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PERTURBED COMPANIONS OF OSTROWSKI TYPE INEQUALITIES FOR N -TIMES DIFFERENTIABLE FUNCTIONS AND APPLICATIONS

Abstract. We firstly examine some inequalities obtained by using sets of complex-valued functions for functions whose high order derivatives are restricted. We also give some approximations for the functions whose derivatives up to the order $n - 1$ ($n \geq 1$) are continuous and whose the n th derivatives are of bounded variation. So, the results provide extensions of those presented in earlier works.

Key words: *Function of bounded variation, Perturbed Ostrowski type inequalities*

2010 Mathematical Subject Classification: *26D15, 26A45, 26D10*

1. Introduction. The inequality discovered by Ostrowski in 1938 has been studied by a large number of researchers due to its comprehensive application fields in numerical analysis and certain special means. This inequality [21], established by using mappings whose first derivatives are bounded, is stated in the following manner.

Theorem 1. *Let $f : [a, b] \rightarrow \mathbb{R}$ be a differentiable mapping on (a, b) , $f' : (a, b) \rightarrow \mathbb{R}$ is bounded on (a, b) , i. e. $\|f'\|_\infty := \sup_{t \in (a, b)} |f'(t)| < \infty$. Then*

we have the inequality

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t) dt \right| \leq \left[\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2} \right] (b-a) \|f'\|_\infty, \quad (1)$$

for all $x \in [a, b]$. The constant $\frac{1}{4}$ is the best possible.

Over the years, interested researchers have studied it to provide novel refinements, improvements, and generalizations of the inequality (1). For

instance, some authors deduced new Ostrowski-type inequalities for differentiable, twice differentiable, or higher-order differentiable functions in [7], [8], [9], and [22] (see also references therein). On the other side, the perturbed method has been much used to generalise integral inequalities. For example, after Dragomir had published his paper [14] involving the perturbed inequality of the Ostrowski type established by utilizing absolutely continuous functions, some authors focused on perturbed integral inequalities for twice and higher order differentiable mappings in [5], [16], [17], [18], and [19]. What is more, some companion perturbed inequalities for various assumptions of the functions are refined by using three- and five-step quadratic kernels in [15], [23], and [24].

In particular, some mathematicians focus on the Ostrowski-type inequalities obtained by using mappings of bounded variation, as well as the other function species. In the reference [11], Dragomir introduced the following useful result for functions of bounded variation:

Theorem 2. *Let $f : [a, b] \rightarrow \mathbb{R}$ be a mapping of bounded variation on $[a, b]$. Then*

$$\left| \int_a^b f(t)dt - (b-a)f(x) \right| \leq \left[\frac{1}{2}(b-a) + \left| x - \frac{a+b}{2} \right| \right] \bigvee_a^b(f) \quad (2)$$

holds for all $x \in [a, b]$. The constant $\frac{1}{2}$ is the best possible.

Moreover, Dragomir indicated the original generalisation of the Ostrowski-type results for functions that are of bounded variation in [10]. Afterwards, results pertaining to the inequality (1) for functions whose first derivatives are of bounded variation, are given in [1], [6], and [20]. Also, certain generalized outcomes for mappings that possess n -th derivatives of bounded variation, are established in [3] and [13]. In addition to all the results, some companion versions of perturbed results concerning Ostrowski's inequality for bounded-variation mappings are examined in [2], [4], and [12].

We also note that Dragomir established the following identity, so as to observe some perturbed outcomes of Ostrowski-type inequalities in [14].

Theorem 3. *Let $f : [a, b] \rightarrow \mathbb{C}$ be absolutely continuous on $[a, b]$ and $x \in [a, b]$. Then, for any complex numbers $\lambda_1(x)$ and $\lambda_2(x)$, we have*

$$\frac{1}{b-a} \int_a^x (t-a) [f'(t) - \lambda_1(x)] dt + \frac{1}{b-a} \int_x^b (t-b) [f'(t) - \lambda_2(x)] dt =$$

$$= f(x) + \frac{1}{2(b-a)} [(b-x)^2 \lambda_2(x) - (x-a)^2 \lambda_1(x)] - \frac{1}{b-a} \int_a^b f'(t) dt$$

where the integrals in the left-hand side are taken in the Lebesgue sense.

The primary purpose of this work is to deduce original inequalities for functions, whose higher-order derivatives are limited. For this, some approximations are examined with the help of the identity obtained by utilizing higher-order differentiable mappings. So, the new companion results are derived, regarding inequality (1) for functions whose n -th derivatives are bounded and of bounded variation. Relations between these results and inequalities given in the earlier works are also examined.

2. The case when $f^{(n)}$ is bounded. Before we can establish the inequalities that will be given in this section, we should mention the following identity.

Lemma 1. *Let $f : [a, b] \rightarrow \mathbb{C}$ be an n -time differentiable function on (a, b) . Then, for any complex numbers $\lambda_i(x)$, $i = 1, 2, 3$ and all $x \in [a, \frac{a+b}{2}]$, we have the identity*

$$\begin{aligned} & \int_a^x \frac{(t-a)^n}{n!} [f^{(n)}(t) - \lambda_1(x)] dt + \\ & + \int_x^{a+b-x} \frac{1}{n!} \left(t - \frac{a+b}{2}\right)^n [f^{(n)}(t) - \lambda_2(x)] dt + \\ & + \int_{a+b-x}^b \frac{(t-b)^n}{n!} [f^{(n)}(t) - \lambda_3(x)] dt = \\ & = S(f : n, x) - R(n, x) + (-1)^n \int_a^b f(t) dt, \quad (3) \end{aligned}$$

where $S(f : n, x)$ and $R(n, x)$ are defined by

$$\begin{aligned} S(f : n, x) = & \sum_{k=0}^{n-1} \frac{(-1)^{n+1} [f^{(k)}(a+b-x) + (-1)^k f^{(k)}(x)]}{(k+1)!} \times \\ & \times \left[(x-a)^{k+1} + (-1)^k \left(\frac{a+b}{2} - x\right)^{k+1} \right] \quad (4) \end{aligned}$$

and

$$R(n, x) = [\lambda_1(x) + (-1)^n \lambda_3(x)] \frac{(x-a)^{n+1}}{(n+1)!} + [1 + (-1)^n] \frac{\lambda_2(x)}{(n+1)!} \left(\frac{a+b}{2} - x\right)^{n+1}.$$

Proof. Combining the resulting identities by using fundamental analysis operators, after applying integration by parts n times to the three integrals in the right-hand side of the equality (3), the required identity can be easily obtained. \square

The expression $S(f : n, x)$ (4) will be used throughout this paper.

Furthermore, we define the sets of complex-valued mappings, for $\gamma, \Gamma \in \mathbb{C}$ and an interval of real numbers $[a, b]$,

$$\bar{U}_{[a,b]}(\gamma, \Gamma) := \left\{ f : [a, b] \rightarrow \mathbb{C} \mid \Re \left[(\Gamma - f(t)) \left(\overline{f(t)} - \bar{\gamma} \right) \right] \geq 0 \right\}$$

for almost every $t \in [a, b]$ and

$$\bar{\Delta}_{[a,b]}(\gamma, \Gamma) := \left\{ f : [a, b] \rightarrow \mathbb{C} \mid \left| f(t) - \frac{\gamma + \Gamma}{2} \right| \leq \frac{1}{2} |\Gamma - \gamma| \right\}$$

for a. e. $t \in [a, b]$

Also, we shall give the following lemma so as to prove the next inequality.

Lemma 2. [14] For any $\gamma, \Gamma \in \mathbb{C}$, $\gamma \neq \Gamma$, the sets $\bar{U}_{[a,b]}(\gamma, \Gamma)$ and $\bar{\Delta}_{[a,b]}(\gamma, \Gamma)$ are nonempty, convex, and closed convex sets and

$$\bar{U}_{[a,]}(\gamma, \Gamma) = \bar{\Delta}_{[a,b]}(\gamma, \Gamma).$$

Theorem 4. Let $f : [a, b] \rightarrow \mathbb{C}$ be an n -time differentiable function on (a, b) and $x \in [a, \frac{a+b}{2}]$. If there exists $\gamma_i, \Gamma_i \in \mathbb{C}$ with $\gamma_i \neq \Gamma_i$, $i = 1, 2, 3$, such that

$$f^{(n)} \in \bar{\Delta}_{[a,x]}(\gamma_1, \Gamma_1) \cap \bar{\Delta}_{[x,a+b-x]}(\gamma_2, \Gamma_2) \cap \bar{\Delta}_{[a+b-x,b]}(\gamma_3, \Gamma_3), \quad (5)$$

then we have the inequality

$$\left| S(f : n, x) - [1 + (-1)^n] \frac{\gamma_2(x) + \Gamma_2(x)}{2(n+1)!} \left(\frac{a+b}{2} - x\right)^{n+1} - \left[\frac{\gamma_1(x) + (-1)^n \gamma_3(x) + \Gamma_1(x) + (-1)^n \Gamma_3(x)}{2} \right] \frac{(x-a)^{n+1}}{(n+1)!} + \right.$$

$$+ (-1)^n \int_a^b f(t) dt \Big| \leq \frac{\varepsilon_1 + \varepsilon_3}{2} \frac{(x-a)^{n+1}}{(n+1)!} + \frac{\varepsilon_2}{(n+1)!} \left(\frac{a+b}{2} - x\right)^{n+1} \quad (6)$$

where $\varepsilon_1 = |\Gamma_1 t(x) - \gamma_1(x)|$, $\varepsilon_2 = |\Gamma_2(x) - \gamma_2(x)|$, $\varepsilon_3 = |\Gamma_3(x) - \gamma_3(x)|$.

Proof. If we take absolute value of both left- and right-hand side of (3) for $\lambda_1(x) = \frac{\gamma_1(x) + \Gamma_1(x)}{2}$, $\lambda_2(x) = \frac{\gamma_2(x) + \Gamma_2(x)}{2}$, $\lambda_3(x) = \frac{\gamma_3(x) + \Gamma_3(x)}{2}$, we get the inequality

$$\begin{aligned} & \left| S(f : n, x) - [1 + (-1)^n] \frac{\gamma_2(x) + \Gamma_2(x)}{2(n+1)!} \left(\frac{a+b}{2} - x\right)^{n+1} - \right. \\ & \quad \left. - \left[\frac{\gamma_1(x) + (-1)^n \gamma_3(x) + \Gamma_1(x) + (-1)^n \Gamma_3(x)}{2} \right] \frac{(x-a)^{n+1}}{(n+1)!} + \right. \\ & \quad \left. + (-1)^n \int_a^b f(t) dt \right| \leq \int_a^x \frac{(t-a)^n}{n!} \left| f^{(n)}(t) - \frac{\gamma_1(x) + \Gamma_1(x)}{2} \right| dt + \\ & \quad + \int_x^{a+b-x} \frac{1}{n!} \left| t - \frac{a+b}{2} \right|^n \left| f^{(n)}(t) - \frac{\gamma_2(x) + \Gamma_2(x)}{2} \right| dt + \\ & \quad + \int_{a+b-x}^b \frac{(b-t)^n}{n!} \left| f^{(n)}(t) - \frac{\gamma_3(x) + \Gamma_3(x)}{2} \right| dt. \end{aligned}$$

Utilizing condition (5), on account of the definition of $\bar{\Delta}_{[a,b]}(\gamma, \Gamma)$, we write the inequality

$$\begin{aligned} \int_a^x \frac{(t-a)^n}{n!} \left| f^{(n)}(t) - \frac{\gamma_1(x) + \Gamma_1(x)}{2} \right| dt & \leq \frac{1}{2} |\Gamma_1(x) - \gamma_1(x)| \int_a^x \frac{(t-a)^n}{n!} dt = \\ & = \frac{1}{2} |\Gamma_1(x) - \gamma_1(x)| \frac{(x-a)^{n+1}}{(n+1)!}. \end{aligned}$$

Similarly, the results of the other integrals can also be obtained. Thus, the proof is completed. \square

Corollary 1. To get the following inequalities, we use:

- the Hölder inequality

$$mn + pq \leq (m^\alpha + p^\alpha)^{\frac{1}{\alpha}} (n^\beta + q^\beta)^{\frac{1}{\beta}},$$

where $m, n, p, q \geq 0$ and $\alpha > 1$ with $\frac{1}{\alpha} + \frac{1}{\beta} = 1$;

- the identity

$$\max \{X, Y\} = \frac{X + Y}{2} + \left| \frac{X - Y}{2} \right|;$$

- the maximum property of $\max \{a^n, b^n\} = (\max \{a, b\})^n$ for $a, b > 0$ and $n \in \mathbb{N}$ in the left-hand side of inequality (6).

The obtained inequalities are

$$\begin{aligned} & \left| S(f : n, x) - [1 + (-1)^n] \frac{\gamma_2(x) + \Gamma_2(x)}{2(n+1)!} \left(\frac{a+b}{2} - x \right)^{n+1} - \right. \\ & \left. - \left[\frac{\gamma_1(x) + (-1)^n \gamma_3(x) + \Gamma_1(x) + (-1)^n \Gamma_3(x)}{2} \right] \frac{(x-a)^{n+1}}{(n+1)!} + (-1)^n \int_a^b f(t) dt \right| \\ & \leq \begin{cases} \frac{1}{(n+1)!} \left[(x-a)^{n+1} + \left(\frac{a+b}{2} - x \right)^{n+1} \right] \max \left\{ \frac{\varepsilon_1 + \varepsilon_3}{2}, \varepsilon_2 \right\}, \\ \frac{1}{t(n+1)!} \left[(x-a)^{(n+1)p} + \left(\frac{a+b}{2} - x \right)^{(n+1)p} \right]^{\frac{1}{p}} \left[\left(\frac{\varepsilon_1 + \varepsilon_3}{2} \right)^q + \varepsilon_2^q \right]^{\frac{1}{q}} \\ \quad \text{for } p > 1, \quad \frac{1}{p} + \frac{1}{q} = 1, \\ \frac{1}{(n+1)!} \left[\frac{b-a}{4} + \left| x - \frac{3a+b}{4} \right| \right]^{n+1} \left[\frac{\varepsilon_1 + \varepsilon_3}{2} + \varepsilon_2 \right], \end{cases} \end{aligned}$$

where $\varepsilon_1 = |\Gamma_1(x) - \gamma_1(x)|$, $\varepsilon_2 = |\Gamma_2(x) - \gamma_2(x)|$, $\varepsilon_3 = |\Gamma_3(x) - \gamma_3(x)|$.

Remark 1. Let f and x be defined as in Theorem 4. If there exists $\gamma_i, \Gamma_i \in \mathbb{C}$ with $\gamma_i \neq \Gamma_i$, $i = 1, 2$, such that

$$f^{(n)} \in \overline{\Delta}_{[a,x]}(\gamma_1, \Gamma_1) \cap \overline{\Delta}_{[x,a+b-x]}(\gamma_2, \Gamma_2) \cap \overline{\Delta}_{[a+b-x,b]}(\gamma_1, \Gamma_1),$$

then we have

$$\left| S(f : n, x) - [1 + (-1)^n] \frac{\gamma_2(x) + \Gamma_2(x)}{2(n+1)!} \left(\frac{a+b}{2} - x \right)^{n+1} - \right.$$

$$\begin{aligned}
 & - \left| \frac{[1 + (-1)^n][\gamma_1(x) + \Gamma_1(x)](x - a)^{n+1}}{2(n + 1)!} + (-1)^n \int_a^b f(t)dt \right| \leq \\
 & \leq \frac{1}{(n + 1)!} \left[\varepsilon_1(x - a)^{n+1} + \varepsilon_2 \left(\frac{a + b}{2} - x \right)^{n+1} \right] \quad (7)
 \end{aligned}$$

where $\varepsilon_1 = |\Gamma_1(x) - \gamma_1(x)|$ and $\varepsilon_2 = |\Gamma_2(x) - \gamma_2(x)|$. Additionally, if there exists $\gamma_1, \Gamma_1 \in \mathbb{C}$ with $\gamma_1 \neq \Gamma_1$ such that $f^{(n)} \in \overline{\Delta}_{[a,b]}(\gamma_1, \Gamma_1)$, then we have the conclusion

$$\begin{aligned}
 & \left| S(f : n, x) + (-1)^n \int_a^b f(t)dt - \right. \\
 & \left. - \frac{[1 + (-1)^n][\gamma_1(x) + \Gamma_1(x)]}{2(n + 1)!} \left[\left(\frac{a + b}{2} - x \right)^{n+1} + (x - a)^{n+1} \right] \right| \leq \\
 & \leq \frac{|\Gamma_1(x) - \gamma_1(x)|}{(n + 1)!} \left[\left(\frac{a + b}{2} - x \right)^{n+1} + (x - a)^{n+1} \right]. \quad (8)
 \end{aligned}$$

Remark 2. If we select $x = a$ in inequality (6), we have

$$\begin{aligned}
 & \left| \sum_{k=0}^{n-1} \frac{(-1)^{n+1} \left[f^{(k)}(b) + (-1)^k f^{(k)}(a) \right]}{(k + 1)!} \left[(-1)^k \left(\frac{b - a}{2} \right)^{k+1} \right] - \right. \\
 & \left. - [1 + (-1)^n] \frac{\gamma_2(x) + \Gamma_2(x)}{2(n + 1)!} \left(\frac{a + b}{2} - x \right)^{n+1} + (-1)^n \int_a^b f(t)dt \right| \leq \\
 & \leq \frac{|\Gamma_2(x) - \gamma_2(x)|}{(n + 1)!} \left(\frac{b - a}{2} \right)^{n+1}.
 \end{aligned}$$

Remark 3. If we take $x = \frac{a + b}{2}$ in the inequality (6), then one concludes the inequality

$$\begin{aligned}
 & \left| \sum_{k=0}^{n-1} \frac{(-1)^{n+1} f^{(k)}\left(\frac{a+b}{2}\right)[1 + (-1)^k]}{(k + 1)!} \left(\frac{b - a}{2} \right)^{k+1} + (-1)^n \int_a^b f(t)dt - \right. \\
 & \left. - \left[\frac{\gamma_1(x) + (-1)^n \gamma_3(x) + \Gamma_1(x) + (-1)^n \Gamma_3(x)}{2} \right] \frac{(b - a)^{n+1}}{2^{n+1}(n + 1)!} \right| \leq
 \end{aligned}$$

$$\leq \frac{|\Gamma_1(x) - \gamma_1(x)| + |\Gamma_3(x) - \gamma_3(x)|}{2(n+1)!} \left(\frac{b-a}{2}\right)^{n+1}.$$

Also, should we use the condition of the result (7) in this inequality, then we can find a new inequality.

Remark 4. Substitution of $x = \frac{3a+b}{4}$ in (6) gives

$$\left| \sum_{k=0}^{n-1} \frac{(-1)^{n+1} [1 + (-1)^k] \left[f^{(k)}\left(\frac{a+3b}{4}\right) + (-1)^k f^{(k)}\left(\frac{3a+b}{4}\right) \right] (b-a)^{k+1}}{4^{k+1} (k+1)!} - \right. \\ \left. - [1 + (-1)^n] \frac{\gamma_2(x) + \Gamma_2(x)}{2(n+1)!} \left(\frac{b-a}{4}\right)^{n+1} + (-1)^n \int_a^b f(t) dt - \right. \\ \left. - \left[\frac{\gamma_1(x) + (-1)^n \gamma_3(x) + \Gamma_1(x) + (-1)^n \Gamma_3(x)}{2} \right] \frac{(b-a)^{n+1}}{4^{n+1} (n+1)!} \right| \leq \\ \leq \frac{1}{(n+1)!} \left(\frac{b-a}{4}\right)^{n+1} \left[\frac{\varepsilon_1 + \varepsilon_3}{2} + \varepsilon_2 \right]$$

where $\varepsilon_1 = |\Gamma_1(x) - \gamma_1(x)|$, $\varepsilon_2 = |\Gamma_2(x) - \gamma_2(x)|$ and $\varepsilon_3 = |\Gamma_3(x) - \gamma_3(x)|$. What is more, applying the condition of the result (8) to this inequality, a new inequality can be found.

In addition to these results, one can deduce some inequalities, taking $n = 1$ in inequality (6) or the other results related to (6); these inequalities were published by Dragomir [15]. Furthermore, if we take $n = 2$ in (6) or the other results connected to (6), then we obtain some inequalities presented in [23] that is published by Sarikaya et. al.

3. The case when $f^{(n)}$ is of Bounded Variation. We begin with the definition of bounded-variation functions and the concept of total variation, which is used throughout this section.

Definition 1. Let $P : a = x_0 < x_1 < \dots < x_n = b$ be any partition of $[a, b]$ and let $\Delta f(x_i) = f(x_{i+1}) - f(x_i)$; then f is said to be of bounded variation, if the sum

$$\sum_{i=1}^m |\Delta f(x_i)|$$

is bounded for all such partitions.

Definition 2. Let f be of bounded variation on $[a, b]$, and $\sum \Delta f(P)$ denote the sum $\sum_{i=1}^n |\Delta f(x_i)|$ corresponding to the partition P of $[a, b]$. The number

$$\bigvee_a^b(f) := \sup \left\{ \sum \Delta f(P) : P \in P([a, b]) \right\},$$

is called the total variation of f on $[a, b]$. Here $P([a, b])$ denotes the family of partitions of $[a, b]$.

Now, a perturbed inequality of the Ostrowski type for functions whose high-order derivatives are of bounded variation, are established in the following theorem.

Theorem 5. Let $f : I \rightarrow \mathbb{C}$ be an n time differentiable function on I° and $[a, b] \subset I^\circ$. If the n -th derivative $f^{(n)}$ is of bounded variation on $[a, b]$, then we have

$$\begin{aligned} & \left| S(f : n, x) + (-1)^n \int_a^b f(t) dt - [f^{(n)}(a) + (-1)^n f^{(n)}(b)] \frac{(x-a)^{n+1}}{(n+1)!} - \right. \\ & \left. - [1 + (-1)^n] \frac{f^{(n)}(x) + f^{(n)}(a+b-x)}{2(n+1)!} \left(\frac{a+b}{2} - x \right)^{n+1} \right| \leq \\ & \leq \frac{(x-a)^{n+1}}{t(n+1)!} \left[\bigvee_a^x(f^{(n)}) + \bigvee_{a+b-x}^b(f^{(n)}) \right] + \\ & \quad + \frac{1}{(n+1)!} \left(\frac{a+b}{2} - x \right)^{n+1} \bigvee_x^{a+b-x}(f^{(n)}) \quad (9) \end{aligned}$$

for any $x \in [a, \frac{a+b}{2}]$.

Proof. Writing $f^{(n)}(a)$, $(f^{(n)}(x) + f^{(n)}(a+b-x))/2$, $f^{(n)}(b)$ instead of $\lambda_1(x)$, $\lambda_2(x)$, $\lambda_3(x)$ in equation (3) respectively, then taking modulus of this equality, we find that

$$\begin{aligned} & \left| S(f : n, x) + (-1)^n \int_a^b f(t) dt - [f^{(n)}(a) + (-1)^n f^{(n)}(b)] \frac{(x-a)^{n+1}}{(n+1)!} - \right. \\ & \left. - [1 + (-1)^n] \frac{f^{(n)}(x) + f^{(n)}(a+b-x)}{2(n+1)!} \left(\frac{a+b}{2} - x \right)^{n+1} \right| \leq \end{aligned}$$

$$\begin{aligned}
&\leq \int_a^x \frac{(t-a)^n}{n!} |f^{(n)}(t) - f^{(n)}(a)| dt + \\
&+ \int_{a+b-x}^b \frac{(b-t)^n}{n!} |f^{(n)}(t) - f^{(n)}(b)| dt + \\
&+ \int_x^{a+b-x} \frac{1}{n!} \left| t - \frac{a+b}{2} \right|^n \left| f^{(n)}(t) - \frac{f^{(n)}(x) + f^{(n)}(a+b-x)}{2} \right| dt.
\end{aligned}$$

Noting that $f^{(n)} : I^\circ \rightarrow \mathbb{C}$ is of bounded variation on $[a, x]$, we get

$$|f^{(n)}(t) - f^{(n)}(a)| \leq \bigvee_a^x(f^{(n)})$$

and observe that

$$\int_a^x \frac{(t-a)^n}{n!} dt = \frac{(x-a)^{n+1}}{(n+1)!}.$$

The other integrals are also examined by noting that $f^{(n)} : I^\circ \rightarrow \mathbb{C}$ is of bounded variation on $[x, a+b-x]$ and $[a+b-x, b]$: we can find the result (9), which finishes the proof. \square

Remark 5. Suppose that all assumptions of Theorem 5 hold. If we take $x = a$ in the inequality given this theorem, we have

$$\begin{aligned}
&\left| \sum_{k=0}^{n-1} \frac{(-1)^{n+1} [f^{(k)}(b) + (-1)^k f^{(k)}(a)]}{(k+1)!} \left[(-1)^k \left(\frac{b-a}{2} \right)^{k+1} \right] - \right. \\
&- [1 + (-1)^n] \frac{f^{(n)}(a) + f^{(n)}(a+b-x)}{2(n+1)!} \left(\frac{b-a}{2} \right)^{n+1} + (-1)^n \int_a^b f(t) dt \left. \right| \leq \\
&\leq \frac{1}{(n+1)!} \left(\frac{a+b}{2} - x \right)^{n+1} \bigvee_a^b(f^{(n)}).
\end{aligned}$$

In addition, if we choose $x = \frac{a+b}{2}$, we get the midpoint inequality

$$\left| \sum_{k=0}^{n-1} \frac{(-1)^{n+1} f^{(k)} \left(\frac{a+b}{2} \right) [1 + (-1)^k]}{(k+1)!} \left(\frac{b-a}{2} \right)^{k+1} - \right.$$

$$\begin{aligned}
& - \left[f^{(n)}(a) + (-1)^n f^{(n)}(b) \right] \frac{(b-a)^{n+1}}{2^{n+1}(n+1)!} + (-1)^n \int_a^b f(t) dt \Big| \leq \\
& \leq \frac{(b-a)^{n+1}}{2^{n+1}(n+1)!} \bigvee_a^b (f^{(n)}).
\end{aligned}$$

Finally, should we take $x = \frac{3a+b}{4}$, we have

$$\begin{aligned}
& \left| \sum_{k=0}^{n-1} \frac{(-1)^{n+1} [1 + (-1)^k] \left[f^{(k)}\left(\frac{a+3b}{4}\right) + (-1)^k f^{(k)}\left(\frac{3a+b}{4}\right) \right] (b-a)^{k+1}}{4^{k+1}(k+1)!} + \right. \\
& \quad \left. + (-1)^n \int_a^b f(t) dt - [f^{(n)}(a) + (-1)^n f^{(n)}(b)] \frac{(b-a)^{n+1}}{4^{n+1}(n+1)!} - \right. \\
& \quad \left. - [1 + (-1)^n] \frac{f^{(n)}\left(\frac{3a+b}{4}\right) + f^{(n)}\left(\frac{a+3b}{4}\right)}{2(n+1)!} \left(\frac{b-a}{4}\right)^{n+1} \right| \leq \\
& \leq \frac{1}{(n+1)!} \left(\frac{b-a}{4}\right)^{n+1} \bigvee_a^b t(f^{(n)}).
\end{aligned}$$

Besides the results that are presented in this section, taking $n = 1$ in the inequality (9) or the other results pertaining to (6), we obtain some inequalities given in [15] by Dragomir. What is more, should we take $n = 2$ in expression (6) or the other results interested in (6), we can find some inequalities presented in [23] by Sarikaya et. al.

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