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NEW WEIGHTED INTEGRAL INEQUALITIES FOR TWICE DIFFERENTIABLE CONVEX FUNCTIONS

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Abstract. In this paper, we establish several new weighted inequalities for some twice differentiable mappings that are connected with the celebrated Hermite-Hadamard type and Ostrowski type integral inequalities. Some of the new inequalities are Hermite-Hadamard-type inequalities involving fractional integrals. The results presented here would provide extensions of those given in earlier works.

1. Introduction

Definition 1.1. The function $f : [a, b] \subset \mathbb{R} \to \mathbb{R}$, is said to be convex if the following inequality holds

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y)$$

for all $x, y \in [a, b]$ and $\lambda \in [0, 1]$. We say that f is concave if $(-f)$ is convex.

The following inequality is well known in the literature as the Hermite-Hadamard integral inequality (see, [6]):

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2},$$

where $f : I \subset \mathbb{R} \to \mathbb{R}$ is a convex function on the interval I of real numbers and $a, b \in I$ with $a < b$.

A largely applied inequality for convex functions, due to its geometrical significance, is Hadamard's inequality, (see [4, 5, 18–23]) which has generated a wide range of directions for extension and a rich mathematical literature.

In 1938, the classical integral inequality established by Ostrowski [8] as follows:

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Theorem 1.1. *Let $f : [a,b] \to \mathbb{R}$ be a differentiable mapping on (a,b) whose derivative $f' : (a,b) \to \mathbb{R}$ is bounded on (a,b), i.e., $\|f'\|_\infty = \sup_{t\in(a,b)} |f'(t)| < \infty$. Then, the inequality holds:*

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)dt \right| \leq \left[\frac{1}{4} + \frac{(x-\frac{a+b}{2})^2}{(b-a)^2} \right](b-a)\|f'\|_\infty$$

for all $x \in [a,b]$. The constant $1/4$ is the best possible.

Inequality (1.1) has wide applications in numerical analysis and in the theory of some special means; estimating error bounds for some special means, some mid-point, trapezoid and Simpson rules and quadrature rules, etc. Hence inequality (1.1) has attracted considerable attention and interest from mathematicans and researchers. Due to this, over the years, the interested reader is also refered to [1–3, 7, 9–16] for integral inequalities in several independent variables. In addition, the current approach of obtaining the bounds, for a particular quadrature rule, have depended on the use of Peano kernel. The general approach in the past has involved the assumption of bounded derivatives of degree greater than one.

In this study, using functions whose twice derivatives absolute values are convex, we obtain new weighted inequalities that are connected with the celebrated Hermite-Hadamard type and Ostrowski type integral inequalities. In addition, we obtain new inequalities of Hermite-Hadamard type and Ostrowski type involving fractional integrals.The results presented here would provide extensions of those given in earlier works.

2. Main Results

Throughout this section, let us define the $S(\alpha; w, f)$ operator as follows:

$$\begin{aligned} S(\alpha; w, f) = & \left[\left(\int_a^x (u-x)\, w(u)\, du \right)^\alpha - \left(\int_b^x (u-x)\, w(u)\, du \right)^\alpha \right] f'(x) \\ & + \alpha \left[\left(\int_a^x (u-x)\, w(u)\, du \right)^{\alpha-1} \left(\int_a^x w(u)\, du \right) \right. \\ & \left. - \left(\int_b^x (u-x)\, w(u)\, du \right)^{\alpha-1} \left(\int_b^x w(u)\, du \right) \right] f(x) \\ & + \alpha(\alpha-1) \left\{ \int_a^x \left(\int_a^t (u-t)\, w(u)\, du \right)^{\alpha-2} \left(\int_a^t w(u)\, du \right)^2 f(t)dt \right. \end{aligned}$$

$$+\int_x^b\left(\int_b^t(u-t)\,w\,(u)\,du\right)^{\alpha-2}\left(\int_b^t w\,(u)\,du\right)^2 f(t)dt\Bigg\}$$

$$-\,\alpha\left[\int_a^x\left(\int_a^t(u-t)\,w\,(u)\,du\right)^{\alpha-1}w\,(t)\,f\,(t)\,dt\right.$$

$$\left.+\int_x^b\left(\int_b^t(u-t)\,w\,(u)\,du\right)^{\alpha-1}w\,(t)\,f\,(t)\,dt\right].$$

In order to prove our main results we need the following lemma.

Lemma 2.1. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be twice differentiable function on I°, $a, b \in I^\circ$ with $a < b$, f'' is absolutely continuous on $[a, b]$ and let $w : [a, b] \to \mathbb{R}$ be nonnegative and continuous on $[a, b]$. Then the following identity holds:*

$$S(\alpha; w, f) = \int_a^b P_w\,(x,t)\,f''\,(t)\,dt, \tag{2.1}$$

where

$$P_w\,(x,t) := \begin{cases} \left(\int_a^t (u-t)\,w\,(u)\,du\right)^\alpha, & a \le t < x, \\[2ex] \left(\int_b^t (u-t)\,w\,(u)\,du\right)^\alpha, & x \le t \le b, \end{cases}$$

for $\alpha \ge 1$.

Proof. By integration by parts, we have the following identity:

$$\int_a^b P_w\,(x,t)\,f''\,(t)\,dt = \int_a^x\left(\int_a^t(u-t)\,w\,(u)\,du\right)^\alpha f''\,(t)\,dt$$

$$+\int_x^b\left(\int_b^t(u-t)\,w\,(u)\,du\right)^\alpha f''\,(t)\,dt$$

$$=\left(\int_a^t(u-t)\,w\,(u)\,du\right)^\alpha f'\,(t)\Bigg|_a^x$$

$$+\,\alpha\int_a^x\left(\int_a^t(u-t)\,w\,(u)\,du\right)^{\alpha-1}\left(\int_a^t w\,(u)\,du\right)f'\,(t)\,dt$$

$$+\left(\int_b^t (u-t)\,w(u)\,du\right)^{\alpha} f'(t)\Bigg|_x^b$$

$$+\alpha\int_x^b \left(\int_b^t (u-t)\,w(u)\,du\right)^{\alpha-1}\left(\int_b^t w(u)\,du\right) f'(t)\,dt$$

$$=\left[\left(\int_a^x (u-t)\,w(u)\,du\right)^{\alpha}-\left(\int_b^x (u-t)\,w(u)\,du\right)^{\alpha}\right] f'(x)$$

$$+\alpha\left\{\left(\int_a^t (u-t)\,w(u)\,du\right)^{\alpha-1}\left(\int_a^t w(u)\,du\right) f(t)\Bigg|_a^x\right.$$

$$+\int_a^x \left[(\alpha-1)\left(\int_a^t (u-t)\,w(u)\,du\right)^{\alpha-2}\left(\int_a^t w(u)\,du\right)^2\right.$$

$$\left.-\left(\int_a^t (u-t)\,w(u)\,du\right)^{\alpha-1} w(t)\right] f(t)\,dt$$

$$+\left(\int_b^t (u-t)\,w(u)\,du\right)^{\alpha-1}\left(\int_b^t w(u)\,du\right) f(t)\Bigg|_x^b$$

$$+\int_x^b \left[(\alpha-1)\left(\int_b^t (u-t)\,w(u)\,du\right)^{\alpha-2}\left(\int_b^t w(u)\,du\right)^2\right.$$

$$\left.\left.-\left(\int_b^t (u-t)\,w(u)\,du\right)^{\alpha-1} w(t)\right] f(t)\,dt\right\}$$

$$=\left[\left(\int_a^x (u-x)\,w(u)\,du\right)^{\alpha}-\left(\int_b^x (u-x)\,w(u)\,du\right)^{\alpha}\right] f'(x)$$

$$+\alpha\left[\left(\int_a^x (u-x)\,w(u)\,du\right)^{\alpha-1}\left(\int_a^x w(u)\,du\right)\right.$$

$$\left.-\left(\int_b^x (u-x)\,w(u)\,du\right)^{\alpha-1}\left(\int_b^x w(u)\,du\right)\right] f(x)$$

$$
\begin{aligned}
&+\alpha(\alpha-1)\left\{\int_a^x\left(\int_a^t (u-t)\,w(u)\,du\right)^{\alpha-2}\right. \\
&\quad\times\left(\int_a^t w(u)\,du\right)^2 f(t)dt \\
&\left.+\int_x^b\left(\int_b^t (u-t)\,w(u)\,du\right)^{\alpha-2}\left(\int_b^t w(u)\,du\right)^2 f(t)dt\right\} \\
&-\alpha\left[\int_a^x\left(\int_a^t (u-t)\,w(u)\,du\right)^{\alpha-1} w(t)\,f(t)\,dt\right. \\
&\left.+\int_x^b\left(\int_b^t (u-t)\,w(u)\,du\right)^{\alpha-1} w(t)\,f(t)\,dt\right],
\end{aligned}
$$

which is the required identity in (2.1) and the proof is completed. □

Remark 2.1. Under the same assumptions of Lemma 2.1 with $\alpha = 1$, then the following identity holds:

$$
S(1;w,f)=\left(\int_a^b (u-x)\,w(u)\,du\right)f'(x)+\left(\int_a^b w(u)\,du\right)f(x)-\int_a^b w(t)\,f(t)\,dt,
$$

which was proved by Sarikaya and Yaldiz in [16].

Definition 2.1. Let $f \in L_1[a,b]$. The Riemann-Liouville integrals $J_{a+}^{\alpha}f$ and $J_{b-}^{\alpha}f$ of order $\alpha > 0$ with $a \geq 0$ are defined by

$$
J_{a+}^{\alpha}f(x)=\frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}f(t)dt, \qquad x>a,
$$

and

$$
J_{b-}^{\alpha}f(x)=\frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1}f(t)dt, \qquad x<b,
$$

respectively. Here, $\Gamma(\alpha)$ is the Gamma function and $J_{a+}^{0}f(x)=J_{b-}^{0}f(x)=f(x)$.

Corollary 2.1. *Under the same assumptions of Lemma* 2.1 *with* $w(s) = 1$, *then the following identity holds:*

$$
\begin{aligned}
S(\alpha;1,f) &= \left(-\frac{1}{2}\right)^{\alpha}\left[(a-x)^{2\alpha}-(b-x)^{2\alpha}\right]f'(x) \\
&\quad+\alpha\left(-\frac{1}{2}\right)^{\alpha-1}\left[(a-x)^{2\alpha-1}-(b-x)^{2\alpha-1}\right]f(x) \qquad (2.2)
\end{aligned}
$$

$$+\frac{\alpha\Gamma(2\alpha)}{2}\left(-\frac{1}{2}\right)^{\alpha-2}\left[J_{x-}^{2\alpha-1}f(a)+J_{x+}^{2\alpha-1}f(b)\right].$$

Remark 2.2. If we take $x=\frac{a+b}{2}$ in (2.2), we get

$$\int_a^{\frac{a+b}{2}}\left(\int_a^t (u-t)\,du\right)^{\alpha} f''(t)\,dt+\int_{\frac{a+b}{2}}^{b}\left(\int_b^t (u-t)\,du\right)^{\alpha} f''(t)\,dt$$
$$=\alpha\left(-\frac{1}{2}\right)^{\alpha-1}\frac{(b-a)^{2\alpha-1}}{2^{2\alpha-2}}f(\frac{a+b}{2})$$
$$+\frac{\alpha\Gamma(2\alpha)}{2}\left(-\frac{1}{2}\right)^{\alpha-2}\left[J_{(\frac{a+b}{2})-}^{2\alpha-1}f(a)+J_{(\frac{a+b}{2})+}^{2\alpha-1}f(b)\right].$$

Theorem 2.1. *Let $f:I\subset\mathbb{R}\to\mathbb{R}$ be twice differentiable function on I°, $a,b\in I^{\circ}$ with $a<b$, f'' is absolutely continuous on $[a,b]$ and let $w:[a,b]\to\mathbb{R}$ be nonnegative and continuous on $[a,b]$. If $|f''|$ is convex on $[a,b]$ then for all $x\in[a,b]$, the following inequalities hold:*

$$\begin{aligned}
|S(\alpha;w,f)| &\leq \frac{\|w\|_{[a,x],\infty}^{\alpha}}{2^{\alpha}(b-a)}\\
&\quad\times\left[\left(\frac{(b-a)(x-a)^{2\alpha+1}}{2\alpha+1}-\frac{(x-a)^{2\alpha+2}}{2\alpha+2}\right)|f''(a)|+\frac{(x-a)^{2\alpha+2}}{2\alpha+2}|f''(b)|\right]\\
&\quad+\frac{\|w\|_{[x,b],\infty}^{\alpha}}{2^{\alpha}(b-a)}\\
&\quad\times\left[\frac{(b-x)^{2\alpha+2}}{2\alpha+2}|f''(a)|+\left(\frac{(b-a)(b-x)^{2\alpha+1}}{2\alpha+1}-\frac{(b-x)^{2\alpha+2}}{2\alpha+2}\right)|f''(b)|\right]\\
&\leq \frac{\|w\|_{[a,b],\infty}^{\alpha}}{2^{\alpha}(b-a)}\left[\left(\frac{(b-a)(x-a)^{2\alpha+1}}{2\alpha+1}+\frac{(b-x)^{2\alpha+2}-(x-a)^{2\alpha+2}}{2\alpha+2}\right)|f''(a)|\right.\\
&\quad\left.+\left(\frac{(b-a)(b-x)^{2\alpha+1}}{2\alpha+1}+\frac{(x-a)^{2\alpha+2}-(b-x)^{2\alpha+2}}{2\alpha+2}\right)|f''(b)|\right],
\end{aligned}\tag{2.3}$$

where $\alpha\geq 1$ and $\|w\|_{\infty}=\sup_{t\in[a,b]}|w(t)|$.

Proof. We take the absolute value of (2.1). Using bounded of the mapping w, we find that

$$|S(\alpha;w,f)|\leq\int_a^b |P(x,t)|\,|f''(t)|\,dt$$
$$=\int_a^x\left(\int_a^t (t-u)\,w(u)\,du\right)^{\alpha}|f''(t)|\,dt$$

$$+\int_x^b\left(\int_t^b (u-t)\,w(u)\,du\right)^{\alpha}|f''(t)|\,dt$$

$$\leq \frac{\|w\|^{\alpha}_{[a,x],\infty}}{2^{\alpha}}\int_a^x (t-a)^{2\alpha}|f''(t)|\,dt+\frac{\|w\|^{\alpha}_{[x,b],\infty}}{2^{\alpha}}\int_x^b (b-t)^{2\alpha}|f''(t)|\,dt.$$

Since $\left|f''(t)\right|$ is convex on $[a,b]=[a,x]\cup[x,b]$, we have

$$\left|f''\left(\frac{b-t}{b-a}a+\frac{t-a}{b-a}b\right)\right|\leq\frac{b-t}{b-a}\left|f''(a)\right|+\frac{t-a}{b-a}\left|f''(b)\right|. \tag{2.4}$$

From (2.4), it follows that

$$|S(\alpha;w,f)|\leq\frac{\|w\|^{\alpha}_{[a,x],\infty}}{2^{\alpha}(b-a)}\int_a^x (t-a)^{2\alpha}\left[(b-t)\left|f''(a)\right|+(t-a)\left|f''(b)\right|\right]dt$$

$$+\frac{\|w\|^{\alpha}_{[x,b],\infty}}{2^{\alpha}(b-a)}\int_x^b (b-t)^{2\alpha}\left[(b-t)\left|f''(a)\right|+(t-a)\left|f''(b)\right|\right]dt$$

$$=\frac{\|w\|^{\alpha}_{[a,x],\infty}}{2^{\alpha}(b-a)}\left[\left(\frac{(b-a)(x-a)^{2\alpha+1}}{2\alpha+1}-\frac{(x-a)^{2\alpha+2}}{2\alpha+2}\right)|f''(a)|\right.$$

$$\left.+\frac{(x-a)^{2\alpha+2}}{2\alpha+2}|f''(b)|\right]+\frac{\|w\|^{\alpha}_{[x,b],\infty}}{2^{\alpha}(b-a)}\left[\frac{(b-x)^{2\alpha+2}}{2\alpha+2}|f''(a)|\right.$$

$$\left.+\left(\frac{(b-a)(b-x)^{2\alpha+1}}{2\alpha+1}-\frac{(b-x)^{2\alpha+2}}{2\alpha+2}\right)|f''(b)|\right]$$

and because of $\|w\|_{[a,x],\infty}\leq\|w\|_{[a,b],\infty}$ and $\|w\|_{[x,b],\infty}\leq\|w\|_{[a,b],\infty}$, we obtain

$$|S(\alpha;w,f)|\leq\frac{\|w\|^{\alpha}_{[a,b],\infty}}{2^{\alpha}(b-a)}\left[\left(\frac{(b-a)(x-a)^{2\alpha+1}}{2\alpha+1}+\frac{(b-x)^{2\alpha+2}-(x-a)^{2\alpha+2}}{2\alpha+2}\right)\left|f''(a)\right|\right.$$

$$\left.+\left(\frac{(b-a)(b-x)^{2\alpha+1}}{2\alpha+1}+\frac{(x-a)^{2\alpha+2}-(b-x)^{2\alpha+2}}{2\alpha+2}\right)\left|f''(b)\right|\right],$$

which completes the proof. □

Remark 2.3. Under the same assumptions of Theorem 2.1 with $\alpha=1$, then the following inequality holds:

$$|S(1;w,f)|\leq\frac{\|w\|_{[a,b],\infty}}{2\,(b-a)}\left[\left(\frac{(b-a)(x-a)^3}{3}+\frac{(b-x)^4-(x-a)^4}{4}\right)\left|f''(a)\right|\right.$$

$$\left.+\left(\frac{(b-a)(b-x)^3}{3}+\frac{(x-a)^4-(b-x)^4}{4}\right)\left|f''(b)\right|\right],$$

which was proved by Sarikaya and Yaldiz in [16].

Corollary 2.2. *Under the same assumptions of Theorem* 2.1 *with* $w(s)=1$, *then the following inequality holds:*

$$|S(\alpha;1,f)| \leq \frac{1}{2^{\alpha}(b-a)}\left[\left(\frac{(b-a)(x-a)^{2\alpha+1}}{2\alpha+1}+\frac{(b-x)^{2\alpha+2}-(x-a)^{2\alpha+2}}{2\alpha+2}\right)\left|f''(a)\right|\right.$$
$$\left.+\left(\frac{(b-a)(b-x)^{2\alpha+1}}{2\alpha+1}+\frac{(x-a)^{2\alpha+2}-(b-x)^{2\alpha+2}}{2\alpha+2}\right)\left|f''(b)\right|\right]. \tag{2.5}$$

Remark 2.4. If we take $\alpha=1$ in (2.5), we get

$$\left|\left(\frac{a+b}{2}-x\right)f'(x)+f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt\right|$$
$$\leq \frac{1}{2(b-a)^2}\left[\left((b-a)\frac{(x-a)^3}{3}+\frac{(b-x)^4-(x-a)^4}{4}\right)\left|f''(a)\right|\right.$$
$$\left.+\left(\frac{(x-a)^4-(b-x)^4}{4}+(b-a)\frac{(b-x)^3}{3}\right)\left|f''(b)\right|\right],$$

which is proved by Sarikaya and Yaldiz in [16].

Corollary 2.3. *Under the same assumptions of Theorem* 2.1 *with* $w(s)=1$ *and* $x=\frac{a+b}{2}$, *then we have*

$$\left|\alpha\left(-\frac{1}{2}\right)^{\alpha-1}\frac{(b-a)^{2\alpha-1}}{2^{2\alpha-2}}f(\frac{a+b}{2})+\frac{\alpha\Gamma(2\alpha)}{2}\left(-\frac{1}{2}\right)^{\alpha-2}\left[J^{2\alpha-1}_{\frac{a+b}{2}-}f(a)+J^{2\alpha-1}_{\frac{2+b}{2}+}f(b)\right]\right|$$
$$\leq \frac{(b-a)^{2\alpha+1}}{(2\alpha+1)2^{3\alpha+1}}\left[\left|f''(a)\right|+\left|f''(b)\right|\right]. \tag{2.6}$$

Remark 2.5. If we take $\alpha=1$ in (2.6), we obtain

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \leq \frac{(b-a)^2}{24}\left(\frac{\left|f''(a)\right|+\left|f''(b)\right|}{2}\right),$$

which is proved by Sarikaya and Yildirim in [17].

Corollary 2.4. *Under the same assumptions of Theorem* 2.1 *with* $\left|f''(a)\right|=\left|f''(b)\right|$ *in* (2.3), *then the following inequalities hold:*

$$|S(\alpha;w,f)| \leq \frac{1}{(2\alpha+1)2^{\alpha}}\left[\|w\|^{\alpha}_{[a,x],\infty}(x-a)^{2\alpha+1}+\|w\|^{\alpha}_{[x,b],\infty}(b-x)^{2\alpha+1}\right]\left|f''(a)\right|$$
$$\leq \frac{\|w\|^{\alpha}_{[a,b],\infty}}{(2\alpha+1)2^{\alpha}}\left[(x-a)^{2\alpha+1}+(b-x)^{2\alpha+1}\right]\left|f''(a)\right|.$$

Theorem 2.2. *Let* $f: I\subset\mathbb{R}\to\mathbb{R}$ *be twice differentiable function on* I°, $a,b\in I^{\circ}$ *with* $a<b$, f'' *is absolutely continuous on* $[a,b]$ *and let* $w:[a,b]\to\mathbb{R}$ *be nonnegative*

and continuous on $[a,b]$*. If* $|f''|^q$ *is convex on* $[a,b]$*,* $q>1$*, then for all* $x\in[a,b]$*, the following inequalities hold:*

$$\begin{aligned}
|S(\alpha;w,f)| &\le \frac{\|w\|_{[a,x],\infty}^{\alpha}(x-a)^{2\alpha+\frac{1}{p}}}{2^{\alpha}(b-a)^{\frac{1}{q}}(2\alpha p+1)^{\frac{1}{p}}} \\
&\quad\times\left[\frac{(b-a)^2-(b-x)^2}{2}\left|f''(a)\right|^q+\frac{(x-a)^2}{2}\left|f''(b)\right|^q\right]^{\frac{1}{q}} \\
&\quad+\frac{\|w\|_{[x,b],\infty}^{\alpha}(b-x)^{2\alpha+\frac{1}{p}}}{2^{\alpha}(b-a)^{\frac{1}{q}}(2\alpha p+1)^{\frac{1}{p}}} \\
&\quad\times\left[\frac{(b-x)^2}{2}\left|f''(a)\right|^q+\frac{(b-a)^2-(x-a)^2}{2}\left|f''(b)\right|^q\right]^{\frac{1}{q}} \\
&\le \frac{\|w\|_{[a,b],\infty}^{\alpha}}{2^{\alpha}(b-a)^{\frac{1}{q}}(2\alpha p+1)^{\frac{1}{p}}} \\
&\quad\times\left\{(x-a)^{2\alpha+\frac{1}{p}}\left[\frac{(b-a)^2-(b-x)^2}{2}\left|f''(a)\right|^q+\frac{(x-a)^2}{2}\left|f''(b)\right|^q\right]^{\frac{1}{q}}\right. \\
&\quad\left.+(b-x)^{2\alpha+\frac{1}{p}}\left[\frac{(b-x)^2}{2}\left|f''(a)\right|^q+\frac{(b-a)^2-(x-a)^2}{2}\left|f''(b)\right|^q\right]^{\frac{1}{q}}\right\},
\end{aligned}$$

where $\alpha\ge1$*,* $\frac{1}{p}+\frac{1}{q}=1$*, and* $\|w\|_{\infty}=\sup\limits_{t\in[a,b]}|w(t)|$*.*

Proof. We take absolute value of (2.1). Using Holder's inequality, we find that

$$\begin{aligned}
|S(\alpha;w,f)| &\le \int_a^x\left(\int_a^t(t-u)\,w(u)\,du\right)^{\alpha}\left|f''(t)\right|dt \\
&\quad+\int_x^b\left(\int_t^b(u-t)\,w(u)\,du\right)^{\alpha}\left|f''(t)\right|dt \\
&\le \frac{\|w\|_{[a,x],\infty}^{\alpha}}{2^{\alpha}}\int_a^x(t-a)^{2\alpha}\left|f''(t)\right|dt+\frac{\|w\|_{[x,b],\infty}^{\alpha}}{2^{\alpha}}\int_x^b(b-t)^{2\alpha}\left|f''(t)\right|dt \\
&\le \frac{\|w\|_{[a,x],\infty}^{\alpha}}{2^{\alpha}}\left(\int_a^x(t-a)^{2\alpha p}\,dt\right)^{\frac{1}{p}}\left(\int_a^x\left|f''(t)\right|^q dt\right)^{\frac{1}{q}}
\end{aligned}$$

$$+\frac{\|w\|_{[x,b],\infty}^{\alpha}}{2^{\alpha}}\left(\int_{x}^{b}(b-t)^{2\alpha p}\,dt\right)^{\frac{1}{p}}\left(\int_{x}^{b}|f''(t)|^{q}\,dt\right)^{\frac{1}{q}}.$$

Since $\left|f''(t)\right|^{q}$ is convex on $[a,b]=[a,x]\cup[x,b]$, we have

$$\left|f''\left(\frac{b-t}{b-a}a+\frac{t-a}{b-a}b\right)\right|^{q}\leq\frac{b-t}{b-a}\left|f''(a)\right|^{q}+\frac{t-a}{b-a}\left|f''(b)\right|^{q}. \tag{2.7}$$

From (2.7), it follows that

$$\begin{aligned}
|S(\alpha;w,f)| &\leq \frac{\|w\|_{[a,x],\infty}^{\alpha}}{2^{\alpha}}\frac{(x-a)^{2\alpha+\frac{1}{p}}}{(2\alpha p+1)^{\frac{1}{p}}}\left(\int_{a}^{x}\left[\frac{b-t}{b-a}\left|f''(a)\right|^{q}+\frac{t-a}{b-a}\left|f''(b)\right|^{q}\right]dt\right)^{\frac{1}{q}} \\
&\quad+\frac{\|w\|_{[x,b],\infty}^{\alpha}}{2^{\alpha}}\frac{(b-x)^{2\alpha+\frac{1}{p}}}{(2\alpha p+1)^{\frac{1}{p}}}\left(\int_{x}^{b}\left[\frac{b-t}{b-a}\left|f''(a)\right|^{q}+\frac{t-a}{b-a}\left|f''(b)\right|^{q}\right]dt\right)^{\frac{1}{q}} \\
&= \frac{\|w\|_{[a,x],\infty}^{\alpha}}{2^{\alpha}(b-a)^{\frac{1}{q}}}\frac{(x-a)^{2\alpha+\frac{1}{p}}}{(2\alpha p+1)^{\frac{1}{p}}} \\
&\quad\times\left[\frac{(b-a)^{2}-(b-x)^{2}}{2}\left|f''(a)\right|^{q}+\frac{(x-a)^{2}}{2}\left|f''(b)\right|^{q}\right]^{\frac{1}{q}} \\
&\quad+\frac{\|w\|_{[x,b],\infty}^{\alpha}}{2^{\alpha}(b-a)^{\frac{1}{q}}}\frac{(b-x)^{2\alpha+\frac{1}{p}}}{(2\alpha p+1)^{\frac{1}{p}}} \\
&\quad\times\left[\frac{(b-x)^{2}}{2}\left|f''(a)\right|^{q}+\frac{(b-a)^{2}-(x-a)^{2}}{2}\left|f''(b)\right|^{q}\right]^{\frac{1}{q}}
\end{aligned}$$

and because of $\|w\|_{[a,x],\infty}\leq\|w\|_{[a,b],\infty}$ and $\|w\|_{[x,b],\infty}\leq\|w\|_{[a,b],\infty}$. Hence, the proof is complete. □

Remark 2.6. Under the same assumptions of Theorem 2.2 with $\alpha=1$, then the following inequality holds:

$$\begin{aligned}
|S(1;w,f)| &\leq \frac{\|w\|_{[a,b],\infty}}{2(b-a)^{\frac{1}{q}}(2p+1)^{\frac{1}{p}}} \\
&\quad\times\left\{(x-a)^{2+\frac{1}{p}}\left[\frac{(b-a)^{2}-(b-x)^{2}}{2}\left|f''(a)\right|^{q}+\frac{(x-a)^{2}}{2}\left|f''(b)\right|^{q}\right]^{\frac{1}{q}}\right. \\
&\quad\left.+(b-x)^{2+\frac{1}{p}}\left[\frac{(b-x)^{2}}{2}\left|f''(a)\right|^{q}+\frac{(b-a)^{2}-(x-a)^{2}}{2}\left|f''(b)\right|^{q}\right]^{\frac{1}{q}}\right\},
\end{aligned}$$

which is given by Sarikaya and Yaldiz in [16].

Corollary 2.5. *Under the same assumptions of Theorem 2.2 with $w(s)=1$, then the following inequality holds:*

$$
\begin{aligned}
|S(\alpha;1,f)| &\leq \frac{1}{2^{\alpha}(b-a)^{\frac{1}{q}}(2\alpha p+1)^{\frac{1}{p}}} \\
&\times\left\{(x-a)^{2\alpha+\frac{1}{p}}\left[\frac{(b-a)^{2}-(b-x)^{2}}{2}\left|f''(a)\right|^{q}+\frac{(x-a)^{2}}{2}\left|f''(b)\right|^{q}\right]^{\frac{1}{q}}\right. \\
&\left.+(b-x)^{2\alpha+\frac{1}{p}}\left[\frac{(b-x)^{2}}{2}\left|f''(a)\right|^{q}+\frac{(b-a)^{2}-(x-a)^{2}}{2}\left|f''(b)\right|^{q}\right]^{\frac{1}{q}}\right\}.
\end{aligned}
\tag{2.8}
$$

Remark 2.7. If we take $\alpha=1$ in (2.8), we get

$$
\begin{aligned}
&\left|\left(\frac{a+b}{2}-x\right)f'(x)+f(x)-\frac{1}{b-a}\int_{a}^{b}f(t)\,dt\right| \\
&\leq \frac{1}{2(b-a)^{1+\frac{1}{q}}(2p+1)^{\frac{1}{p}}} \\
&\times\left\{(x-a)^{2+\frac{1}{p}}\left[\frac{(b-a)^{2}-(b-x)^{2}}{2}\left|f''(a)\right|^{q}+\frac{(x-a)^{2}}{2}\left|f''(b)\right|^{q}\right]^{\frac{1}{q}}\right. \\
&\left.+(b-x)^{2+\frac{1}{p}}\left[\frac{(b-x)^{2}}{2}\left|f''(a)\right|^{q}+\frac{(b-a)^{2}-(x-a)^{2}}{2}\left|f''(b)\right|^{q}\right]^{\frac{1}{q}}\right\},
\end{aligned}
$$

which is given by Sarikaya and Yaldiz in [16].

Corollary 2.6. *Under the same assumptions of Theorem 2.2 with $w(s)=1$ and $x=\frac{a+b}{2}$, then, we have*

$$
\begin{aligned}
&\left|\alpha\left(-\frac{1}{2}\right)^{\alpha-1}\frac{(b-a)^{2\alpha-1}}{2^{2\alpha-2}}f\left(\frac{a+b}{2}\right)\right. \\
&\left.+\frac{\alpha\Gamma(2\alpha)}{2}\left(-\frac{1}{2}\right)^{\alpha-2}\left[J^{2\alpha-1}_{\left(\frac{a+b}{2}\right)-}f(a)+J^{2\alpha-1}_{\left(\frac{a+b}{2}\right)+}f(b)\right]\right| \\
&\leq \frac{(b-a)^{2\alpha+1}}{(2\alpha p+1)^{\frac{1}{p}}2^{3\alpha+1}} \\
&\times\left\{\left[\frac{3\left|f''(a)\right|^{q}+\left|f''(b)\right|^{q}}{4}\right]^{\frac{1}{q}}+\left[\frac{\left|f''(a)\right|^{q}+3\left|f''(b)\right|^{q}}{4}\right]^{\frac{1}{q}}\right\}.
\end{aligned}
\tag{2.9}
$$

Remark 2.8. If we take $\alpha = 1$ in (2.9), we obtain

$$\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(t)\,dt \right|$$
$$\leq \frac{(b-a)^2}{16(2p+1)^{\frac{1}{p}}}\left\{\left[\frac{3\left|f''(a)\right|^q + \left|f''(b)\right|^q}{4}\right]^{\frac{1}{q}} + \left[\frac{\left|f''(a)\right|^q + 3\left|f''(b)\right|^q}{4}\right]^{\frac{1}{q}}\right\},$$

which is proved by Sarikaya and Yildirim in [17].

Theorem 2.3. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be twice differentiable function on I°, $a, b \in I^\circ$ with $a < b$, f'' is absolutely continuous on $[a, b]$ and let $w : [a, b] \to \mathbb{R}$ be nonnegative and continuous on $[a, b]$. If $|f''|^q$ is convex on $[a, b]$, $q > 1$, then for all $x \in [a, b]$, the following inequalities hold:*

$$|S(\alpha; w, f)| \leq \frac{(b-a)^{\frac{1}{q}}}{2^\alpha (2\alpha p+1)^{\frac{1}{p}}}\left[\|w\|_{[a,x],\infty}^{\alpha p}(x-a)^{2\alpha p+1} + \|w\|_{[x,b],\infty}^{\alpha p}(b-x)^{2\alpha p+1}\right]^{\frac{1}{p}}$$
$$\times\left[\frac{\left|f''(a)\right|^q + \left|f''(b)\right|^q}{2}\right]^{\frac{1}{q}}$$
$$\leq \frac{\|w\|_{[a,b],\infty}^{\alpha}(b-a)^{\frac{1}{q}}}{2^\alpha (2\alpha p+1)^{\frac{1}{p}}}\left[(x-a)^{2\alpha p+1} + (b-x)^{2\alpha p+1}\right]^{\frac{1}{p}} \tag{2.10}$$
$$\times\left[\frac{\left|f''(a)\right|^q + \left|f''(b)\right|^q}{2}\right]^{\frac{1}{q}},$$

where $\alpha \geq 1$, $\frac{1}{p} + \frac{1}{q} = 1$, and $\|w\|_\infty = \sup_{t\in[a,b]} |w(t)|$.

Proof. We take absolute value of (2.1). Using Holder's inequality, we find that

$$|S(\alpha; w, f)| \leq \int_a^b |P(x,t)|\,|f''(t)|\,dt$$
$$\leq \left[\int_a^x\left(\int_a^t (t-u)\,w(u)\,du\right)^{\alpha p} dt + \int_x^b\left(\int_t^b (u-t)\,w(u)\,du\right)^{\alpha p} dt\right]^{\frac{1}{p}}$$
$$\times\left(\int_a^b |f''(t)|^q\,dt\right)^{\frac{1}{q}}$$

$$\leq \left[\frac{\|w\|_{[a,x],\infty}^{\alpha p}}{2^{\alpha p}} \frac{(x-a)^{2\alpha p+1}}{2\alpha p+1} + \frac{\|w\|_{[x,b],\infty}^{\alpha p}}{2^{\alpha p}} \frac{(b-x)^{2\alpha p+1}}{2\alpha p+1} \right]^{\frac{1}{p}}$$
$$\times \left(\int_a^b |f''(t)|^q \, dt \right)^{\frac{1}{q}}.$$

From (2.7), it follows that

$$|S(\alpha; w, f)| \leq \left[\frac{\|w\|_{[a,x],\infty}^{\alpha p}}{2^{\alpha p}} \frac{(x-a)^{2\alpha p+1}}{2\alpha p+1} + \frac{\|w\|_{[x,b],\infty}^{\alpha p}}{2^{\alpha p}} \frac{(b-x)^{2\alpha p+1}}{2\alpha p+1} \right]^{\frac{1}{p}}$$
$$\times \left[\int_a^b \left[\frac{b-t}{b-a} \left| f''(a) \right|^q + \frac{t-a}{b-a} \left| f''(b) \right|^q \right] dt \right]^{\frac{1}{q}}$$
$$= \frac{(b-a)^{\frac{1}{q}}}{2^{\alpha} (2\alpha p+1)^{\frac{1}{p}}} \left[\|w\|_{[a,x],\infty}^{\alpha p} (x-a)^{2\alpha p+1} + \|w\|_{[x,b],\infty}^{\alpha p} (b-x)^{2\alpha p+1} \right]^{\frac{1}{p}}$$
$$\times \left[\frac{\left| f''(a) \right|^q + \left| f''(b) \right|^q}{2} \right]^{\frac{1}{q}}$$

and because of $\|w\|_{[a,x],\infty} \leq \|w\|_{[a,b],\infty}$ and $\|w\|_{[x,b],\infty} \leq \|w\|_{[a,b],\infty}$. Hence, the proof is complete. □

Remark 2.9. Under the same assumptions of Theorem 2.3 with $\alpha = 1$, then the following inequality holds:

$$|S(1; w, f)| \leq \frac{\|w\|_{[a,b],\infty} (b-a)^{\frac{1}{q}}}{2 (2p+1)^{\frac{1}{p}}} \left[(x-a)^{2p+1} + (b-x)^{2p+1} \right]^{\frac{1}{p}}$$
$$\times \left[\frac{\left| f''(a) \right|^q + \left| f''(b) \right|^q}{2} \right]^{\frac{1}{q}}$$

which is proved by Sarikaya and Yaldiz in [16].

Corollary 2.7. *Under the same assumptions of Theorem* 2.3 *with* $w(s) = 1$, *then the following inequality holds:*

$$|S(\alpha; 1, f)| \leq \frac{(b-a)^{\frac{1}{q}}}{2^{\alpha} (2\alpha p+1)^{\frac{1}{p}}} \left[(x-a)^{2\alpha p+1} + (b-x)^{2\alpha p+1} \right]^{\frac{1}{p}}$$
$$\times \left[\frac{\left| f''(a) \right|^q + \left| f''(b) \right|^q}{2} \right]^{\frac{1}{q}}. \tag{2.11}$$

Remark 2.10. If we take $\alpha = 1$ in (2.11), we get

$$\left|\left(\frac{a+b}{2}-x\right)f'(x)+f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt\right|$$
$$\leq \frac{\left[(x-a)^{2p+1}+(b-x)^{2p+1}\right]^{\frac{1}{p}}}{2(b-a)^{\frac{1}{p}}(2p+1)^{\frac{1}{p}}}\left[\frac{\left|f''(a)\right|^q+\left|f''(b)\right|^q}{2}\right]^{\frac{1}{q}}.$$

Corollary 2.8. *Under the same assumptions of Theorem* 2.3 *with* $w(s) = 1$ *and* $x = \frac{a+b}{2}$, *then we have*

$$\left|\alpha\left(-\frac{1}{2}\right)^{\alpha-1}\frac{(b-a)^{2\alpha-1}}{2^{2\alpha-2}}f\left(\frac{a+b}{2}\right)\right.$$
$$\left.+\frac{\alpha\Gamma(2\alpha)}{2}\left(-\frac{1}{2}\right)^{\alpha-2}\left[J^{2\alpha-1}_{\left(\frac{a+b}{2}\right)-}f(a)+J^{2\alpha-1}_{\left(\frac{a+b}{2}\right)+}f(b)\right]\right| \tag{2.12}$$
$$\leq \frac{(b-a)^{2\alpha+1}}{(2\alpha p+1)^{\frac{1}{p}}2^{3\alpha}}\left[\frac{\left|f''(a)\right|^q+\left|f''(b)\right|^q}{2}\right].$$

Remark 2.11. If we take $\alpha = 1$ in (2.12), we obtain

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \leq \frac{(b-a)^2}{8(2p+1)^{\frac{1}{p}}}\left[\frac{\left|f''(a)\right|^q+\left|f''(b)\right|^q}{2}\right]^{\frac{1}{q}},$$

which is proved by Sarikaya and Yildirim in [17].

Corollary 2.9. *Under the same assumptions of Theorem* 2.3 *with* $\left|f''(a)\right|^q = \left|f''(b)\right|^q$ *in* (2.10), *then the following inequality holds:*

$$|S(\alpha;w,f)| \leq \frac{(b-a)^{\frac{1}{q}}\left|f''(a)\right|}{2^\alpha(2\alpha p+1)^{\frac{1}{p}}}\left[\|w\|^{\alpha p}_{[a,x],\infty}(x-a)^{2\alpha p+1}+\|w\|^{\alpha p}_{[x,b],\infty}(b-x)^{2\alpha p+1}\right]^{\frac{1}{p}}$$
$$\leq \frac{\|w\|^{\alpha}_{[a,b],\infty}(b-a)^{\frac{1}{q}}}{2^\alpha(2\alpha p+1)^{\frac{1}{p}}}\left|f''(a)\right|\left[(x-a)^{2\alpha p+1}+(b-x)^{2\alpha p+1}\right]^{\frac{1}{p}}.$$

Theorem 2.4. *Let* $f : I \subset \mathbb{R} \to \mathbb{R}$ *be twice differentiable function on* I°, $a, b \in I^\circ$ *with* $a < b$, f'' *is absolutely continuous on* $[a, b]$ *and let* $w : [a, b] \to \mathbb{R}$ *be nonnegative and continuous on* $[a, b]$. *If* $|f''|^q$ *is convex on* $[a, b]$, $q \geq 1$, *then for all* $x \in [a, b]$, *the following inequalities hold:*

$$|S(\alpha;w,f)| \leq \frac{\|w\|^{\alpha}_{[a,x],\infty}(x-a)^{2\alpha+1}}{2^{2\alpha}(2\alpha+1)^{\frac{1}{p}}(b-a)^{\frac{1}{q}}}$$
$$\times\left[\frac{(b-a)+(2\alpha+1)(b-x)}{(2\alpha+1)(2\alpha+2)}\left|f''(a)\right|^q+\frac{(x-a)}{2\alpha+2}\left|f''(b)\right|^q\right]^{\frac{1}{q}}$$

$$+\frac{\|w\|_{[x,b],\infty}^{\alpha}(b-x)^{2\alpha+1}}{2^{2\alpha}(2\alpha+1)^{\frac{1}{p}}(b-a)^{\frac{1}{q}}}$$

$$\times\left[\frac{(b-a)+(2\alpha+1)(x-a)}{(2\alpha+1)(2\alpha+2)}\left|f''(b)\right|^{q}+\frac{(b-x)}{2\alpha+2}\left|f''(a)\right|^{q}\right]^{\frac{1}{q}}$$

$$\leq\frac{\|w\|_{[a,b],\infty}^{\alpha}}{2^{2\alpha}(2\alpha+1)^{\frac{1}{p}}(b-a)^{\frac{1}{q}}} \tag{2.13}$$

$$\times\left\{(x-a)^{2\alpha+1}\left[\frac{(b-a)+(2\alpha+1)(b-x)}{(2\alpha+1)(2\alpha+2)}\left|f''(a)\right|^{q}\right.\right.$$

$$\left.+\frac{(x-a)}{2\alpha+2}\left|f''(b)\right|^{q}\right]^{\frac{1}{q}}+(b-x)^{2\alpha+1}$$

$$\left.\times\left[\frac{(b-a)+(2\alpha+1)(x-a)}{(2\alpha+1)(2\alpha+2)}\left|f''(b)\right|^{q}+\frac{(b-x)}{2\alpha+2}\left|f''(a)\right|^{q}\right]^{\frac{1}{q}}\right\},$$

where $\alpha\geq 1$, $\frac{1}{p}+\frac{1}{q}=1$, *and* $\|w\|_{\infty}=\sup_{t\in[a,b]}|w(t)|$.

Proof. We take absolute value of (2.1). Because of $\frac{1}{p}+\frac{1}{q}=1$, $\alpha\left(\frac{1}{p}+\frac{1}{q}\right)$ can be written instead of α. Using Holder's inequality and the convexity of $\left|f'\right|^{q}$, we find that

$$|S(\alpha;w,f)|\leq\left(\int_{a}^{x}\left(\int_{a}^{t}(t-u)\,w(u)\,du\right)^{\alpha}dt\right)^{\frac{1}{p}}$$

$$\times\left(\int_{a}^{x}\left(\int_{a}^{t}(t-u)\,w(u)\,du\right)^{\alpha}\left|f''(t)\right|^{q}dt\right)^{\frac{1}{q}}$$

$$+\left(\int_{x}^{b}\left(\int_{t}^{b}(u-t)\,w(u)\,du\right)^{\alpha}dt\right)^{\frac{1}{p}}$$

$$\times\left(\int_{x}^{b}\left(\int_{t}^{b}(u-t)\,w(u)\,du\right)^{\alpha}\left|f''(t)\right|^{q}dt\right)^{\frac{1}{q}}.$$

From (2.7), it follows that

$$|S(\alpha;w,f)|\leq\frac{\|w\|_{[a,x],\infty}^{\alpha}}{2^{2\alpha}}\left(\frac{(x-a)^{2\alpha+1}}{2\alpha+1}\right)^{\frac{1}{p}}$$

$$\times\left(\int_a^x (t-a)^{2\alpha}\left[\frac{b-t}{b-a}\left|f''(a)\right|^q+\frac{t-a}{b-a}\left|f''(b)\right|^q\right]dt\right)^{\frac{1}{q}}$$

$$+\frac{\|w\|_{[x,b],\infty}^{\alpha}}{2^{2\alpha}}\left(\frac{(b-x)^{2\alpha+1}}{2\alpha+1}\right)^{\frac{1}{p}}$$

$$\times\left(\int_x^b (b-t)^{2\alpha}\left[\frac{b-t}{b-a}\left|f''(a)\right|^q+\frac{t-a}{b-a}\left|f''(b)\right|^q\right]dt\right)^{\frac{1}{q}}$$

$$=\frac{\|w\|_{[a,x],\infty}^{\alpha}}{2^{2\alpha}(b-a)^{\frac{1}{q}}}\left(\frac{(x-a)^{2\alpha+1}}{2\alpha+1}\right)^{\frac{1}{p}}$$

$$\times\left[\left((b-a)\frac{(x-a)^{2\alpha+1}}{2\alpha+1}-\frac{(x-a)^{2\alpha+2}}{2\alpha+2}\right)\left|f''(a)\right|^q\right.$$

$$\left.+\frac{(x-a)^{2\alpha+2}}{2\alpha+2}\left|f''(b)\right|^q\right]^{\frac{1}{q}}$$

$$+\frac{\|w\|_{[x,b],\infty}^{\alpha}}{2^{2\alpha}(b-a)^{\frac{1}{q}}}\left(\frac{(b-x)^{2\alpha+1}}{2\alpha+1}\right)^{\frac{1}{p}}$$

$$\times\left[\left((b-a)\frac{(b-x)^{2\alpha+1}}{2\alpha+1}-\frac{(b-x)^{2\alpha+2}}{2\alpha+2}\right)\left|f''(b)\right|^q\right.$$

$$\left.+\frac{(b-x)^{2\alpha+2}}{2\alpha+2}\left|f''(a)\right|^q\right]^{\frac{1}{q}}$$

$$=\frac{\|w\|_{[a,x],\infty}^{\alpha}(x-a)^{2\alpha+1}}{2^{2\alpha}(2\alpha+1)^{\frac{1}{p}}(b-a)^{\frac{1}{q}}}$$

$$\times\left[\left(\frac{(b-a)}{2\alpha+1}-\frac{(x-a)}{2\alpha+2}\right)\left|f''(a)\right|^q+\frac{(x-a)}{2\alpha+2}\left|f''(b)\right|^q\right]^{\frac{1}{q}}$$

$$+\frac{\|w\|_{[x,b],\infty}^{\alpha}(b-x)^{2\alpha+1}}{2^{2\alpha}(2\alpha+1)^{\frac{1}{p}}(b-a)^{\frac{1}{q}}}$$

$$\times\left[\left(\frac{(b-a)}{2\alpha+1}-\frac{(b-x)}{2\alpha+2}\right)\left|f''(b)\right|^q+\frac{(b-x)}{2\alpha+2}\left|f''(a)\right|^q\right]^{\frac{1}{q}}$$

and because of $\|w\|_{[a,x],\infty}\leq\|w\|_{[a,b],\infty}$ and $\|w\|_{[x,b],\infty}\leq\|w\|_{[a,b],\infty}$, therefore, the proof is complete. □

Corollary 2.10. *Under the same assumptions of Theorem 2.4 with $\alpha = 1$, then the following inequality holds:*

$$|S(1; w, f)| \le \frac{\|w\|_{[a,b],\infty}}{4 \cdot 3^{\frac{1}{p}} (b-a)^{\frac{1}{q}}} \left\{ (x-a)^3 \left[\frac{4b-a-3x}{12} \left| f''(a) \right|^q + \frac{(x-a)}{4} \left| f''(b) \right|^q \right]^{\frac{1}{q}} \right.$$
$$\left. + (b-x)^3 \left[\frac{3x-4a+b}{12} \left| f''(b) \right|^q + \frac{(b-x)}{4} \left| f''(a) \right|^q \right]^{\frac{1}{q}} \right\}.$$

Corollary 2.11. *Under the same assumptions of Theorem 2.4 with $w(s) = 1$, then the following inequality holds:*

$$\begin{aligned} |S(\alpha; 1, f)| \le & \frac{1}{2^{2\alpha} (2\alpha+1)^{\frac{1}{p}} (b-a)^{\frac{1}{q}}} \\ & \times \left\{ (x-a)^{2\alpha+1} \left[\frac{(b-a) + (2\alpha+1)(b-x)}{(2\alpha+1)(2\alpha+2)} \left| f''(a) \right|^q \right. \right. \\ & \left. + \frac{(x-a)}{2\alpha+2} \left| f''(b) \right|^q \right]^{\frac{1}{q}} \\ & + (b-x)^{2\alpha+1} \left[\frac{(b-a) + (2\alpha+1)(x-a)}{(2\alpha+1)(2\alpha+2)} \left| f''(b) \right|^q \right. \\ & \left. \left. + \frac{(b-x)}{2\alpha+2} \left| f''(a) \right|^q \right]^{\frac{1}{q}} \right\}. \end{aligned} \tag{2.14}$$

Corollary 2.12. *If we take $\alpha = 1$ in (2.14), we get*

$$\left| \left(\frac{a+b}{2} - x \right) f'(x) + f(x) - \frac{1}{b-a} \int_a^b f(t)\, dt \right|$$
$$\le \frac{1}{4 \cdot 3^{\frac{1}{p}} (b-a)^{\frac{1}{q}+1}} \left\{ (x-a)^3 \left[\frac{4b-a-3x}{12} \left| f''(a) \right|^q + \frac{(x-a)}{4} \left| f''(b) \right|^q \right]^{\frac{1}{q}} \right.$$
$$\left. + (b-x)^3 \left[\frac{3x-4a+b}{12} \left| f''(b) \right|^q + \frac{(b-x)}{4} \left| f''(a) \right|^q \right]^{\frac{1}{q}} \right\}.$$

Corollary 2.13. *Under the same assumptions of Theorem 2.4 with $w(s) = 1$ and $x = \frac{a+b}{2}$, we obtain*

$$\left| \alpha \left(-\frac{1}{2} \right)^{\alpha-1} \frac{(b-a)^{2\alpha-1}}{2^{2\alpha-2}} f\left(\frac{a+b}{2} \right) \right.$$
$$\left. + \frac{\alpha \Gamma(2\alpha)}{2} \left(-\frac{1}{2} \right)^{\alpha-2} \left[J^{2\alpha-1}_{\left(\frac{a+b}{2}\right)-} f(a) + J^{2\alpha-1}_{\left(\frac{a+b}{2}\right)+} f(b) \right] \right|$$

$$\leq \frac{(b-a)^{2\alpha+1}}{(2\alpha+1)^{\frac{1}{p}}2^{4\alpha+1+\frac{1}{q}}}\left\{\left[\frac{(2\alpha+3)}{(2\alpha+1)(2\alpha+2)}\left|f''(a)\right|^q+\frac{1}{2\alpha+2}\left|f''(b)\right|^q\right]^{\frac{1}{q}}\right.$$
$$\left.+\left[\frac{(2\alpha+3)}{(2\alpha+1)(2\alpha+2)}\left|f''(b)\right|^q+\frac{1}{2\alpha+2}\left|f''(a)\right|^q\right]^{\frac{1}{q}}\right\}.$$

Corollary 2.14. *Under the same assumptions of Corollary* 2.13 *with* $\alpha = 1$, *then we have*

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(t)\,dt\right|$$
$$\leq \frac{(b-a)^2}{2^{\frac{1}{q}+5}\times 3^{\frac{1}{p}}}\left\{\left[\frac{5\left|f''(a)\right|^q+3\left|f''(b)\right|^q}{12}\right]^{\frac{1}{q}}+\left[\frac{3\left|f''(a)\right|^q+5\left|f''(b)\right|^q}{12}\right]^{\frac{1}{q}}\right\}.$$

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