# A numerical investigation of VMS-POD model for Darcy-Brinkman equations

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***Abstract*—We extend the variational multiscale proper orthogonal decomposition reduced order modeling (VMSPOD) to flows governed by double diffusive convection. We present stability and convergence analyses for it, and give results for numerical tests on a benchmark problem which show it is an effective approach.**

***Index Terms*—variational multiscale, proper orthogonal decomposition, double-diffusive, reduced order models.**

## I. Introduction

We consider the Darcy-Brinkman equations with double diffusive convection, the dimensionless form of which is given as:

$$\begin{aligned}
\mathbf{u}_t - 2\nu\nabla\cdot\mathbb{D}u + (\mathbf{u}\cdot\nabla)\mathbf{u} + Da^{-1}\mathbf{u} \\
+\nabla p = (\beta_T T + \beta_C C)\ \mathbf{g} \text{ in } (0,\tau]\times\Omega, \\
\nabla\cdot\mathbf{u} = 0 \text{ in } (0,\tau]\times\Omega, \\
\mathbf{u} = \mathbf{0} \text{ in } (0,\tau]\times\partial\Omega, \\
T_t + \mathbf{u}\cdot\nabla T = \gamma\Delta T \text{ in } (0,\tau]\times\partial\Omega, \\
C_t + \mathbf{u}\cdot\nabla C = D_c\Delta C \text{ in } (0,\tau]\times\partial\Omega, \\
T, C = 0 \text{ on } \Gamma_D, \\
\nabla T\cdot n = \nabla C\cdot n = 0 \text{ on } \Gamma_N, \\
\mathbf{u}(0,\mathbf{x}) = \mathbf{u}_0,\ T(0,\mathbf{x}) = T_0,\ C(0,\mathbf{x}) = C_0 \text{ in } \Omega,
\end{aligned} \tag{1}$$

where $\mathbf{u}(t,\mathbf{x})$, $p(t,\mathbf{x})$, $T(t,\mathbf{x})$, $C(t,\mathbf{x})$ are the fluid velocity, the pressure, the temperature, and the concentration fields, respectively. Let $\Omega\subset\mathbb{R}^d, d\in\{2,3\}$ be a confined porous enclosure with polygonal boundary $\partial\Omega$ and $\Gamma_N$ be a regular open subset of the boundary and $\Gamma_D = \partial\Omega\setminus\Gamma_N$. The initial velocity, temperature and concentration fields are given as $\mathbf{u}_0,\ T_0,\ C_0$. The parameters in (1) are the kinematic viscosity $\nu > 0$, inversely proportional to $Re$, the thermal diffusivity $\gamma > 0$, the velocity deformation tensor $\mathbb{D}\mathbf{u} = (\nabla\mathbf{u} + \nabla\mathbf{u}^T)/2$, the mass diffusivity $D_c > 0$, the Darcy number $Da$, and the gravitational acceleration vector $\mathbf{g}$. The solutal and the thermal expansion coefficients are $\beta_C$, and $\beta_T$, respectively. The dimensionless parameters are the Prandtl number $Pr$, the Darcy number $Da$, the buoyancy ratio $N$, the Lewis number $Le$, the Schmidt number $Sc$, and the thermal and solutal Grashof numbers $Gr_T$ and $Gr_C$, respectively. Here $H$ is the cavity height, $k$ the permeability, and $\Delta T$ and $\Delta C$ are the temperature and the concentration differences, respectively.

Double diffusive convection drives a flow with two potentials that have different diffusion rates. A common example occurs in oceanography, where temperature and salt concentration gradients and diffusivity drive the flow of salt water. The physical model uses that momentum is forced by both heat and mass transfer, and a Darcy term accounts for the porous boundary. Since simulation of the double-diffusive system (1) can be very expensive as in all multiphysics flow problems, practitioners need efficient methods to approximate solutions. One efficient method is reduced order modeling (ROM) using proper orthogonal decomposition (POD). This method is highly efficient and has been found to be successful for many different types of flow problems. In particular, recent work with POD-ROM has shown that the approach can work well on multiphysics flow problems such as the Boussinesq system for fluids driven by a single potential, and also for magnetohydrodynamics flow [1], [2], [3]. Hence it is a natural and important next step to extend this methodology to flows governed by the system (1), as such tools will prove useful in the coming years as simulations of ocean water flows become more prevalent.

However, in turbulent flows POD does not work well. In this case, a stabilization method is required. The combining of POD with the VMS method has been successful to solve this challenge. VMS aims to model unresolved scales by adding an artificial viscosity to only resolved small-scales. Hence, the oscillations in small scales can be removed. In POD, basis functions are ordered with respect to their kinetic energy content. Hence the hierarchy of small and large scales is presented naturally. Thus, the POD and VMS is particularly suitable. Using VMS in POD was pioneered in [4], [5], [6], [7], and their studies showed this could increase numerical accuracy.

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This work is arranged as follows. Section 2 presents the continuous variational formulation of the double diffusive Darcy-Brinkman system (1) and its discretization, and here the VMSPOD variational formulation is defined. Section 3 is devoted to the numerical analysis of the VMSPOD formulation. Finally, Section 4 concludes the work with a summary.

## II. Full order model for the double diffusive Darcy-Brinkman system

Throughout the work standard notations for Sobolev spaces and their norms will be used. The norm in $(H^k(\Omega))^d$ is denoted by $\|\cdot\|_k$ and the norms in Lebesgue spaces $(L^p(\Omega))^d$, $1 \le p < \infty$, $p \ne 2$ by $\|\cdot\|_{L^p}$. The space $L^2(\Omega)$ is equipped with the norm and inner product $\|\cdot\|$ and $(\cdot,\cdot)$, respectively, and for these we drop the subscripts. The continuous velocity, pressure, temperature and concentration spaces are denoted by

$$\begin{aligned} \mathbf{X} &:= (\mathbf{H}_0^1(\Omega))^d, Q := L_0^2(\Omega), \\ W &:= \{S \in H^1(\Omega) : S = 0 \text{ on } \Gamma_D\}, \\ \Psi &:= \{\Phi \in H^1(\Omega) : \Phi = 0 \text{ on } \Gamma_D\}, \end{aligned}$$

and the divergence free space given as

$$\mathbf{V} := \{\mathbf{v} \in \mathbf{X} : (\nabla \cdot \mathbf{v}, q) = 0, \ \forall q \in Q\}.$$

We denote the dual space of $\mathbf{X}$ by $\mathbf{H}^{-1}$ with norm

$$\|\mathbf{f}\|_{-1} = \sup_{\mathbf{v}\in X} \frac{|(\mathbf{f},\mathbf{v})|}{\|\nabla \mathbf{v}\|}.$$

The variational formulation of (1) reads as follows: Find $\mathbf{u} : (0,\tau] \to \mathbf{X}$, $p : (0,\tau] \to Q$, $T : [0,\tau] \to W$ and $C : [0,\tau] \to \Psi$ satisfying

$$(\mathbf{u}_t,\mathbf{v}) + 2\nu(\mathbb{D}\mathbf{u},\mathbb{D}\mathbf{v}) + b_1(\mathbf{u},\mathbf{u},\mathbf{v}) + (Da^{-1}\mathbf{u},\mathbf{v}) - (p,\nabla\cdot\mathbf{v}) = \beta_T(\mathbf{g}T,\mathbf{v}) + \beta_C(\mathbf{g}C,\mathbf{v}), \tag{2}$$

$$(T_t,S) + b_2(\mathbf{u},T,S) + \gamma(\nabla T,\nabla S) = 0, \tag{3}$$

$$(C_t,\phi) + b_3(\mathbf{u},C,\Phi) + D_c(\nabla C,\nabla\phi) = 0, \tag{4}$$

for all $(\mathbf{v},q,S,\Phi) \in (X,Q,W,\Psi)$, where

$$\begin{aligned} b_1(\mathbf{u},\mathbf{v},\mathbf{w}) &:= \frac{1}{2}\left(((\mathbf{u}\cdot\nabla)\mathbf{v},\mathbf{w}) - ((\mathbf{u}\cdot\nabla)\mathbf{w},\mathbf{v})\right), \\ b_2(\mathbf{u},T,S) &:= \frac{1}{2}\left(((\mathbf{u}\cdot\nabla)T,S) - ((\mathbf{u}\cdot\nabla)S,T)\right), \\ b_3(\mathbf{u},C,\Phi) &:= \frac{1}{2}\left(((\mathbf{u}\cdot\nabla)C,\Phi) - ((\mathbf{u}\cdot\nabla)\Phi,C)\right), \end{aligned}$$

represent the skew-symmetric forms of the convective terms.

We consider a conforming finite element method for (2)-(4), with spaces $\mathbf{X}_h \subset \mathbf{X}$, $Q_h \subset Q$, $W_h \subset W$ and $\Psi_h \subset \Psi$. We also assume that the pair $(\mathbf{X}_h, Q_h)$ satisfies the discrete inf-sup condition. It will also be assumed for simplicity that the finite element spaces $\mathbf{X}_h$, $W_h$, $\Psi_h$ are composed of piecewise polynomials of degree at most $m$ and $Q_h$ is composed of piecewise polynomials of degree at most $m-1$. In addition, we assume that the spaces satisfy the interpolation approximation properties. The discretely divergence free space for $(\mathbf{X}_h, Q_h)$ pairs is given by

$$\mathbf{V}_h = \{\mathbf{v}_h \in \mathbf{X}_h : (\nabla\cdot\mathbf{v}_h, q_h) = 0, \forall q_h \in Q_h\}. \tag{5}$$

The inf-sup condition implies that the space $\mathbf{V}_h$ is a closed subspace of $\mathbf{X}_h$ and the formulation above involving $\mathbf{X}_h$ and $Q_h$ is equivalent to the following $\mathbf{V}_h$ formulation: Find $(\mathbf{u}_h, T_h, C_h) \in (\mathbf{V}_h, W_h, \Psi_h)$ satisfying

$$(\mathbf{u}_{h,t},\mathbf{v}_h) + 2\nu(\mathbb{D}\mathbf{u}_h,\mathbb{D}\mathbf{v}_h) + b_1(\mathbf{u}_h,\mathbf{u}_h,\mathbf{v}_h) + (Da^{-1}\mathbf{u}_h,\mathbf{v}_h) = \beta_T(\mathbf{g}T_h,\mathbf{v}_h) + \beta_C(\mathbf{g}C_h,\mathbf{v}_h), \tag{6}$$

$$(T_{h,t},S_h) + b_2(\mathbf{u}_h,T_h,S_h) + \gamma(\nabla T_h,\nabla S_h) = 0, \tag{7}$$

$$(C_{h,t},\Phi_h) + b_3(\mathbf{u}_h,C_h,\Phi_h) + D_c(\nabla C_h,\nabla\Phi_h) = 0, \tag{8}$$

for all $(\mathbf{v}_h, S_h, \Phi_h) \in (\mathbf{V}_h, W_h, \psi_h)$.

The goal of the POD is to find low dimensional bases for velocity, temperature, concentration by solving the minimization problems. The solution of the problem is obtained by using the method of snapshots. We note that all eigenvalues are sorted in descending order. Thus, the basis functions $\{\boldsymbol{\psi}_i\}_{i=1}^{r_1}$, $\{\phi_i\}_{i=1}^{r_2}$ and $\{\eta_i\}_{i=1}^{r_3}$ correspond to the first $r_1, r_2$ and $r_3$ largest eigenvalues $\{\lambda_i\}_{i=1}^{r_1}$, $\{\mu_i\}_{i=1}^{r_2}$, $\{\xi_i\}_{i=1}^{r_3}$ of the velocity, the temperature, the concentration, respectively. For simplicity, we will denote POD-ROM spaces using just $r$ instead of $r_1$, $r_2$ and $r_3$. However, in the analysis, we are careful to distinguish that these parameters can be chosen independently.

Let $\mathbf{X}_r$, $W_r$ and $\Psi_r$ be the POD-ROM spaces spanned by POD basis functions:

$$\mathbf{X}_r = span\{\boldsymbol{\psi}_1, \boldsymbol{\psi}_2, \ldots, \boldsymbol{\psi}_{r_1}\}, \tag{9}$$

$$W_r = span\{\phi_1, \phi_2, \ldots, \phi_{r_2}\}, \tag{10}$$

$$\Psi_r = span\{\eta_1, \eta_2, \ldots, \eta_{r_3}\}. \tag{11}$$

Note that by construction $\mathbf{X}_r \subset \mathbf{V}_h \subset \mathbf{X}$, $W_r \subset W_h \subset W$ and $\Psi_r \subset \Psi_h \subset \Psi$.

Now, we state the POD-Galerkin (POD-G) formulation of the Darcy-Brinkman double diffusive system. Given

$$\mathbf{g} \in L^2(0,k;H^{-1}(\Omega)) \text{ and } \mathbf{u}_0 \in (L^2(\Omega))^d, \quad T_0, C_0 \in L^2(\Omega),$$

Find $(\mathbf{u}_r, T_r, C_r) \in (\mathbf{X}_r, W_r, \Psi_r)$ satisfying

$$(\mathbf{u}_{r,t}, \mathbf{v}_r) + 2\nu(\mathbb{D}\mathbf{u}_r, \mathbb{D}\mathbf{v}_r) + b_1(\mathbf{u}_r, \mathbf{u}_r, \mathbf{v}_r) + (Da^{-1}\mathbf{u}_r, \mathbf{v}_r) = \beta_T(\mathbf{g}T_r, \mathbf{v}_r) + \beta_C(\mathbf{g}C_r, \mathbf{v}_r), \quad (12)$$

$$(T_{r,t}, S_r) + b_2(\mathbf{u}_r, T_r, S_r) + \gamma(\nabla T_r, \nabla S_r) = 0, \quad (13)$$

$$(C_{r,t}, \Phi_r) + b_3(\mathbf{u}_r, C_r, \Phi_r) + D_c(\nabla C_r, \nabla \Phi_r) = 0, \quad (14)$$

for all $(\mathbf{v}_r, S_r, \Phi_r) \in (\mathbf{X}_r, W_r, \Psi_r)$.

For simplicity, we equip this system (12)-(14) with a backward Euler temporal discretization. We consider adding the decoupled VMS-ROM stabilization from [7], where in effect additional viscosity gets added to the smaller $R$ velocity modes in a post-processing step. Specifically, we post-process $\mathbf{u}_r^{n+1}$ by solving the algorithm:

**Algorithm II.1.** *The post-processing VMS-POD approximation for double diffusive system (1) given as:* ***Step 1****: Find* $(\mathbf{w}_r^{n+1}, T_r^{n+1}, C_r^{n+1}) \in (\mathbf{X}_r, W_r, \Psi_r)$ *satisfying*

$$\begin{aligned}(\frac{\mathbf{w}_r^{n+1} - \mathbf{u}_r^n}{\Delta t}, \mathbf{v}_r) + 2\nu(\mathbb{D}\mathbf{w}_r^{n+1}, \mathbb{D}\mathbf{v}_r) \\ +b_1(\mathbf{w}_r^{n+1}, \mathbf{w}_r^{n+1}, \mathbf{v}_r) + (Da^{-1}\mathbf{w}_r^{n+1}, \mathbf{v}_r) \\ = \beta_T(\mathbf{g}T_r^{n+1}, \mathbf{v}_r) + \beta_C(\mathbf{g}C_r^{n+1}, \mathbf{v}_r), \end{aligned} \quad (15)$$

$$\begin{aligned}(\frac{T_r^{n+1} - T_r^n}{\Delta t}, S_r) + b_2(\mathbf{w}_r^{n+1}, T_r^{n+1}, S_r) \\ +\gamma(\nabla T_r^{n+1}, \nabla S_r) = 0, \end{aligned} \quad (16)$$

$$\begin{aligned}(\frac{C_r^{n+1} - C_r^n}{\Delta t}, \Phi_r) + b_3(\mathbf{w}_r^{n+1}, C_r^{n+1}, \Phi_r) \\ +D_c(\nabla C_r^{n+1}, \nabla \Phi_r) = 0, \end{aligned} \quad (17)$$

*for all* $(\mathbf{v}_r, S_r, \Phi_r) \in (\mathbf{X}_r, W_r, \Psi_r)$.
***Step 2****: Find* $\mathbf{u}_r^{n+1} \in \mathbf{X}_r$, $\forall \mathbf{v}_r \in \mathbf{X}_r$*:*

$$\left(\frac{\mathbf{u}_r^{n+1} - \mathbf{w}_r^{n+1}}{\Delta t}, \mathbf{v}_r\right) = \left(\nu_T(I - P_R)\nabla \frac{(\mathbf{u}_r^{n+1} + \mathbf{w}_r^{n+1})}{2}, (I - P_R)\nabla \mathbf{v}_r\right), \quad (18)$$

*where* $P_R$ *is the* $L^2$ *projection into* $\mathbf{X}_R$*, which is the subset of* $\mathbf{X}_r$ *that is the span of the first* $R$ $(< r)$ *velocity modes.*

### *A. Projection Error*

This subsection starts with the estimations for the $L^2$ projection error. In order to prove an error estimate for the error between the true solution and the POD solution of the double diffusive Darcy-Brinkman system, we first recall the main estimates for projections. For the error assessment, we use the $L^2$ projections of $\mathbf{u}_r, T_r$ and $C_r$, respectively. The $L^2$ projection operators $P_{u,r} : L^2 \to \mathbf{X}_r$, $P_{T,r} : L^2 \to W_r$, $P_{C,r} : L^2 \to \Psi_r$ are defined by

$$\begin{aligned} (\mathbf{u} - P_{u,r}\mathbf{u}, \boldsymbol{\psi}_r) &= 0, \quad \forall \boldsymbol{\psi}_r \in \mathbf{X}_r \\ (T - P_{T,r}T, \phi_r) &= 0, \quad \forall \phi_r \in W_r \\ (C - P_{C,r}C, \eta_r) &= 0, \quad \forall \eta_r \in \Psi_r \end{aligned} \quad (19)$$

## III. Numerical analysis of double diffusive Darcy-Brinkman system

This section is devoted to a derivation of the a priori error estimation of (15)-(18). We first give the stability of the solutions of (15)-(18).

**Lemma III.1.** *(Stability) The post-processed VMS-POD approximation (15)-(18) is unconditionally stable in the following sense: for any* $\Delta t > 0$*,*

$$\begin{aligned}\|\mathbf{u}_r^M\|^2 + \sum_{n=0}^{M-1} \Bigg[ 2\nu_T \Delta t \left\| (I - P_R)\nabla \frac{(\mathbf{w}_r^{n+1} + \mathbf{u}_r^{n+1})}{2} \right\|^2 \\ + \|\mathbf{w}_r^{n+1} - \mathbf{u}_r^n\|^2 + \nu\Delta t \|\nabla \mathbf{w}_r^{n+1}\|^2 + Da^{-1}\Delta t \|\mathbf{w}_r^{n+1}\|^2 \Bigg] \\ \leq \|\mathbf{u}_0\|^2 + C^* \|\mathbf{g}\|_\infty^2 (\beta_T^2 \gamma^{-1} \|T_0\|^2 + \beta_C^2 D_c^{-1} \|C_0\|^2). \end{aligned} \quad (20)$$

$$\|T_r^M\|^2 + \sum_{n=0}^{M-1} 2\Delta t \gamma \|\nabla T_r^n\|^2 \leq \|T_0\|^2, \quad (21)$$

$$\|C_r^M\|^2 + \sum_{n=0}^{M-1} 2\Delta t D_c \|\nabla C_r^n\|^2 \leq \|C_0\|^2, \quad (22)$$

*where* $C^* = \min\{\nu^{-1}, Da\}$.

The optimal asymptotic error estimation requires the following regularity assumptions for the true solution:

$$\begin{aligned} \mathbf{u} &\in L^\infty(0, k; \mathbf{H}^{m+1}(\Omega)) \\ \mathbf{u}_{tt} &\in L^2(0, T; \mathbf{H}^1(\Omega)) \\ T,\ C &\in L^\infty(0, k; H^{m+1}(\Omega)) \\ T_{tt},\ C_{tt} &\in L^2(0, T; H^1(\Omega)) \\ p &\in L^\infty(0, k; H^m(\Omega)) \end{aligned} \quad (23)$$

We define the discrete norms for $\mathbf{v}^n \in \mathbf{H}^p(\Omega)$, $n = 0, 1, 2, ..., M$ as the following:

$$|||\mathbf{v}|||_{\infty,p} := \max_{0 \leq n \leq M} \|\mathbf{v}^n\|_p,$$

$$|||\mathbf{v}|||_{m,p} := (\Delta t \sum_{n=0}^{M} \|\mathbf{v}^n\|_p^m)^{1/m}.$$

**Theorem III.1.** *(Error Estimation) Suppose regularity assumptions holds. Then for the sufficiently small* $\Delta t$*,* *the error satisfies*

$$\begin{aligned}\|\mathbf{u}^M - \mathbf{u}_r^M\|^2 + &\|T^M - T_r^M\|^2 + \|C^M - C_r^M\|^2 \\ \leq \quad K\Bigg( & 1 + h^{2m} + (\Delta t)^2 + (1 + \|S_{u,r}\|_2 \\ & + \|S_{u,R}\|_2 + \|S_{T,r}\|_2 + \|S_{C,r}\|_2) h^{2m+2} \\ & + \sum_{i=r_1+1}^{d} (\|\boldsymbol{\psi}_i\|_1^2 + 1)\lambda_i + \sum_{i=r_2+1}^{d} (\|\phi_i\|_1^2 + 1)\mu_i \\ & + \sum_{i=r_3+1}^{d} (\|\eta_i\|_1^2 + 1)\xi_i + \sum_{i=R+1}^{d} \|\boldsymbol{\psi}_i\|_1^2 \lambda_i \Bigg). \quad (24)\end{aligned}$$

## IV. Numerical Studies

In this section we present results of numerical tests using the POD-ROM studied above. In the following tests, we fix $\Delta t = 0.0025$, $T = 1$.

### A. Test 1:

In this test, to create POD basis, 4000 snapshots is used in the time interval [0, 1]. We construct the correlation matrix by using these snapshots. The largest eigenvalues of the correlation matrix are illustrated in Figure 1. We see that the eigenvalues show a rapid

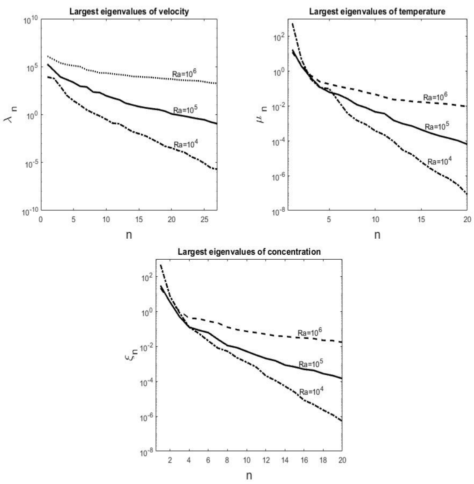


Fig. 1: The largest eigenvalues for the velocity, temperature and concentration for different $Ra$

decrease for $Ra = 10^4$, $Ra = 10^5$, and a slow decrease for $Ra = 10^6$. Captured energy for velocity ($E_u$), temperature ($E_T$), concentration ($E_C$) can be defined as

$$E_u = \frac{\sum_{j=1}^{r} \lambda_j}{\sum_{j=1}^{M} \lambda_j} \times 100, \quad E_T = \frac{\sum_{j=1}^{r} \mu_j}{\sum_{j=1}^{M} \mu_j} \times 100,$$

$$E_C = \frac{\sum_{j=1}^{r} \xi_j}{\sum_{j=1}^{M} \xi_j} \times 100.$$

Process time and percent of captured energy with respect to the different POD modes number for velocity, temperature and concentration are shown in Table I-II. DNS process time is 80000 seconds even using a super computer. As seen in the tables, process time is

TABLE I: Percent of captured energy for velocity, temperature and concentration with $Ra = 10^5$ varying $r$

| $r$ | $E_u$ | $E_T$ | $E_C$ | CPU(s) |
|---|---|---|---|---|
| 4 | 98.0576 | 99.1554 | 99.3996 | 35.260284 |
| 8 | 99.8173 | 99.8942 | 99.9304 | 68.713127 |
| 12 | 99.9781 | 99.9845 | 99.9852 | 64.741996 |
| 16 | 99.9939 | 99.9966 | 99.9955 | 71.881944 |
| 20 | 99.9985 | 99.9992 | 99.9985 | 78.383086 |

TABLE II: Percent of captured energy for velocity, temperature and concentration with $Ra = 10^6$ varying $r$

| $r$ | $E_u$ | $E_T$ | $E_C$ | CPU(s) |
|---|---|---|---|---|
| 8 | 93.0866 | 97.6547 | 97.4737 | 896.318716 |
| 16 | 97.4949 | 99.1031 | 98.9494 | 837.966706 |
| 24 | 98.9831 | 99.5758 | 99.4581 | 833.658012 |
| 32 | 99.4980 | 99.7838 | 99.6876 | 903.636491 |
| 40 | 99.7096 | 99.8793 | 99.8065 | 974.256394 |

reduced with POD method. In this way, computational cost decrease remarkably. In addition, when we select POD modes number $r = 12$ for $Ra = 10^4$ and $r = 20$ for $Ra = 10^5$, these capture $\%99.999$ of the system's kinetic energy. On the other hand, for $Ra = 10^6$, it needs more modes to capture a large part of the system's energy.Thus a stabilization method is needed to obtain good numerical results for this test.

### B. Test 2:

In this test, we check the accuracy of the method for different $Ra$. The variation of $L^2$ error with respect to time are shown for $Ra = 10^4$ and $Ra = 10^5$ in Figure 2 for $Ra = 10^6$ in Figure 3.

As seen in the figure 2-3, the $L^2$ error and the $H^1$ error become close to zero as the time increase. It gives that our solution matches DNS for $Ra = 10^4$ and $Ra = 10^5$. However, the POD method does not work well for large $Ra$. Hence, we need a stabilization method for $Ra = 10^6$. When we used VMS type stabilization method results match DNS.

## V. Conclusions

We proposed a modular regularization with VMS-POD for double diffusive system. We proved stability and convergence results for the VMSPOD scheme, and gave results of several numerical tests. Our tests showed POD gives very good results for Rayleigh numbers $Ra = 10^4,\ 10^5$ without VMS-type stabilization, which were accurately simulated with $r = 10$

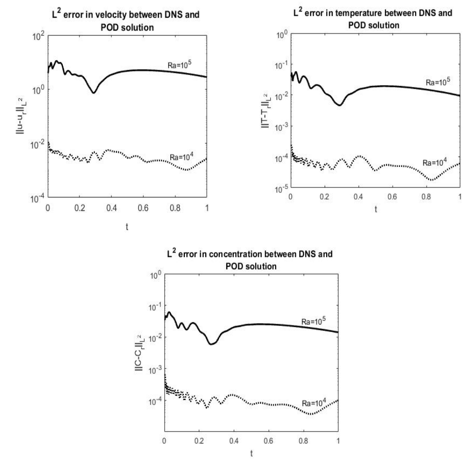


Fig. 2: The $L^2$ error in the velocity, temperature and concentration for $Ra = 10^4$ and $Ra = 10^5$.

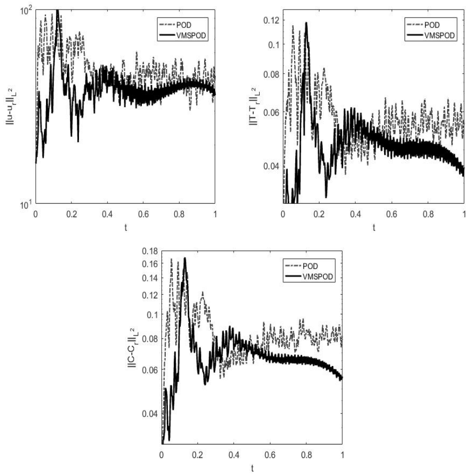


Fig. 3: The $L^2$ error in the velocity, temperature and concentration for $Ra = 10^6$.

and $r = 20$, respectively. For higher $Ra$, POD did not perform well without stabilization, but adding VMS-type stabilization, the approach gave good qualitative results.

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