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Discrete Burr-Hatke Distribution With Properties, Estimation Methods and Regression Model

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ABSTRACT A new one-parameter discrete distribution, namely discrete Burr-Hatke distribution is introduced and its mathematical properties are studied comprehensively. The main properties of the discrete Burr-Hatke distribution such as mean, variance, skewness and kurtosis measures are obtained in explicit forms. Several parameter estimation methods are used to obtain unknown model parameters and these estimation methods are compared via simulation study. The discrete Burr-Hatke distribution is over-dispersed since its variance is greater than its mean. This property of the proposed distribution opens a new opportunity to model over-dispersed data sets. To show the importance of the proposed distribution against the existing discrete probability distributions, three data sets in different fields are analyzed. Additionally, count regression model based on the discrete Burr-Hatke distribution is introduced with its residual analysis.

INDEX TERMS Burr-Hatke distribution, L-moment statistics, regression model, estimation methods, simulation.

I. INTRODUCTION

Modeling the number of occurrences of events is an important issue and gains much attention in recent years. These types of data sets are modeled by discrete probability distributions such as Poisson, negative-binomail, geometric, Poisson-Lindley etc. In the last decade, several discrete distributions have been introduced such as discrete Lindley distribution by Gómez-Déniz and Calderín-Ojeda [6], generalized geometric by Gómez-Déniz [5], discrete generalized exponential type II by Nekoukhou *et al.* [16], discrete Rayleigh by Roy [20], two-parameter discrete Lindley by Hussain *et al.* [10] and exponentiated discrete Lindley by El-Morshedy *et al.* (2020), new discrete extended Weibull by Jia *et al.* [12], new Poisson-weighted exponential by Altun [1] and among others. The main goal of these studies is to provide an alternative model in modeling the over-dispersed or under-dispersed count data sets. These distributions are obtained by using the survival discretization method. Let the random variable X has the survival function such as $S(x) = Pr(X > x)$. The probability mass function (pmf) of the discrete random variable X is

$$P(X = x) = S(x) - S(x + 1), \quad x = 0, 1, 2, 3, \ldots \tag{1}$$

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In this study, using the survival discretization method, a new one-parameter discrete distribution, named discrete Burr-Hatke (DBH), is proposed. The DBH distribution can be a good choice for modeling the right-skewed and over-dispersed counts data sets with decreasing failure rate. The statistical properties of the DBH distribution, including the moments, order statistics, L-moments, are studied in detail. Three parameter estimation methods, maximum likelihood (ML), method of moments (MM) and proportion estimation (PE), are discussed to estimate the unknown parameter of the DBH distribution via Monte-Carlo simulation study. In addition to these, a new count regression model for over-dispersed response variable is introduced with its residual analysis and compared with Poisson regression model.

The rest of the paper is organized as follows: In Section 2, DBH distribution is introduced and its statistical properties are obtained. In Section 3, different methods of parameter estimation are given for DBH distribution. In Section 4, DBH regression model is introduced. In Section 5, two simulation studies are performed to investigate the finite sample performance of parameter estimation methods and also brief simulation study is given to evaluate the ML estimators of the parameters of DBH regression model. Section 6 is devoted to demonstrate usefulness of proposed models by means of real data applications. Section 7 contains the conclusion remarks of this study.

II. THE DBH DISTRIBUTION

Let the random variable X follows a Burr-Hatke distribution, introduced by Maniu and Voda [15]. The cumulative distribution function (cdf) and probability density function (pdf) are given, respectively, by

$$\Pi(x;\beta) = 1 - \frac{e^{-\beta x}}{x+1}; \quad x > 0, \tag{2}$$

and

$$\pi(x;\beta) = \frac{1+\beta(x+1)}{(x+1)^2} e^{-\beta x}; \quad x > 0, \tag{3}$$

where $\beta > 0$ is a scale parameter. Using Equation (1), the cdf of the DBH distribution is given by

$$F(x;\lambda) = 1 - \frac{\lambda^{x+1}}{x+2}; \quad x \in \mathbb{N}_0, \tag{4}$$

where $0 < \lambda = e^{-\beta} < 1$ and $\mathbb{N}_0 = 0, 1, 2, 3, \ldots$. The corresponding pmf to (4) is given by

$$f(x;\lambda) = \left(\frac{1}{x+1} - \frac{\lambda}{x+2}\right)\lambda^x; \; x \in \mathbb{N}_0, \tag{5}$$

where λ controls the shape of the distribution. The pmf in Equation (5) is log-convex for all values of $0 < \lambda < 1$, where $\frac{f(x+1;\lambda)}{f(x;\lambda)}$ is an increasing function in x for all values of the parameter λ, and therefore, the pmf is always decreasing function in x. Figure 1 shows the pmf plots for different values of the parameter λ.

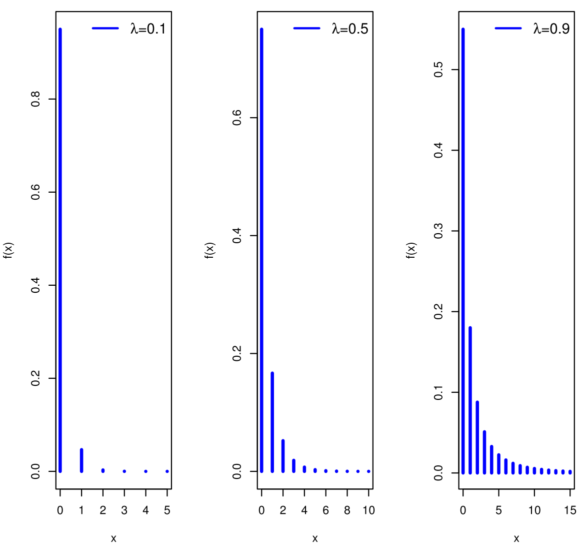

FIGURE 1. **The pmf plots of the DBH model.**

The hazard rate function (hrf) is given by

$$h(x;\lambda) = 1 - \frac{x+1}{x+2}\lambda; \; x \in \mathbb{N}_0, \tag{6}$$

where $h(x;\lambda) = \frac{f(x;\lambda)}{1-F(x-1;\lambda)}$. The hrf is always decreasing function in x for all values of the parameter λ, where $\frac{d}{dx}h(x;\lambda) < 0$. Figure 2 displays some possible shapes of hrf for selected parameter values.

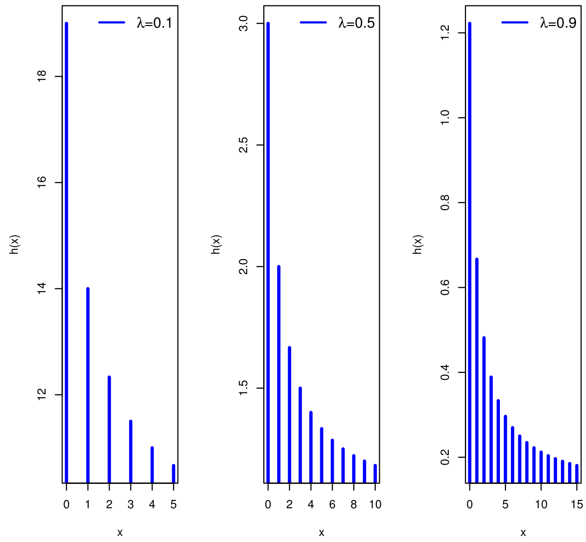

FIGURE 2. **The hrf plots of DBH model.**

A. MOMENTS AND RELATED CONCEPTS

The rth raw moments of the DBH distribution can be obtained by using

$$\begin{aligned}\mathbf{E}(X^r) &= \sum_{x=0}^{\infty}\left\{([x+1]^r - x^r)(1-F(x;\lambda))\right\} \\ &= \lambda\sum_{x=0}^{\infty}\frac{[x+1]^r - x^r}{x+2}\lambda^x. \end{aligned}\tag{7}$$

Using Equation (7), the first four moments of the DBH distribution are

$$\mathbf{E}(X) = -\frac{\ln(1-\lambda)}{\lambda} - 1, \tag{8}$$

$$\mathbf{E}(X^2) = \frac{\lambda}{2}\left(\frac{2(\lambda-3)}{\lambda(\lambda-1)} + \frac{6\ln(1-\lambda)}{\lambda^2}\right), \tag{9}$$

$$\mathbf{E}(X^3) = -\frac{\left(7\lambda^2 - 14\lambda + 7\right)\ln(1-\lambda) + \lambda(\lambda^2 - 11\lambda + 7)}{\lambda(\lambda-1)^2}, \tag{10}$$

and

$$\mathbf{E}(X^4) = \frac{\lambda}{2}\left(\frac{2(\lambda^3-31\lambda^2+37\lambda-15)}{\lambda(\lambda-1)^3} + \frac{30\ln(1-\lambda)}{\lambda^2}\right). \tag{11}$$

Using Equations (8-11), the variance, skewness and kurtosis can be derived in closed forms where

$$\mathbf{Var}(X) = \mathbf{E}(X^2) - [\mathbf{E}(X)]^2, \tag{12}$$

$$\mathbf{S} = \frac{\mathbf{E}(X^3) - 3\mathbf{E}(X^2)\mathbf{E}(X) + 2\,[\mathbf{E}(X)]^3}{[\mathbf{Var}(X)]^{3/2}}, \tag{13}$$

and

$$\mathbf{K} = \frac{\mathbf{E}(X^4) - 4\mathbf{E}(X^2)\mathbf{E}(X) + 6\mathbf{E}(X^2)[\mathbf{E}(X)]^2 - 3[\mathbf{E}(X)]^4}{[\mathbf{Var}(X)]^2}. \tag{14}$$

TABLE 1. Some descriptive statistics for the DBH distribution.

Measure $\downarrow \lambda \longrightarrow$	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
Mean	0.0536	0.1157	0.1889	0.2770	0.3862	0.5271	0.7199	1.0117	1.5584
Var	0.0585	0.1394	0.2547	0.4253	0.6918	1.1406	1.9884	3.9408	10.896
Skewness	4.8987	3.7935	3.3966	3.2365	3.2051	3.2749	3.4612	3.8455	4.7872
Kurtosis	30.5254	21.1540	18.5921	17.9041	18.2020	19.3689	21.7466	26.6450	40.4690

TABLE 2. The IOD and COV statistics for the DBH distribution.

Measure $\downarrow \lambda \longrightarrow$	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
IOD	1.0919	1.2051	1.3482	1.5353	1.7911	2.1638	2.7618	3.8949	6.9916
COV	4.5133	3.2271	2.6714	2.3540	2.1532	2.0260	1.9586	1.9620	2.1181

Table 1 presents some numerical results of the mean, Var, skewness and kurtosis for the DBH distribution for different values of the model parameter.

From Table 1, it is clear that:

1) When the parameter λ approaches to unity, the mean and variance of the DBH distribution increase,
2) The proposed model is appropriate only for modelling positive skewed data,
3) The skewness and kurtosis have a bathtub-shaped,
4) The proposed model is leptokurtic in nature.

B. INDEX OF DISPERSION AND COEFFICIENT OF VARIATION

The index of dispersion (IOD) is a measure to decide the possible over-dispersion (under-dispersion) in the used data set. When the IOD is higher than one, it indicates the over-dispersion, opposite case indicates the under-dispersion. When the IOD is equal to one, it indicates the equi-dispersion. The IOD of the DBH distribution is

$$\mathbf{IOD}(X) = \frac{\mathrm{Var}\,(X)}{\mathrm{E}\,(X)} = \frac{-\lambda}{\ln(1-\lambda)+\lambda} \times \left\{\frac{\lambda}{2}\left(\frac{2(\lambda-3)}{\lambda(\lambda-1)}+\frac{6\ln(1-\lambda)}{\lambda^2}\right)-\left(\frac{\ln(1-\lambda)+\lambda}{\lambda}\right)^2\right\}. \tag{15}$$

Further, the coefficient of variation (COV) is a measure of variability in the data. If X has a DBH model, then the COV can be expressed as

$$\mathbf{COV}(X) = \frac{\lambda}{2\,|\ln(1-\lambda)+\lambda|} \times \sqrt{2\lambda\left(\frac{2(\lambda-3)}{\lambda(\lambda-1)}+\frac{6\ln(1-\lambda)}{\lambda^2}\right)-4\left(\frac{\ln(1-\lambda)+\lambda}{\lambda}\right)^2}. \tag{16}$$

The COV measure is generally used to compare to independent samples based on their variability. The higher COV value indicates the higher variability. Table 2 lists some numerical results of the IOD and COV for different values of the model parameter.

From Table 2, it is clear that:

1) The IOD increases, whereas the COV can be take inverse J-shaped with $\lambda \longrightarrow 1$.
2) The proposed model is appropriate only for modelling over-dispersed data where IOD > 1.

C. ORDER STATISTICS AND L-MOMENT STATISTICS

Assume that the random variables $X_1, X_2, \ldots, X_n$ follow a DBH distribution and $X_{1:n}, X_{2:n}, \ldots, X_{n:n}$ are the corresponding order statistics (ORST) of these random variables. The cdf of ith ORST is

$$\begin{aligned} F_{i:n}(x;\lambda) &= \sum_{k=i}^{n}\binom{n}{k}[F_i(x;\lambda]^k\,[1-F_i(x;\lambda)]^{n-k} \\ &= \sum_{k=i}^{n}\sum_{j=0}^{n-k}\Psi_{(m)}^{(n,k)}\,[F_i(x;\lambda)]^{k+j} \\ &= \sum_{k=i}^{n}\sum_{j=0}^{n-k}\Psi_{(m)}^{(n,k)}F_i(x;\lambda,k+j), \end{aligned} \tag{17}$$

where $\Psi_{(m)}^{(n,k)} = (-1)^j\binom{n}{k}\binom{n-k}{j}$ and $F_i(x;\lambda,k+j) = [F_i(x;\lambda)]^{k+j}$ represents the cdf of the exponentiated DBH (EDBH) model with power parameter $k+j$. Further, the corresponding pmf of the ith ORST is

$$\begin{aligned} f_{i:n}(x;\lambda) &= F_{i:n}(x;\lambda) - F_{i:n}(x-1;\lambda) \\ &= \sum_{k=i}^{n}\sum_{j=0}^{n-k}\Psi_{(m)}^{(n,k)}f_i(x;\lambda,k+j), \end{aligned} \tag{18}$$

where $f_i(x;\lambda,k+j)$ represents the pmf of the EDBH model with power parameter $k+j$. Thus, the uth moments of $X_{i:n}$ can be expressed as

$$\mathbf{E}(X_{i:n}^u) = \sum_{x=0}^{\infty}\sum_{k=i}^{n}\sum_{j=0}^{n-k}\ominus_{(m)}^{(n,k)}x^u f_i(x;\lambda,k+j). \tag{19}$$

The L-moment statistics (LMST), introduced by Hosking [8] can be used to summarize the theoretical distribution. LMST is an expectation of linear combinations of ORST. The

LMST of X is given by

$$\Upsilon_\delta = \frac{1}{\delta}\sum_{j=0}^{\delta-1}(-1)^j\binom{\delta-1}{j}\mathbf{E}\left(X_{\delta-j:\delta}\right). \tag{20}$$

Using (20), some statistical measures based on the LMST can be defined such as mean $= \Upsilon_1$, COV $= \frac{\Upsilon_2}{\Upsilon_1}$, coefficient of skewness $= \frac{\Upsilon_3}{\Upsilon_2}$ and coefficient of kurtosis $= \frac{\Upsilon_4}{\Upsilon_2}$.

III. ESTIMATION METHODS

In this section, we consider three parameter estimation methods to obtain the unknown model parameters. These are maximum likelihood, method of moment and proportion estimation methods.

A. MAXIMUM LIKELIHOOD ESTIMATION

Let $X_1, X_2, \ldots, X_n$ be a random variables follow a DBH distribution. The log-likelihood function (L) is given by

$$L(x;\lambda) = \ln(\lambda)\sum_{i=1}^{n}x_i + \sum_{i=1}^{n}\ln\left[\frac{1}{x_i+1} - \frac{\lambda}{x_i+2}\right]. \tag{21}$$

By differentiating Equation (21) with respect to the parameter λ, we get the normal non-linear likelihood equation as follows

$$\frac{1}{\lambda}\sum_{i=1}^{n}x_i - \sum_{i=1}^{n}\frac{x_i+1}{x_i+2-\lambda(x_i+1)} = 0. \tag{22}$$

The solution of (22) gives maximum likelihood estimator of λ. However, there is no explicit form for the solution of (22). Therefore, (22) has to be solved by using iterative methods such as Newton-Raphson, Nelder-Mead etc. The other choice is to direct minimization of negative log-likelihood function. The optim and nlm functions of R software (see, [21] can be used for this purpose.

B. MOMENT ESTIMATION

While using the MME for the parameter of DBH distribution, λ, we have to first equate the population moment to the corresponding sample moment than solve the non-linear equation

$$\frac{1}{n}\sum_{i=1}^{n}x_i + \frac{\ln(1-\lambda)}{\lambda} + 1 = 0, \tag{23}$$

with respect to λ. The **nleqslv** function of R software could be used to solve.

C. PROPORTION ESTIMATION

Let $x_1, x_2, \ldots, x_n$ be a random sample from the DBH distribution. DBH distribution has a single parameter. Therefore, one indicator function is defined as follows.

$$I(x_i) = \begin{cases} 1 & \text{if } x_i = 0 \\ 0 & \text{if otherwise.} \end{cases} \tag{24}$$

Assume, $W = \sum_{i=1}^{n} I(x_i)$ denotes the number of zero observations in the sample. Using (4) and (24), we get $\mathbf{P}(X \leq 0) = \frac{W}{n}$. Thus, the PE of the λ is

$$\widehat{\lambda} = \frac{2}{n}(n-W). \tag{25}$$

Since $\frac{W}{n}$ is unbiased and consistent empirical estimator of probability $\mathbf{P}(X \leq 0)$, the $\widehat{\lambda}$ is also unbiased and consistent estimator of λ (see, Khan *et al.* 1989 for details).

IV. DBH REGRESSION MODEL

Poisson regression model is widely used to model count response variable with some covariates. However, when the response variable displays over-dispersion, Poisson regression does not work. Here, we introduce a new regression model for modeling these kind of data sets.

Proposition 1: Let $\lambda = 1 + W\left(-e^{-\mu-1}(\mu+1)\right)/(\mu+1)$. Then, the pmf of DBH distribution is

$$\begin{aligned} P(Y=y) = &\left(\frac{1}{y+1} - \frac{1+W\left(-e^{-\mu-1}(\mu+1)\right)/(\mu+1)}{y+2}\right) \\ &\times\left\{1+W\left(-e^{-\mu-1}(\mu+1)\right)/(\mu+1)\right\}^y, \\ &y = 0,1,2,\ldots, \end{aligned} \tag{26}$$

where $E(Y|\mu) = \mu$ and $W(\cdot)$ is the Lambert-W function (see, `http://mathworld.wolfram.com/LambertW-Function.html` for details).

Assume that the random variable Y represents the counted number of occurrences of an event and is distributed as DBH distribution with the parameter μ, given in (26). The mean of the random variable Y can be modeled by explanatory variables using the appropriate link functions. The log-link function can be used to link the covariates to the mean of dependent variable, as follows

$$\mu_i = E(Y_i) = \exp\left(\boldsymbol{x}_i^T\boldsymbol{\beta}\right), \quad i = 1,\ldots,n, \tag{27}$$

where $\boldsymbol{x}_i^T = (x_{i1},\ldots,x_{ik})$ is the vector of explanatory variables and $\boldsymbol{\beta} = (\beta_0,\beta_1,\ldots,\beta_k)^T$ is the vector of unknown regression coefficients. Inserting Equation (27) in Equation (26), the log-likelihood function of DBH regression model is

$$\begin{aligned} \ell(\boldsymbol{\beta}) = &\sum_{i=1}^{n} \ln\left(\frac{1}{y_i+1} - \frac{1+W\left(-e^{-e^{\boldsymbol{x}_i^T\boldsymbol{\beta}}-1}\left(e^{\boldsymbol{x}_i^T\boldsymbol{\beta}}+1\right)\right)/\left(e^{\boldsymbol{x}_i^T\boldsymbol{\beta}}+1\right)}{y_i+2}\right) \\ &+ \sum_{i=1}^{n} y_i \ln\left(1+W\left(-e^{-e^{\boldsymbol{x}_i^T\boldsymbol{\beta}}-1}\left(e^{\boldsymbol{x}_i^T\boldsymbol{\beta}}+1\right)\right)/\left(e^{\boldsymbol{x}_i^T\boldsymbol{\beta}}+1\right)\right). \end{aligned} \tag{28}$$

The MLE of $\boldsymbol{\beta}$, say $\hat{\boldsymbol{\beta}}$ can be obtained by taking the partial derivatives of (28) with respect to the vector of unknown regression coefficients and solving them simultaneously for zero. However, there is no closed form expressions for the

TABLE 3. Simulation results of DBH distribution for several parameter values.

Parameters	Sample sizes	AE			Bias			MSE			MRE		
		MLE	MME	PE	MLE	MME	PE	MLE	MM	PE	MLE	MM	PE
$\lambda = 0.7$	25	0.6667	0.6706	0.7114	-0.0333	-0.0294	0.0114	0.0197	0.0200	0.0451	0.9524	0.9580	1.0163
	50	0.6839	0.6854	0.7003	-0.0161	-0.0146	0.0003	0.0076	0.0078	0.0187	0.9770	0.9792	1.0005
	100	0.6939	0.6946	0.7026	-0.0061	-0.0054	0.0026	0.0036	0.0036	0.0078	0.9912	0.9923	1.0038
	200	0.6953	0.6955	0.6997	-0.0047	-0.0045	-0.0003	0.0018	0.0018	0.0045	0.9932	0.9936	0.9995
	500	0.6980	0.6982	0.6996	-0.0020	-0.0018	-0.0004	0.0007	0.0007	0.0017	0.9971	0.9974	0.9995
$\lambda = 0.5$	25	0.4716	0.4731	0.4944	-0.0284	-0.0269	-0.0056	0.0196	0.0198	0.0302	0.9433	0.9461	0.9888
	50	0.4872	0.4877	0.4978	-0.0128	-0.0123	-0.0022	0.0098	0.0098	0.0150	0.9744	0.9754	0.9955
	100	0.4983	0.4986	0.5036	-0.0017	-0.0014	0.0036	0.0044	0.0044	0.0075	0.9966	0.9971	1.0072
	200	0.4959	0.4959	0.4954	-0.0041	-0.0041	-0.0046	0.0021	0.0021	0.0032	0.9918	0.9918	0.9908
	500	0.4985	0.4985	0.4997	-0.0015	-0.0015	-0.0003	0.0010	0.0010	0.0015	0.9969	0.9971	0.9994
$\lambda = 0.3$	25	0.3757	0.3765	0.3942	-0.0243	-0.0235	-0.0058	0.0198	0.0199	0.0259	0.9393	0.9413	0.9856
	50	0.3902	0.3907	0.4010	-0.0098	-0.0093	0.0010	0.0090	0.0091	0.0128	0.9755	0.9766	1.0026
	100	0.3982	0.3984	0.4036	-0.0018	-0.0016	0.0036	0.0045	0.0045	0.0061	0.9956	0.9960	1.0089
	200	0.3991	0.3992	0.4020	-0.0009	-0.0008	0.0020	0.0021	0.0021	0.0029	0.9977	0.9980	1.0050
	500	0.3984	0.3984	0.3982	-0.0016	-0.0016	-0.0018	0.0010	0.0010	0.0014	0.9960	0.9960	0.9956

MLEs of the parameters of the DBH regression model. Therefore, (28) have to be solved iteratively by using the statistical or mathematical software such as MATLAB, R or S-PLUS. Here, we use the statistical software, R, to obtain the MLE of $\boldsymbol{\beta}$. To construct the asymptotic confidence intervals of the regression coefficients, the asymptotic covariance matrix $K(\boldsymbol{\beta})^{-1}$ of $\widehat{\boldsymbol{\beta}}$ is used. The asymptotic covariance matrix is approximated by the inverse of the $(k+1)\times(k+1)$ observed information matrix whose elements are evaluated numerically via most statistical packages such as **hess** and **fdHess** functions of R software.

A. RESIDUAL ANALYSIS

Residual analysis is an essential tool to check the adequacy of fitted model on the used data set. Here, the randomized quantile residuals are used to check the model assumption. Let $F\,(y;\mu)$ is the cdf of DBH distribution. The randomized quantile residuals of DBH regression model are

$$r_{q,i} = \Phi^{-1}\left(u_i\right), \tag{29}$$

where $u_i = F\left(y_i;\hat{\mu}_i\right)$ is uniformly distributed random variable between $a_i = \lim_{y\uparrow y_i} F\left(y;\hat{\mu}_i\right)$ and $b_i = F\left(y;\hat{\mu}_i\right)$. The randomization strategy is used to prevent masses of overlapping points. When the fitted model is correct, the randomized quantile residuals are normally distributed with zero mean and unit variance.

TABLE 4. The simulations results of the DBH regression model.

Sample size	Parameters	$\beta_0 = 0.5$	$\beta_1 = 0.5$	$\beta_2 = 0.5$	$\beta_3 = 2$
		β_0	β_1	β_2	β_3
n=50	AE	0.4776	0.4833	0.5110	2.7929
	Bias	-0.0224	-0.0167	0.0110	0.7929
	MSE	0.1711	0.2801	0.2520	1.2784
n=250	AE	0.4882	0.4991	0.5109	2.4293
	Bias	-0.0118	-0.0009	0.0109	0.4293
	MSE	0.0352	0.0580	0.0547	0.8258
n=500	AE	0.4955	0.4892	0.5125	2.2712
	Bias	-0.0045	-0.0108	0.0125	0.2712
	MSE	0.0160	0.0250	0.0239	0.4794
n=1000	AE	0.4930	0.5053	0.5049	2.1749
	Bias	-0.0070	0.0053	0.0049	0.1749
	MSE	0.0079	0.0131	0.0125	0.3017
Sample size	Parameters	$\beta_0 = 0.5$	$\beta_1 = 2$	$\beta_2 = 0.5$	$\beta_3 = 4$
		β_0	β_1	β_2	β_3
n=50	AE	0.4778	2.0207	0.4682	4.1155
	Bias	-0.0222	0.0207	-0.0318	0.1155
	MSE	0.1314	0.2058	0.1886	1.1047
n=250	AE	0.4981	1.9882	0.5012	3.9653
	Bias	-0.0019	-0.0118	0.0012	-0.0347
	MSE	0.0266	0.0408	0.0325	0.5779
n=500	AE	0.4946	2.0014	0.5067	4.0264
	Bias	-0.0054	0.0014	0.0067	0.0264
	MSE	0.0120	0.0174	0.0177	0.2622
n=1000	AE	0.4993	1.9969	0.5008	4.0079
	Bias	-0.0007	-0.0031	0.0008	0.0079
	MSE	0.0061	0.0094	0.0096	0.1183

V. SIMULATION RESULTS

Two simulation studies are carried out to compare the finite sample performance of parameter estimation methods for the unknown parameters of DBH models.

A. SIMULATION OF DBH DISTRIBUTION

Here, a simulation study is given to compare the finite sample behaviour of the MLE, MME and PE methods. The simulation procedure is given below.

1) Generate 1000 samples of size $n = 20, 22, 24, \ldots, 100$ from DBH(0.7), DBH(0.5) and DBH(0.3), respectively.
2) Compute the MLEs, MMEs and PEs for the 1000 samples, say $\widehat{\lambda}_j$ for $j = 1, 2, \ldots, 1000$.
3) Compute the average of estimates (AEs), biases, mean-squared errors (MSEs) and mean relative

TABLE 5. The competitive models of the DBH distribution.

Distribution	Abbreviation	Author(s)
Geometric	Geo	-
Generalized geometric	GGeo	Gómez-Déniz (2010)
Discrete generalized exponential type II	DGE-II	Nekoukhou et al. (2013)
Discrete Rayleigh	DR	Roy (2004)
Discrete inverse Rayleigh	DIR	Hussain and Ahmad (2014)
Discrete linear failure rate	DLFR	Chandrakant et al. (2017)
Discrete Weibull	DW	Toshio and Shunji (1975)
Discrete inverse Weibull	DIW	Jazi et al. (2010)
Discrete Lindley	DLi	Gómez-Déniz and Calderín-Ojeda (2011)
Discrete Lindley-Two Parameter	DLi-II	Hussain et al. (2016)
Exponentiated discrete Lindley	EDLi	El-Morshedy et al. (2020)
Poisson	Poi	Poisson (1837)
Discrete Poisson-Lindley	PoiLi	Sankaran, M. (1970)
Discrete Pareto	DPa	Krishna and Pundir (2009)
Discrete Burr type II	DB-II	Para and Jan (2016a)
Discrete log-logistic	DLog-L	Para and Jan (2016b)
Discrete Lomax distribution	DLo	Para and Jan (2016a)

errors MREs) by using the below quantities.

$$\mathrm{Bias}(\lambda) = \frac{1}{1000}\sum_{j=1}^{1000}\left(\widehat{\lambda_j}-\lambda\right),$$
$$\mathrm{MSE}(\lambda) = \frac{1}{1000}\sum_{j=1}^{1000}\left(\widehat{\lambda_j}-\lambda\right)^2. \tag{30}$$
$$\mathrm{AE}(\lambda) = \frac{1}{1000}\sum_{j=1}^{1000}\widehat{\lambda_j},$$
$$\mathrm{MRE}(\lambda) = \frac{1}{1000}\sum_{j=1}^{1000}\frac{\widehat{\lambda_j}}{\lambda}. \tag{31}$$

The empirical results are given in Tables. From Table 3 the following observations can be noted:

1) The magnitude of bias of the parameter approaches zero when the sample size increases.
2) The MSEs of the parameter approaches zero when the sample size increases. This shows the consistency of the estimators.
3) The performance of all estimation methods is quite well for both small and large samples.

We have presented results only for $\lambda = 0.7, 0.5$ and 0.3. But, the results are similar for other choices for λ.

B. SIMULATION OF DBH REGRESSION MODEL

In this section, simulation study is given to evaluate the performance of the MLEs of the parameters of DBH regression model. We generate $N = 10,000$ samples of size $n = 50, 250, 500$ and 1000 by using the $\mu_i = \exp(\beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2} + \beta_3 x_{i3})$. The independent variables x_1, x_2 and x_3 are generated from standard uniform distribution, $U(0,1)$. The following vectors of parameters are used to implement the simulation study: $\boldsymbol{\beta} = (\beta_0 = 0.5, \beta_1 = 0.5, \beta_2 = 0.5, \beta_3 = 2)$ and $\boldsymbol{\beta} = (\beta_0 = 0.5, \beta_1 = 2, \beta_2 = 0.5, \beta_3 = 4)$

The simulation results are discussed based on the AEs, biases and MSEs. Table 4 lists the simulation results. Based on the results in Table 4, when the n increases, the AEs of the parameters are near the their nominal values and biases approach to zero. Similarly, the MSEs of the parameters approach zero when the sample size increases. The consistency property of the MLE is proved by these results.

VI. EMPIRICAL STUDIES

In this section, the importance of DBH distribution is demonstrated based on the applications to real data sets. The computational codes can be found in `https://github.com/emrahaltun/DBH-paper`. The fitted models are compared using some criteria, namely, Bayesian information criterion (BIC), Hannan-Quinn information criterion (HQIC) and Chi-square (χ^2) with its corresponding P-value. We compare the fits of the DBH distribution with some competitive distributions having one or two parameters which are listed in Table 5.

A. DATA SET I: CARIOUS TEETH

The first data set consists of the number of carious teeth among the four deciduous molars. The sample size is 100. The detail information on the used data set can be found in Krishna and Pundir [14]. The fits of the DBH distribution is compared with some competitive models having one or two parameters such as DPa, Geo, DR, DIR, DLi, Poi, PoiLi, EDLi, DLi-II, DGE-II, GGeo, DLFR, DW, DIW and DLogL models. The MLEs with their corresponding standard errors (Std-er), confidence intervals (C. I) for the parameter(s) and goodness-of-fit test for data set I are listed in Tables 6 and 7, where Table 6 lists the competitive models having only one parameter, whereas Table 7 reports the competitive models having two-parameter.

According to Tables 6 and 7, it is clear that some competitive models (with significance greater than 0.05) work quite well besides the DBH model. But, the DBH model is the best model for evaluation of this data, because it has the smallest value of BIC and HQIC as well as the highest p-value among all tested models. Figures 3 and 4 show the fitted pmf plots for data set I which support our results reported in Tables 6 and 7.

TABLE 6. The MLE, C. I, χ^2 and P-values for the competitive models having only one parameter for data set I.

X	Observed frequency	Expected frequency DBH	DPa	Geo	DR	DIR	DLi	Poi	PoiLi
0	64	66.44	69.04	59.88	33.50	62.50	57.13	51.17	37.5
1	17	18.54	15.37	24.02	46.94	26.41	26.88	34.28	25.00
2	10	7.46	6.01	9.64	17.01	5.99	10.45	11.49	15.63
3	6	3.50	3.01	3.87	2.39	2.19	3.71	2.57	9.38
≥ 4	3	4.06	6.57	2.59	0.16	2.91	1.83	0.49	12.49
Total	100	100	100	100	100	100	100	100	100
λ	**MLE**	0.671	0.184	0.401	0.665	0.625	0.274	0.670	1.998
	Std-er	0.062	0.032	0.038	0.029	0.049	0.029	0.082	0.263
	L.C.I	0.549	0.121	0.327	0.608	0.529	0.217	0.509	1.481
	U.C.I	0.792	0.247	0.475	0.722	0.721	0.331	0.831	2.514
χ^2		1.357	3.225	3.347	66.07	9.056	6.638	23.65	30.889
Degree of freedom		2	2	2	2	2	2	2	2
P. value		0.507	0.199	0.188	< 0.001	0.011	0.036	< 0.001	< 0.001

TABLE 7. The MLE, C. I, χ^2 and P-values for the competitive models having only two-parameter for data set I.

X	Observed frequency	Expected frequency DBH	EDLi	DLi-II	DGE-II	GGeo	DLFR	DW	DIW	DLogL
0	64	66.44	63.57	59.88	63.51	62.73	59.90	62.58	63.30	62.73
1	17	18.54	19.75	24.02	20.19	21.36	24.01	21.35	22.48	22.42
2	10	7.46	9.09	9.64	8.81	8.76	9.63	8.85	6.44	7.01
3	6	3.50	4.19	3.87	4.01	3.86	3.86	3.88	2.76	2.98
≥ 4	3	4.06	3.4	2.59	3.48	3.29	2.60	3.34	5.02	4.86
Total	100	100	100	100	100	100	100	100	100	100
λ	**MLE**	0.671	0.379	0.401	0.468	0.467	0.401	0.374	0.633	0.745
	Std-er	0.062	0.065	0.269	0.072	0.089	0.056	0.049	0.049	0.101
	L.C.I	0.549	0.252	0.000	0.327	0.293	0.291	0.278	0.537	0.546
	U.C.I	0.792	0.506	0.928	0.609	0.641	0.511	0.470	0.729	0.944
θ	**MLE**	–	0.543	0.478	0.718	0.678	1.00	0.895	1.576	1.768
	Std-er	–	0.158	0.529	0.206	0.302	0.044	0.119	0.251	0.267
	L.C.I	–	0.234	0.000	0.314	0.086	0.913	0.662	1.084	1.244
	U.C.I	–	0.852	1.514	1.122	1.270	1.000	1.128	2.067	2.292
BIC		230.52	232.12	234.16	232.82	233.35	234.15	233.41	241.76	240.15
HQIC		228.97	229.02	231.06	229.72	230.25	231.05	230.30	238.66	237.04
χ^2		1.357	0.739	3.347	0.973	1.570	3.340	1.507	3.503	2.783
Degree of freedom		2	1	1	1	1	1	1	1	1
P. value		0.507	0.390	0.067	0.324	0.210	0.068	0.219	0.061	0.095

TABLE 8. Estimation methods and goodness-of-fit for data set I.

Method ↓ Measure →	λ	χ^2	P-value
MME	0.677	1.219	0.544
PE	0.720	0.698	0.705

It is clear that the data set plausibly came from some competitive models. But, the DBH model is the best. Table 8 shows two different estimation methods of the DBH parameter for data set I.

Depending on P-value, it is observed that the MME and PE methods work quite well besides the MLE method for estimating the unknown parameter. But, the PE method is the best among all estimation methods for this data. Table 9 lists some statistics for data set I using the estimators of the DBH parameter in Tables 6 and 7.

Regarding Table 9, the following observations can be made:

1) The estimates of the MLE, MME and PE methods give the same results approximately.
2) This data is suffering from over dispersion phenomena where IOD > 1.
3) This data has a long right tail as compared to its left tail where a positive value of skewness with leptokurtic.

B. DATA SET II: CYSTS OF KIDNEYS

The second data set consists of the counts of cysts of kidneys using steroids (see, Chan et al. [3]). The fits of the DBH distribution is compared with some competitive models Geo, DR, DIR, DLi, Poi, PoiLi, DLi-II, GGeo, DLFR, DIW, DLogL, DLo and DB-II models. The MLEs with their corresponding Std-er, C. I for the parameter(s) and goodness of fit test for data set II are reported in Tables 10 and 11.

According to Tables 10 and 11, it is clear that the GGeo, DLogL, DLo and DB-II work quit well besides the DBH model. But, the DBH model is the best model for this data.

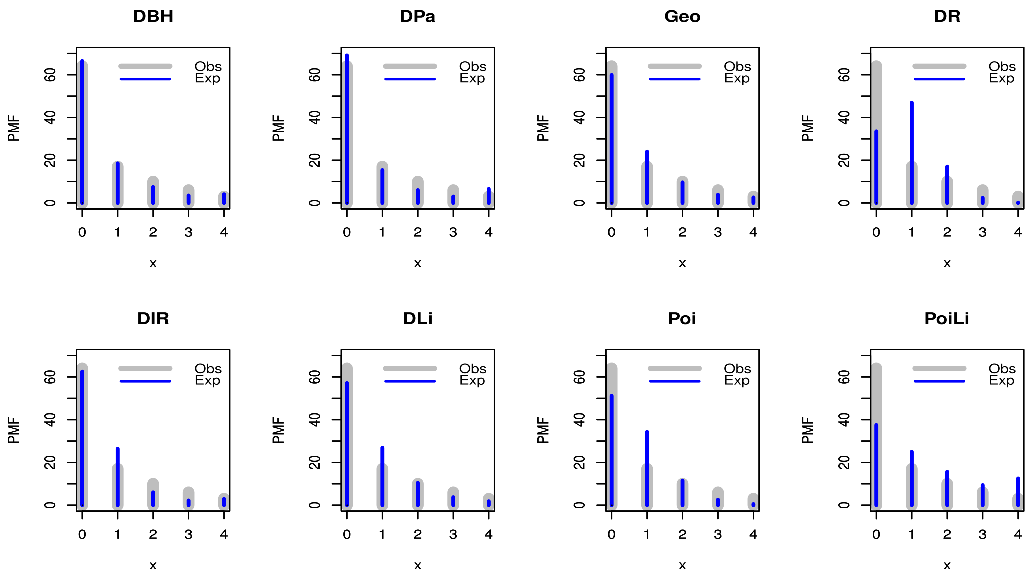

FIGURE 3. The fitted pmfs of the models having only one parameter for data set I.

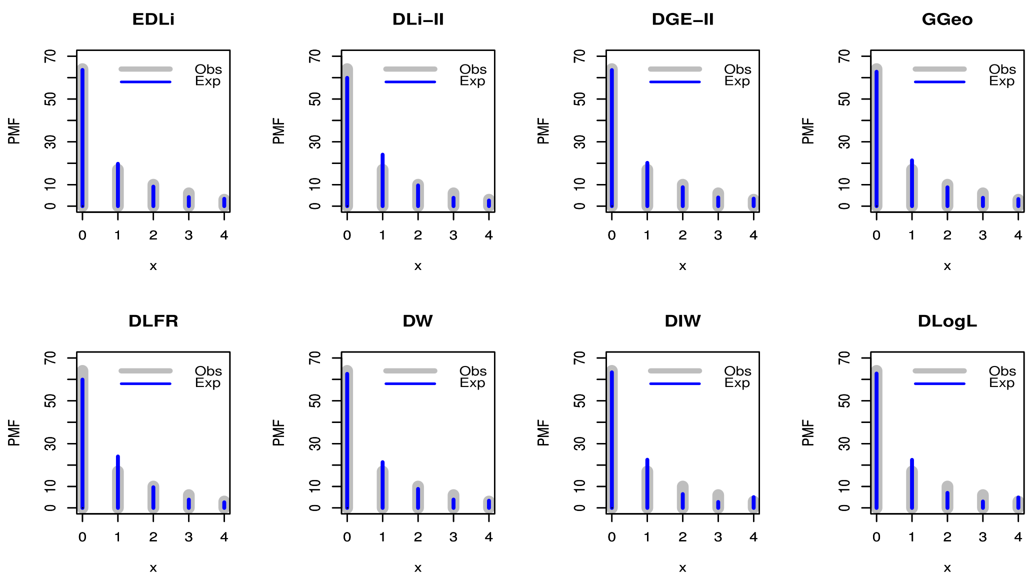

FIGURE 4. The fitted pmfs of the models having only two-parameter for data set I.

TABLE 9. Some statistics for data set I.

Method ↓ Measure →	Mean	Var	IOD	COV	Skewness	Kurtosis
MLE	0.6567	1.6773	2.5538	1.9719	3.3924	20.8869
MME	0.6692	1.7361	2.5941	1.9687	3.4055	21.0512
PE	0.7680	2.2489	2.9283	1.9526	3.5175	22.4488

Figures 5 and 6 show the fitted pmfs plot for data set II which support our results reported in Tables 10 and 11.

It is clear that the data set plausibly came from the DBH, GGeo, DLogL, DLo and DB-II distributions. But, the DBH model is the best. Table 12 shows two different estimation methods of the DBH parameter for data set II.

Depending on P-value, it is observed that the MME method works quite well besides the MLE method for estimating the unknown parameter. But, the MME method is the best among all estimation methods for this data. Table 13 lists some statistics for data set II using the estimators of the DBH parameters in Tables 10 and 11.

TABLE 10. The MLE, C. I, χ^2 and P-values for the competitive models having only one parameter using data set II.

X	Observed frequency	Expected frequency DBH	Geo	DR	DIR	DLi	Poi	PoiLi
0	65	61.94	45.98	11.00	60.94	40.25	27.42	44.14
1	14	20.06	26.76	26.83	33.96	29.83	38.08	28.00
2	10	9.65	15.57	29.55	8.11	18.36	26.47	16.70
3	6	5.52	9.06	22.23	3.00	10.35	12.26	9.57
4	4	3.49	5.28	12.49	1.42	5.53	4.26	5.34
5	2	2.34	3.07	5.42	0.78	2.86	1.18	2.92
6	2	1.65	1.79	1.85	0.47	1.44	0.27	1.57
7	2	1.19	1.04	0.52	0.31	0.71	0.05	0.84
8	1	0.89	0.61	0.11	0.21	0.35	0.01	0.44
9	1	0.67	0.35	0.02	0.15	0.17	0	0.23
10	1	0.52	0.21	0	0.11	0.08	0	0.12
11	2	2.08	0.28	0	0.54	0.07	0	0.13
Total	**110**	**110**	**110**	**110**	**110**	**110**	**110**	**110**
λ	**MLE**	0.874	0.582	0.900	0.554	0.436	1.390	1.087
	Std-er	0.041	0.030	0.009	0.049	0.026	0.112	0.109
	L.C.I	0.794	0.523	0.882	0.458	0.385	1.170	0.873
	U.C.I	0.954	0.641	0.918	0.649	0.487	1.609	1.301
χ^2		2.613	22.84	321.1	51.047	43.48	294.1	31.151
Degree of freedom		4	4	4	4	4	4	4
P. value		0.625	< 0.001	< 0.001	< 0.001	< 0.001	< 0.001	< 0.001

TABLE 11. The MLE, C. I, χ^2 and P-values for the competitive models having only two-parameter using data set II.

X	Observed frequency	Expected frequency DBH	DLi-II	GGeo	DLFR	DIW	DLogL	DLo	DB-II
0	65	61.94	46.03	62.79	45.98	44.81	63.19	61.62	64.71
1	14	20.06	26.77	19.66	26.76	26.87	20.10	21.02	19.18
2	10	9.65	15.57	9.43	15.57	15.90	8.64	9.69	8.49
3	6	5.52	9.05	5.43	9.06	9.34	4.66	5.28	4.64
4	4	3.49	5.27	3.46	5.28	5.46	2.86	3.20	2.87
5	2	2.34	3.06	2.35	3.07	3.18	1.92	2.09	1.92
6	2	1.65	1.78	1.66	1.79	1.85	1.37	1.44	1.37
7	2	1.19	1.04	1.21	1.04	1.08	1.02	1.04	1.01
8	1	0.89	0.60	0.90	0.61	0.63	0.79	0.77	0.78
9	1	0.67	0.35	0.68	0.35	0.36	0.62	0.59	0.61
10	1	0.52	0.20	0.52	0.21	0.21	0.50	0.46	0.49
11	2	2.08	0.28	1.91	0.28	0.31	4.33	2.80	3.93
Total	**110**	**110**	**110**	**110**	**110**	**110**	**110**	**110**	**110**
α	**MLE**	0.874	0.581	0.800	0.582	0.581	0.780	0.152	0.278
	Std-er	0.041	0.045	0.064	0.038	0.048	0.136	0.098	0.045
	L.C.I	0.794	0.494	0.674	0.507	0.489	0.514	0.000	0.189
	U.C.I	0.954	0.669	0.927	0.656	0.675	1.046	0.345	0.366
β	**MLE**	–	0.001	0.188	1.000	1.049	1.208	1.830	1.053
	Std-er	–	0.058	0.089	0.010	0.146	0.159	0.953	0.167
	L.C.I	–	0.000	0.013	0.980	0.763	0.895	0.000	0.725
	U.C.I	–	0.115	0.362	1.020	1.335	1.520	3.698	1.381
BIC		342.491	366.9	346.514	366.934	355.27	352.835	350.362	351.679
HQIC		340.8858	363.7	343.303	363.724	352.06	349.625	347.152	348.469
χ^2		2.613	22.89	2.461	22.845	24.135	2.830	3.242	2.453
Degree of freedom		4	3	3	3	3	3	3	3
P. value		0.625	< 0.001	0.482	< 0.001	< 0.001	0.419	0.356	0.484

TABLE 12. Estimation methods and goodness-of-fit for data set II.

Method ↓ Measure →	λ	χ^2	P-value
MME	0.877	2.512	0.642
PE	0.818	6.239	0.182

Regarding Table 13, the following observations can be made:

1) The estimates of the MLE and MME methods give the same results approximately.
2) This data is suffering from over dispersion phenomena where IOD > 1.
3) This data has a long right tail as compared to its left tail with leptokurtic.

C. DATA SET III: NUMBER OF DOCTOR VISITS

The data set comes from the Medicaid Consumer Survey sponsored by the Health Care Financing Administration in 1986. The detail information on the data set can be found

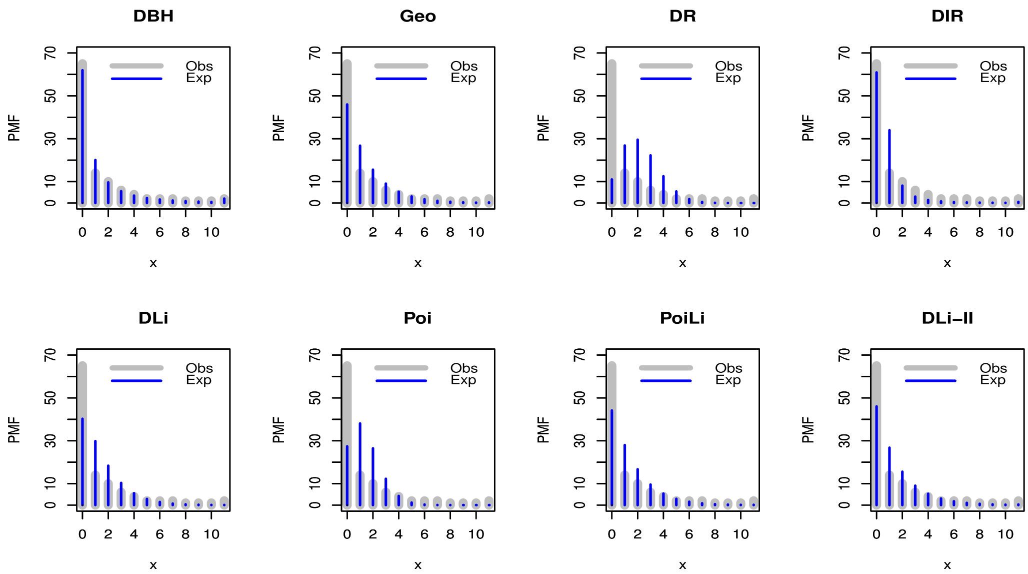

FIGURE 5. The fitted pmfs of the models having only one parameter for data set II.

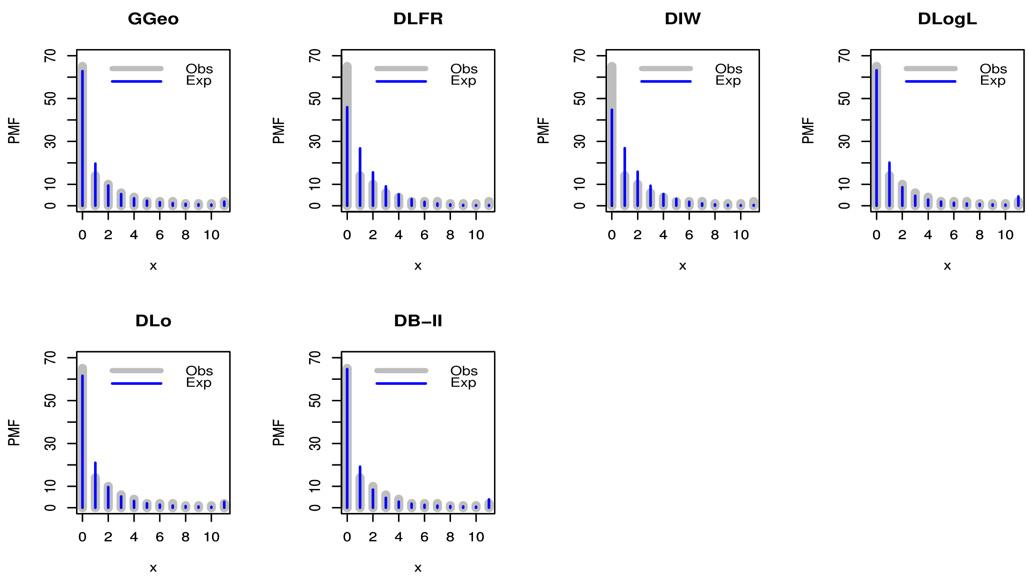

FIGURE 6. The fitted pmfs of the models having only two-parameter for data set II.

TABLE 13. Some statistics for data set II.

Method ↓ Measure →	Mean	Var	IOD	COV	Skewness	Kurtosis
MLE	1.3701	7.8855	5.7553	2.0495	4.4321	34.9137
MME	1.3894	8.1610	5.8734	2.0559	4.4671	35.4426
PE	1.0828	4.5680	4.2186	1.9738	3.9524	28.0728

in Gurmu [7]. The data can be found in **Ecdat** package of R software. The aim of the study is to investigate the effects of number of children in the household (x_{1i}) and health status (x_{2i}) on the number of doctor visits (y_i). Note that the higher positive values of the health status indicate the poorer health status. The below regression structure is fitted by Poisson and DBH regression models.

$$\ln(\mu_i) = \beta_0 + \beta_1 x_{1i} + \beta_2 x_{2i}. \tag{32}$$

Table 14 lists the estimated parameters and corresponding Std-er of fitted regression models. Since the DBH regression model has lower values of the AIC and BIC statistics,

TABLE 14. Estimated parameters of Poisson and DBH regression models.

Parameters	Poisson				DBH			
	Estimates	Std-er	z-value	p-value	Estimates	Std-er	z-value	p-value
β_0	-0.007	0.089	-0.081	0.936	0.335	0.260	1.286	0.198
β_1	0.955	0.195	4.905	<0.001	-0.012	0.518	-0.023	0.981
β_2	0.293	0.018	15.895	<0.001	0.333	0.123	2.703	0.007
ℓ	-1102.647				-802.148			
AIC	2211.294				1610.296			
BIC	2223.846				1622.849			

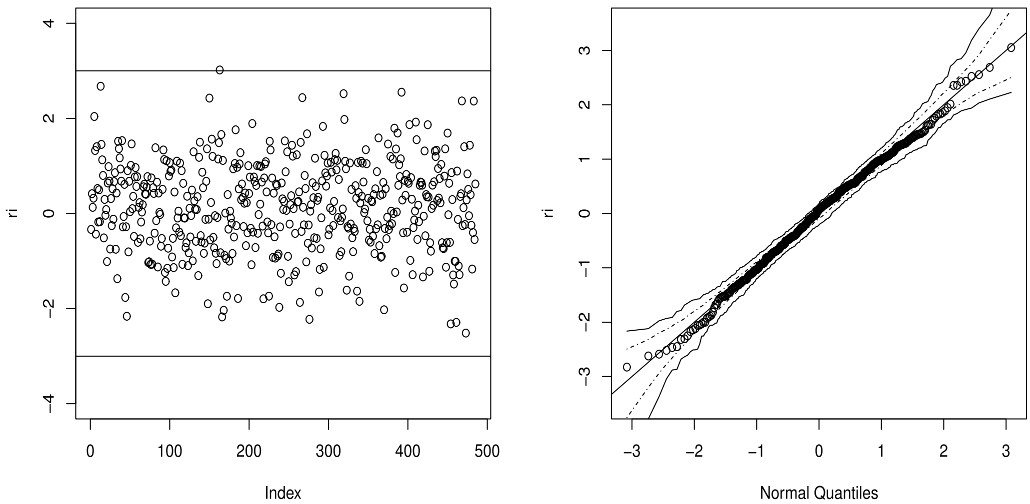

FIGURE 7. The randomized quantile residuals and corresponding Q-Q plot.

it is concluded that the DBH regression model provides higher modeling accuracy than the Poisson regression model. As seen from the estimated parameters of DBH regression model, we conclude that the number of children has no statistically significant effect on the number of doctor visits. However, the health status is statistically significant effect on the number of doctor visits. The individuals having poor health status have the larger number of doctor visits increase.

Figure 7 displays the randomized quantile residuals and corresponding quantile-quantile (Q-Q) plot. These figures reveal that none of the observations can be evaluated as a possible outlier. Moreover, these figures prove that the DBH regression model fits well to the current data set.

VII. CONCLUSION

Discrete probability distributions play an important role in modeling the counts. The count data sets are generally over-dispersed. This study introduces a flexible discrete distribution to model these kind of data sets. The main advantage of the DBH distribution against the existing ones is that the statistical properties of the DBH distribution are in explicit forms which are important in statistical inference. The importance of the DBH distribution is demonstrated via two real data sets and compared with seventeen competitive models. More importantly, a new regression model based on the DBH distribution is introduced and compared with famous Poisson regression model for the number of doctor visits data set. We believe that the DBH model will increase its popularity in the near future, especially in modeling the over-dispersed count data sets.

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