Article

Implicit Equations of the Henneberg-Type Minimal Surface in the Four-Dimensional Euclidean Space

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Abstract: Considering the Weierstrass data as \((\psi, f, g) = (2, 1 - z^{-m}, z^n)\), we introduce a two-parameter family of Henneberg-type minimal surface that we call \(\mathcal{H}_{m,n}\) for positive integers \((m, n)\) by using the Weierstrass representation in the four-dimensional Euclidean space \(\mathbb{E}^4\). We define \(\mathcal{H}_{m,n}\) in \((r, \theta)\) coordinates for positive integers \((m, n)\) with \(m \neq 1, n \neq -1, -m + n \neq -1\), and also in \((u, v)\) coordinates, and then we obtain implicit algebraic equations of the Henneberg-type minimal surface of values \((4, 2)\).

Keywords: Henneberg-type minimal surface; Weierstrass representation; four-dimensional space; implicit equation; degree

1. Introduction

The theory of surfaces has an important role in mathematics, physics, biology, architecture, see e.g., the classical books [1,2] and papers [3–9].

A minimal surface in the three-dimensional Euclidean space \(\mathbb{E}^3\), also in higher dimensions, is a regular surface for which the mean curvature vanishes identically. See [10–27] for details. On the other hand, a Henneberg surface [4–6], also obtained by the Weierstrass representation [8,9] is well-known classical minimal surface in \(\mathbb{E}^3\).

In the four-dimensional Euclidean space \(\mathbb{E}^4\), a general definition of rotation surfaces was given by Moore in [28] as follows

\[
X(u, t) = \begin{pmatrix}
x_1(u) \cos(at) - x_2(u) \sin(at) \\
x_1(u) \cos(at) + x_2(u) \sin(at) \\
x_3(u) \cos(bt) - x_4(u) \sin(bt) \\
x_3(u) \cos(bt) + x_4(u) \sin(bt)
\end{pmatrix},
\]

A more restricted case can be found in [29]:

\[
W(u, t) = (x_1(u), x_2(u), r(u) \cos(t), r(u) \sin(t)).
\]

It is a bit too general since the curve is not located in any subspace before rotation.

Güler and Kişi [30] studied the Weierstrass representation, the degree and the classes of surfaces in \(\mathbb{E}^4\), see [31–38] for some previous work.

In this paper, we study a two-parameter family of Henneberg-type minimal surfaces using the Weierstrass representation in \(\mathbb{E}^4\). We give the Weierstrass equations for a minimal surface in \(\mathbb{E}^4\), and obtain two normals of the surface in Section 2.