Earthline Journal of Mathematical Sciences
ISSN (Online): 2581-8147
Volume 4, Number 2, 2020, Pages 425-431
https://doi.org/10.34198/ejms.4220.425431

The Fourth Fundamental Form of the Torus Hypersurface

Erhan Güler

Department of Mathematics, Faculty of Sciences, Bartın University, 74100 Bartın, Turkey
e-mail: eguler@bartin.edu.tr

Abstract

We introduce the fourth fundamental form of the torus hypersurface in the four dimensional Euclidean space. We also compute I, II, III and IV fundamental forms of a torus hypersurface.

1. Introduction

Surfaces and hypersurfaces have been worked by the mathematicians for centuries. We see some new papers about torus surfaces and torus hypersurfaces in the literature such as [2-15].

Aminov [1] gave the three dimensional submanifold M^3 in $\mathbb{E}^4$, homeomorphic to $S^1 \times S^2$, considering in a similar way to the construction of an ordinary torus in $\mathbb{E}^3$.

Let γ be a circle of radius R with the center at the origin O in a coordinate plane $\mathbb{E}^2$, and P be a point of γ. Spanning $\mathbb{E}^3$ on vectors OP, e_3, e_4, we consider the sphere $S^2(P)$ of radius r with the center at P. When P moves along γ, then all points of $S^2(P)$ form the submanifold M^3 in $\mathbb{E}^4$, and then a torus hypersurface in $\mathbb{E}^4$ can be parametrized by:

$$\mathbf{x}(u,v,w) = \begin{pmatrix} (R + r\cos u \cos v)\cos w \\ (R + r\cos u \cos v)\sin w \\ \mathrm{r}\cos u \sin v \\ \mathrm{r}\sin u \end{pmatrix}, \tag{1.1}$$

where $u, v, w \in I \subset \mathbb{R}$.

In this paper, we study the fourth fundamental form of the torus hypersurface in the four dimensional Euclidean space $\mathbb{E}^4$. We present fundamental notions of four

Received: October 19, 2020; Accepted: October 26, 2020

2010 Mathematics Subject Classification: 53A35, 53C42.

Keywords and phrases: four space, torus hypersurface, fourth fundamental form.

*Corresponding author

dimensional Euclidean geometry. Moreover, we give fundamental forms I, II, III, and IV of torus hypersurface.

2. Preliminaries

We consider characteristic polynomial of shape operator **S**:

$$P_{\mathbf{S}}(\lambda) = 0 = \det(\mathbf{S} - \lambda I_n) = \sum_{k=0}^{n} (-1)^k \, s_k \lambda^{n-k}, \tag{2.1}$$

where I_n denotes the identity matrix of order n in $\mathbb{E}^{n+1}$. Then, we get curvature formulas

$$\binom{n}{i} \mathfrak{C}_i = s_i.$$

Here, $\binom{n}{0} \mathfrak{C}_0 = s_0 = 1$ by definition. So, k-th fundamental form of hypersurface M^n is defined by

$$\mathrm{I}\big(\mathbf{S}^{k-1}(X), Y\big) = \langle \mathbf{S}^{k-1}(X), Y\rangle.$$

Then, we get

$$\sum_{i=0}^{n} (-1)^i \binom{n}{i} \mathfrak{C}_i \, \mathrm{I}\big(\mathbf{S}^{k-1}(X), Y\big) = 0. \tag{2.2}$$

In the rest of this paper, we shall identify a vector $(\mathrm{a}, \mathrm{b}, \mathrm{c}, \mathrm{d})$ with its transpose $(\mathrm{a}, \mathrm{b}, \mathrm{c}, \mathrm{d})^t$.

Let $\mathbf{M} = \mathbf{M}(u, v, w)$ be an isometric immersion of a hypersurface M^3 in $\mathbb{E}^4$. Inner product of vectors $\vec{x} = (x_1, x_2, x_3, x_4)$ and $\vec{y} = (y_1, y_2, y_3, y_4)$ in $\mathbb{E}^4$ is given by as follows:

$$\langle \vec{x}, \vec{y} \rangle = x_1 y_1 + x_2 y_2 + x_3 y_3 + x_4 y_4.$$

Vector product $\vec{x} \times \vec{y} \times \vec{z}$ of $\vec{x} = (x_1, x_2, x_3, x_4)$, $\vec{y} = (y_1, y_2, y_3, y_4)$, $\vec{z} = (z_1, z_2, z_3, z_4)$in $\mathbb{E}^4$ is defined by as follows:

$$\vec{x} \times \vec{y} \times \vec{z} = \det \begin{pmatrix} e_1 \, e_2 \, e_3 \, e_4 \\ x_1 x_2 x_3 x_4 \\ y_1 y_2 y_3 y_4 \\ z_1 \, z_2 \, z_3 \, z_4 \end{pmatrix}.$$

The Gauss map of a hypersurface **M** is given by

$$e = \frac{\mathbf{M}_u \times \mathbf{M}_v \times \mathbf{M}_w}{\|\mathbf{M}_u \times \mathbf{M}_v \times \mathbf{M}_w\|},$$

where $\mathbf{M}_u = d\mathbf{M}/du$. For a hypersurface $\mathbf{M}$ in $\mathbb{E}^4$, we have following fundamental form matrices

$$\mathrm{I} = \begin{pmatrix} E & F & A \\ F & G & B \\ A & B & C \end{pmatrix},$$

$$\mathrm{II} = \det \begin{pmatrix} L & M & P \\ M & N & T \\ P & T & V \end{pmatrix},$$

$$\mathrm{III} = \begin{pmatrix} X & Y & O \\ Y & Z & R \\ O & R & S \end{pmatrix}.$$

Here, the coefficients are given by

$E = \langle \mathbf{M}_u, \mathbf{M}_u \rangle$, $F = \langle \mathbf{M}_u, \mathbf{M}_v \rangle$, $G = \langle \mathbf{M}_v, \mathbf{M}_v \rangle$, $A = \langle \mathbf{M}_u, \mathbf{M}_w \rangle$, $B = \langle \mathbf{M}_v, \mathbf{M}_{\boldsymbol{w}} \rangle$, $C = \langle \mathbf{M}_w, \mathbf{M}_w \rangle$,

$L = \langle \mathbf{M}_{uu}, e \rangle$, $M = \langle \mathbf{M}_{uv}, e \rangle$, $N = \langle \mathbf{M}_{vv}, e \rangle$, $P = \langle \mathbf{M}_{uw}, e \rangle$, $T = \langle \mathbf{M}_{vw}, e \rangle$, $V = \langle \mathbf{M}_{ww}, e \rangle$,

$X = \langle e_u, e_u \rangle$, $Y = \langle e_u, e_{\boldsymbol{v}} \rangle$, $Z = \langle e_v, e_{\boldsymbol{v}} \rangle$, $O = \langle e_u, e_{\boldsymbol{w}} \rangle$, $R = \langle e_v, e_{\boldsymbol{w}} \rangle$, $S = \langle e_w, e_{\boldsymbol{w}} \rangle$,

and e is the Gauss map (i.e. the unit normal vector field).

3. The Fourth Fundamental Form

Next, we obtain the fourth fundamental form matrix for a hypersurface $\mathbf{M}(u, v, w)$ in $\mathbb{E}^4$. Using characteristic polynomial $P_{\mathbf{S}}(\lambda) = a\lambda^3 + b\lambda^2 + c\lambda + d = 0$, we obtain curvature formulas: $\mathfrak{C}_0 = 1$ (by definition),

$$\mathfrak{C}_1 = -\frac{b}{\binom{3}{1} a}, \quad \mathfrak{C}_2 = \frac{c}{\binom{3}{2} a}, \quad \mathfrak{C}_3 = -\frac{d}{\binom{3}{3} a}.$$

Theorem 3.1. *For any hypersurface M^3 in $\mathbb{E}^4$, the fourth fundamental form is related by*

$$\mathfrak{C}_0 \mathrm{IV} - 3\mathfrak{C}_1 \mathrm{III} + 3\mathfrak{C}_2 \mathrm{II} - \mathfrak{C}_3 \mathrm{I} = 0. \tag{3.1}$$

Proof. Taking $n = 3$ in (2.2), then some computing, we get the fourth fundamental form matrix as follows

$$\mathrm{IV} = \begin{pmatrix} \zeta & \eta & \delta \\ \eta & \phi & \sigma \\ \delta & \sigma & \xi \end{pmatrix}, \tag{3.2}$$

where

$$\zeta = -\frac{\left\{\begin{array}{c} CL^2N - CLM^2 - GLP^2 + B^2LX + A^2NX + GL^2V + F^2VX + NP^2E + M^2VE \\ -CNXE - GVXE - CGLX + 2(BTXE - BL^2T - MPTE + ABMX - ALNP \\ +BLMP + ALMT + CFMX + AGPX - BFPX - AFTX - FLMV + FLPT) \end{array}\right\}}{\det \mathrm{I}},$$

$$\eta = \frac{\left\{\begin{array}{c} CM^3 - FNP^2 - GMP^2 - FLT^2 - B^2LY - A^2NY + FM^2V - F^2VY + MT^2E \\ +CNYE - MNVE + GVYE - CLMN + CGLY + FLNV - GLMV + 2(AFPY \\ -BTYE + ABMY + ANMP - BLMT - CFMY - AGPY + BFPY - TM^2A - BM^2P) \end{array}\right\}}{\det \mathrm{I}},$$

$$\delta = \frac{\left\{\begin{array}{c} GP^3 - B^2LO - A^2NO + ANP^2 + CM^2P - ALT^2 - AVM^2 - F^2OV + PT^2E \\ +CNOE + GOVE - NPVE + CGLO - CLNP + ALNV - GLPV + 2(ABMO \\ -BOTE - CFMO - AGOP + BFOP + AFOT - BLPT + FMPV - BMP^2 - FP^2T) \end{array}\right\}}{\det \mathrm{I}},$$

$$\phi = -\frac{\left\{\begin{array}{c} CLN^2 - CM^2N - GLT^2 + B^2LZ + A^2NZ + GM^2V + F^2VZ - NT^2E + N^2VE \\ -CNZE - GVZE - CGLZ + 2(-AN^2P + BTZE - ABMZ + BMNP + ANMT \\ -BLNT + CFMZ + AGPZ - BFPZ - AFTZ + FMNV + FNPT - GMPT) \end{array}\right\}}{\det \mathrm{I}},$$

$$\sigma = \frac{\left\{\begin{array}{c} ET^3 - BNP^2 - B^2LR - A^2NR + BLT^2 + CM^2T - BM^2V + GP^2T - F^2RV \\ +CNRE + GRVE - NTVE + CGLR - CLNT + BLNV - GLTV + 2(ABMR \\ -BRTE - CFMR - AGPR + BFPR + AFRT + ANPT + FMTV - AT^2M - FT^2P) \end{array}\right\}}{\det \mathrm{I}},$$

$$\xi = -\frac{\left\{\begin{array}{c} CNP^2 - B^2LS - A^2NS + CLT^2 + GLV^2 - GP^2V + F^2SV + NV^2E - T^2VE \\ -CNSE - GSVE - CGLS + 2(-FMV^2 + BSTE - ABMS + CFMS + AGPS \\ -BFPS - AFST - CMPT - ANPV + BMPV + ATMV - BLTV - FPTV) \end{array}\right\}}{\det \mathrm{I}}.$$

4. Curvatures of Torus Hypersurface

In this section, we compute curvatures of torus hypersurface (1.1).

With the first differentials of (1.1) depends on u, v, w, we get the Gauss map of (1.1):

$$e = -\begin{pmatrix} \cos u \cos v \cos w \\ \cos u \cos v \sin w \\ \cos u \sin v \\ \sin u \end{pmatrix}. \tag{4.1}$$

We get the first and the second fundamental form matrices of (1.1), respectively,

$$\mathrm{I} = \begin{pmatrix} r^2 & 0 & 0 \\ 0 & r^2 \cos^2 u & 0 \\ 0 & 0 & (R + r \cos u \cos v)^2 \end{pmatrix},$$

$$\mathrm{II} = \begin{pmatrix} r & 0 & 0 \\ 0 & r \cos^2 u & 0 \\ 0 & 0 & (R + r \cos u \cos v) \cos u \cos v \end{pmatrix}.$$

Using $\mathrm{I}^{-1}.\mathrm{II}$, torus hypersurface (1.1) in $\mathbb{E}^4$ has following shape operator

$$\mathbf{S} = \begin{pmatrix} k_1 & 0 & 0 \\ 0 & k_2 & 0 \\ 0 & 0 & k_3 \end{pmatrix} = \begin{pmatrix} \frac{1}{r} & 0 & 0 \\ 0 & \frac{1}{r} & 0 \\ 0 & 0 & \frac{\cos u \cos v}{R + r \cos u \cos v} \end{pmatrix}.$$

So, we compute the third fundamental form matrix using (4.1) of (1.1):

$$\mathrm{III} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos^2 u & 0 \\ 0 & 0 & \cos^2 u \ \cos^2 v \end{pmatrix}.$$

Finally, using (3.2) on (1.1), we obtain the fourth quantities of (1.1), i.e., symmetric matrix, as follows

$$\mathrm{IV} = \begin{pmatrix} \frac{1}{r} & 0 & 0 \\ 0 & \frac{\cos^2 u}{r} & 0 \\ 0 & 0 & \frac{\cos^3 u \ \cos^3 v}{R + r \cos u \cos v} \end{pmatrix}.$$

Corollary 4.1. *Torus hypersurface* (1.1) *in* $\mathbb{E}^4$ *has following relations*

$$\mathrm{IV} = \mathrm{III}.\mathbf{S},$$

$$\mathrm{III} = \mathrm{II}.\mathbf{S},$$

$$\mathrm{II} = \mathrm{I}.\mathbf{S}.$$

Proof. Considering $\mathrm{I}, \mathrm{II}, \mathrm{III}, \mathrm{IV}$ and $\mathbf{S}$ of (1.1), we obtain all quantities.

Corollary 4.2. *Torus hypersurface* (1.1) *in* $\mathbb{E}^4$ *has following relations*

$$\frac{(\det \mathrm{II})(\det \mathrm{III})^2}{(\det \mathrm{I})(\det \mathrm{IV})^2} = \det \mathbf{S} = k_1 k_2 k_3 = \frac{\cos u \cos v}{r^2(R + r\cos u \cos v)} = \mathfrak{C}_3.$$

Proof. Using $\mathrm{I}, \mathrm{II}, \mathrm{III}, \mathrm{IV}$ and $\mathbf{S}$ of (1.1), it is clear.

5. Conclusion

Torus hypersurfaces have been recently worked by a number of authors. We extend some well-known results of the torus hypersurfaces with the help of the fourth fundamental form

References

[1] Yu. Aminov, *The Geometry of Submanifolds*, Gordon and Breach Science Publishers, Amsterdam, 2001.

[2] V.A. Borovitskiĭ, K-closedness for weighted Hardy spaces on the torus T^2, *Zap. Nauchn. Sem.* (*POMI*) 456 (2017), 25-36 (in Russian); translation in *J. Math. Sci.* (*N.Y.*) 234(3) (2018), 282-289. https://doi.org/10.1007/s10958-018-4004-9

[3] J. Dasgupta, B. Khan and V. Uma, Cohomology of torus manifold bundles, *Math. Slovaca* 69(3) (2019), 685-698. https://doi.org/10.1515/ms-2017-0257

[4] C.L. Duston, Torus solutions to the Weierstrass-Enneper representation of surfaces, *J. Math. Phys.* 60(8) (2019), 1-5. https://doi.org/10.1063/1.5097669

[5] J. Harvey and C. Searle, Almost non-negatively curved 4-manifolds with torus symmetry, *Proc. Amer. Math. Soc.* 148(11) (2020), 4933-4950. https://doi.org/10.1090/proc/15093

[6] M. Hasegawa and D. Ida, Instability of stationary closed strings winding around flat torus in five-dimensional Schwarzschild spacetimes, *Phys. Rev. D* 98(4) (2018), 1-7. https://doi.org/10.1103/PhysRevD.98.044045

[7] S. Hirose and E. Kin, On hyperbolic surface bundles over the circle as branched double covers of the 3-sphere, *Proc. Amer. Math. Soc.* 148(4) (2020), 1805-1814. https://doi.org/10.1090/proc/14825

[8] Y. Kamiyama, The orbit space of a hypersurface of a torus by an involution, *JP J. Geom. Top.* 21(4) (2018), 365-372. https://doi.org/10.17654/GT021040365

[9] E. Krasko and A. Omelchenko, Enumeration of *r*-regular maps on the torus. Part I: rooted maps on the torus, the projective plane and the Klein bottle, Sensed maps on the torus, *Discrete Math.* 342(2) (2019), 584-599. https://doi.org/10.1016/j.disc.2018.07.013

[10] E. Krasko and A. Omelchenko, Enumeration of *r*-regular maps on the torus. Part II: Unsensed maps, *Discrete Math.* 342(2) (2019), 600-614. https://doi.org/10.1016/j.disc.2018.09.004

[11] L.M. Lerman and K.N. Trifonov, The topology of symplectic partially hyperbolic automorphisms of the 4-torus, *Mat. Zametki* 108(3) (2020), 474-476 (in Russian). https://doi.org/10.1134/S0001434620090175

[12] M. Mase, Families of *K*3 surfaces and curves of (2,3)-torus type, *Kodai Math. J.* 42(3) (2019), 409-430. https://doi.org/10.2996/kmj/1572487224

[13] S. Nakamura, The orthonormal Strichartz inequality on torus, *Trans. Amer. Math. Soc.* 373(2) (2020), 1455-1476. https://doi.org/10.1090/tran/7982

[14] Mauricio Poletti, Geometric growth for Anosov maps on the 3 torus, *Bull. Braz. Math. Soc.* (*N.S.*) 49(4) (2018), 699-713. https://doi.org/10.1007/s00574-018-0079-7

[15] T. Sakajo, Vortex crystals on the surface of a torus, *Philos. Trans. Roy. Soc. A* 377(2158) (2019), 1-17. https://doi.org/10.1098/rsta.2018.0344