitionistic Fuzzy Normed Linear Spaces

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ABSTRACT

The main goal of this article is to present the notion of Fibonacci I-convergence of sequences on intuitionistic fuzzy normed linear space. To accomplish this goal, we mainly investigate some fundamental properties of the newly introduced notion. Then, we examine the Fibonacci I-Cauchy sequences and Fibonacci I completeness in the construction of an intuitionistic fuzzy normed linear space. Some intuitionistic fuzzy Fibonacci ideal convergent spaces have been established. Further, we prove on some algebraic and topological features of these convergent sequence spaces.

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1. Introduction and Background

The initial work on the statistical convergence of sequences was carried out by Fast [1]. Schoenberg [2] validated a number of elementary properties of statistical convergence and represented this notion as a method of summability.

The notion of $\mathcal{I}$-convergence initially originated in the study of Kostyrko et al. [3]. Kostyrko et al. [4] proposed and proved some new properties of $\mathcal{I}$-convergence and introduced extremal $\mathcal{I}$-limit points. Further, the study was extended by Šalát et al. [5], Tripathy and Hazarika [6] and many others.

Fibonacci sequences were published by Fibonacci in the book 'Liber Abaci'. The Fibonacci sequences were earlier stated as Virahanka numbers by Indian mathematics [7]. The sequence

$$(1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, \dots)$$

is known as the Fibonacci sequence [8]. The Fibonacci numbers may be given by the following relation:

$$f_n = f_{n+1} - f_{n-2}$$

for some integers $n \geq 2$.

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Some properties of Fibonacci numbers are given by

$$\lim_{n\to\infty}\frac{f_{n+1}}{f_n}=\frac{1+\sqrt{5}}{2}=\alpha, \quad \text{(Golden ratio)}$$

$$\sum_{k=0}^{n} f_k = f_{n+2}-1 \quad (n\in\mathbb{N}),$$

$$\sum_k \frac{1}{f_k} \quad \text{converges,}$$

$$f_{n-1}f_{n+1}-f_n^2=(-1)^{n+1}, \quad n\geq 1. \ \text{(Cassini formula)}$$

The first application of Fibonacci sequence in the sequence spaces was given by Kara and Başarır [9]. Then, Kara [10] obtained the Fibonacci difference matrix $\widehat{F}$ via Fibonacci sequence (f_n) for $n\in\{1,2,3,\ldots\}$, and studied some new sequence spaces in this connection. The definition of statistical convergence using the Fibonacci sequence was introduced in [11]. Some works on spaces connected Fibonacci sequence can be found in [12–15].

Kara [10] defined the infinite matrix $\widehat{F}=(\widehat{f}_{kn})$ by

$$\widehat{f}_{kn}=\begin{cases}-\dfrac{f_{k+1}}{f_k}, & n=k-1\\ \dfrac{f_k}{f_{k+1}}, & n=k\\ 0, & 0\leq n<k-1 \text{ or } n>k,\end{cases}$$

where f_k is the kth Fibonacci number for every $k\in\mathbb{N}$.

The Fibonacci sequence of numbers and the associated 'Golden Ratio' are observed in nature. We examine that various natural things follow the Fibonacci sequence. It appears in biological settings such as branching in trees, the flowering of an artichoke and the arrangement of a pine cone's bracts etc. Nowadays Fibonacci numbers play a very significant role in coding theory. Fibonacci numbers in different forms are extensively applied in constructing security coding. The Fibonacci Numbers are also applied in Pascal's Triangle. Amazing applications can be examined in [16].

After the advent of fuzzy set theory by Zadeh [17], fuzzy logic has found its applications in some subbranches of mathematics like topological spaces [18–20], theory of functions [21,22] and approximation theory [23].

Fuzzy set theory has found large-scale applications in many fields of science and engineering, such as computer programming [24], non-linear operators [25], population changes [26], control of chaos [27], and quantum physics [28].

The intuitionistic fuzzy sets were focused on by Atanassov [29], and it has been utilized in decision-making problems [30], E-infinity theory of high-energy physics [31]. In intuitionistic fuzzy sets (IFSs) the 'degree of non-belongingness' is not independent but it is dependent on the 'degree of belongingness'. Fuzzy sets (FSs) can be thought as a remarkable case of an IFS where the 'degree of non-belongingness' of an element is absolutely equal to '1-degree of belongingness'. Uncertainty is based on the belongingness degree in IFSs. An intuitionistic fuzzy metric space was considered by Park [32]. Saadati and Park [33] obtained an intuitionistic fuzzy normed linear space (IFNLS for short). Karakuş et al.

[34] studied statistical convergence in IFNLS and Mursaleen et al. [35] studied the statistical convergence of double sequences in IFNLS. Some works related to the convergence of sequences in a few IFNLS can be found in [36–44].

Recently, Kirişci [45] studied the Fibonacci statistical convergence on IFNLS. He defined the Fibonacci statistically Cauchy sequences in an IFNLS and investigated the Fibonacci statistical completeness of the space.

Firstly, some basic definitions of this paper can be seen in [3,33,41,45].

2. Main Results

In this section, we give the Fibonacci $\mathcal{I}$-convergence in an IFNLS.

Definition 2.1: Let $(\mathcal{X}, \phi, \psi, *, \Diamond)$ be an IFNLS and $\mathcal{I} \subset P(\mathbb{N})$ be a nontrivial ideal. A sequence $x = (x_k)$ in $\mathcal{X}$ is said to be Fibonacci $\mathcal{I}$-convergence with regards to the intuitionistic fuzzy norm (IFN) (ϕ, ψ) (briefly, F$\mathcal{I}$C-IFN), if there is a number $\xi \in \mathcal{X}$ such that for every $p > 0$ and $\varepsilon \in (0, 1)$, the set

$$\mathcal{K}_\varepsilon(\widehat{F}) := \left\{k \in \mathbb{N} : \phi\left(\widehat{F}x_k - \xi, p\right) \leq 1 - \varepsilon \text{ or } \psi\left(\widehat{F}x_k - \xi, p\right) \geq \varepsilon\right\} \in \mathcal{I}.$$

We write $\mathcal{I}_{F\mathcal{I}_{(\phi,\psi)}} - \lim x_k = \xi$. The set of F$\mathcal{I}$C-IFN will be demonstrated by $\mathcal{I}(\widehat{F})_{IFN}$.

Example 2.1: Taking $\mathcal{I} = \{A \subset \mathbb{N} : \delta(A) = 0\}$, $\mathcal{I}$ is an admissible ideal in $\mathbb{N}$ and so Fibonacci $\mathcal{I}$-convergence coincides with Fibonacci statistical convergence in an IFNLS.

Example 2.2: Let $(\mathcal{X}, \|.\|)$ be a normed space and $k * l = kl$ and $k \Diamond l = \min\{k + l, 1\}$, $k, l \in [0, 1]$. Any $x \in X$ and $p > 0$, consider

$$\phi\,(x, p) := \frac{p}{p + \|x\|}, \quad \psi\,(x, p) := \frac{\|x\|}{p + \|x\|}.$$

Then, $(\mathcal{X}, \phi, \psi, *, \Diamond)$ be an IFNLS. Define the $\widehat{F}x_k = (f^2_{k+1}) = (1, 2^2, 3^2, 5^2, \ldots)$. Since $f^2_{k+1} \to \infty$ as $k \to \infty$ and $\widehat{F}x = (1, 0, 0, \ldots)$, then $\widehat{F}x \in \mathcal{I}(\widehat{F})_{IFN}$. Consider

$$\mathcal{A}_k\,(\varepsilon, p) := \left\{k \in \mathbb{N} : \phi\left(\widehat{F}x_k, p\right) \leq 1 - \varepsilon \text{ or } \psi\left(\widehat{F}x_k, p\right) \geq \varepsilon\right\}$$

for $\varepsilon \in (0, 1)$ and for all $p > 0$. When k becomes sufficiently large, the quantity $\phi(\widehat{F}x_k - \xi, p)$ becomes less than $1 - \varepsilon$ and similarly the quantity $\psi(\widehat{F}x_k - \xi, p)$ becomes greater than ε. So, for $\varepsilon > 0$ and $p > 0$, $A_\varepsilon(\widehat{F}) \in \mathcal{I}$.

Now, we investigate the sequence spaces in IFNLS as the sets of sequences whose $\widehat{F}$-transforms are in the spaces $c_0^{\mathcal{I}}(\phi, \psi)$, $c^{\mathcal{I}}(\phi, \psi)$ and $l_\infty^{\mathcal{I}}(\phi, \psi)$. In addition, we put forward some inclusion theorems and obtain various topological and algebraic features from these results. Assume that a sequence $x = (x_k) \in \omega$ and $\mathcal{I}$ is an admissible ideal of a subset of $\mathbb{N}$. We identify

$$c^{\mathcal{I}}_{0(\phi,\psi)}\left(\widehat{F}\right) = \left\{x = (x_k) \in \omega : \left\{k \in \mathbb{N} : \phi\left(\widehat{F}x_k, p\right) \leq 1 - \varepsilon \text{ or } \psi\left(\widehat{F}x_k, p\right) \geq \varepsilon\right\} \in \mathcal{I}\right\},$$

$$c^{\mathcal{I}}_{(\phi,\psi)}\left(\widehat{F}\right) = \left\{\begin{array}{l} x = (x_k) \in \omega : \left\{k \in \mathbb{N} : \phi\left(\widehat{F}x_k - \xi, p\right) \leq 1 - \varepsilon \text{or} \right. \\ \quad \left. \psi\left(\widehat{F}x_k - \xi, p\right) \geq \varepsilon \text{ for some } \xi \in \mathbb{R}\right\} \in \mathcal{I} \end{array}\right\},$$

$$l^{\mathcal{I}}_{\infty(\phi,\psi)}\left(\widehat{F}\right) = \left\{\begin{array}{l} x = (x_k) \in \omega : \exists M > 0 \text{ so that } \left\{k \in \mathbb{N} : \phi\left(\widehat{F}x_k, p\right) \leq 1 - M \text{ or} \right. \\ \quad \left. \psi\left(\widehat{F}x_k, p\right) \geq M\right\} \in \mathcal{I} \end{array}\right\}.$$

Theorem 2.1: *Let $(\mathcal{X}, \phi, \psi, *, \Diamond)$ be an IFNLS. The inclusion relation $c_{0(\phi,\psi)}^{\mathcal{I}}(\widehat{F}) \subset c_{(\phi,\psi)}^{\mathcal{I}}(\widehat{F}) \subset l_{\infty(\phi,\psi)}^{\mathcal{I}}(\widehat{F})$ supplies.*

Proof: It can be observed that $c_{0(\phi,\psi)}^{\mathcal{I}}(\widehat{F}) \subset c_{(\phi,\psi)}^{\mathcal{I}}(\widehat{F})$. We only denote that $c_{(\phi,\psi)}^{\mathcal{I}}(\widehat{F}) \subset l_{\infty(\phi,\psi)}^{\mathcal{I}}(\widehat{F})$. Take $x = (x_k) \in c_{(\phi,\psi)}^{\mathcal{I}}(\widehat{F})$. Then, there is $\xi \in \mathcal{X}$ so that $\mathcal{I}_{F\mathcal{I}_{(\phi,\psi)}} - \lim x_k = \xi$. So, for all $p > 0$ and $\varepsilon \in (0,1)$, the set

$$K = \left\{k \in \mathbb{N} : \phi\left(\widehat{F}x_k - \xi, \frac{p}{2}\right) > 1 - \varepsilon \text{ and } \psi\left(\widehat{F}x_k - \xi, \frac{p}{2}\right) < \varepsilon\right\} \in \mathcal{F}(\mathcal{I}).$$

$\phi(\xi, \frac{p}{2}) = s$ and $\psi(\xi, \frac{p}{2}) = t$ for all $p > 0$. As $s, t \in (0,1)$ and $\varepsilon \in (0,1)$, there exist $u_1, u_2 \in (0,1)$ such that $(1-\varepsilon) * s > 1 - u_1$ and $\varepsilon \Diamond t < u_2$. As a result, for $p > 0$ and $\varepsilon \in (0,1)$, we obtain

$$\begin{aligned} \phi\left(\widehat{F}x_k, p\right) &= \phi\left(\widehat{F}x_k + \xi - \xi, p\right) \geq \phi\left(\widehat{F}x_k - \xi, \frac{p}{2}\right) * \phi\left(\xi, \frac{p}{2}\right) \\ &> (1-\varepsilon) * s > 1 - u_1 \end{aligned}$$

and

$$\begin{aligned} \psi\left(\widehat{F}x_k, p\right) &= \psi\left(\widehat{F}x_k + \xi - \xi, p\right) \leq \psi\left(\widehat{F}x_k - \xi, \frac{p}{2}\right) \Diamond \psi\left(\xi, \frac{p}{2}\right) \\ &< \varepsilon \Diamond t < u_2. \end{aligned}$$

Taking $u = \max\{u_1, u_2\}$, we get the set

$$\left\{\begin{array}{l} x = (x_k) \in \omega : \exists u > 0 \text{ so that } \left\{k \in \mathbb{N} : \phi\left(\widehat{F}x_k, p\right) > 1 - u \text{ and} \right. \\ \quad \left. \psi\left(\widehat{F}x_k, p\right) < u\right\} \in \mathcal{F}(\mathcal{I}). \end{array}\right\}$$

Hence, $x = (x_k) \in l_{\infty(\phi,\psi)}^{\mathcal{I}}(\widehat{F})$ implies $c_{(\phi,\psi)}^{\mathcal{I}}(\widehat{F}) \subset l_{\infty(\phi,\psi)}^{\mathcal{I}}(\widehat{F})$. ■

The converse of the inclusion relation does not supply. We establish the following example to prove our claim.

Example 2.3: Assume $(\mathcal{X} = \mathbb{R}, \|.\|)$ be a normed space such that $\|x\| = \sup_k |x_k|$. Suppose $k * l = \min\{k, l\}$ and $k \Diamond l = \max\{k, l\}$ for each $k, l \in [0,1]$. Identify the norm (ϕ, ψ) on $\mathcal{X}^2 \times (0, \infty)$ as follows

$$\phi(x, p) := \frac{p}{p + \|x\|}, \quad \psi(x, p) := \frac{\|x\|}{p + \|x\|}.$$

Then, $(\mathcal{X}, \phi, \psi, *, \Diamond)$ is an IFNS. Define the sequence $\widehat{F}x = (1, 0, 0, \ldots)$, it can be easily observed that $(x_k) \in c_{(\phi,\psi)}^{\mathcal{I}}(\widehat{F})$ and $\mathcal{I}_{F\mathcal{I}_{(\phi,\psi)}} - \lim x_k = 1$, but $(x_k) \notin c_{0(\phi,\psi)}^{\mathcal{I}}(\widehat{F})$.

Example 2.4: Suppose $(\mathcal{X} = \mathbb{R}, \|.\|)$ be a normed space and (ϕ, ψ) be the IFN as determined in the above example. Examine the sequence $(x_k) = (-1)^k$. Then $(x_k) \in l_{\infty(\phi,\psi)}^{\mathcal{I}}(\widehat{F})$, but $(x_k) \notin c_{(\phi,\psi)}^{\mathcal{I}}(\widehat{F})$.

Lemma 2.1: *Let $(\mathcal{X}, \phi, \psi, *, \Diamond)$ be an IFNLS. For all $\varepsilon > 0$ and $p > 0$, the following statements are equivalent:*

(a) $\mathcal{I}_{F\mathcal{I}_{(\phi,\psi)}} - \lim x_k = \xi$;
(b) $\{k \in \mathbb{N} : \phi(\widehat{F}x_k - \xi, p) \le 1 - \varepsilon\} \in \mathcal{I}$ and $\{k \in \mathbb{N} : \psi(\widehat{F}x_k - \xi, p) \ge \varepsilon\} \in \mathcal{I}$;
(c) $\{k \in \mathbb{N} : \phi(\widehat{F}x_k - \xi, p) > 1 - \varepsilon \text{and} \psi(\widehat{F}x_k - \xi, p) < \varepsilon\} \in \mathcal{F}(\mathcal{I})$,
(d) $\{k \in \mathbb{N} : \phi(\widehat{F}x_k - \xi, p) > 1 - \varepsilon\} \in \mathcal{F}(\mathcal{I})$ and $\{k \in \mathbb{N} : \psi(\widehat{F}x_k - \xi, p) < \varepsilon\} \in \mathcal{F}(\mathcal{I})$ and
(e) $\mathcal{I}_{F\mathcal{I}_{(\phi,\psi)}} - \lim \phi(\widehat{F}x_k - \xi, p) = 1$ and $\mathcal{I}_{F\mathcal{I}_{(\phi,\psi)}} - \lim \omega(\widehat{F}x_k - \xi, p) = 0$.

Proof: It is easy to demonstrate the equivalence of (a)–(d). Here, we just prove the equivalence of (b) and (e). Let (b) holds. For every $\varepsilon > 0$ and $p > 0$, we get

$$\begin{aligned}&\left\{k \in \mathbb{N} : \left|\phi\left(\widehat{F}x_k - \xi, p\right) - 1\right| \ge \varepsilon\right\}\\ &\quad = \left\{k \in \mathbb{N} : \phi\left(\widehat{F}x_k - \xi, p\right) \ge 1 + \varepsilon\right\} \cup \left\{k \in \mathbb{N} : \phi\left(\widehat{F}x_k - \xi, p\right) \le 1 - \varepsilon\right\}\end{aligned}$$

and for every $\varepsilon > 0$ the set $\{k \in \mathbb{N} : \phi(\widehat{F}x_k - \xi, p) \ge 1 + \varepsilon\} = \emptyset \in \mathcal{I}$, it follows together with (b) that $\{k \in \mathbb{N} : |\phi(\widehat{F}x_k - \xi, p) - 1| \ge \varepsilon\} \in \mathcal{I}$. Hence, we have $\mathcal{I}_{F\mathcal{I}_{(\mu,\nu)}} - \lim \phi(\widehat{F}x_k - \xi, p) = 1$. In a similar way, for all $\varepsilon > 0$ and $p > 0$,

$$\begin{aligned}&\left\{k \in \mathbb{N} : \left|\psi\left(\widehat{F}x_k - \xi, p\right) - 0\right| \ge \varepsilon\right\}\\ &\quad = \left\{k \in \mathbb{N} : \psi\left(\widehat{F}x_k - \xi, p\right) \ge \varepsilon\right\} \cup \left\{k \in \mathbb{N} : \psi\left(\widehat{F}x_k - \xi, p\right) \le -\varepsilon\right\}\end{aligned}$$

and $\{k \in \mathbb{N} : \psi(\widehat{F}x_k - \xi, p) \le -\varepsilon\} = \emptyset \in \mathcal{I}$, implies that $\mathcal{I}_{F\mathcal{I}_{(\mu,\nu)}} - \lim \psi(\widehat{F}x_k - \xi, p) = 0$. Also, it is clear that (e) implies (b). ■

Theorem 2.2: *Let $(\mathcal{X}, \phi, \psi, *, \Diamond)$ be an IFNLS. If (x_k) is Fibonacci $\mathcal{I}$-convergent with regards to the IFN (ϕ, ω), then $\mathcal{I}_{F\mathcal{I}_{(\mu,\nu)}} - \lim x$ is unique.*

Proof: Assume that there exist two distinct elements $\xi_1, \xi_2 \in X$ such that $\mathcal{I}_{F\mathcal{I}_{(\phi,\psi)}} - \lim x_k = \xi_1$ and $\mathcal{I}_{F\mathcal{I}_{(\phi,\psi)}} - \lim x_k = \xi_2$. Given $\varepsilon \in (0, 1)$, choose $\gamma > 0$ such that $(1 - \gamma) * (1 - \gamma) > 1 - \varepsilon$ and $\gamma \Diamond \gamma < \varepsilon$. So, for any $p > 0$, we determine the following:

$$\begin{aligned}\mathcal{K}_{\phi,1}(\gamma, p) &= \left\{k \in \mathbb{N} : \phi\left(\widehat{F}x_k - \xi_1, \frac{p}{2}\right) \le 1 - \gamma\right\},\\ \mathcal{K}_{\psi,1}(\gamma, p) &= \left\{k \in \mathbb{N} : \psi\left(\widehat{F}x_k - \xi_1, \frac{p}{2}\right) \ge \gamma\right\},\\ \mathcal{K}_{\phi,2}(\gamma, p) &= \left\{k \in \mathbb{N} : \phi\left(\widehat{F}x_k - \xi_2, \frac{p}{2}\right) \le 1 - \gamma\right\},\\ \mathcal{K}_{\psi,2}(\gamma, p) &= \left\{k \in \mathbb{N} : \psi\left(\widehat{F}x_k - \xi_2, \frac{p}{2}\right) \ge \gamma\right\}.\end{aligned}$$

and

$$\mathcal{K}_{\phi,\psi}(\gamma, p) = \left(\mathcal{K}_{\phi,1}(\gamma, p) \cup \mathcal{K}_{\phi,2}(\gamma, p)\right) \cap \left(\mathcal{K}_{\psi,1}(\gamma, p) \cup \mathcal{K}_{\psi,2}(\gamma, p)\right).$$

Since $\mathcal{I}_{F\mathcal{I}_{(\phi,\psi)}} - \lim x_k = \xi_1$ and $\mathcal{I}_{F\mathcal{I}_{(\phi,\psi)}} - \lim x_k = \xi_2$, all the sets $\mathcal{K}_{\phi,1}(\gamma, p)$, $\mathcal{K}_{\psi,1}(\gamma, p)$, $\mathcal{K}_{\phi,2}(\gamma, p)$, $\mathcal{K}_{\psi,2}(\gamma, p)$ and $\mathcal{K}_{\phi,\psi}(\gamma, p)$ belongs to $\mathcal{I}$. This implies that its complement $\mathcal{K}^c_{\phi,\psi}(\gamma, p)$ is a non-empty set in $\mathcal{F}(\mathcal{I})$. Let $m \in K^c_{\phi,\psi}(\gamma, p)$. Then we have $m \in \mathcal{K}^c_{\phi,1}(\gamma, p) \cap \mathcal{K}^c_{\phi,2}(\gamma, p)$ or $m \in \mathcal{K}^c_{\psi,1}(\gamma, p) \cap \mathcal{K}^c_{\psi,2}(\gamma, p)$.

Case (i): Suppose that $m \in \mathcal{K}^c_{\phi,1}(\gamma,p) \cap \mathcal{K}^c_{\phi,2}(\gamma,p)$. Then we have $\phi(\widehat{F}x_m - \xi_1, \frac{p}{2}) > 1 - r$, $\phi(\widehat{F}x_m - \xi_2, \frac{p}{2}) > 1 - r$ and therefore

$$\begin{aligned}\phi\left(\xi_1 - \xi_2, p\right) &\geq \phi\left(\widehat{F}x_m - \xi_1, \frac{p}{2}\right) * \phi\left(\widehat{F}x_m - \xi_2, \frac{p}{2}\right)\\ &> (1-\gamma) * (1-\gamma) > 1 - \varepsilon.\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we get $\phi(\xi_1 - \xi_2, p) = 1$ for all $p > 0$, which yields $\xi_1 = \xi_2$.

Case (ii): Suppose that $m \in \mathcal{K}^c_{\psi,1}(\gamma,p) \cap K^c_{\psi,2}(\gamma,p)$. Then, we have $\psi(\widehat{F}x_m - \xi_1, \frac{p}{2}) < \gamma$, $\psi(\widehat{F}x_m - \xi_2, \frac{p}{2}) < \gamma$ and therefore

$$\begin{aligned}\psi\left(\xi_1 - \xi_2, p\right) &< \psi\left(\widehat{F}x_m - \xi_1, \frac{p}{2}\right) \Diamond \psi\left(\widehat{F}x_m - \xi_2, \frac{p}{2}\right)\\ &< \gamma \Diamond \gamma < \varepsilon.\end{aligned}$$

Since arbitrary $\varepsilon > 0$, we get $\psi(\xi_1 - \xi_2, p) = 0$ for all $p > 0$. This occurs that $\xi_1 = \xi_2$. So, we conclude that $\mathcal{I}_{F\mathcal{I}_{(\phi,\psi)}} - \lim x$ is unique. ∎

Theorem 2.3: *Suppose $(\mathcal{X}, \phi, \psi, *, \Diamond)$ be an IFNLS, and $x = (x_k)$, $y = (y_k)$ be two sequences in X.*

(a) If $(\phi,\psi) - \lim x_k = \xi$, then $\mathcal{I}_{F\mathcal{I}_{(\phi,\psi)}} - \lim \widehat{F}x_k = \xi$.
(b) If $\mathcal{I}_{F\mathcal{I}_{(\phi,\psi)}} - \lim \widehat{F}x_k = \xi_1$ and $\mathcal{I}_{F\mathcal{I}_{(\phi,\psi)}} - \lim \widehat{F}y_k = \xi_2$, then $\mathcal{I}_{F\mathcal{I}_{(\phi,\psi)}} - \lim(\widehat{F}x_k + \widehat{F}y_k) = (\xi_1 + \xi_2)$;
(c) If $\mathcal{I}_{F\mathcal{I}_{(\phi,\psi)}} - \lim \widehat{F}x_k = \xi$ and α be any real number, then $\mathcal{I}_{F\mathcal{I}_{(\phi,\psi)}} - \lim \alpha\widehat{F}x_k = \alpha\xi$.

Proof: (a) As $(\phi,\psi) - \lim x_k = \xi$, so for each $\varepsilon > 0$ and $p > 0$ there exists $r_0 \in \mathbb{N}$ such that $\phi(x_k - \xi, p) > 1 - \varepsilon$ and $\psi(x_k - \xi, p) < \varepsilon$ for all $k \geq r_0$. The set

$$\mathcal{A} = \{k \in \mathbb{N} : \phi\left(x_k - \xi, p\right) \leq 1 - \varepsilon \text{ or } \psi\left(x_k - \xi, p\right) \geq \varepsilon\}$$

is contained in $\{1, 2, \ldots, r_0 - 1\}$, then

$$\left\{k \in \mathbb{N} : \phi\left(\widehat{F}x_k - \xi, p\right) \leq 1 - \varepsilon \text{ or } \psi\left(\widehat{F}x_k - \xi, p\right) \geq \varepsilon\right\} \in \mathcal{I},$$

since $\mathcal{I}$ is admissible. This shows that $\mathcal{I}_{F\mathcal{I}_{(\phi,\psi)}} - \lim x_k = \xi$.

(b) Let $\varepsilon > 0$ be given. Choose $\gamma > 0$ such that $(1-\gamma) * (1-\gamma) > 1 - \varepsilon$ and $\gamma \Diamond \gamma < \varepsilon$. For any $p > 0$, give

$$\begin{aligned}\mathcal{K}_{\phi,1}\left(\gamma,p\right) &= \left\{k \in \mathbb{N} : \phi\left(\widehat{F}x_k - \xi_1, \frac{p}{2}\right) \leq 1 - \gamma\right\},\\ \mathcal{K}_{\psi,1}\left(\gamma,p\right) &= \left\{k \in \mathbb{N} : \psi\left(\widehat{F}x_k - \xi_1, \frac{p}{2}\right) \geq \gamma\right\},\\ \mathcal{K}_{\phi,2}\left(\gamma,p\right) &= \left\{k \in \mathbb{N} : \phi\left(\widehat{F}y_k - \xi_2, \frac{p}{2}\right) \leq 1 - \gamma\right\},\\ \mathcal{K}_{\psi,2}\left(\gamma,p\right) &= \left\{k \in \mathbb{N} : \psi\left(\widehat{F}y_k - \xi_2, \frac{p}{2}\right) \geq \gamma\right\}\end{aligned}$$

and

$$\mathcal{K}_{\phi,\psi}\left(\gamma,p\right) = \left(\mathcal{K}_{\phi,1}\left(\gamma,p\right) \cup \mathcal{K}_{\phi,2}\left(\gamma,p\right)\right) \cup \left(\mathcal{K}_{\psi,1}\left(\gamma,p\right) \cup \mathcal{K}_{\psi,2}\left(\gamma,p\right)\right).$$

Since $\mathcal{I}_{F\mathcal{I}_{(\phi,\psi)}} - \lim x_k = \xi_1$ and $\mathcal{I}_{F\mathcal{I}_{(\phi,\psi)}} - \lim y_k = \xi_2$, so for $p > 0$, $\mathcal{K}_{\phi,1}(\gamma,p)$, $\mathcal{K}_{\psi,1}(\gamma,p)$, $\mathcal{K}_{\phi,2}(\gamma,p)$, $\mathcal{K}_{\psi,2}(\gamma,p)$ and $\mathcal{K}_{\phi,\psi}(\gamma,p)$ belongs to $\mathcal{I}$. So, $\mathcal{K}^c_{\phi,\psi}(\gamma,p)$ is a non-empty set in $\mathcal{F}(\mathcal{I})$. We show that

$$\mathcal{K}^c_{\phi,\psi}(\gamma,p) \subset \left\{ \begin{array}{l} k \in \mathbb{N} : \phi\left(\widehat{F}(x_k + y_k) - (\xi_1 + \xi_2), p\right) > 1 - \varepsilon \text{ and} \\ \psi\left(\widehat{F}(x_k + y_k) - (\xi_1 + \xi_2), p\right) < \varepsilon \end{array} \right\}.$$

Let $m \in \mathcal{K}^c_{\phi,\psi}(\gamma,p)$. Then, we get

$$\phi\left(\widehat{F}x_m - \xi_1, \frac{p}{2}\right) > 1 - \gamma, \quad \phi\left(\widehat{F}y_m - \xi_2, \frac{p}{2}\right) > 1 - \gamma$$
$$\psi\left(\widehat{F}x_m - \xi_1, \frac{p}{2}\right) < \gamma, \quad \psi\left(\widehat{F}y_m - \xi_2, \frac{p}{2}\right) < \gamma.$$

Now, we have

$$\begin{aligned}\phi\left(\widehat{F}(x_m + y_m) - (\xi_1 + \xi_2), p\right) &\geq \phi\left(\widehat{F}x_m - \xi, \frac{p}{2}\right) * \phi\left(\widehat{F}y_m - \xi_2, \frac{p}{2}\right) \\ &> (1-\gamma) * (1-\gamma) > 1 - \varepsilon\end{aligned}$$

and

$$\begin{aligned}\psi\left(\widehat{F}(x_m + y_m) - (\xi_1 + \xi_2), p\right) &\leq \psi\left(\widehat{F}x_m - \xi, \frac{p}{2}\right) \Diamond \psi\left(\widehat{F}y_m - \xi_2, \frac{p}{2}\right) \\ &< \gamma \Diamond \gamma < \varepsilon.\end{aligned}$$

This shows that

$$\mathcal{K}^c_{\phi,\psi}(\gamma,p) \subset \left\{ \begin{array}{l} k \in \mathbb{N} : \phi\left(\widehat{F}(x_k + y_k) - (\xi_1 + \xi_2), p\right) > 1 - \varepsilon \text{ and} \\ \psi\left(\widehat{F}(x_k + y_k) - (\xi_1 + \xi_2), p\right) < \varepsilon \end{array} \right\}.$$

Since $\mathcal{K}^c_{\phi,\psi}(\gamma,p) \in \mathcal{F}(\mathcal{I})$. Hence $\mathcal{I}_{F\mathcal{I}_{(\phi,\psi)}} - \lim(x_k + y_k) = (\xi_1 + \xi_2)$.

(c) *Case* (i): When $\alpha = 0$, for all $\varepsilon > 0$ and $p > 0$, $\phi(\widehat{F}0x_k - 0\xi, p) = \phi(0,p) = 1 > 1 - \varepsilon$ and $\psi(\widehat{F}0x_k - 0\xi, p) = \omega(0,p) = 0 < \varepsilon$. It gives us $(\phi,\psi) - \lim 0x_k = \theta$, and by part (*i*), we get $\mathcal{I}_{F\mathcal{I}_{(\phi,\psi)}} - \lim \widehat{F}0x_k = \theta$.

Case (ii): When $\alpha \neq 0$. As $\mathcal{I}_{F\mathcal{I}_{(\phi,\psi)}} - \lim x_k = \xi$, for each $\varepsilon > 0$ and $p > 0$,

$$\mathcal{A} = \left\{k \in \mathbb{N} : \phi\left(\widehat{F}x_k - \xi, p\right) > 1 - \varepsilon \text{ and } \psi\left(\widehat{F}x_k - \xi, p\right) < \varepsilon\right\} \in \mathcal{F}(\mathcal{I}). \tag{1}$$

To show the result it is enough to prove that for each $\varepsilon > 0$ and $p > 0$,

$$\mathcal{A} \subset \left\{k \in \mathbb{N} : \phi\left(\alpha\widehat{F}x_k - \alpha\xi, p\right) > 1 - \varepsilon \text{ and } \psi\left(\alpha\widehat{F}x_k - \alpha\xi, p\right) < \varepsilon\right\}.$$

Let $m \in \mathcal{A}$. Then, we get $\phi(\widehat{F}x_m - \xi, p) > 1 - \varepsilon$ and $\psi(\widehat{F}x_m - \xi, p) < \varepsilon$. Now,

$$\begin{aligned}\phi\left(\alpha\widehat{F}x_m - \alpha\xi, p\right) &= \phi\left(\left(\widehat{F}x_m - \xi\right), \frac{p}{|\alpha|}\right) \geq \phi\left(\widehat{F}x_m - \xi, p\right) * \phi\left(0, \frac{p}{|\alpha|} - p\right) \\ &= \phi\left(\widehat{F}x_m - \xi, p\right) * 1 = \phi\left(\widehat{F}x_m - \xi, p\right) > 1 - \varepsilon\end{aligned}$$

and

$$\begin{aligned}\psi\left(\alpha\widehat{F}x_m - \alpha\xi, t\right) &= \psi\left(\left(\widehat{F}x_m - \xi\right), \frac{p}{|\alpha|}\right) \leq \psi\left(\widehat{F}x_m - \xi, p\right) \Diamond \psi\left(0, \frac{p}{|\alpha|} - p\right) \\ &= \psi\left(\widehat{F}x_m - \xi, p\right) \Diamond 0 = \psi\left(\widehat{F}x_m - \xi, p\right) < \varepsilon\end{aligned}$$

Hence, we have

$$\mathcal{A} \subset \left\{k \in \mathbb{N} : \phi\left(\alpha\widehat{F}x_k - \alpha\xi, p\right) > 1 - \varepsilon \text{ and } \psi\left(\alpha\widehat{F}x_k - \alpha\xi, p\right) < \varepsilon\right\}.$$

But (1) shows that $\mathcal{I}_{F\mathcal{I}_{(\phi,\psi)}} - \lim \alpha\widehat{F}x_k = \alpha\xi$. ■

Before the next theorem, we recall the following:

Let $(\mathcal{X}, \phi, \psi, *, \Diamond)$ be an IFNLS. The open ball $\mathcal{B}_x^{\mathcal{I}}(p, \varepsilon)(\widehat{F})$ with center at x and radius p w.r.t. parameter of fuzziness $0 < \varepsilon < 1$ is given as

$$\mathcal{B}_x^{\mathcal{I}}(p, \varepsilon)\left(\widehat{F}\right) = \left\{y = (y_k) \in \mathcal{X} : \phi\left(\widehat{F}(x) - \widehat{F}(y), p\right) \leq 1 - \varepsilon \text{ or } \psi\left(\widehat{F}(x) - \widehat{F}(y), p\right) \geq \varepsilon\right\} \in \mathcal{I}$$

where $p > 0$. A subset $\mathcal{A}$ of $\mathcal{X}$ is called $\mathcal{I}F$-bounded if there exists $p > 0$ and $0 < \varepsilon < 1$ such that $\phi(\widehat{F}y, p) > 1 - \varepsilon$ and $\psi(\widehat{F}y, p) < \varepsilon$ for all $y \in \mathcal{A}$.

Let $l^{\infty}_{(\phi,\psi)}(\mathcal{X})$ denotes the space of all $\mathcal{I}F$-bounded sequences whereas by $\mathcal{I}^{\infty}_{(\phi,\psi)}(\mathcal{X})$ we denote the space of all *IF*-bounded and $\mathcal{I}$-convergent sequences in $(\mathcal{X}, \phi, \omega, *, \Diamond)$. Now, we have the following theorem.

Theorem 2.4: *Let $(\mathcal{X}, \phi, \psi, *, \Diamond)$ be an IFNLS. Then $\mathcal{I}_{F\mathcal{I}_{(\phi,\psi)}}(\mathcal{X})$ is a closed linear space of $l^{\infty}_{(\phi,\psi)}(\mathcal{X})$.*

Proof: It is clear that $\mathcal{I}^{\infty}_{(\phi,\psi)}(\mathcal{X})$ is a subspace of $l^{\infty}_{(\phi,\psi)}(\mathcal{X})$. Next, we prove the closedness of $\mathcal{I}^{\infty}_{(\phi,\psi)}(\mathcal{X})$. As $\mathcal{I}^{\infty}_{(\phi,\psi)}(\mathcal{X}) \subset \overline{\mathcal{I}^{\infty}_{(\phi,\psi)}(\mathcal{X})}$ provides, so we show that $\overline{\mathcal{I}^{\infty}_{(\phi,\psi)}(\mathcal{X})} \subset \mathcal{I}^{\infty}_{(\phi,\psi)}(\mathcal{X})$. Let $x \in \overline{\mathcal{I}^{\infty}_{(\phi,\psi)}(\mathcal{X})}$. Then, $\mathcal{B}_x^{\mathcal{I}}(p, \varepsilon) \cap \mathcal{I}^{\infty}_{(\phi,\psi)}(\mathcal{X}) \neq \emptyset$, for each open ball $\mathcal{B}_x^{\mathcal{I}}(p, \varepsilon)$ centered at x and radius p w.r.t. parameter of fuzziness $0 < \varepsilon < 1$. Taking $y \in \mathcal{B}_x^{\mathcal{I}}(p, \varepsilon) \cap \mathcal{I}^{\infty}_{(\phi,\psi)}(\mathcal{X})$, $p > 0$ and $\varepsilon \in (0, 1)$. Choosing $\gamma \in (0, 1)$ such that $(1 - \gamma) * (1 - \gamma) > 1 - \varepsilon$ and $\gamma \Diamond \gamma < \varepsilon$. As $y \in \mathcal{B}_x^{\mathcal{I}}(p, \varepsilon) \cap \mathcal{I}^{\infty}_{(\phi,\psi)}(\mathcal{X})$, there exists a subset $\mathcal{K} \subset \mathbb{N}$ such that $\mathcal{K} \in \mathcal{F}(\mathcal{I})$ and for all $k \in \mathcal{K}$, we get $\phi(\widehat{F}x_k - \widehat{F}y_k, \frac{p}{2}) > 1 - \gamma, \psi(\widehat{F}x_k - \widehat{F}y_k, \frac{p}{2}) < \gamma, \phi(\widehat{F}y_k, \frac{p}{2}) > 1 - \gamma, \psi(\widehat{F}y_k, \frac{p}{2}) < \gamma$. But for all $k \in \mathcal{K}$, we get

$$\begin{aligned}\phi\left(\widehat{F}x_k, p\right) &= \phi\left(\widehat{F}x_k - \widehat{F}y_k + \widehat{F}y_k, p\right) \\ &\geq \phi\left(\widehat{F}x_k - \widehat{F}y_k, \frac{p}{2}\right) * \phi\left(\widehat{F}y_k, \frac{p}{2}\right) > (1 - \gamma) * (1 - \gamma) > 1 - \varepsilon\end{aligned}$$

and

$$\begin{aligned}\psi\left(\widehat{F}x_k, p\right) &= \psi\left(\widehat{F}x_k - \widehat{F}y_k + \widehat{F}y_k, p\right) \\ &\leq \psi\left(\widehat{F}x_k - \widehat{F}y_k, \frac{p}{2}\right) \Diamond \psi\left(\widehat{F}y_k, \frac{p}{2}\right) < \gamma \Diamond \gamma < \varepsilon.\end{aligned}$$

It gives

$$\mathcal{K} \subset \left\{k \in \mathbb{N} : \phi\left(\widehat{F}x_k, p\right) > 1 - \varepsilon \text{ and } \psi\left(\widehat{F}x_k, p\right) < \varepsilon\right\}.$$

Since $\mathcal{K} \in \mathcal{F}(\mathcal{I})$, it concludes that

$$\left\{k \in \mathbb{N} : \phi\left(\widehat{F}x_k, p\right) > 1 - \varepsilon \text{ and } \psi\left(\widehat{F}x_k, p\right) < \varepsilon\right\} \in \mathcal{F}(\mathcal{I}).$$

Therefore, we get $x \in \mathcal{I}^{\infty}_{(\phi,\psi)}(\mathcal{X})$. ■

Theorem 2.5: *All open ball with center at x and radius p w.r.t. parameter of fuzziness* $0 < \varepsilon < 1$, *i.e.* $\mathcal{B}_x^{\mathcal{I}}(p,\varepsilon)(\widehat{F})$ *is an open set in* $c^{\mathcal{I}}_{(\phi,\psi)}(\widehat{F})$.

Proof: Examine the open ball $\mathcal{B}_x^{\mathcal{I}}(p,\varepsilon)(\widehat{F})$ with center at x and radius p w.r.t. parameter of fuzziness $0 < \varepsilon < 1$,

$$\mathcal{B}_x^{\mathcal{I}}(p,\varepsilon)\left(\widehat{F}\right) = \left\{y = (y_k) \in \mathcal{X} : \phi\left(\widehat{F}(x) - \widehat{F}(y), p\right) \leq 1 - \varepsilon \text{ or } \psi\left(\widehat{F}(x) - \widehat{F}(y), p\right) \geq \varepsilon\right\} \in \mathcal{I}.$$

Then

$$\left(\mathcal{B}_x^{\mathcal{I}}\right)^c(p,\varepsilon)\left(\widehat{F}\right) = \left\{y = (y_k) \in \mathcal{X} : \phi\left(\widehat{F}(x) - \widehat{F}(y), p\right) > 1 - \varepsilon \text{ and } \psi\left(\widehat{F}(x) - \widehat{F}(y), p\right) < \varepsilon\right\} \in \mathcal{F}(\mathcal{I}).$$

Assume $y = (y_k) \in (\mathcal{B}_x^{\mathcal{I}})^c(p,\varepsilon)(\widehat{F})$. Then, the set

$$\left\{y = (y_k) \in \mathcal{X} : \phi\left(\widehat{F}(x) - \widehat{F}(y), p\right) > 1 - \varepsilon \text{ and } \psi\left(\widehat{F}(x) - \widehat{F}(y), p\right) < \varepsilon\right\} \in \mathcal{F}(\mathcal{I}).$$

For

$$\phi\left(\widehat{F}(x) - \widehat{F}(y), p\right) > 1 - \varepsilon \text{ and } \psi\left(\widehat{F}(x) - \widehat{F}(y), p\right) < \varepsilon$$

there is a $p_0 \in (0,p)$ so that

$$\phi\left(\widehat{F}(x) - \widehat{F}(y), p_0\right) > 1 - \varepsilon \text{ and } \psi\left(\widehat{F}(x) - \widehat{F}(y), p_0\right) < \varepsilon.$$

Taking $\varepsilon_0 = \phi(\widehat{F}(x) - \widehat{F}(y), p_0)$ means $\varepsilon_0 > 1 - \varepsilon$. Then, there exists $u \in (0,1)$ so that $\varepsilon_0 > 1 - u > 1 - \varepsilon$. For $\varepsilon_0 > 1 - u$, we get $\varepsilon_1, \varepsilon_2 \in (0,1)$ such that $\varepsilon_0 * \varepsilon_1 > 1 - u$ and $(1 - \varepsilon_0)\Diamond(1 - \varepsilon_0) < u$. Take $\varepsilon_3 = \max\{\varepsilon_1, \varepsilon_2\}$. Consider the open ball $\mathcal{B}_y^{\mathcal{I}}(p - p_0, 1 - \varepsilon_3)(\widehat{F})$. We have to denote $\mathcal{B}_y^{\mathcal{I}}(p - p_0, 1 - \varepsilon_3)(\widehat{F}) \subset \mathcal{B}_x^{\mathcal{I}}(p,\varepsilon)(\widehat{F})$.

Assume $z = (z_k) \in (\mathcal{B}_y^{\mathcal{I}})^c(p - p_0, 1 - \varepsilon_3)(\widehat{F})$, then

$$\left\{k \in \mathbb{N} : \phi\left(\widehat{F}(x_k) - \widehat{F}(z_k), p - p_0\right) > \varepsilon_3 \text{ and } \psi\left(\widehat{F}(x_k) - \widehat{F}(z_k), p - p_0\right) < 1 - \varepsilon_3\right\} \in \mathcal{F}(\mathcal{I}).$$

So

$$\begin{aligned}\phi\left(\widehat{F}(x) - \widehat{F}(z), p\right) &\geq \phi\left(\widehat{F}(x) - \widehat{F}(y), p_0\right) * \phi\left(\widehat{F}(y) - \widehat{F}(z), p - p_0\right)\\ &\geq \varepsilon_0 * \varepsilon_3 \geq \varepsilon_0 * \varepsilon_1 > 1 - u > 1 - \varepsilon,\end{aligned}$$

hence

$$\left\{k \in \mathbb{N} : \phi\left(\widehat{F}(x_k) - \widehat{F}(z_k), p\right) > 1 - \varepsilon\right\} \in \mathcal{F}(\mathcal{I}),$$

and

$$\begin{aligned}\psi\left(\widehat{F}(x) - \widehat{F}(z), p\right) &\leq \psi\left(\widehat{F}(x) - \widehat{F}(y), p_0\right) \Diamond \psi\left(\widehat{F}(y) - \widehat{F}(z), p - p_0\right)\\ &\leq (1 - \varepsilon_0) \Diamond (1 - \varepsilon_3) \leq (1 - \varepsilon_0) \Diamond (1 - \varepsilon_2) < u < \varepsilon,\end{aligned}$$

hence

$$\left\{k \in \mathbb{N} : \psi\left(\widehat{F}(x) - \widehat{F}(z), p\right) < \varepsilon\right\} \in \mathcal{F}(\mathcal{I}).$$

Therefore the set

$$\left\{k \in \mathbb{N} : \phi\left(\widehat{F}(x_k) - \widehat{F}(z_k), p\right) > 1 - \varepsilon \text{ and } \psi\left(\widehat{F}(x_k) - \widehat{F}(z_k), p\right) < \varepsilon\right\} \in \mathcal{F}(\mathcal{I}).$$

So $z = (z_k) \in (\mathcal{B}_y^{\mathcal{I}})^c(p,\varepsilon)(\widehat{F})$. As a result, we get $(\mathcal{B}_y^{\mathcal{I}})^c(p - p_0, 1 - \varepsilon_3)(\widehat{F}) \subset (\mathcal{B}_y^{\mathcal{I}})^c(p,\varepsilon)(\widehat{F})$. We prove $\mathcal{B}_x^{\mathcal{I}}(p,\varepsilon)(\widehat{F})$ is an open set in $c^{\mathcal{I}}_{(\phi,\psi)}(\widehat{F})$. ∎

Theorem 2.6: *The spaces $c^{\mathcal{I}}_{(\phi,\psi)}(\widehat{F})$ and $c^{\mathcal{I}}_{0(\phi,\psi)}(\widehat{F})$ are Hausdorff spaces.*

Proof: It is clear that $c^{\mathcal{I}}_{0(\phi,\psi)}(\widehat{F}) \subset c^{\mathcal{I}}_{(\phi,\psi)}(\widehat{F})$. We have to prove the result for only $c^{\mathcal{I}}_{(\phi,\psi)}(\widehat{F})$. Assume $x = (x_k), y = (y_k) \in c^{\mathcal{I}}_{(\phi,\psi)}(\widehat{F})$ such that $x \neq y$. At that time, for all $p \in \mathbb{N}$, we get

$$0 < \phi\left(\widehat{F}x_k - \widehat{F}y_k, p\right) < 1, \quad 0 < \psi\left(\widehat{F}x_k - \widehat{F}y_k, p\right) < 1.$$

Presume

$$\varepsilon_1 = \phi\left(\widehat{F}x_k - \widehat{F}y_k, p\right), \quad \varepsilon_2 = \psi\left(\widehat{F}x_k - \widehat{F}y_k, p\right),$$

and $\varepsilon = \max\{\varepsilon_1, 1 - \varepsilon_2\}$. Afterwards, for all $\varepsilon_0 > \varepsilon$ there are $\varepsilon_3, \varepsilon_4 \in (0, 1)$ so that $\varepsilon_3 * \varepsilon_3 \geq \varepsilon_0$, $(1 - \varepsilon_4)\diamondsuit(1 - \varepsilon_4) \leq (1 - \varepsilon_0)$. Again suppose $\varepsilon_5 = \max\{\varepsilon_3, \varepsilon_4\}$ and contemplate the open balls $\mathcal{B}^{\mathcal{I}}_x(1 - \varepsilon_5, \frac{p}{2})(\widehat{F})$ and $\mathcal{B}^{\mathcal{I}}_y(1 - \varepsilon_5, \frac{p}{2})(\widehat{F})$ centered at x and y respectively. Then, we demonstrate that

$$\mathcal{B}^{\mathcal{I}}_x\left(1 - \varepsilon_5, \frac{p}{2}\right)(\widehat{F}) \cap \mathcal{B}^{\mathcal{I}}_y\left(1 - \varepsilon_5, \frac{p}{2}\right)(\widehat{F}) = \emptyset.$$

If possible assume

$$z = (z_k) \in \mathcal{B}^{\mathcal{I}}_x\left(1 - \varepsilon_5, \frac{p}{2}\right)(\widehat{F}) \cap \mathcal{B}^{\mathcal{I}}_y\left(1 - \varepsilon_5, \frac{p}{2}\right)(\widehat{F}).$$

Then, we obtain

$$\begin{aligned}
\varepsilon_1 &= \phi\left(\widehat{F}x_k - \widehat{F}y_k, p\right) \\
&\geq \phi\left(\widehat{F}x_k - \widehat{F}z_k, \frac{p}{2}\right) * \phi\left(\widehat{F}z_k - \widehat{F}y_k, \frac{p}{2}\right) \\
&> \varepsilon_5 * \varepsilon_5 \\
&\geq \varepsilon_3 * \varepsilon_3 \\
&\geq \varepsilon_0 > \varepsilon_1, \\
\varepsilon_2 &= \psi\left(\widehat{F}x_k - \widehat{F}y_k, p\right) \\
&\leq \psi\left(\widehat{F}x_k - \widehat{F}z_k, p\right) \diamondsuit \psi\left(\widehat{F}z_k - \widehat{F}y_k, \frac{p}{2}\right) \\
&< (1 - \varepsilon_5) \diamondsuit (1 - \varepsilon_5) \\
&\leq (1 - \varepsilon_4) \diamondsuit (1 - \varepsilon_4) \\
&< (1 - \varepsilon_0) < \varepsilon_2.
\end{aligned}$$

From the above equations we obtain a contradiction. So,

$$\mathcal{B}^{\mathcal{I}}_x\left(1 - \varepsilon_5, \frac{p}{2}\right)(\widehat{F}) \cap \mathcal{B}^{\mathcal{I}}_y\left(1 - \varepsilon_5, \frac{p}{2}\right)(\widehat{F}) = \emptyset.$$

As a result, the space $c^{\mathcal{I}}_{(\phi,\psi)}(\widehat{F})$ is a Hausdorff space. ■

Definition 2.2: Let $(\mathcal{X}, \phi, \psi, *, \diamondsuit)$ be an IFNLS and $\mathcal{I} \subset P(\mathbb{N})$ be a nontrivial ideal. A sequence $x = (x_k)$ in $\mathcal{X}$ is named Fibonacci $\mathcal{I}$-Cauchy with regards to the IFN (ϕ, ψ) or $\mathcal{I}_{F\mathcal{I}_{(\phi,\psi)}}$-Cauchy sequence if, for all $\varepsilon > 0$ and $p > 0$, there exists a positive integer N so that

$$\mathcal{K}_\varepsilon(\widehat{F}) := \left\{k \in \mathbb{N} : \phi\left(\widehat{F}x_k - \widehat{F}x_N, p\right) \leq 1 - \varepsilon \text{ or } \psi\left(\widehat{F}x_k - \widehat{F}x_N, p\right) \geq \varepsilon\right\} \in \mathcal{I}.$$

Theorem 2.7: *Let $(\mathcal{X}, \phi, \psi, *, \Diamond)$ be an IFNLS. Then a sequence $x = (x_k)$ in X Fibonacci $\mathcal{I}$-convergent with regards to the IFN (ϕ, ψ) iff it is Fibonacci $\mathcal{I}$-Cauchy with regards to the IFN (ϕ, ψ).*

Proof: *Necessity*. Let $x = (x_k)$ in $\mathcal{X}$ Fibonacci $\mathcal{I}$-convergent to ξ with regards to the IFN (ϕ, ψ), i.e. $\mathcal{I}_{F\mathcal{I}_{(\phi,\psi)}} - \lim x_k = \xi$. For a given $\varepsilon > 0$, choose $\gamma > 0$ such that $(1-\gamma) * (1-\gamma) > 1-\varepsilon$ and $\gamma \Diamond \gamma < \varepsilon$. Since $\mathcal{I}_{F\mathcal{I}_{(\phi,\psi)}} - \lim x_k = \xi$, we get

$$\mathcal{K}(\widehat{F}) = \left\{k \in \mathbb{N} : \phi\left(\widehat{F}x_k - \xi, p\right) \leq 1-\gamma \text{ or } \psi\left(\widehat{F}x_k - \xi, p\right) \geq \gamma\right\} \in \mathcal{I} \tag{2}$$

for all $p > 0$, which implies that

$$\emptyset \neq \mathcal{K}^c(\widehat{F}) = \left\{k \in \mathbb{N} : \phi\left(\widehat{F}x_k - \xi, p\right) > 1-\gamma \text{ or } \psi\left(\widehat{F}x_k - \xi, p\right) < \gamma\right\} \in \mathcal{F}(\mathcal{I}).$$

Let $m \in \mathcal{K}^c(\widehat{F})$. But for $p > 0$, we have $\phi(\widehat{F}x_m - \xi, p) > 1-\gamma$ or $\psi(\widehat{F}x_m - \xi, p) < \gamma$. Taking

$$\mathcal{B}(\widehat{F}) = \left\{k \in \mathbb{N} : \phi\left(\widehat{F}x_k - \widehat{F}x_m, p\right) \leq 1-\varepsilon \text{ or } \psi\left(\widehat{F}x_k - \widehat{F}x_m, p\right) \geq \varepsilon\right\}; \quad p > 0,$$

to show the result it is sufficient to prove $\mathcal{B}(\widehat{F})$ is contained in $\mathcal{K}(\widehat{F})$. Let $k \in \mathcal{B}(\widehat{F})$, then we have $\phi(\widehat{F}x_k - \widehat{F}x_m, \frac{p}{2}) \leq 1-\gamma$ or $\psi(\widehat{F}x_k - \widehat{F}x_m, \frac{p}{2}) \geq \gamma$, for $p > 0$. We have two possible cases.

Case (i): We consider $\phi(\widehat{F}x_k - \widehat{F}x_m, p) \leq 1-\varepsilon$. So, we have $\phi(\widehat{F}x_k - \xi, \frac{p}{2}) \leq 1-\gamma$ and then, $k \in \mathcal{K}(\widehat{F})$. As otherwise i.e. if $\phi(\widehat{F}x_k - \xi, \frac{p}{2}) > 1-\gamma$, then we have

$$\begin{aligned} 1-\varepsilon \geq \phi\left(\widehat{F}x_k - \widehat{F}x_m, p\right) &\geq \phi\left(\widehat{F}x_k - \xi, \frac{p}{2}\right) * \phi\left(\widehat{F}x_m - \xi, \frac{p}{2}\right) \\ &> (1-\gamma) * (1-\gamma) > 1-\varepsilon; \end{aligned}$$

which is impossible. Hence, $\mathcal{B}(\widehat{F}) \subset \mathcal{K}(\widehat{F})$.

Case (ii): If $\psi(\widehat{F}x_k - \widehat{F}x_m, p) \geq \varepsilon$, we have $\psi(\widehat{F}x_k - \xi, \frac{p}{2}) > \gamma$ and therefore $k \in \mathcal{K}(\widehat{F})$. As otherwise i.e. if $\psi(\widehat{F}x_k - \xi, \frac{t}{2}) < \gamma$, we get

$$\begin{aligned} \varepsilon \leq \psi\left(\widehat{F}x_k - \widehat{F}x_m, p\right) &\geq \psi\left(\widehat{F}x_k - \xi, \frac{p}{2}\right) \Diamond \psi\left(\widehat{F}x_m - \xi, \frac{p}{2}\right) \\ &< \gamma \Diamond \gamma < \varepsilon; \end{aligned}$$

which is impossible. Hence, $\mathcal{B}(\widehat{F}) \subset \mathcal{K}(\widehat{F})$. Thus, in all cases, we get $\mathcal{B}(\widehat{F}) \subset \mathcal{K}(\widehat{F})$. By (2) $\mathcal{B}(\widehat{F}) \in \mathcal{I}$. This shows that (x_k) in X Fibonacci $\mathcal{I}$-Cauchy sequence.

Sufficiency. Let $x = (x_k)$ in $\mathcal{X}$ Fibonacci $\mathcal{I}$-Cauchy with respect to the IFN (ϕ, ψ) but not Fibonacci $\mathcal{I}$-convergent with regards to the IFN (ϕ, ψ). Then there exists r such that

$$A_{(\varepsilon,p)}(\widehat{F}) := \left\{k \in \mathbb{N} : \phi\left(\widehat{F}x_k - \widehat{F}x_r, p\right) \leq 1-\varepsilon \text{ or } \psi\left(\widehat{F}x_k - \widehat{F}x_r, p\right) \geq \varepsilon\right\} \in \mathcal{I}$$

and

$$B_{(\varepsilon,p)}(\widehat{F}) = \left\{k \in \mathbb{N} : \phi\left(\widehat{F}x_k - \xi, \frac{p}{2}\right) > 1-\varepsilon \text{ or } \psi\left(\widehat{F}x_k - \xi, \frac{p}{2}\right) < \varepsilon\right\} \in \mathcal{I}$$

equivalently, $\mathcal{B}^c_{(\varepsilon,p)}(\widehat{F}) \in \mathcal{F}(\mathcal{I})$. Since

$$\phi\left(\widehat{F}x_k - \widehat{F}x_r, p\right) \geq 2\phi\left(\widehat{F}x_k - \xi, \frac{p}{2}\right) > 1-\varepsilon,$$

and

$$\psi\left(\widehat{F}x_k - \widehat{F}x_r, p\right) \leq 2\psi\left(\widehat{F}x_k - \xi, \frac{p}{2}\right) < \varepsilon,$$

If $\phi(\widehat{F}x_k - \xi, \frac{p}{2}) > \frac{(1-\varepsilon)}{2}$ and $\psi(\widehat{F}x_k - \xi, \frac{p}{2}) < \frac{\varepsilon}{2}$, respectively, we have $\mathcal{A}^c_{(\varepsilon,p)}(\widehat{F}) \in \mathcal{I}$, and so $\mathcal{A}_{(\varepsilon,p)}(\widehat{F}) \in \mathcal{F}(\mathcal{I})$, which is a contradiction, as $x = (x_k)$ was Fibonacci $\mathcal{I}$-Cauchy with respect to the IFN (ϕ, ψ). Hence, $x = (x_k)$ must be Fibonacci $\mathcal{I}$-convergent with regards to the IFN (ϕ, ψ). ■

Definition 2.3: Assume that $(X, \phi, \psi, *, \lozenge)$ is an IFNLS. A sequence $x = (x_k)$ in X is called Fibonacci $\mathcal{I}^*$-convergent to $\xi \in X$ with regards to IFN (ϕ, ψ) if there exists a subset

$$\mathcal{M} = \{k_1, k_2, \ldots : k_1 < k_2 < \cdots\}$$

of $\mathbb{N}$ such that $\mathcal{M} \in \mathcal{F}(\mathcal{I})$ and $\phi, \psi - \lim_{n\to\infty} x_{k_n} = \xi$. The element ξ is called the Fibonacci $\mathcal{I}^*$-limit of the sequence (x_k) with regards to IFN (ϕ, ψ) and it is demonstrated by $\mathcal{I}^*_{F\mathcal{I}_{(\phi,\psi)}} - \lim x_k = \xi$.

Theorem 2.8: *Let $(\mathcal{X}, \phi, \psi, *, \lozenge)$ be an IFNLS and $\mathcal{I} \subset P(\mathbb{N})$ be a nontrivial ideal. If $\mathcal{I}^*_{F\mathcal{I}_{(\phi,\psi)}} - \lim x_k = \xi$ then $\mathcal{I}_{F\mathcal{I}_{(\phi,\psi)}} - \lim x_k = \xi$.*

Proof: Suppose that $\mathcal{I}^*_{F\mathcal{I}_{(\phi,\psi)}} - \lim x_k = \xi$. Then $\mathcal{M} = \{k_1, k_2, \ldots : k_1 < k_2 < \cdots\} \in \mathcal{F}(\mathcal{I})$ such that $(\phi, \psi) - \lim_{n\to\infty} x_{k_n} = \xi$. For all $\varepsilon > 0$ and $p > 0$ there exists an integer $N > 0$ such that $\phi(x_{k_n} - \xi, p) > 1 - \varepsilon$ and $\psi(x_{k_n} - \xi, p) < \varepsilon$ for all $n > N$. Since

$$\left\{n \in \mathbb{N} : \phi\left(x_{k_n} - \xi, p\right) > 1 - \varepsilon \text{ or } \psi\left(x_{k_n} - \xi, p\right) < \varepsilon\right\} \in \mathcal{I}.$$

Hence,

$$\left\{k \in \mathbb{N} : \phi\left(\widehat{F}x_k - \xi, p\right) > 1 - \varepsilon \text{ or } \psi\left(\widehat{F}x_k - \xi, p\right) < \varepsilon\right\}$$
$$\subseteq H \cup \{k_1 < k_2 < \cdots < \cdots < k_{N-1}\} \in \mathcal{I}.$$

for all $\varepsilon > 0$ and $p > 0$. As a result, we conclude that $\mathcal{I}_{F\mathcal{I}_{(\phi,\psi)}} - \lim x_k = \xi$. ■

3. Conclusion

In the current study, using the concept of Fibonacci sequence, we have introduced the new notion of Fibonacci ideal convergent sequence in IFNLS. We have shown that these sequences follow many properties similar to that of classical real-valued sequences. Further, Fibonacci $\mathcal{I}$-Cauchy sequences have been introduced and the Fibonacci $\mathcal{I}$-completeness of an IFNLS has been established. Finally, the concept of Fibonacci $\mathcal{I}^*$-convergence, which is stronger than Fibonacci ideal convergence, has been investigated. Several intuitionistic fuzzy Fibonacci ideal convergent spaces have been established and significant features of these spaces have been obtained.

Disclosure statement

No potential conflict of interest was reported by the author(s).

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