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Research Article

Application of Sinc-Galerkin Method for Solving Space-Fractional Boundary Value Problems

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We employ the sinc-Galerkin method to obtain approximate solutions of space-fractional order partial differential equations (FPDEs) with variable coefficients. The fractional derivatives are used in the Caputo sense. The method is applied to three different problems and the obtained solutions are compared with the exact solutions of the problems. These comparisons demonstrate that the sinc-Galerkin method is a very efficient tool in solving space-fractional partial differential equations.

1. Introduction

Fractional calculus, which might be considered as an extension of classical calculus, is as old as the classical calculus and fractional differential equations have been often used to describe many scientific phenomena in biomedical engineering, image processing, earthquake engineering, signal processing, physics, statistics, electrochemistry, and control theory.

Because finding the exact or analytical solutions of fractional order differential equations is not an easy task, several different numerical solution techniques have been developed for the approximate solutions of these types of equations. Some of the well-known numerical techniques might be listed as generalized differential transform method [1, 2], finite difference method [3], Adomian decomposition method [4, 5], homotopy perturbation method [6–8], Haar wavelet method [9, 10], differential transform method [11–13], and Adams-Bashforth-Moulton scheme [14]. A detailed and informative study on fractional calculus can be found in [15]. Furthermore a relatively new analytical method was presented in [16] to solve time "The Time-Fractional Coupled-Korteweg-de-Vries Equations" via homotopy decomposition method by the same authors. The sinc methods were introduced in [17] and expanded in [18] by Frank Stenger. Sinc functions were firstly analyzed in [19, 20]. In [21], the sinc-Galerkin method is used to approximate solutions of nonlinear differential equations with homogeneous and nonhomogeneous boundary conditions. In [22], the sinc-Galerkin method is applied to nonlinear fourth-order differential equations with nonhomogeneous and homogeneous boundary conditions. In the paper at [23], the numerical solutions of Troesch's problem are obtained by the sinc-Galerkin method and the results are compared with methods of Laplace, homotopy perturbation, splines, and perturbation. Reference [24] which contains short abstract version of current paper has been presented in an International Conference and Workshop on Mathematical Analysis 2014, Malaysia. In [25], the authors present a comparison between sinc-Galerkin method and sinc-collocation method to obtain approximate solutions of linear and nonlinear boundary value problems. Similarly, the wavelet-Galerkin method and the sinc-Galerkin method for solving nonhomogeneous heat equations are compared in [26]. The paper [27] offers an application of the sinc-Galerkin method for solving second-order singular Dirichlet-type boundary value problems. In [28], the sinc-Galerkin method is used to approximate solutions of fractional order ordinary differential equations in Caputo sense.

In this paper we propose a new solution technique for approximate solution of space-fractional order partial differential equations (FPDEs) with variable coefficients and boundary conditions by using the sinc-Galerkin method that has almost not been employed for the space-fractional order partial differential equations in the form

$$u_{tt} = a(x)u_{xx} + b(x)\, {}_0^C D_x^{\beta} u + c(x)\, u + f(x,t), \quad 0 < \beta < 1 \tag{1}$$

with boundary conditions

$$\begin{aligned} u(0,t) &= u(1,t) = 0, \\ u(x,0) &= u(x,1) = 0, \end{aligned} \tag{2}$$

where ${}_0^C D_x$ is Caputo fractional derivative operator.

The paper is organized as follows. Section 2 presents basic theorems of fractional calculus and sinc-Galerkin method. In Section 3, we use the sinc-Galerkin method to obtain an approximate solution of a general space-fractional partial differential equation. In Section 4, we present three examples in order to illustrate the effectiveness and accuracy of the present method. The obtained results are compared with the exact results.

2. Preliminaries

2.1. Fractional Calculus. In this section, we present the definitions of the fractional Riemann-Liouville derivative and the Caputo of fractional derivatives. By using these definitions, we give the definition of the integration by parts of fractional order.

Definition 1 (see [29]). Let $f : [a,b] \times [c,d] \to \mathbb{R}$ be a function; α is a positive real number, and n is the integer. α, n satisfy the inequality $n-1 \leq \alpha < n$ and Γ the Euler gamma function. Then,

(i) the left and right Riemann-Liouville fractional derivatives of order α with respect to x of $f(x,t)$ function are given as

$${}_a D_x^{\alpha} f(x,t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_a^x (x-s)^{n-\alpha-1} f(s,t)\, ds, \tag{3}$$

$${}_x D_b^{\alpha} f(x,t) = \frac{(-1)^n}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_x^b (s-x)^{n-\alpha-1} f(s,t)\, ds, \tag{4}$$

respectively;

(ii) the left and right Caputo fractional derivatives of order α with respect to x of $f(x,t)$ function are given as

$${}_a^C D_x^{\alpha} f(x,t) = \frac{1}{\Gamma(n-\alpha)} \int_a^x (x-s)^{n-\alpha-1} \frac{\partial^n f(s,t)}{\partial s^n} ds, \tag{5}$$

$${}_x^C D_b^{\alpha} f(x,t) = \frac{1}{\Gamma(n-\alpha)} \int_x^b (-1)^n (s-x)^{n-\alpha-1} \frac{\partial^n f(s,t)}{\partial s^n} ds, \tag{6}$$

respectively.

Now, we can write the definition of integration by parts of fractional order by using the relations given in (3)–(6).

Definition 2. If $0 < \alpha < 1$ and f is a function such that $f(a,t) = f(b,t) = 0$, one can write

$$\begin{aligned} \int_a^b g(x,t)\, {}_a^C D_x^{\alpha} f(x,t)\, dx &= \int_a^b f(x,t)\, {}_x D_b^{\alpha} g(x,t)\, dx, \\ \int_a^b g(x,t)\, {}_x^C D_b^{\alpha} f(x,t)\, dx &= \int_a^b f(x,t)\, {}_a D_x^{\alpha} g(x,t)\, dx. \end{aligned} \tag{7}$$

2.2. Properties of Sinc Basis Functions and Quadrature Interpolations. In this section, we recall notations and definitions of the sinc function state some known results and derive some useful formulas to be used in the next sections of the present paper.

2.2.1. The Sinc Basis Functions

Definition 3 (see [30]). The function which defined all $z \in \mathbb{C}$ by

$$\operatorname{sinc}(z) = \begin{cases} \dfrac{\sin(\pi z)}{\pi z}, & z \neq 0, \\ 1, & z = 0 \end{cases} \tag{8}$$

is called the *sinc function*.

Definition 4 (see [30]). Let f be a function defined on $\mathbb{R}$ and let $h > 0$. Define the series

$$C(f,h)(x) = \sum_{k=-\infty}^{\infty} f(kh) \operatorname{sinc}\left(\frac{x-kh}{h}\right), \tag{9}$$

where from (8) we have

$$\begin{aligned} S(k,h)(x) &= \operatorname{sinc}\left(\frac{x-kh}{h}\right) \\ &= \begin{cases} \dfrac{\sin(\pi((x-kh)/h))}{\pi((x-kh)/h)} & x \neq kh, \\ 1, & x = kh. \end{cases} \end{aligned} \tag{10}$$

If the series in (9) converges, it is called the *Whittaker cardinal function* of f. They are based on the infinite strip D_s in the complex plane

$$D_s \equiv \left\{ w = u + iv : |v| < d \leq \frac{\pi}{2} \right\}. \tag{11}$$

Generally, approximations can be constructed for infinite, semi-infinite, and finite intervals. Define the function

$$w = \phi(z) = \ln\left(\frac{z}{1-z}\right) \tag{12}$$

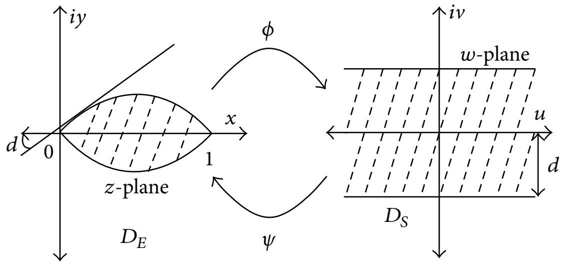

Figure 1: The domains D_E and D_S.

which is a conformal mapping from D_E, the eye-shaped domain in the z-plane, onto the infinite strip D_S, where

$$D_E = z = \left\{ x + iy : \left| \arg \left(\frac{z}{1-z} \right) \right| < d \le \frac{\pi}{2} \right\}. \tag{13}$$

This is shown in Figure 1. For the sinc-Galerkin method, the bases functions are derived from the composite translated sinc functions

$$S_k(z) = S(k,h)(z) \, o \phi(z) = \operatorname{sinc}\left(\frac{\phi(z) - kh}{h} \right), \tag{14}$$

where $z \in D_E$. The function $z = \phi^{-1}(w) = e^w/(1+e^w)$ is an inverse mapping of $w = \phi(z)$. We may define the range of ϕ^{-1} on the real line as

$$\Gamma = \left\{ \phi^{-1}(u) \in D_E : -\infty < u < \infty \right\} \tag{15}$$

evenly spaced nodes $\{kh\}_{k=-\infty}^{\infty}$ on the real line. The image which corresponds to these nodes is denoted by

$$x_k = \phi^{-1}(kh) = \frac{e^{kh}}{1+e^{kh}}. \tag{16}$$

2.2.2. Sinc Function Interpolation and Quadrature

Definition 5 (see [21]). Let D_E be a simply connected domain in the complex plane C and let ∂D_E denote the boundary of D_E. Let a, b be points on ∂D_E and let ϕ be a conformal map D_E onto D_S such that $\phi(a) = -\infty$ and $\phi(b) = \infty$. If the inverse map of ϕ is denoted by φ, define

$$\Gamma = \left\{ \phi^{-1}(u) \in D_E : -\infty < u < \infty \right\}, \tag{17}$$

and $z_k = \varphi(kh)$, $k = \pm 1, \pm 2, \ldots$.

Definition 6 (see [21]). Let $B(D_E)$ be the class of functions F that are analytic in D_E and satisfy

$$\int_{\psi(L+u)} |F(z)| \, dz \longrightarrow 0, \quad \text{as } u = \mp\infty, \tag{18}$$

in which

$$L = \left\{ iy : |y| < d \le \frac{\pi}{2} \right\}, \tag{19}$$

and those on the boundary of D_E satisfy

$$T(F) = \int_{\partial D_E} |F(z) \, dz| < \infty. \tag{20}$$

Theorem 7 (see [21]). *Let Γ be $(0,1)$, $F \in B(D_E)$; then, for $h > 0$ sufficiently small,*

$$\begin{aligned} \int_\Gamma F(z) \, dz - h \sum_{j=-\infty}^{\infty} \frac{F(z_j)}{\phi'(z_j)} \\ = \frac{i}{2} \int_{\partial D} \frac{F(z) \, k(\phi, h)(z)}{\sin(\pi \phi(z)/h)} dz \equiv I_F, \end{aligned} \tag{21}$$

where

$$\left| k(\phi, h) \right|_{z \in \partial D} = \left| e^{[(i\pi\phi(z)/h)\operatorname{sgn}(Im\phi(z))]} \right|_{z \in \partial D} = e^{-\pi d/h}. \tag{22}$$

For the sinc-Galerkin method, the infinite quadrature rule must be truncated to a finite sum. The following theorem indicates the conditions under which an exponential convergence results.

Theorem 8 (see [21]). *If there exist positive constants α, β, and C such that*

$$\left| \frac{F(x)}{\phi'(x)} \right| \le C \begin{cases} e^{-\alpha|\phi(x)|}, & x \in \psi((-\infty, \infty)) \\ e^{-\beta|\phi(x)|}, & x \in \psi((0, \infty)), \end{cases} \tag{23}$$

then the error bound for the quadrature rule (21) *is*

$$\left| \int_\Gamma F(x) \, dx - h \sum_{j=-M}^{N} \frac{F(x_j)}{\phi'(x_j)} \right| \le C \left(\frac{e^{-\alpha M h}}{\alpha} + \frac{e^{-\beta N h}}{\beta} \right) + |I_F|. \tag{24}$$

The infinite sum in (21) *is truncated with the use of* (23) *to arrive at inequality* (24). *Making the selections*

$$\begin{aligned} h &= \sqrt{\frac{\pi d}{\alpha M}}, \\ N &\equiv \left[\!\left[\frac{\alpha M}{\beta} + 1 \right]\!\right], \end{aligned} \tag{25}$$

where $[\![\cdot]\!]$ is an integer part of the statement and M is the integer value which specifies the grid size, then

$$\int_\Gamma F(x) \, dx = h \sum_{j=-M}^{N} \frac{F(x_j)}{\phi'(x_j)} + O\left(e^{-(\pi \alpha d M)^{1/2}} \right). \tag{26}$$

We used these theorems to approximate the integrals that arise in the formulation of the discrete systems corresponding to a second-order boundary value problem.

3. The Sinc-Galerkin Method

Consider fractional boundary value problem

$$u_{tt} = a(x) u_{xx} + b(x) \, {}_0^C D_x^{\beta} u + c(x) u + f(x,t), \quad 0 < \beta < 1 \tag{27}$$

with boundary conditions

$$\begin{aligned} u(0,t) &= u(1,t) = 0, \\ u(x,0) &= u(x,1) = 0, \end{aligned} \tag{28}$$

where ${}_0^C D_t$ is Caputo fractional derivative operator. An approximate solution for $u(x,t)$ is represented by the formula

$$u_{m_x,m_t}(x,t) = \sum_{j=-M_t}^{N_t} \sum_{i=-M_x}^{N_x} u_{ij} S_{ij}(x,t), \quad (29)$$

where $m_x = M_x + N_x + 1$ and $m_t = M_t + N_t + 1$. The basis functions $\{S_{ij}(x,t)\}$ are given by

$$S_{ij}(x,t) = S_i(x) S_j(t) = [S(i,h_x) o\phi(x)] [S(j,h_t) o\gamma(t)], \quad (30)$$

where

$$\phi(x) = \ln\left(\frac{x}{1-x}\right), \quad \gamma(t) = \ln\left(\frac{t}{1-t}\right). \quad (31)$$

The unknown coefficients u_{ij} in (29) are determined by orthogonalizing the residual with respect to the functions $\{S_{kl}(x,t)\}$, $-M_x \le k \le N_x$, $-M_t \le l \le N_t$. This yields the discrete Galerkin system

$$\left\langle Lu_{m_x,m_t} - f(x,t), S_{kl} \right\rangle = 0, \quad -M_x \le k \le N_x, \quad -M_t \le l \le N_t, \quad (32)$$

where inner product is defined by

$$\langle f, g\rangle = \iint_0^1 f(x,t)\, g(x,t)\, W(x,t)\, dx\, dt, \quad (33)$$

where $W(x)$ is *weight function* and it is convenient to take

$$W(x,t) = w(x)\, v(t) = \left[\frac{1}{[\phi'(x)]^{(1/2)}}\right] [\gamma'(t)]^{-1/2} \quad (34)$$

for the problem (27)-(28).

Lemma 9 (see [23]). *Let ϕ be the conformal one-to-one mapping of the simply connected domain D_E onto D_S, given by* (12). *Then*

$$\delta_{jk}^{(0)} = [S(j,h) o\phi(x)]\big|_{x=x_k} = \begin{cases} 1, & j = k, \\ 0, & j \ne k, \end{cases}$$

$$\delta_{jk}^{(1)} = h \frac{d}{d\phi} [S(j,h) o\phi(x)]\Big|_{x=x_k} = \begin{cases} 0, & j = k, \\ \dfrac{(-1)^{k-j}}{k-j}, & j \ne k, \end{cases}$$

$$\delta_{jk}^{(2)} = h^2 \frac{d^2}{d\phi^2} [S(j,h) o\phi(x)]\Big|_{x=x_k} = \begin{cases} -\dfrac{\pi^2}{3}, & j = k, \\ \dfrac{-2(-1)^{k-j}}{(k-j)^2}, & j \ne k. \end{cases} \quad (35)$$

The following theorems which can easily be proven by using Lemma 9 and definitions are used to solve (27).

Theorem 10 (see [31]). *The following relations hold:*

$$\langle u_{tt}, S_k S_l\rangle \cong h_t h_x \frac{w(x_k)}{\phi'(x_k)} \sum_{j=-M_t}^{N_t} \sum_{i=0}^{2} \frac{u(x_k,t_j)}{\gamma'(t_j)} \left[\frac{1}{h_t^i} \delta_{lj}^{(i)} \eta_i\right],$$

$$\langle a(x) u_{xx}, S_k S_l\rangle \cong h_x h_t \frac{v(t_l)}{\gamma'(t_l)} \sum_{i=-M_x}^{N_x} \sum_{j=0}^{2} \frac{u(x_i,t_l)}{\phi'(x_i)} \left[\frac{1}{h_x^j} \delta_{ki}^{(j)} \rho_j\right],$$

$$\langle c(x) u, S_k S_l\rangle \cong h_t h_x \frac{w(x_k)\, c(x_k)\, u(x_k,t_l)\, v(t_l)}{\phi'(x_k)\, \gamma'(t_l)},$$

$$\langle f(x,t), S_k S_l\rangle \cong h_t h_x \frac{w(x_k)\, f(x_k,t_l)\, v(t_l)}{\phi'(x_k)\, \gamma'(t_l)}, \quad (36)$$

where

$$\eta_2 = (\gamma')^2 v, \quad \eta_1 = \gamma'' v + 2v'\gamma', \quad \eta_0 = v'', \quad \rho_2 = (\phi')^2 aw, \quad \rho_1 = \phi'' aw + 2\phi'(aw' + a'w), \quad \rho_0 = a'' w + 2a'w' + aw''. \quad (37)$$

Proof. See [31]. □

Theorem 11. *For $0 < \beta < 1$, the following relations hold:*

$$\left\langle b(x)\, {}_0^C D_x^\beta u, S_k S_l \right\rangle \cong \frac{-h_x h_t}{\Gamma(1-\beta)} \frac{v(t_l)}{\gamma'(t_l)} \times \sum_{i=-M_x}^{N_x} \frac{u(x_i,t_l)}{\phi'(x_i)} \frac{d}{dx} \left[h_P \sum_{r=-P}^{P} \frac{(y_r - x)^{-\beta} R(y_r)}{\mu'(y_r)}\right]\Bigg|_{x=x_i}, \quad (38)$$

where $R(x) = b(x) S_k(x) w(x)$, $\mu(s) = \ln((s-x)/(1-s))$, and $h_P = \pi/\sqrt{P}$.

Proof. The inner product with sinc basis elements of $b(x)\, {}_0^C D_x^\beta u$ is given by

$$\left\langle b(x)\, {}_0^C D_x^\beta u, S_k S_l \right\rangle = \iint_0^1 b(x)\, {}_0^C D_x^\beta u(x,t)\, S_k(x)\, S_l(t)\, w(x)\, v(t)\, dx\, dt. \quad (39)$$

Using Definition 2, we can write

$$\left\langle b(x)\,{}_{0}^{C}D_{x}^{\beta}u, S_kS_l\right\rangle = \iint_0^1 b(x)\,{}_{0}^{C}D_{x}^{\beta}u(x,t)\,S_k(x)\,S_l(t)\,w(x)\,v(t)\,dx\,dt = \iint_0^1 u(x,t)\,S_l(t)\,v(t)\,{}_{x}D_{1}^{\beta}\left(b(x)\,S_k(x)\,w(x)\right)dx\,dt, \quad (40)$$

where $R(x) = b(x)S_k(x)w(x)$. By the definition of the Riemann-Liouville fractional derivative, we have

$$ {}_{x}D_{1}^{\beta}R(x) = -\frac{1}{\Gamma(1-\beta)}\frac{d}{dx}\int_x^1 (s-x)^{-\beta}R(s)\,ds. \quad (41)$$

We will use the sinc quadrature rule given with (26) to compute it because the integral given in last equality is divergent on the interval $[t,1]$. For this purpose, a conformal map and its inverse image that denotes the sinc grid points are given by

$$\mu(s) = \ln\left(\frac{s-x}{1-s}\right), \qquad y_r = \mu^{-1}(rh_P) = \frac{e^{rh_P}+x}{1+e^{rh_P}}, \quad (42)$$

respectively. Then, according to equality (26) we can write

$$-\frac{1}{\Gamma(1-\beta)}\frac{d}{dx}\int_x^1 (s-x)^{-\beta}R(s)\,ds \cong -\frac{1}{\Gamma(1-\beta)}\frac{d}{dx}\left[h_P\sum_{r=-P}^{P}\frac{(y_r-x)^{-\beta}R(y_r)}{\mu'(y_r)}\right] = N(x), \quad (43)$$

where $h_P = \pi/\sqrt{P}$. As a result, it can be written in the following way:

$$\left\langle b(x)\,{}_{0}^{C}D_{x}^{\beta}u, S_kS_l\right\rangle \cong \iint_0^1 u(x,t)\,S_l(t)\,v(t)\,N(x)\,dx\,dt. \quad (44)$$

Now, applying the sinc quadrature rule with respect to x and t in last integral, we obtain

$$\iint_0^1 u(x,t)\,S_l(t)\,v(t)\,N(x)\,dx\,dt \cong h_xh_t\sum_{i=M_x}^{N_x}\sum_{n=-M_t}^{N_t}\frac{u(x_i,t_n)\,v(t_n)\,S_l(t_n)\,N(x_i)}{\phi'(x_i)\,\gamma'(t_n)}. \quad (45)$$

Consequently, using $S_l(t)|_{t=t_n} = \delta_{\ln}^{(0)} = \delta_{\ln}$ we obtain

$$\left\langle b(x)\,{}_{0}^{C}D_{x}^{\beta}u, S_kS_l\right\rangle \cong \frac{-h_xh_t}{\Gamma(1-\beta)}\frac{v(t_l)}{\gamma'(t_l)} \times \sum_{i=-M_x}^{N_x}\frac{u(x_i,t_l)}{\phi'(x_i)}\frac{d}{dx}\left[h_P\sum_{r=-P}^{P}\frac{(y_r-x)^{-\beta}R(y_r)}{\mu'(y_r)}\right]\Bigg|_{x=x_i}. \quad (46)$$

This completes the proof. □

Replacing each term of (32) with the approximation defined in Theorems 10 and 11, replacing $u(x_k,t_j)$ by u_{kj}, and dividing by h_th_x we obtain the following theorem.

Theorem 12. *If the assumed approximate solution of the boundary-value problem (27)-(28) is (29), then the discrete sinc-Galerkin system for the determination of the unknown coefficients $\{u_{kj}, -M_x \le k \le N_x, -M_t \le j \le N_t\}$ is given by*

$$\begin{aligned}\frac{w(x_k)}{\phi'(x_k)}\sum_{j=-M_t}^{N_t}\sum_{i=0}^{2}\frac{u_{kj}}{\gamma'(t_j)}\left[\frac{1}{h_t^i}\delta_{lj}^{(i)}\eta_i\right] &= \frac{v(t_l)}{\gamma'(t_l)}\sum_{i=-M_x}^{N_x}\sum_{j=0}^{2}\frac{u_{il}}{\phi'(x_i)}\left[\frac{1}{h_x^j}\delta_{ki}^{(j)}\rho_j\right] \\ &\quad - \frac{1}{\Gamma(1-\beta)}\frac{v(t_l)}{\gamma'(t_l)} \times \sum_{i=-M_x}^{N_x}\frac{u_{il}}{\phi'(x_i)}\frac{d}{dx}\left[h_P\sum_{r=-P}^{P}\frac{(y_r-x)^{-\beta}R(y_r)}{\mu'(y_r)}\right]\Bigg|_{x=x_i} \\ &\quad + \frac{w(x_k)\,c(x_k)\,u_{kl}\,v(t_l)}{\phi'(x_k)\,\gamma'(t_l)} + \frac{w(x_k)\,f(x_k,t_l)\,v(t_l)}{\phi'(x_k)\,\gamma'(t_l)}.\end{aligned} \quad (47)$$

We introduce the following notations in order the write the system above in a matrix-vector form. Let $I_{m_x}^{(P)}$, $P = 0,1,2$ be the $m_x \times m_x$ matrices $I^{(P)}$, with jkth entry $\delta_{jk}^{(P)}$ as given by Lemma 9. Further, $D(g_x)$ is an $m_x \times m_x$ diagonal matrix whose diagonal entries are

$$\left[g(x_{-M_x}), g(x_{-M_x+1}), \ldots, g(x_0), \ldots, g(x_{N_x})\right]^T. \quad (48)$$

The matrices $I_{m_t}^{(P)}$, $P = 0, 1, 2$, *and* $D(g_t)$ *are similarly defined though of size* $m_t \times m_t$. *Introducing this notation in* (47) *leads to the matrix form*

$$\begin{aligned} & D\left(\frac{1}{\phi'}\right) D(w) U D(v) \left[\sum_{j=0}^{2} \frac{1}{h_t^j} I_{m_t}^{(j)} D\left(\frac{\eta_j}{\gamma' v}\right)\right]^t \\ & - \left[\sum_{i=0}^{2} \frac{1}{h_x^i} I_{m_x}^{(i)} D\left(\frac{\rho_i}{(\phi')^2 w}\right)\right] D(\phi') D(w) U D\left(\frac{v}{\gamma'}\right) \\ & - \left[B D\left(\frac{1}{(\phi')^2 w}\right)\right] D(\phi') D(w) U D\left(\frac{v}{\gamma'}\right) \\ & - D\left(\frac{w}{\phi'}\right) D(c) U D\left(\frac{v}{\gamma'}\right) = D\left(\frac{w}{\phi'}\right) F D\left(\frac{v}{\gamma'}\right), \end{aligned} \tag{49}$$

where

$$B = -\frac{1}{\Gamma(1-\beta)} \frac{d}{dx} \left[h_P \sum_{r=-P}^{P} \frac{(y_r - x)^{-\beta} R(y_r)}{\mu'(y_r)} \right] \Bigg|_{x=x_i}. \tag{50}$$

Firstly multiplying this term with $D(\phi')$ *and secondly multiplying it with* $D(\gamma')$ *yield the equivalent system*

$$\Phi X + X \Psi = G, \tag{51}$$

where

$$\begin{aligned} \Phi &= -A1 - A2 - C, \\ \Psi &= B1 \end{aligned} \tag{52}$$

for

$$\begin{aligned} A1 &= D(\phi') \left[\sum_{i=0}^{2} \frac{1}{h_x^i} I_{m_x}^{(i)} D\left(\frac{\rho_i}{(\phi')^2 w}\right)\right] D(\phi'), \\ A2 &= D(\phi') \left[B D\left(\frac{1}{(\phi')^2 w}\right)\right] D(\phi'), \\ B1 &= \left[\sum_{j=0}^{2} \frac{1}{h_t^j} I_{m_t}^{(j)} D\left(\frac{\eta_j}{\gamma' v}\right)\right]^t D(\gamma'), \\ C &= D(c). \end{aligned} \tag{53}$$

Furthermore,

$$\begin{aligned} G &= D(w) F D(v), \\ X &= D(w) U D(v). \end{aligned} \tag{54}$$

Φ, Ψ, X, *and* G *have dimensions* $m_x \times m_x$, $m_t \times m_t$, $m_x \times m_t$, *and* $m_x \times m_t$, *respectively. At last, the* $m_x \times m_t$ *matrices* U *and* F *have klth entries given by* u_{kl} *and* $f(x_k, t_l) = f(e^{kh}/(1 + e^{kh}), e^{lh}/(1 + e^{lh}))$, *respectively.*

To obtain the approximate solution equation (29), *we need to solve the system for* U *which requires solving* (51) *for* X. *Solution of* (51) *for* X *is shown in [26].*

4. Examples

In this section, the present method will be tested on three different problems.

Example 1. Consider fractional boundary value problem

$$\begin{aligned} u_{tt} &= u_{xx} + {}_0^C D_x^{0.7} u + f(x, t), \\ u(0, t) &= u(1, t) = 0, \\ u(x, 0) &= u(x, 1) = 0 \end{aligned} \tag{55}$$

which has the following exact solution:

$$u(x, t) = x^2 (1 - x) t^3 (1 - t)^2 \tag{56}$$

for

$$\begin{aligned} f(x, t) = & -2(1-t)^2 t^3 (1-x) + 4(1-t)^2 t^3 x \\ & + 6(1-t)^2 t (1-x) x^2 - 12(1-t) t^2 \\ & \times (1-x) x^2 + 2t^3 (1-x) x^2 + 0.334273 \\ & \times (-1+t)^2 t^3 x^{1.3} (-5.12821 + 6.68896x). \end{aligned} \tag{57}$$

The numerical solutions which are obtained by using the sinc-Galerkin method (SGM) for this problem are presented in Tables 1 and 2 for different values. Also, the graphs of exact and approximate solutions for different values are presented in Figures 2 and 3.

Example 2. Consider fractional boundary value problem

$$\begin{aligned} u_{tt} &= e^x u_{xx} + (x^2 + 1) {}_0^C D_x^{0.3} u - \frac{1}{x+1} u + f(x, t) \\ u(0, t) &= u(1, t) = 0 \\ u(x, 0) &= u(x, 1) = 0 \end{aligned} \tag{58}$$

which has the following exact solution:

$$u(x, t) = x^2 (1 - x) \sin(\pi t) \tag{59}$$

for

$$\begin{aligned} & f(x, t) \\ & = -\pi^2 (1-x) x^2 \sin(\pi t) + \frac{(1-x) x^2 \sin(\pi t)}{1+x} \\ & - 0.770383 (1 + x^2) (1.68067 x^{1.7} - 1.86741 x^{2.7}) \\ & \times \sin(\pi t) - e^x (2(1-x) \sin(\pi t) - 4x \sin(\pi t)). \end{aligned} \tag{60}$$

The numerical solutions which are obtained by using the sinc-Galerkin method (SGM) for this problem are presented in Tables 3 and 4. In addition, in Figures 4 and 5, the graphs of exact and approximate solutions for different values are presented.

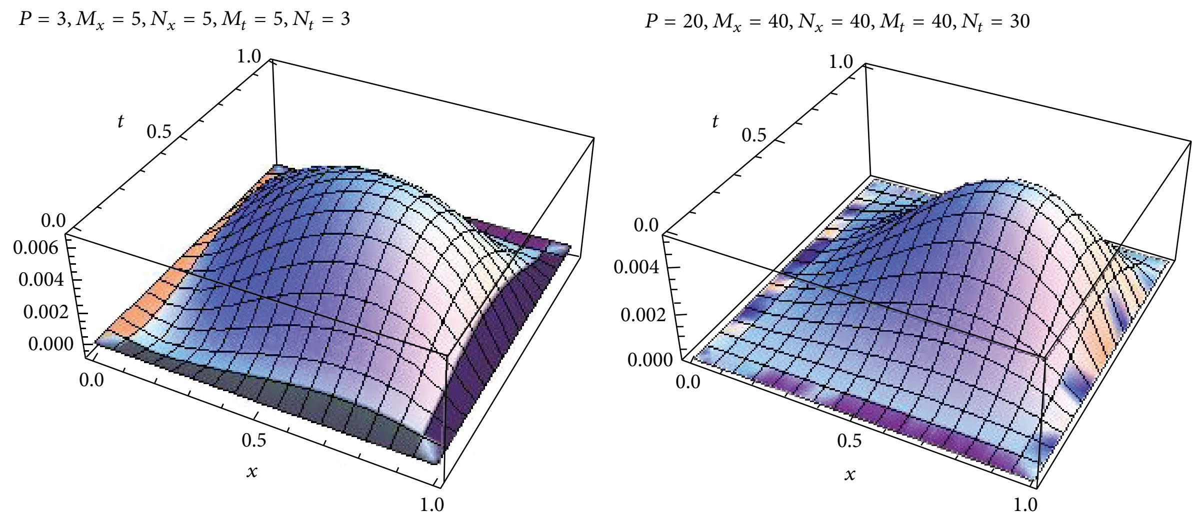

Figure 2: Graphs of approximate solutions for different values.

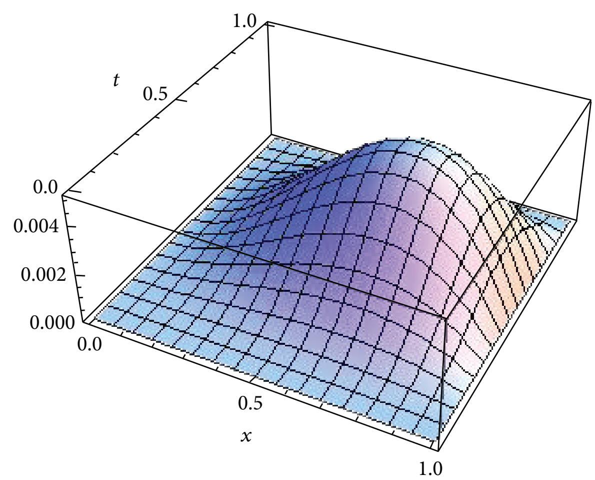

Figure 3: Graph of exact solution.

Table 1: Numerical results for $P = 3$, $M_x = 5$, $N_x = 5$, $M_t = 5$, $N_t = 3$.

t	x	Exact sol.	Num. sol.	Error
0.03	0.3	0.00000160	0.000069824	0.000068223
	0.6	0.00000365	0.000557026	0.000553368
	0.9	0.00000205	0.000286490	0.000284433
0.06	0.3	0.00001202	−0.000160475	0.000172499
	0.6	0.00002748	0.000688434	0.000660950
	0.9	0.00001545	0.000405155	0.000389695
0.09	0.3	0.00003803	−0.000317899	0.000355931
	0.6	0.00008693	0.000879725	0.000792795
	0.9	0.00004889	0.000577303	0.000528404

Table 2: Numerical results for $P = 20$, $M_x = 40$, $N_x = 40$, $M_t = 40$, $N_t = 30$.

t	x	Exact sol.	Num. sol.	Error
0.03	0.3	0.00000160	0.00000376	$2.16753 * 10^{-6}$
	0.6	0.00000365	0.00001237	$8.71244 * 10^{-6}$
	0.9	0.00000205	0.00000667	$4.61338 * 10^{-6}$
0.06	0.3	0.00001202	0.00001645	$4.43271 * 10^{-6}$
	0.6	0.00002748	0.00004476	$1.72861 * 10^{-5}$
	0.9	0.00001545	0.00002455	$9.09322 * 10^{-6}$
0.09	0.3	0.00003803	0.00004497	$6.93846 * 10^{-6}$
	0.6	0.00008693	0.00011262	$2.56904 * 10^{-5}$
	0.9	0.00004889	0.00006228	$1.33882 * 10^{-5}$

Example 3 (see [32]). Consider the fractional convection-diffusion equation

$$\begin{gathered} \frac{\partial^\gamma u(x,t)}{\partial t^\gamma} = -a(x)\frac{\partial^\alpha u(x,t)}{\partial x^\alpha} + b(x)\frac{\partial^\beta u(x,t)}{\partial x^\beta} + f(x,t) \\ 0 < \gamma \le 2, \quad 1 < \alpha \le 2, \quad 0 < \beta \le 1, \\ u(0,t) = u(1,t) = 0, \\ u(x,0) = u(x,1) = 0. \end{gathered} \tag{61}$$

In particular, if $\gamma = 2$, $\alpha = 2$, $\beta = 0.35$, $a(x) = \Gamma(2.35)\Gamma(2.65)x^{0.35}$, $b(x) = \Gamma(0.7)\Gamma(1)x^2$.

The problem has the following exact solution:

$$u(x,t) = \left(x^{1.7} - x^2\right)\sin(2\pi t) \tag{62}$$

for

$$\begin{aligned} f(x,t) = {} & \left(2\pi x^{1.7} - x^2\right)t^{-1}E_{2,0}\left(-(2\pi t)^2\right) \\ & + \left\{\Gamma(2.7)\left(\Gamma(2.65) - \Gamma(1)\right)x^{1.7}\right. \\ & \left. + \Gamma(3)\left(\Gamma(0.7) - \Gamma(2.35)\right)x^2\right\}\sin(2\pi t). \end{aligned} \tag{63}$$

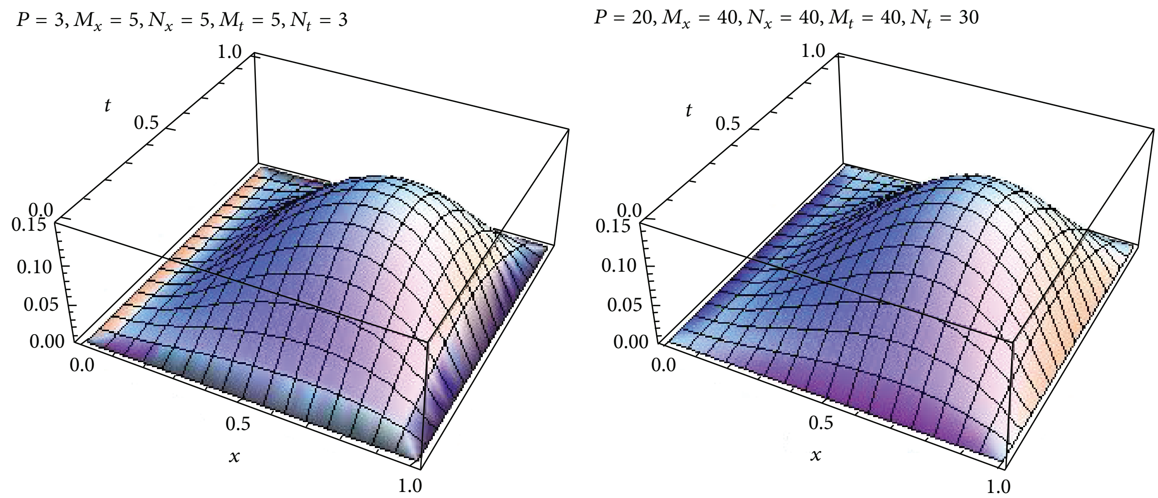

Figure 4: Graphs of approximate solutions for different values.

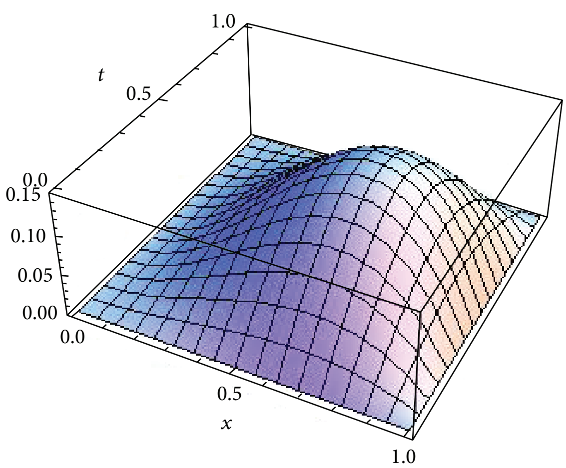

Figure 5: Graphs of exact solution.

Table 3: Numerical results for $P = 3$, $M_x = 5$, $N_x = 5$, $M_t = 5$, $N_t = 3$.

t	x	Exact sol.	Num. sol.	Error
0.03	0.3	0.0059288	0.0071562	0.00122741
	0.6	0.0135516	0.0149881	0.00143653
	0.9	0.0076227	0.0083725	0.00074978
0.06	0.3	0.0118050	0.0131991	0.00139409
	0.6	0.0269829	0.0254147	0.00156823
	0.9	0.0151779	0.0145261	0.00065176
0.09	0.3	0.0175764	0.0193179	0.00174151
	0.6	0.0401747	0.0362484	0.00392631
	0.9	0.0225983	0.0208906	0.00170767

The numerical solutions which are obtained by using the sinc-Galerkin method (SGM) for this problem are presented in Tables 5 and 6. In addition, in Figures 6 and 7, the graphs of exact and approximate solutions for different values are presented.

Table 4: Numerical results for $P = 20$, $M_x = 40$, $N_x = 40$, $M_t = 40$, $N_t = 30$.

t	x	Exact sol.	Num. sol.	Error
0.03	0.3	0.0059288	0.0059287	$1.16151 * 10^{-7}$
	0.6	0.0135516	0.0135515	$1.41292 * 10^{-7}$
	0.9	0.0076227	0.0076227	$1.58382 * 10^{-8}$
0.06	0.3	0.0118050	0.0118048	$2.68162 * 10^{-7}$
	0.6	0.0269829	0.0269826	$3.57381 * 10^{-7}$
	0.9	0.0151779	0.0151778	$7.75388 * 10^{-8}$
0.09	0.3	0.0175764	0.0175761	$3.24333 * 10^{-7}$
	0.6	0.0401747	0.0401744	$3.45727 * 10^{-7}$
	0.9	0.0225983	0.0225983	$1.68470 * 10^{-8}$

Table 5: Numerical results for $P = 3$, $M_x = 5$, $N_x = 5$, $M_t = 5$, $N_t = 3$.

t	x	Exact sol.	Num. sol.	Error
0.03	0.3	0.0073366	0.0043780	0.00295855
	0.6	0.0111718	0.0092590	0.00191284
	0.9	0.0048740	0.0045355	0.00033852
0.06	0.3	0.0144134	0.0107858	0.00362752
	0.6	0.0219479	0.0228301	0.00088218
	0.9	0.0095754	0.0110942	0.00151873
0.09	0.3	0.0209795	0.0167664	0.00421307
	0.6	0.0319465	0.0354700	0.00352348
	0.9	0.0139377	0.0172001	0.00326239

5. Conclusion

In this study, we use the sinc-Galerkin method to obtain approximate solutions of boundary value problems for space-fractional partial differential equations with variable coefficients. In order to illustrate the efficiency and accuracy of the present method, the method is applied to three examples in the literature and the obtained results are compared with exact solutions. As a result, it is shown that sinc-Galerkin

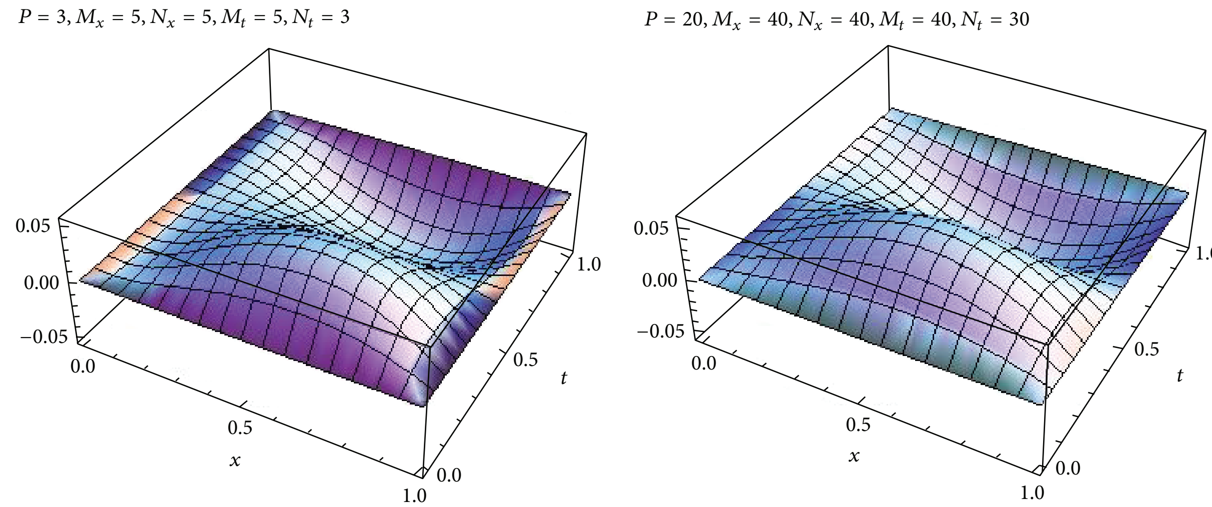

Figure 6: Graphs of approximate solutions for different values.

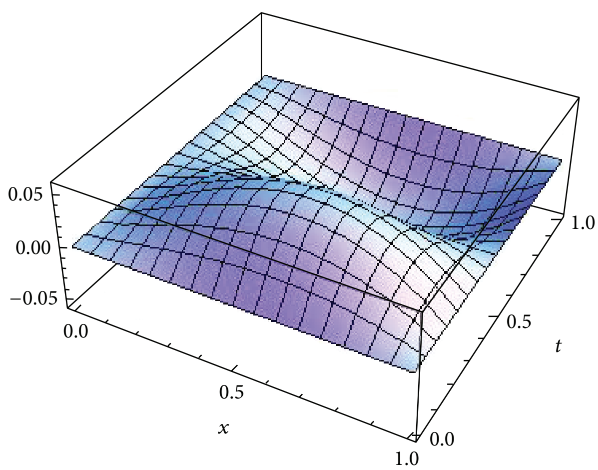

Figure 7: Graphs of exact solution.

Table 6: Numerical results for $P = 20, M_x = 40, N_x = 40, M_t = 40, N_t = 30$.

t	x	Exact sol.	Num. sol.	Error
0.03	0.3	0.0073366	0.0073365	$6.80615 * 10^{-8}$
	0.6	0.0111718	0.0111718	$2.45596 * 10^{-8}$
	0.9	0.0048740	0.0048740	$1.74662 * 10^{-8}$
0.06	0.3	0.0144134	0.0144139	$5.33686 * 10^{-7}$
	0.6	0.0219479	0.0219489	$9.67538 * 10^{-7}$
	0.9	0.0095754	0.0095759	$4.80610 * 10^{-7}$
0.09	0.3	0.0209795	0.0209786	$8.62732 * 10^{-7}$
	0.6	0.0319465	0.0319454	$1.08969 * 10^{-6}$
	0.9	0.0139377	0.0139373	$3.85036 * 10^{-7}$

method is very effective and accurate for obtaining approximate solutions of space-fractional differential equations with variable coefficients. In the future, we plan to extend the present numerical solution technique to nonlinear space-fractional partial differential equations.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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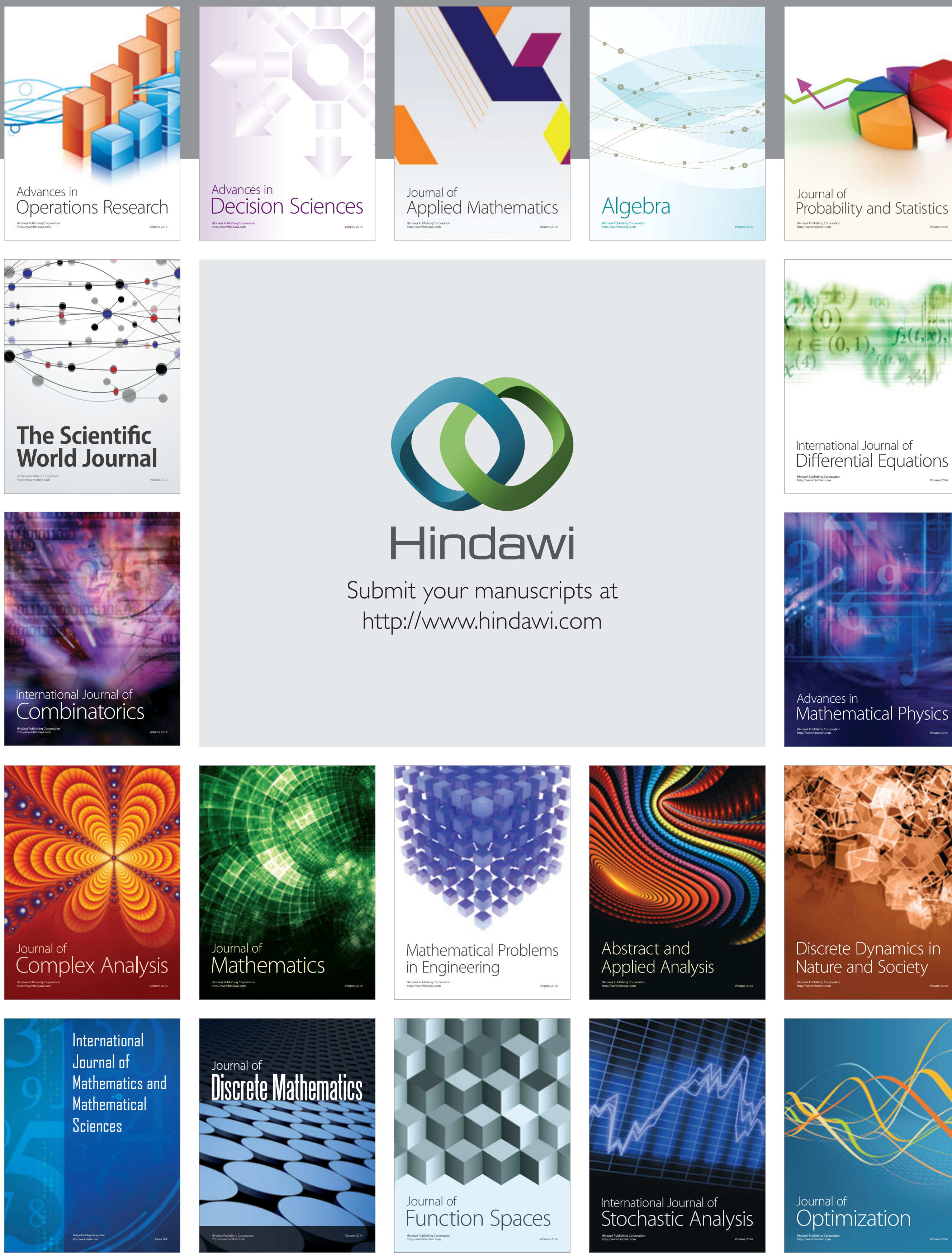
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