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TORSION OF A CIRCULAR ANISOTROPIC BEAM WITH TWO LINEAR CRACKS WEAKENED WITH A CIRCULAR CAVITY

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Various components made of anisotropic materials (plast-mass, glass material, etc.) have been widely used in the production of modern mechanisms and machinery. Precise calculation of these elements, constituting the design, holds great importance. In general, fracture and distribution are essential issues in safety calculations. In this study, torsion of a beam with an S oblast having outer and inner constraints as L_2 and L_1 circles with R_2 and R_1 radii, respectively, is investigated.

Keywords: anisotropic medium, orthotropic beams, isotropic beams, affine connections, conformal mapping functions

1. Introduction

There is no study available in the literature a solution of the mentioned in the abstract cases, since the contour L_1 is void of the mapping function. In this study, the solution of the problem is presented with numerical values.

Here, torsion of an area limited with an outer circle L_2 with radius R_2 and inner circle L_1 with radius R_1, having two linear cracks, is investigated. The coordinates of the end points of these cracks are taken as $\pm e$. Volumetric forces are neglected.

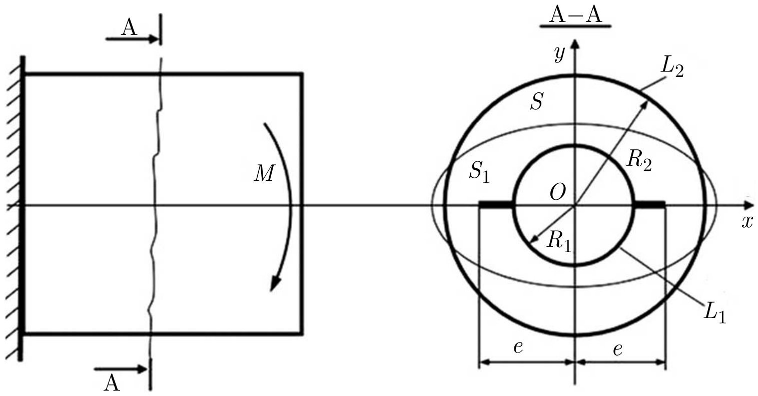

Fig. 1. Anisotropic beam and its cross-section after affine transformation

The beam is twisted by means of a torsional moment applied to the edges (Fig. 1). Here, the coordinate origin is taken as the center of cross-section. The beam is assumed to be made of a homogeneous anisotropic material. At least one elastic plane of symmetry is available on each point of the beam. In this case, all stresses except τ_{xz} and τ_{yz} are zero.

As known (Kosmodamianskii, 1976; Lethniskii, 1971; Kuliyev, 1991), solutions to torsion problems related to orthotropic beams are found using the solutions to torsion problems of beams with other cross-sections. In this case, affine connections are used

$$x_1 = x \qquad \text{and} \qquad y_1 = \beta y \tag{1.1}$$

It should be noted that if the affine connection is assumed as below, then the beam will not be orthotropic

$$x_1 = x + \alpha y \qquad \text{and} \qquad y_1 = \beta y \tag{1.2}$$

With expression (1.1), none of the horizontal values in the beam cross-section changes (since the horizontal axis does not change) as for the vertical values (i.e. those on Oy axis), they will either increase or decrease depending on the coefficient β that characterizes anisotropy of the beam.

Thereby, in order to evaluate stresses on orthotropic bars, firstly, the torsion problem of an isotropic beam with S_1 cross-section (which is obtained with affine connection $= x_1$ and $y_1 = \beta y$) should be solved.

According to the previous studies, $\tau_{x_1y_1}$ and $\tau_{y_1z_1}$ stresses are found from the following equation

$$\tau_{x_1z_1} - \mathrm{i}\tau_{y_1z_1} = \mathrm{i}[2F'(z_1) - \overline{z}_1] \tag{1.3}$$

where $\tau_{x_1z_1}$ and $\tau_{y_1z_1}$ are components of the tangential stresses on the cross-section S_1, and i – imaginary unit, $F(z_1)$ is the regular function on the cross-section S_1, $z_1 = x_1 + \mathrm{i}y_1$ and $\overline{z}_1 = x_1 - \mathrm{i}y_1$ are complex variables. $F(z_1)$ is calculated from boundary conditions, i.e., from the states of equilibrium and the equation of deformations on the boundaries.

These boundaries can be written as follows

$$\begin{array}{lll} \varepsilon_x = 0 & \varepsilon_y = 0 & \varepsilon_z = 0 \\ \gamma_{xy} = 0 & \gamma_{xz} = a_{55}\tau_{xz} & \gamma_{yz} = a_{44}\tau_{yz} \end{array}$$

where a_{44} and a_{55} are elastic constants that characterize anisotropy of the material.

Equations (1.1) with affine connections, semi axes of L_2 contour with radius R_2 are transformed into an ellipse $a = x_1$ and $b = \beta y_1$, on the other hand, the contour L_1 with radius R_1 on the Ox axis with two cracks, is transformed into an ellipse with two cracks (here it is assumed that $\beta \neq 1$).

If $\beta < 1$, then the linear values decline along the Oy axis, in the case of $\beta > 1$ the same values increase.

$F(z_1)$ regular function within the enclosed area S_1 can be evaluated using the below given boundary conditions (Kosmodamianskii, 1976; Kuliyev, 1991, 2004; Sherman, 1992)

$$F(z_1) - \overline{F(z_1)} = \mathrm{i}t_1\overline{t}_1 + C_k \tag{1.4}$$

where t_1 are affixes of the points on one of the contours of the cross-section S_1. C_k is an arbitrary constant.

Components of $\tau_{x_1z_1}$ and $\tau_{y_1z_1}$ tangential stresses on characteristic points of the cross--section S_1 can be calculated by equation (1.4) (here, the end points of the cracks are also included).

Afterwards, τ_{xz} and τ_{yz} can be calculated for an orthotropic beam by the following equation (Kosmodamianskii, 1976; Lethniskii, 1971; Kuliyev, 1991)

$$\tau_{xz} = \beta\tau_{x_1z_1} \qquad \tau_{yz} = \tau_{y_1z_1}$$

As indicated by these equations, τ_{yz} and $\tau_{y_1z_1}$ stresses do not vary on isotropic and anisotropic beams. Here, τ_{xz} tangential stress varies depending on the parameter β. It increases or decreases depending on β ($\beta = \sqrt{a_{44}/a_{55}}$).

$F(z_1)$ regular function can be expressed as follows for the contour L_2 within the enclosed S_1 area (Kosmodamianskii, 1976; Kuliyev, 1991, 2004; Sherman, 1992)

$$F(t_2) = \mathrm{i}\sum_{k=0}^{\infty} \alpha_k\Big(\frac{A_2}{t_2}\Big)^k + \mathrm{i}\sum_{k=0}^{\infty} b_k\Big(\frac{t_2}{A_2}\Big)^k \qquad \text{on} \qquad L_2 \tag{1.5}$$

where

$$\alpha_k = \sum_{\nu=k-2E(k/2)}^{k}{}^{*} \alpha_\nu L_{(k-\nu)/2} \qquad\qquad b_k = \sum_{n=k}^{\infty} \beta_k a^{(2)}_{(n-k)/2} \tag{1.6}$$

$F(z_1)$ function can be defined as follows in the inner L_1 contour (an ellipse with two linear cracks) (Kosmodamianskii, 1976; Kuliyev, 1991; Sherman, 1992)

$$F(t_1) = i\sum_{k=0}^{\infty} \alpha_k \xi_1^{-k} + \mathrm{i}\sum_{k=1}^{\infty} H_1(k)\xi_1^{k} + \mathrm{i}\sum_{k=0}^{\infty} H_2(k)\xi_1^{-k} \qquad \text{on} \qquad L_1 \tag{1.7}$$

The following notations are given in equation (1.7)

$$\begin{aligned}
&H_1(k) = \sum_{\nu=k}^{\infty} b_\nu\Big(\frac{A_1}{A_2}\Big)^\nu m_1^{\frac{\nu-k}{2}} C_\nu^{\frac{\nu-k}{2}} \qquad\qquad H_2(k) = \sum_{\nu=\varepsilon}^{\infty}{}^{*} b_\nu\Big(\frac{A_1}{A_2}\Big)^\nu m_1^{\frac{\nu+k}{2}} C_\nu^{\frac{\nu+k}{2}} \\
&\varepsilon' = \varepsilon + \frac{1}{2}(k+\varepsilon) \qquad \varepsilon = 0 \qquad \varepsilon = 1 \\
&b_k = \sum_{n=k}^{\infty}{}^{*} \beta_n a_{(n-k)/2}
\end{aligned} \tag{1.8}$$

The outer circle with contour L_2 (with semi axes $a_2 = R_2$ and $b_2 = \beta R_2$) is mapped on the circle with the unit diameter (equals to 1) using the mapping function (Kosmodamianskii, 1976; Kuliyev, 1991, 2004; Sherman, 1992)

$$t_2 = A_2\Big(\tau + \frac{m_2}{\tau}\Big) \qquad A_2 = \frac{R_2 + \beta R_2}{2} \qquad m_2 = \frac{R_2 - \beta R_2}{R_2 + \beta R_2} \tag{1.9}$$

The inner contour L_1 is mapped onto the circle with the unit diameter (equals to 1) using the mapping function (Kuliyev, 1991, 2004)

$$t_1 = A_1\tau\sum_{n=0}^{\infty} \tau^{-n} \varPi_n \tag{1.10}$$

where

$$\begin{aligned}
&A_1 = \frac{R_1 + \beta R_1}{2} \qquad m_1 = \frac{R_1 - \beta R_1}{R_1 + \beta R_1} \\
&\varPi_n = \sum_{k=0}^{\infty} \gamma_{k-1} T_{n-k} \qquad T_n = \sum_{\nu=n-2E(n/2)}^{n}{}^{*} m_1^{\frac{n-\nu}{2}} \gamma_{-1}^{\frac{n-\nu}{2}} l_\nu^{n-\nu}
\end{aligned}$$

The inverse functions of (1.9) and (1.10) mapping functions are as bellow (Kosmodamianskii, 1976; Kuliyev, 2004; Sherman, 1992)

$$\xi_2 = \frac{z_2}{A_2}\sum_{n=0}^{\infty} a_n^{(2)}\Big(\frac{A_2}{z_2}\Big)^{2n} \qquad \xi_1 = \chi(z) = \frac{z_1}{A_1}\sum_{n=0}^{\infty} E_k\Big(\frac{A_1}{z_1}\Big)^k \tag{1.11}$$

The coefficients of the serial elements of the analytical functions $F_1(z)$ and $F_2(z)$ can be found using proper boundary conditions given below: (Kosmodamianskii, 1976; Kuliyev, 1991; Sherman, 1992)

$$
\begin{aligned}
&F_1(t_1) + \overline{F_1(t_1)} = \mathrm{i}\bar{t}_1 + C_1 \\
&F_1(t_2) + \overline{F_1(t_2)} = \mathrm{i}\bar{t}_2 + C_2
\end{aligned}
\tag{1.12}
$$

where t_1 and t_2 variables are properly found from (1.9) and (1.10) equations.

As we place equations (1.6) and (1.7) into boundary condition (1.12), we obtain a linear algebraic system depending on two unknown coefficients following some mathematical connections and remarks by Kosmodamianskii (1976), Kuliyev (1991, 2004). Here, we proceed with the variable τ since $\tau\overline{\tau} = 1$ on the unit circle)

$$
\begin{aligned}
&\alpha_k + H_1(k) + H_2(k) = \sum_{n=k}^{\infty} \varPi_{n-k}\varPi \qquad \text{on} \qquad L_1 \\
&V_1(k) + V_2(k) + V_3(k) = A_2^2 m_2 \varepsilon \qquad \text{on} \qquad L_1
\end{aligned}
\tag{1.13}
$$

where $V_1(k)$, $V_2(k)$, $V_3(k)$ are respectively found from the following equations (Kuliyev, 1991, 2004)

$$
\begin{aligned}
&V_1(k) = \sum_{\nu=0}^{k}{}^{*} \Big(\frac{A_1}{A_2}\Big)^k C_{-\nu}^{\frac{k-\nu}{2}} m_2^{\frac{k-\nu}{2}} \alpha_\nu \qquad\qquad V_2(k) = \sum_{\nu=k}^{\infty}{}^{*} b_\nu C_\nu^{\frac{k+\nu}{2}} m_2^{\frac{k+\nu}{2}} \\
&V_3(k) = \sum_{\nu=k}^{\infty}{}^{*} b_\nu C_\nu^{\frac{k-\nu}{2}} m_2^{\frac{k-\nu}{2}}
\end{aligned}
\tag{1.14}
$$

From the first terms of these equations, a system of equations is obtained. α_k and β_k are coefficients that can be found using these equations.

This is presented with the following numerical example.

Cross-sectional dimensions of the beam are assumed in accordance with the following ratio for numerical calculations.

1. In the case of $\beta = 1/2$, the semi axes of the outer contour (curvilinear line, the circle with radius R_2) are transformed into the ellipse with semi axes $a_2 = R_2$; $b_2 = \beta R_2$, and the inner contour (the one with two cracks and radius R_1) is transformed into the ellipse with two cracks. The semi axes of such ellipses can be defined as $a_1 = R_1$; $b_1 = \beta R_1$. Accordingly, the problem related with torsion of the beam depending on the parameter β is calculated using the torsion problem of another beam with a different cross section.

Tangential τ_{xz} and τ_{yz} stresses, given in Table 1, are found using equation (1.3).

Table 1. Tangential τ_{xz} and τ_{yz} stresses for choice 1

Choice 1	Points	$\tau_{xz}/(\mu\tau a_2)$	$\tau_{yz}/(\mu\tau a_2)$
$\beta = 1/2$	$z = 0.65$	–	24.58
$a_1 = 2b_1$	$z = 0.70$	–	13.29
$a_1 = 0.25a_2$	$z = 0.75$	–	8.02
	$z = 1.00$	–	2.92
	$z = 0.5\mathrm{i}a_1$	0.548	–
	$z = 0.7\mathrm{i}a_1$	0.141	–
	$z = 1.0\mathrm{i}a_1$	0.118	–

2. In the case of $\beta = 1/2$, $R_2 = 0$. The inner contour transforms into the linear crack with length $l = 2e$; and the outer contour transforms into the ellipse with semi axes $a_2 = R_2$, $b_2 = \beta R_2$. This way, the problem, the subject of the current study, is solved by means of the solution to the torsion problem of the elliptical beam with a central linear crack.

The values of τ_{xz} and τ_{yz} tangential stresses for choice 2 are given in Table 2.

Table 2. Values of τ_{xz} and τ_{yz} tangential stresses

Choice 1	Points	$\tau_{xz}/(\mu\tau a_2)$	$\tau_{yz}/(\mu\tau a_2)$
$\beta = 1/2$	$z = 0.65$	–	14.17
$b_1 = 0$	$z = 0.70$	–	7.76
$a_1 = 0.5a_2$	$z = 0.75$	–	4.77
	$z = 1.00$	–	1.56
	$z = 0.5\mathrm{i}a_1$	-1.01	–
	$z = 0.7\mathrm{i}a_1$	-0.031	–
	$z = 1.0\mathrm{i}a_1$	-0.48	–

2. Conclusion

Calculations of torsion of orthotropic beams can be performed using calculations of isotropic beams with different cross-sections (the cross-section S is obtained with affine connection $x_1 = x$ and $y_1 = \beta y$). Here, the linear values on the x axis do not vary with the varying parameter β), the ones on the y axis increase or decrease in direct proportion with β.

The stresses on orthotropic beams can be calculated using the equations given below ($\tau_{x_1z_1}$ and $\tau_{y_1z_1}$ are known)

$$\tau_{xz} = \beta^2\tau\tau_{x_1y_1} \qquad \tau_{yz} = \beta\tau\tau_{y_1z_1} \tag{2.1}$$

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