Article

Family of Enneper Minimal Surfaces

Erhan Güler

Department of Mathematics, Faculty of Sciences, Bartın University, Bartın 74100, Turkey; eguler@bartin.edu.tr; Tel.: +90-378-501-1000-1521

Received: 18 October 2018; Accepted: 22 November 2018; Published: 26 November 2018

Abstract: We consider a family of higher degree Enneper minimal surface $E_m$ for positive integers $m$ in the three-dimensional Euclidean space $\mathbb{E}^3$. We compute algebraic equation, degree and integral free representation of Enneper minimal surface for $m = 1, 2, 3$. Finally, we give some results and relations for the family $E_m$.

Keywords: Enneper minimal surface family; Weierstrass representation; algebraic surface; degree; integral free representation

1. Introduction

Minimal surfaces have an important role in the mathematics, physics, biology, architecture, etc. These kinds of surfaces have been studied over the centuries by many mathematicians and also geometers. A minimal surface in $\mathbb{E}^3$ is a regular surface for which the mean curvature vanishes identically.

There are many important classical works on minimal surfaces in the literature such as [1–10]. However, we only see a few notable works about algebraic minimal surfaces, including general results and the properties. They were given by Enneper [11,12], Henneberg [13,14] and Weierstrass [9,15].

One of them is the classical Enneper minimal surface that was given by Enneper. See [11,12] for details. About Enneper minimal surface, many nice papers were done such as [16–24] in the last few decades.

In this paper, we introduce a family of higher degree Enneper minimal surface $E_m$ for positive integers $m$ in the three-dimensional Euclidean space $\mathbb{E}^3$. In Section 2, we give the family of Enneper minimal surfaces $E_m$. We obtain the algebraic equation and degree of surface $E_1$ (resp., $E_2$, $E_3$). Using the integral free form of Weierstrass, we find some algebraic functions for $E_m$ ($m \geq 1$, $m \in \mathbb{Z}$) in Section 3. Finally, we give some general findings for a family of higher degree Enneper minimal surface $E_m$ with a table in the last section.

2. The Family of Enneper Minimal Surfaces $E_m$

We will often identify $\mathbb{C}$ and $\mathbb{C}^3$ without further comment. Let $\mathbb{E}^3$ be a three-dimensional Euclidean space with natural metric $\langle \cdot, \cdot \rangle = dx^2 + dy^2 + dz^2$.

Let $\mathcal{U}$ be an open subset of $\mathbb{C}$. A minimal (or isotropic) curve is an analytic function $\Psi : \mathcal{U} \rightarrow \mathbb{C}^n$ such that $\Psi' (\zeta) \cdot \Psi'' (\zeta) = 0$, where $\zeta \in \mathcal{U}$, and $\Psi' := \frac{\partial \Psi}{\partial \zeta}$. In addition, if $\Psi' \cdot \Psi'' = |\Psi'|^2 \neq 0$, then $\Psi$ is a regular minimal curve.

Thus, let see the following lemma for complex minimal curves.

Lemma 1. Let $\Psi : \mathcal{U} \rightarrow \mathbb{C}^3$ be a minimal curve and write $\Psi' = (\varphi_1, \varphi_2, \varphi_3)$. Then,

$$\mathcal{F} = \frac{\varphi_1 - i\varphi_2}{2} \quad \text{and} \quad \mathcal{G} = \frac{\varphi_3}{\varphi_1 - i\varphi_2}$$