A Generalized Statistical Convergence via Ideals Defined by Folner Sequence on Amenable Semigroup

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Abstract: The purpose of this study is to extend the notions of $\mathscr{I}$-convergence, $\mathscr{I}$-limit superior and $\mathscr{I}$-limit inferior, $\mathscr{I}$-cluster point and $\mathscr{I}$-limit point to functions defined on discrete countable amenable semigroups.

1. Introduction

Let G be a discrete countable amenable semigroup with identity in which both right and left cancelation laws hold, and $w(G)$ and $m(G)$ denote the spaces of all real valued functions and all bounded real functions f on G respectively. $m(G)$ is a Banach space with the supremum norm $\|f\|_\infty = sup\{|f(g)| : g \in G\}$. Nomika [17] showed that, if G is countable amenable group, there exists a sequence $\{S_n\}$ of finite subsets of G such that (i) $G = \cup_{i=1}^{\infty} S_n$, (ii) $S_n \subset S_{n+1}$, $n = 1,2,3,...,$ (iii) $\lim_{n\to\infty} \frac{|S_n g \cap S_n|}{|S_n|} = 1$, $\lim_{n\to\infty} \frac{|gS_n \cap S_n|}{|S_n|} = 1$ for all $g \in G$. Here $|A|$ denotes the number of elements in the finite set A. Any sequence of finite subsets of G satisfying (i), (ii) and (iii) is called a Folner sequence for G.
The sequence $S_n = \{0,1,2,...,n-1\}$ is a familiar Folner sequence giving rise to the classical Cesàro method of summability.
The concept of summability in amenable semigroups was introduced in [14], [15]. In [3], Douglass extended the notion of arithmetic mean to amenable semigroups and obtained a characterization for almost convergence in amenable semigroups.
In [16], the notions of convergence and statistical convergence, statistical limit point and statistical cluster point to functions on discrete countable amenable semigroups were introduced.
Fast [5] presented an interesting generalization of the usual sequential limit which he called statistical convergence for number sequences.
After studies about statistical convergence, Kostyrko, Macaj and Wilczyński defined $\mathscr{I}$-convergence in a metric space by using the notion of an ideal of the set of positive integers.(see [10]) Later, it was further studied by Salát, Tripathy and Ziman ([12], [13]) and many others. $\mathscr{I}$-convergence is a generalization of statistical convergence.
We recall the concept of asymptotic and logarithmic density of a set $A \subset \mathbb{N}$ (see [19] pp. 71, 95-96). Let $A \subset \mathbb{N}$. Put $d_n(A) = \frac{1}{n}\sum_{k=1}^{n} \chi_A(k)$ and $\delta_n(A) = \frac{1}{p_n}\sum_{k=1}^{n} \frac{\chi_A(k)}{k}$ for $n \in \mathbb{N}$, where $p_n = \sum_{k=1}^{n} \frac{1}{k}$. The numbers $\underline{d}(A) = \limsup_{n\to\infty} d_n(A)$ and $\overline{d}(A) = \liminf_{n\to\infty} d_n(A)$ are called the lower and upper asymptotic density of A, respectively. Similarly, the numbers $\underline{\delta}(A) = \liminf_{n\to\infty} \delta_n(A)$ and $\overline{\delta}(A) = \limsup_{n\to\infty} \delta_n(A)$ are called the lower and upper logarithmic density of A, respectively. If $\underline{d}(A) = \overline{d}(A)$ $(\underline{\delta}(A) = \overline{\delta}(A))$, then $d(A) = \underline{d}(A)$ is called the asymptotic density of A $(\delta(A) = \underline{\delta}(A)$ is called the logarithmic density of A, respectively). It is well known that for each $A \subset \mathbb{N}$, $\underline{d}(A) \leq \underline{\delta}(A) \leq \overline{\delta}(A) \leq \overline{d}(A)$.
Denote by $\mathscr{I}_d$, $\mathscr{I}_\delta$ the class of all A with $d(A) = 0\,(\delta(A) = 0$, respectively$)$. Then $\mathscr{I}_d$ and $\mathscr{I}_\delta$ are non-trivial admissible ideals, $\mathscr{I}_d$-convergence concides with the statistical convergence and $\mathscr{I}_\delta$-convergence is said to be logarithmic statistical convergence.

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Recently, Das, Savas and Ghosal [2] introduced new notions, namely $\mathscr{I}$-statistical convergence and $\mathscr{I}$-lacunary statistical convergence by using ideal.
In [8], he extended the concepts of statistical limit superior and inferior (as introduced by Fridy and Orhan) to $\mathscr{I}$-limit superior and inferior and give some $\mathscr{I}$-analogue of properties of statistical limit superior and inferior for a sequence of real numbers.
The purpose of this study is to extend the notions of $\mathscr{I}$-convergence, $\mathscr{I}$-limit superior and $\mathscr{I}$-limit inferior, $\mathscr{I}$-cluster point and $\mathscr{I}$-limit point to functions defined on discrete countable amenable semigroups. Also, in this paper, we make a new approach to the notions of $[V,\lambda]$-summability and λ-statistical convergence by using ideals and introduce new notions, namely, $\mathscr{I}$-$[V,\lambda]$-summability and $\mathscr{I}$-λ-statistical convergence to functions defined on discrete countable amenable semigroups. For the particular case when the amenable semigroup is the additive positive integers, our definition and theorems yield the results of [8], [10], [14].

2. Definitions and Notations

Definition 2.1 ([16]) Let G be a discrete countable amenable semigroup with identity in which both right and left cancelation laws hold. $f\in w(G)$ is said to be convergent to s, for any Folner sequence $\{S_n\}$ for G, if for each $\varepsilon>0$ there exists $k_0\in\mathbb{N}$ such that $|f(g)-s|<\varepsilon$ for all $m>k_0$ and $g\in G\backslash S_m$.

Definition 2.2 ([16]) Let G be a discrete countable amenable semigroup with identity in which both right and left cancelation laws hold. $f\in w(G)$ is said to be a Cauchy sequence for any Folner sequence $\{S_n\}$ for G, if for each $\varepsilon>0$ there exists $k_0\in\mathbb{N}$ such that $|f(g)-f(h)|<\varepsilon$ for all $m>k_0$ and $g\in G\backslash S_m$.

Definition 2.3 ([16]) Let G be a discrete countable amenable semigroup with identity in which both right and left cancelation laws hold. $f\in w(G)$ is said to be strongly summable to s, for any Folner sequence $\{S_n\}$ for G, if

$$\lim_{n\to\infty}\frac{1}{|S_n|}\sum_{g\in S_n}|f(g)-s|=0,$$

where $|S_n|$ denotes the cardinality of the set S_n.
The upper and lower Folner densities of a a set $S\subset G$ are defined by

$$\overline{\delta}(S)=\lim_{n\to\infty}\sup\frac{1}{|S_n|}|\{g\in S_n:g\in S\}|$$

and

$$\underline{\delta}(S)=\lim_{n\to\infty}\inf\frac{1}{|S_n|}|\{g\in S_n:g\in S\}|$$

respectively $\overline{\delta}(S)=\underline{\delta}(S)$, then

$$\delta(S)=\lim_{n\to\infty}\frac{1}{|S_n|}|\{g\in S_n:g\in S\}|$$

is called Folner density of S. Take $G=\mathbb{N}$, $S_n=\{0,1,2,...,n-1\}$ and S be the set of positive integers with leading digit 1 in the decimal expansion. The set S has no Folner density with respect to the Folner sequence $\{S_n\}$, since $\underline{\delta}(S)=\frac{1}{9}$, $\overline{\delta}(S)=\frac{5}{9}$. To facililate this idea we introduce the following notion: If f is function such that $f(g)$ satisfies property P for all g expect a set of Folner density zero, we say that $f(g)$ satisfies P for "almost all g", and abbreviate this by "$a.a.g$".

Definition 2.4 ([16]) Let G be a discrete countable amenable semigroup with identity in which both right and left cancelation laws hold. $f\in w(G)$ is said to be statistically convergent to s, for any Folner sequence $\{S_n\}$ for G, if for every $\varepsilon>0$

$$\lim_{n\to\infty}\frac{1}{|S_n|}|\{g\in S_n:|f(g)-s|\}|=0.$$

The set of all statistically convergent functions will be denoted by $S(G)$.

Definition 2.5 ([16]) Let G be a discrete countable amenable semigroup with identity in which both right and left cancelation laws hold. $f\in w(G)$ is said to be statistical Cauchy function for any Folner sequence $\{S_n\}$ for G, if for every $\varepsilon>0$ and $l\geq 0$, then there exists an $m\in G\backslash S_l$ such that

$$\lim_{n\to\infty}\frac{1}{|S_n|}|\{g\in S_n:|f(g)-f(m)|\geq\varepsilon\}|=0.$$

3. Main Results

Definition 3.1 Let G be a discrete countable amenable semigroup with identity in which both right and left cancelation laws hold. $f \in w(G)$ is said to be $\mathscr{I}$-convergent to s for any Folner sequence $\{S_n\}$ for G, if for every $\varepsilon > 0$;

$$\{g \in S_n : |f(g) - s| \geq \varepsilon\} \in \mathscr{I};$$

i.e., $|f(g) - s| < \varepsilon \; a.a.g.$ The set of all $\mathscr{I}$-convergent sequences will be denoted by $\mathscr{I}(G)$.
In this section, we study the concepts of $\mathscr{I}$-limit superior and $\mathscr{I}$-limit inferior for a Folner sequence, give the relationship between them, and prove some basic properties of these concepts.
For any Folner sequence $\{S_n\}$ for G and for $f \in w(G)$ let B_f denote the set,

$$B_f := \{b \in \mathbb{R} : \{g \in S_n : f(g) > b\} \notin \mathscr{I}\}$$

and similarly,

$$A_f := \{a \in \mathbb{R} : \{g \in S_n : f(g) < a\} \notin \mathscr{I}\}$$

We begin with a definition.
Definition 3.2 If $f \in w(G)$, then the $\mathscr{I}$-limit superior of $f \in w(G)$, for any Folner sequence $\{S_n\}$ for G, is given by

$$\mathscr{I}\text{-}\limsup f : \begin{cases} \sup B_f, & \text{if } B_f \neq \emptyset, \\ -\infty, & \text{if } B_f = \emptyset, \end{cases}$$

Similarly, the $\mathscr{I}$-limit inferior for any Folner sequence $\{S_n\}$ for G is given by

$$\mathscr{I}\text{-}\liminf f : \begin{cases} \inf A_f, & \text{if } A_f \neq \emptyset, \\ \infty, & \text{if } A_f = \emptyset, \end{cases}$$

It is easy to see that for any $f \in w(G)$ and for any Folner sequence $\{S_n\}$ for G, $\mathscr{I}\text{-}\liminf f \leq \mathscr{I} - \limsup f$.
Definition 3.3 The function $f \in w(G)$ is said to be $\mathscr{I}$-bounded for any Folner sequence $\{S_n\}$ for G, if there is a number M such that

$$\{g \in S_n : |f(g)| \geq M\} \in \mathscr{I}.$$

Note that $\mathscr{I}$-boundedness implies that $\mathscr{I}\text{-}\limsup f$ and $\mathscr{I}\text{-}\liminf f$ are finite. The following theorem can be proved by a straightforward least upper bound argument.
Theorem 3.4 *For any Folner sequence $\{S_n\}$ for G, if $\mu = \mathscr{I}\text{-}\limsup f$ is finite, then for each $\varepsilon > 0$*

$$\{g \in S_n : f(g) > \mu - \varepsilon\} \notin \mathscr{I} \text{ and } \{g \in S_n : f(g) > \mu + \varepsilon\} \in \mathscr{I}. \tag{1.1}$$

Conversely, if (1.1) holds for every $\varepsilon > 0$ then $\mu = \mathscr{I} - \limsup f$.

Proof. Let $\varepsilon > 0$. Since $\mu + \varepsilon > \mu = \sup\{f : \sup B_f \notin \mathscr{I}\}$, the number $\mu + \varepsilon$ is not in $\{f : \sup B_f \notin \mathscr{I}\}$ and $\{g \in S_n : f(g) > \mu + \varepsilon\} \in \mathscr{I}$. Further $\mu - \varepsilon < \mu$ and there exists $t' \in \mathbb{R}$ such that $\mu - \varepsilon < t' < \mu$, $t' \in \{f : \sup B_f \notin \mathscr{I}\}$. Hence $\{g \in S_n : f(g) > t'\} \notin \mathscr{I}$ and also $\{g \in S_n : f(g) > \mu - \varepsilon\} \notin \mathscr{I}$. Consequently (1.1) holds.

On the other hand, suppose that the number μ fulfils (1.1) for every $\varepsilon > 0$. Then, if $\varepsilon > 0$, we have $\mu + \varepsilon \notin \{f : \sup B_f \notin \mathscr{I}\}$ and $\mathscr{I} - \limsup f \leq \mu + \varepsilon$. Since this holds for every $\varepsilon > 0$, we have $\mathscr{I} - \limsup f \leq \mu$. The first condition in (1.1) implies $\mathscr{I} - \limsup f \geq \mu - \varepsilon$ for each $\varepsilon > 0$, so we have $\mathscr{I} - \limsup f \geq \mu$. Inequalities $\mathscr{I} - \limsup f \leq \mu$ and $\mathscr{I} - \limsup f \geq \mu$ imply $\mu = \mathscr{I} - \limsup f$. □

The dual statement for $\mathscr{I}\text{-}\limsup f$ is as follows.
Theorem 3.5 *For any Folner sequence $\{S_n\}$ for G, if $\lambda = \mathscr{I}\text{-}\liminf f$ is finite, then for each $\varepsilon > 0$*

$$\{g \in S_n : f(g) < \lambda + \varepsilon\} \notin \mathscr{I} \text{ and } \{g \in S_n : f(g) < \lambda - \varepsilon\} \in \mathscr{I}. \tag{2.1}$$

Conversely, if (2.1) holds for every $\varepsilon > 0$ then $\lambda = \mathscr{I}\text{-}\liminf f$.

Proof. The proof of this theorem is similar to proof of the theorem 1. □

Theorem 3.6 *For any Folner sequence $\{S_n\}$ for G,*

$$\mathscr{I} - \liminf f \leq \mathscr{I} - \limsup f.$$

Proof. First consider the case in which $\mathscr{I}$-$\limsup f = -\infty$. Hence we have $B_f = \emptyset$, so for every b in $\mathbb{R}$, $\{g \in S_n : f(g) > b\} \in \mathscr{I}$ which implies that $\{g \in S_n : f(g) \leq b\} \in \mathscr{F}(\mathscr{I})$ so for every a in in $\mathbb{R}$, $\{g \in S_n : f(g) \leq a\} \notin \mathscr{I}$. Hence $\mathscr{I}$-$\liminf f = -\infty$.

The case in which $\mathscr{I}$-$\limsup f = +\infty$ needs no proof, so we next assume that $\mu := \mathscr{I}$-$\limsup f$ is finite and $\lambda := \mathscr{I}$-$\liminf f$. Given $\varepsilon > 0$ we show that $\mu + \varepsilon \in A_f$, so that $\lambda \leq \mu + \varepsilon$. By theorem 1, $\{g \in S_n : f(g) > \mu + \varepsilon\} \in \mathscr{I}$ because $\mu = \sup B_f$. This implies $\{g \in S_n : f(g) \leq \mu + \frac{\varepsilon}{2}\} \in \mathscr{F}(\mathscr{I})$. Since

$$\left\{g \in S_n : f(g) \leq \mu + \frac{\varepsilon}{2}\right\} \subseteq \{g \in S_n : f(g) < \mu + \varepsilon\}$$

and $\mathscr{F}(\mathscr{I})$ is a filter on $\mathbb{N}$,

$$\{g \in S_n : f(g) < \mu + \varepsilon\} \in \mathscr{F}(\mathscr{I}).$$

This implies

$$\{g \in S_n : f(g) < \mu + \varepsilon\} \notin \mathscr{I}.$$

Hence $\mu + \varepsilon \in A_f$. By the definition $\lambda := \mathscr{I}$-$\liminf f$, so we conclude that $\lambda \leq \mu + \varepsilon$; and since ε is arbitrary this proves that $\lambda \leq \mu$. □

Theorem 3.7 *For any Folner sequence $\{S_n\}$ for G, $\mathscr{I}$-bounded function f is $\mathscr{I}$-convergent if and only if $\mathscr{I}$-$\limsup f = \mathscr{I}$-$\liminf f$.*

Proof. For any Folner sequence $\{S_n\}$ for G, let $\lambda := \mathscr{I}$-$\liminf f$ and $\mu := \mathscr{I} - \limsup f$. First assume that $\mathscr{I}$-$\lim f = s$ and $\varepsilon > 0$. Then $\{g \in S_n : |f(g) - s| \geq \varepsilon\} \in \mathscr{I}$, so that $\{g \in S_n : f(g) > s + \varepsilon\} \in \mathscr{I}$ which implies that $\mu \leq s$. We also have $\{g \in S_n : f(g) < s - \varepsilon\} \in \mathscr{I}$, which yields that $s \leq \lambda$. Therefore $\mu \leq \lambda$. Combining this with $Theorem$ 3 we conclude that $\mu = \lambda$.

Now assume that for any Folner sequence $\{S_n\}$ for G, $\mathscr{I}$-$\limsup f = \mathscr{I}$-$\liminf f$. If $\varepsilon > 0$, then (1.1) and (2.1) imply that

$$\left\{g \in S_n : f(g) > s + \frac{\varepsilon}{2}\right\} \in \mathscr{I}$$

and

$$\left\{g \in S_n : f(g) < s - \frac{\varepsilon}{2}\right\} \in \mathscr{I}.$$

Hence, for any Folner sequence $\{S_n\}$ for G, $\mathscr{I}$-$\lim f = s$. □

Definition 3.8 Let G be a discrete countable amenable semigroup with identity in which both right and left cancelation laws hold. $f \in w(G)$ is said to be $\mathscr{I}$-Cauchy function for any Folner sequence $\{S_n\}$ for G if, for each $\varepsilon > 0$ and $l \geq 0$, then there exists an $m \in G/S_l$ such that

$$\{g \in S_n : |f(g) - f(m)| \geq \varepsilon\} \in \mathscr{I}$$

i.e., $|f(g) - f(m)| < \varepsilon$ a.a.g.

Theorem 3.9 *The following statements are equivalent:*
(i) $f \in w(G)$ is $\mathscr{I}$-convergent function
(ii) $f \in w(G)$ is $\mathscr{I}$-Cauchy function.

Proof. $(i) \Rightarrow (ii)$ To prove that (i) implies (ii) we assume that $\mathscr{I}$-$\lim f(g) = s$. Let $\varepsilon > 0$. Then $|f(g) - s| < \frac{\varepsilon}{2}$ a.a.g and, if g_0 is chosen so that $|f(g_0) - s| < \frac{\varepsilon}{2}$ a.a.g, then we have

$$|f(g) - f(g_0)| < |f(g) - s| + |f(g_0) - s| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

a.a.g. Hence f is $\mathscr{I}$-Cauchy function.
$(ii) \Rightarrow (i)$ Next $\{g \in S_n : |f(g) - f(m)| < \varepsilon\} \in \mathscr{F}(\mathscr{I})$ holds for all $\varepsilon > 0$. Then the set

$$C_\varepsilon = \{g \in S_n : f(g) \in [f(m) - \varepsilon, f(m) + \varepsilon]\} \in \mathscr{F}(\mathscr{I})$$

for all $\varepsilon > 0$. Denote $J_\varepsilon = [f(m) - \varepsilon, f(m) + \varepsilon]$.
Fix an $\varepsilon > 0$. Then $C_\varepsilon \in \mathscr{F}(\mathscr{I})$ and $C_{\frac{\varepsilon}{2}} \in \mathscr{F}(\mathscr{I})$. Hence $C_\varepsilon \cap C_{\frac{\varepsilon}{2}} \in \mathscr{F}(\mathscr{I})$. This implies

$$J = J_\varepsilon \cap J_{\frac{\varepsilon}{2}} \neq \emptyset,$$
$$\{g \in S_n : f(g) \in J\} \in \mathscr{F}(\mathscr{I}),$$
$$\operatorname{diam}(J) \leq \tfrac{1}{2}\operatorname{diam}(J_\varepsilon).$$

(diam(J) denotes the lenght of the interval J.) This way, by induction, we can construct the sequence of (closed) intervals $J_\varepsilon = I_0 \supseteq I_1 \supseteq \ldots \supseteq I_n \supseteq \ldots$ with the property diam$(I_n) \leq \frac{1}{2}$diam(I_{n-1}) $(n = 2, 3, \ldots)$. Then there exists a $s \in \bigcap_{n \in \mathbb{N}} I_n$ and it is routine work to verify $\mathscr{I}$-$\lim f(g) = s$. □

4. $\mathscr{I}$-Limit Points and $\mathscr{I}$-Cluster points

In [10] Koystrkro introduced the concepts of $\mathscr{I}$ limit point and $\mathscr{I}$ cluster point. In this section we extend these concepts of $\mathscr{I}$ limit point and $\mathscr{I}$ cluster point to the functions defined on discrete countable amenable semigroups. If $f \in w(G)$ and $H \subset G$, we write $R_f(G)$ to denote the range of $f \in w(G)$. If $R_f(H)$ is a subset of $R_f(G)$ and $\lim_{n\to\infty} \frac{|H\cap S_n|}{|S_n|} = 0$ then $R_f(H)$ is called a subset of Folner density zero for any Folner sequence $\{S_n\}$ for G, or a thin subset. On the other hand $R_f(H)$ is a nonthin subset of $R_f(G)$ if $\lim_{n\to\infty} \frac{|H\cap S_n|}{|S_n|} \neq 0$.

Definition 4.1 The number s is a $\mathscr{I}$ limit point for an $f \in w(G)$, for any Folner sequence $\{S_n\}$ for G, provided that there is nonthin subset of $R_f(G)$ that f $\mathscr{I}$-converges to s in it.

Definition 4.2 The number c is a $\mathscr{I}$ cluster point for an $f \in w(G)$, for any Folner sequence $\{S_n\}$ for G, provided that for each $\varepsilon > 0$ the set $\{g \in S_n : |f(g) - c| < \varepsilon\} \notin \mathscr{I}$.

For $f \in w(G)$, let $L_f(G)$, $\mathscr{I}(\Lambda_f(G))$, $\mathscr{I}(\Gamma_f(G))$ denote the sets of all ordinary limit points, $\mathscr{I}$ limit points and $\mathscr{I}$ cluster points of f, respectively. It is clear that $\mathscr{I}(\Lambda_f(G)) \subseteq \mathscr{I}(\Gamma_f(G)) \subseteq L_f(G)$.

Theorem 4.3 *Let $f \in w(G)$ be $\mathscr{I}$-bounded for any Folner sequence $\{S_n\}$ for G and let $\mathscr{I}(\Gamma_f(G))$ be the set of all $\mathscr{I}$ cluster points of f, for any Folner sequence $\{S_n\}$ for G. Then*

$$\mathscr{I}\text{-}\limsup f = \max \mathscr{I}(\Gamma_f(G)).$$

Proof. Put $\mathscr{I}$-$\limsup f = \mu$. Suppose $\mu' > \mu$. First we show that μ' is not in $\mathscr{I}(\Gamma_f(G))$. We have

$$\mu = \sup S,\ S = \{t : \{g \in S_n : f(g) > t\} \notin \mathscr{I}\}. \tag{7.1}$$

Choose $\varepsilon > 0$ such that $\mu < \mu' - \varepsilon < \mu'$. Then $\mu' - \varepsilon \notin S$ and

$$\{g \in S_n : f(g) > \mu' - \varepsilon\} \in \mathscr{I}.$$

It follows from the definition of $\mathscr{I}$ cluster point for an $f \in w(G)$ that $\mu' \notin \mathscr{I}(\Gamma_f(G))$.

We show $\mu \in \mathscr{I}(\Gamma_f(G))$. Let $\eta > 0$. It follows from (7.1) that there is a $t_0 \in \mathbb{R}$ such that $\mu - \eta < t_0 \leq \mu$, $t_0 \in S$. Hence

$$\{g \in S_n : f(g) > t_0\} \notin \mathscr{I}. \tag{7.2}$$

Simultaneously, since $\mu + \frac{\eta}{2} \notin S$, we have

$$\left\{g \in S_n : f(g) > \mu + \frac{\eta}{2}\right\} \in \mathscr{I}. \tag{7.2'}$$

It follows from (7.2) and (7.2′) $\{g \in S_n : f(g) \in (\mu - \eta, \mu + \eta)\} \notin \mathscr{I}$ and $\mu \in \mathscr{I}(\Gamma_f(G))$. □

Remark 4.4 It can be shown for a $\mathscr{I}$-bounded sequence $\{S_n\}$ for G the equality

$$\mathscr{I}\text{-}\liminf f = \min \mathscr{I}(\Gamma_f(G)).$$

Example 4.5 Take $G = \mathbb{Z}$, $H = \{0, \pm1, \pm3, \pm5, \pm7, \ldots\}$, $S_n = [-n, n]$ and define f as follows:

$$f(g) = \begin{cases} 0, & \text{if } g \in H, \\ 1, & \text{if } g \notin G \backslash H. \end{cases}$$

Then $L_f(G) = \{0, 1\}$ and $\mathscr{I}(\Lambda_f(G)) = \{0\}$.

5. Relationship between $\mathscr{I}_d$ and $\mathscr{I}_\delta$-Convergence and Cesàro summability

Recall that the Folner sequence $\{S_n\}$ for G is said to be strongly $(C,1)$-summable to s if and only if $\lim_{n\to\infty} \frac{1}{|S_n|} \sum_{g\in S_n} |f(g) - s| = 0$.

If the Folner sequence $\{S_n\}$ for G is bounded, then $\mathscr{I}_d$-$\lim f = s$ implies $(C,1)$-$\lim f(g) = s$. The converse is obviously not true. (e.g. $\{S_n\} = \{0, 1, 0, 1, \ldots\}$). However $f \in m(G)$ is bounded, the $\mathscr{I}_d$-convergence to some number is equivalent to strongly Cesàro-summability to the same number. But, for $\mathscr{I}_\delta$-convergence the situation is different.

Proposition 5.1 *Let $f \in w(G)$ be $\mathscr{I}$-bounded for any Folner sequence $\{S_n\}$ for G such that $\mathscr{I}_\delta$-$\lim f = 0$ and $(C,1)$-$\lim f$ does not exist.*

Proof. Put $S = \bigcup_{n=2}^{\infty} S_n$, where $S_n = \left\{ n^{n^2}+1, n^{n^2}+2, \ldots, n^{n^2+1} \right\}$ for $n \in \mathbb{N}$, $n \geq 2$. If $S(k) = d_k(S)$ for $k \in \mathbb{N}$, then

$$\overline{d}(S) \geq \lim_{n\to\infty} \sup \frac{S\left(n^{n^2+1}\right)}{n^{n^2+1}} \geq \lim_{n\to\infty} \sup \frac{n^{n^2+1} - n^{n^2}}{n^{n^2+1}} = 1.$$

Hence $\overline{d}(S) = 1$. Simultaneously $\sum_{k=1}^{n} \frac{1}{k} = \ln n + \gamma + O\left(\frac{1}{n}\right)$, where γ is an Euler constant, we have $\sum_{j \in S_n} \frac{1}{j} = \ln n + O\left(\frac{1}{n^{n^2}}\right)$ for all $n \in \mathbb{N}$, $n \geq 2$. From this by a simple calculation we get

$$\overline{\delta}(S) \leq \lim_{n\to\infty} \frac{\sum_{k=1}^{n} \ln k + O(1)}{\sum_{j=1}^{n^{n^2+1}} \frac{1}{j}} \leq \lim_{n\to\infty} \frac{n \ln n + O(1)}{(n^2+1)\ln n + O(1)} = 0.$$

So we have $\delta(S) = 0$ and consequently $\underline{d}(S) = 0$. So $d(S)$ does not exist.
Define f as follows:

$$f(g) = \begin{cases} 0, & \text{if } g \in G \backslash S, \\ 1, & \text{if } g \in S. \end{cases}$$

Since $\delta(S) = 0$ we have $\mathscr{I}_\delta$-$\lim f = 0$. But $(C,1)$-$\lim f(g)$ does not exist. □

6. Conclusion

The convergence of folner sequences on amenable semigroups has been recently studied by several authors. In this study, we extend concepts of statistical limit superior and inferior (as introduced by Nuray and Rhoades) to $\mathscr{I}$-limit superior and inferior and give some $\mathscr{I}$-analogue of properties of statistical limit superior and inferior for folner seqeunces on amenable semigroup. We investigate some properties of the new concepts. So, we have extended some well-known results.

References

[1] Day M., Amenable semigroups, *Illinois J. Math.* 1 (1957) 509–544.

[2] Das P., Savaş E., Ghosal S. Kr., On generalized of certain summability methods using ideals, *Appl. Math. Letter,* 36 (2011) 1509-1514.

[3] Douglass S.A., On a concept of summability in amenable semigroups, *Math. Scand.* 28 (1968) 96–102.

[4] Douglass S.A., Summing sequences for amenable semigroups, *Michigan Math* J. 20 (1973) 169–179.

[5] Fast H., Sur la convergence statistique, *Colloq. Math.* 2 (1951) 241–244.

[6] Fridy J. A., Statistical limit points, *Proc. Amer. Math. Soc.* 118(1993) 1187–1192.

[7] Fridy J. A., Orhan C., Statistical limit superior and limit inferior, *Proc. Amer. Math. Soc.* 125 (1993) 3625-3631.

[8] Demirci K., $\mathscr{I}$-limit superior and limit inferior, *Math. Commun.* 6(2) (2001) 165–172.

[9] Freedman A. R., Sember J. J., Densities and summability, *Pacitific J. Math.* 95(1981) 10–11.

[10] Kostyrko P., SalátT., Wilezyński W., $\mathscr{I}$-Convergence, *Real Analysis Exchange* 26(2) (2000) 669-686.

[11] Kostyrko P., Macaj M., SalátT., Sleziak M., $\mathscr{I}$-Convergence and Extremal $\mathscr{I}$-Limit Points, *Math. Slovaca* 55 (2005) 443–464.

[12] Salát T., Tripathy B.C., Ziman M., On Some Properties of $\mathscr{I}$-Convergence, *Tatra Mt. Math. Publ.* 28 (2004) 279-286 .

[13] Salát T., Tripathy B.C., Ziman M., $\mathscr{I}$-Convergence Field, *Tatra Mt. Math. Publ.* 28 (2005) 279-286.

[14] Mah P.F., Summability in amenable semigroups, *Trans. Amer. Math. Soc.* 156 (1971) 391–403.

[15] Mah P.F., Matrix summability in amenable semigroups, *Proc. Amer. Math. Soc.* 36 (1972) 414–420.

[16] Nuray F., Rhoades B.E., Some kinds of convergence defined by Folner sequences, *Analysis* 31(4) (2011) 381–390.

[17] Nomika I., Folner's conditions for amenable semigroups, *Math. Scand.* 15 (1964) 18–28.

[18] Ornstein D.S., B. Weiss, Entropy and isomorphism theorems for actions of amenable groups, *J. Anal. Math.* 48 (1987) 1–141.

[19] Ostmann H. H., Additive Zahlentheorie I , Springer-Verlag, Berlin-Gottingen-Heidelberg, 1956.