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$S_\lambda(\mathcal{I})$-CONVERGENCE OF COMPLEX UNCERTAIN SEQUENCE

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This study introduces the $\lambda_{\mathcal{I}}$-statistically convergence concepts of complex uncertain sequences: $\lambda_{\mathcal{I}}$-statistically convergence almost surely ($S_\lambda(\mathcal{I}).a.s.$), $\lambda_{\mathcal{I}}$-statistically convergence in measure, $\lambda_{\mathcal{I}}$-statistically convergence in mean, $\lambda_{\mathcal{I}}$-statistically convergence in distribution and $\lambda_{\mathcal{I}}$-statistically convergence uniformly almost surely ($S_\lambda(\mathcal{I}).u.a.s.$). In addition, decomposition theorems and relationships among them are discussed.

1. Introduction and background. Freedman and Sember introduced the concept of a lower asymptotic density and defined the concept of convergence in density, in [3]. Taking this definition, we can give the definition of statistical convergence which has been formally introduced by Fast [1] and Steinhaus [21]. Schoenberg reintroduced this concept independently [2]. A number sequence (x_k) is statistically convergent to L provided that for every $\varepsilon > 0$, $d\big\{k \in \mathbb{N}\colon |x_k - L| \geq \varepsilon\big\} = 0$ or equivalently there exists a subset $K \subseteq \mathbb{N}$ with $d\big(K\big) = 1$ and $n_0(\varepsilon)$ such that $k > n_0(\varepsilon)$ and $k \in K$ imply that $|x_k - L| < \varepsilon$. In this case we write $st - \lim x_k = L$. From the definition, we can easily show that any convergent sequence is statistically convergent, but not conversely.

Let $\lambda = \big(\lambda_n\big)$ is a non-decreasing sequence of positive numbers tending to ∞ such that $\lambda_{n+1} \leq \lambda_n + 1$, $\lambda_1 = 1$. Mursaleen [29] defined λ-statistical convergence by using the λ sequence. He denoted this new method by S_λ. A number sequence $x = \big(x_k\big)$ is said to be λ-statistically convergent to the number L if for every $\varepsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{\lambda_n} |\{k \in I_n\colon |x_k - L| \geq \varepsilon\}| = 0,$$

where $I_n = \big[n - \lambda_n + 1, n\big]$. It is denoted by st-$\lim x_k = L$. Let Λ denote the set of all non-decreasing sequences $\lambda = \big(\lambda_n\big)$ of positive numbers tending to ∞ such that $\lambda_{n+1} \leq \lambda_n$ and $\lambda_1 = 1$.

The concept of $\mathcal{I}$-convergence of real sequences is a generalization of statistical convergence which is based on the structure of the ideal $\mathcal{I}$ of subsets of the set of natural numbers. P. Kostyrko et al. [26] introduced the concept of $\mathcal{I}$-convergence of sequences in a metric space and studied some properties of this convergence. Later, it was further studied by Salát, Tripathy and Ziman ([23], [24]) and many others. Recently, Das, Savas and Ghosal [6] introduced new notions, namely $\mathcal{I}$-statistical convergence and $\mathcal{I}$-lacunary statistical convergence by using ideal.

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However, in our daily life, we often encounter the case that there are lack of or no observed data about the events, not only for economic reasons or technical difficulties, but also for influence of unexpected events.

In order to deal with belief degree, an uncertainty theory was founded by Liu [10] in 2007, and refined by Liu [11] in 2010 which based on an uncertain measure which satisfies normality, duality, subadditivity, and product axioms. Thereafter, a concept of uncertain variable was proposed to represent the uncertain quantity and a concept of uncertainty distribution to describe uncertain variables. Up to now, uncertainty theory has successfully been applied to uncertain programming (Liu [12], Liu and Chen [13]), uncertain risk analysis and uncertain reliability analysis (Liu [15]), uncertain logic (Liu [16]), uncertain differential equation (Liu [17], Yao and Chen [19]), uncertain graphs (Gao and Gao [8], Zhang and Peng [20]), uncertain calculus (Liu [14]) and uncertain finance (Chen [5], Liu [14], Liu [18]), etc.

In real life, uncertainty not only appears in real quantities but also in complex quantities. In order to model complex uncertain quantities, Peng [30] presented the concepts of complex uncertain variable and complex uncertainty distribution, and also the expected value was proposed to measure a complex uncertain variable in 2012. Since sequence convergence plays an important role in the fundamental theory of mathematics, there are also many convergence concepts in uncertainty theory. In 2007, Liu [10] first introduced convergence in measure, convergence in mean, convergence almost surely (a.s.) and convergence in distribution and their relationships were also discussed. You [31] introduced another type of convergence named convergence uniformly almost surely and showed the relationships among those convergence concepts. Zhang [33] proved some theorems on the convergence of a sequence of uncertain complex variables. After that, Guo and Xu [9] gave the concept of convergence in mean square for a sequence of uncertain complex variables and showed that an uncertain sequence converged in mean square if and only if it was a Cauchy sequence. Tripathy and Kumar [32] introduced statistical convergence of complex uncertain sequence. Kişi and Güler [27] defined λ-statistically convergence of a sequence of uncertain complex variables. Inspired by these, we study the convergence concepts of $\lambda_{\mathcal{I}}$-statistically convergence of a sequence of uncertain complex variables by using ideal and discuss the relationships among them in this paper.

2. Definitions and notations. In this section, some fundamental concepts in uncertainty theory are introduced, which were used throughout the study.

Definition 1 ([26]). A family of sets $\mathcal{I} \subseteq 2^{\mathbb{N}}$ is called an *ideal* if and only if

(i) $\varnothing \in \mathcal{I}$,

(ii) for each $A, B \in \mathcal{I}$ we have $A \cup B \in \mathcal{I}$,

(iii) for each $A \in \mathcal{I}$ and each $B \subseteq A$ we have $B \in \mathcal{I}$.

Definition 2 ([10]). Let L be a σ-algebra on a non-empty set Γ. A set function M is called an *uncertain measure* if it satisfies the following axioms:

Axiom 1. (Normality Axiom) $M\{\Gamma\} = 1$;
Axiom 2. (Duality Axiom) $M\{\Lambda\} + M\{\Lambda^c\} = 1$ for any $\Lambda \in L$;
Axiom 3. (Subadditivity Axiom) For every countable sequence of $\{\lambda_j\} \in L$, we have

$$M\left(\bigcup_{j=1}^{\infty} \lambda_j\right) \leq \sum_{j=1}^{\infty} M\{\lambda_j\}.$$

The triplet $\big(\Gamma, L, M\big)$ is called an *uncertainty space*, and each element Λ in L is called an event.

Definition 3 ([10])**.** An *uncertain variable* ξ is a measurable function from an uncertainty space $\big(\Gamma, L, M\big)$ to the set of real numbers, i.e. for any Borel set B of real numbers, the set

$$\big\{\xi \in B\big\} = \big\{\gamma \in \Gamma \colon \xi\big(\gamma\big) \in B\big\}$$

is an event.

A *uncertainty distribution* Φ of uncertain variable ξ is defined by $\Phi\big(x\big) = M\big\{\xi \leq x\big\}$, $\forall x \in \mathbb{R}$.

Considering the importance of the role of convergence of sequence in mathematics, some concepts of convergence for uncertain sequences were introduced by Liu [10] as follows:

Definition 4 ([10])**.** The uncertain sequence $\big\{\zeta_n\big\}$ is said to be *convergent almost surely* (a.s.) to ζ if there exists an event Λ with $M\big(\Lambda\big) = 1$ such that

$$\lim_{n\to\infty} \big|\zeta_n\big(\gamma\big) - \zeta\big(\gamma\big)\big| = 0,$$

for every $\gamma \in \Lambda$. In this case we write $\zeta_n \to \zeta$, a.s.

Definition 5 ([10])**.** The uncertain sequence $\big\{\zeta_n\big\}$ is said to be *convergent in measure* to ζ if

$$\lim_{n\to\infty} M\big\{\big|\zeta_n - \zeta\big| \geq \varepsilon\big\} = 0,$$

for every $\varepsilon > 0$.

Definition 6 ([10])**.** The uncertain sequence $\big\{\zeta_n\big\}$ is said to be *convergent in mean to* ζ if

$$\lim_{n\to\infty} E\big[\big|\zeta_n - \zeta\big|\big] = 0.$$

Definition 7 ([10])**.** Let $\Phi, \Phi_1, \Phi_2, \ldots$ be the uncertainty distributions of uncertain variables $\zeta, \zeta_1, \zeta_2, \ldots$, respectively. We say the sequence $\big\{\zeta_n\big\}$ *converges in distribution* to ζ if

$$\lim_{n\to\infty} \big\|\Phi_n\big(x\big)\big\| = \Phi\big(x\big)$$

for all x at which $\Phi\big(x\big)$ is continuous.

Definition 8 ([10])**.** The uncertain sequence $\big\{\zeta_n\big\}$ is said to be *convergent uniformly almost surely* (u.a.s.) to ζ if there exists a sequence of events $\big\{E_k\big\}$, $M\big\{E_k\big\} \to 0$ such that $\big\{\zeta_n\big\}$ converges uniformly to ζ in $\Gamma - E_k$, for any fixed k.

3. Main results. In this study, we give new concepts and study their certain properties. In addition, decomposition theorems and relationships among the concepts are discussed.

Let $\zeta_1, \zeta_2, \ldots, \zeta_n, \ldots$ be complex uncertain variables. The sequence $\big\{\zeta_n\big\}$ is said to be *$\lambda_{\mathcal{I}}$-statistically convergent almost surely ($S_\lambda\big(\mathcal{I}\big)$.a.s.)* to ζ if for every $\varepsilon, \delta > 0$ there exists an event Λ with $M\big(\Lambda\big) = 1$ such that

$$\Big\{n \in \mathbb{N} \colon \frac{1}{\lambda_n}\big|\big\{k \in I_n \colon \big\|\zeta_k\big(\gamma\big) - \zeta\big(\gamma\big)\big\| \geq \varepsilon\big\}\big| \geq \delta\Big\} \in \mathcal{I},$$

for every $\gamma \in \Lambda$. In this case we write $\zeta_n \to \zeta \ \Big(S_\lambda\Big(\mathcal{I}\Big).a.s.\Big)$.

The sequence $\{\zeta_n\}$ is said to be $\lambda_{\mathcal{I}}$-*statistically convergent in measure* to ζ if

$$\Big\{n \in \mathbb{N}\colon \frac{1}{\lambda_n}\big|\big\{k \in I_n\colon M\big(\|\zeta_k - \zeta\| \geq \varepsilon\big) \geq \delta\big\}\big| \geq \vartheta\Big\} \in \mathcal{I},$$

for every $\varepsilon, \delta, \vartheta > 0$.

The sequence $\{\zeta_n\}$ is said to be $\lambda_{\mathcal{I}}$-*statistically convergent in mean* to ζ if

$$\Big\{n \in \mathbb{N}\colon \frac{1}{\lambda_n}\big|\big\{k \in I_n\colon E\big(\|\zeta_k - \zeta\|\big) \geq \varepsilon\big\}\big| \geq \delta\Big\} \in \mathcal{I},$$

for every $\varepsilon, \delta > 0$.

Let $\Phi, \Phi_1, \Phi_2, \dots$ be the complex uncertainty distributions of complex uncertain variables $\zeta, \zeta_1, \zeta_2, \dots$, respectively. We say the complex uncertain sequence $\{\zeta_n\}$ $\lambda_{\mathcal{I}}$-*statistically converges in distribution* to ζ if for every $\varepsilon, \delta > 0$,

$$\Big\{n \in \mathbb{N}\colon \frac{1}{\lambda_n}\big|\big\{k \in I_n\colon \|\Phi_k(c) = \Phi(c)\| \geq \varepsilon\big\}\big| \geq \delta\Big\} \in \mathcal{I},$$

for all c at which $\Phi(c)$ is continuous.

The sequence $\{\zeta_n\}$ is said to be $\lambda_{\mathcal{I}}$-*statistically convergent uniformly almost surely ($S_\lambda(\mathcal{I})$. u. a. s.)* to ζ if for every $\varepsilon, \sigma > 0$, $\exists \delta > 0$, $\forall x \in \mathbb{R}$ and a sequence of events $\{E'_k\}$ such that

$$\{n \in \mathbb{N}\colon \frac{1}{\lambda_n}\big|\big\{k \in I_n\colon \big|M\big(E'_k\big) - 0\big| \geq \varepsilon\big\}\big| \geq \sigma\} \in \mathcal{I} \Longrightarrow$$
$$\Longrightarrow \{n \in \mathbb{N}\colon \frac{1}{\lambda_n}\big|\big\{k \in I_n\colon \big|\zeta_k(x) - \zeta(x)\big| \geq \delta\big\}\big| \geq \sigma\} \in \mathcal{I}.$$

The sequence $\{\zeta_n\}$ is said to be $\lambda_{\mathcal{I}}$-*statistically convergent* to ζ if for every $\varepsilon > 0$,

$$\Big\{n \in \mathbb{N}\colon \frac{1}{\lambda_n}\big|\big\{k \in I_n\colon \|\zeta_k(\gamma) - \zeta(\gamma)\| \geq \varepsilon\big\}\big| \geq \delta\Big\} \in \mathcal{I},$$

for every $\gamma \in \Lambda$.

4. Relationships between convergence concepts. In this section, we give the relationships among the convergence concepts of a sequence of uncertain complex variables.

4.1. $\lambda_{\mathcal{I}}$-statistically convergence in mean and $\lambda_{\mathcal{I}}$-statistically convergence in measure.

Proposition 1. *If the sequence $\{\zeta_n\}$ of complex uncertain variables $\lambda_{\mathcal{I}}$-statistically converges in mean to ζ, then $\{\zeta_n\}$ $\lambda_{\mathcal{I}}$-statistically converges in measure to ζ.*

Proof. It follows from the Markov inequality that for any given $\varepsilon, \delta, \vartheta > 0$, we have

$$\Big\{n \in \mathbb{N}\colon \frac{1}{\lambda_n}\Big|\big\{k \in I_n\colon M\big(\|\zeta_k - \zeta\| \geq \varepsilon\big) \geq \delta\big\}\Big| \geq \vartheta\Big\} \subseteq$$
$$\subseteq \Big\{n \in \mathbb{N}\colon \frac{1}{\lambda_n}\Big|\Big\{k \in I_n\colon \Big(\frac{E(\|\zeta_k - \zeta\|)}{\varepsilon}\Big) \geq \delta\Big\}\Big| \geq \vartheta\Big\} \in \mathcal{I}.$$

Thus $\{\zeta_n\}$ $\lambda_{\mathcal{I}}$-statistically converges in measure to ζ and the theorem is thus proved. □

Remark 1. Converse of above theorem is not true. i.e. $\lambda_{\mathcal{I}}$-statistical convergence in measure does not imply $\lambda_{\mathcal{I}}$-statistical convergence in mean. The following example illustrates this.

Example 1. Consider the uncertainty space (Γ, L, M) to be $\gamma_1, \gamma_2, \dots$ with

$$M\{\Lambda\} = \begin{cases} \sup\limits_{\gamma_n \in \Lambda} \frac{1}{n+1}, & \text{if } \sup\limits_{\gamma_n \in \Lambda} \frac{1}{n+1} < 0.5; \\ 1 - \sup\limits_{\gamma_n \in \Lambda^c} \frac{1}{n+1}, & \text{if } \sup\limits_{\gamma_n \in \Lambda^c} \frac{1}{n+1} < 0.5; \\ 0.5, & \text{otherwise,} \end{cases}$$

and the complex uncertain variables be defined by

$$\zeta_n(\gamma) = \begin{cases} (n+1)i, & \text{if } \gamma = \gamma_n; \\ 0, & \text{otherwise,} \end{cases}$$

for $n = 1, 2, \dots$ and $\zeta \equiv 0$. For some small number $\varepsilon, \delta, \vartheta > 0$ and $n \geq 2$, we have

$$\begin{aligned} &\Big\{n \in \mathbb{N} \colon \frac{1}{\lambda_n}\big|\{k \in I_n \colon M(\|\zeta_k - \zeta\| \geq \varepsilon) \geq \delta\}\big| \geq \vartheta\Big\} = \\ &= \Big\{n \in \mathbb{N} \colon \frac{1}{\lambda_n}\big|\{k \in I_n \colon M(\gamma \colon \|\zeta_k(\gamma) - \zeta(\gamma)\| \geq \varepsilon) \geq \delta\}\big| \geq \vartheta\Big\} = \\ &= \Big\{n \in \mathbb{N} \colon \frac{1}{\lambda_n}\big|\{k \in I_n \colon M\{\gamma_n\} \geq \delta\}\big| \geq \vartheta\Big\} \in \mathcal{I}. \end{aligned}$$

Thus, the sequence $\{\zeta_n\}$ $\lambda_{\mathcal{I}}$-statistically converges in measure to ζ. However, for each $n \geq 2$, we have the uncertainty distribution of uncertain variable $\|\zeta_n - \zeta\| = \|\zeta_n\|$ is

$$\Phi_n(x) = \begin{cases} 0, & \text{if } x < 0; \\ 1 - \frac{1}{n+1}, & \text{if } 0 \leq x < n+1; \\ 1, & \text{if } x \geq n+1. \end{cases}$$

So, for each $n \geq 2$, and for every $\varepsilon, \delta > 0$, we have

$$\begin{aligned} &\Big\{n \in \mathbb{N} \colon \frac{1}{\lambda_n}\big|\{k \in I_n \colon E(\|\zeta_k - \zeta\| - 1) \geq \varepsilon\}\big| \geq \delta\Big\} = \\ &= \left\{n \in \mathbb{N} \colon \frac{1}{\lambda_n}\left|\left\{k \in I_n \colon \left(\left[\int_0^{n+1} 1 - \left(1 - \frac{1}{n+1}\right) dx\right] - 1\right) \geq \varepsilon\right\}\right| \geq \delta\right\}, \end{aligned}$$

which is impossible. That is, the sequence $\{\zeta_n\}$ does not $\lambda_{\mathcal{I}}$-statistically converge in mean to ζ.

4.2. $\lambda_{\mathcal{I}}$-statistically convergence in measure and $\lambda_{\mathcal{I}}$-statistically convergence in distribution.

Proposition 2. *Let $\{\zeta_n\}$ be a sequence of complex uncertain variables with real part $\{\xi_n\}$ and imaginary part $\{\gamma_n\}$, respectively, for $n = 1, 2, \dots$. If the sequences $\{\xi_n\}$ and $\{\gamma_n\}$ $\lambda_{\mathcal{I}}$-statistically converge in measure to ξ and γ, respectively, then the sequence $\{\zeta_n\}$ $\lambda_{\mathcal{I}}$-statistically converges in measure to $\zeta = \xi + i\gamma$.*

Proof. It follows from the definition of $\lambda_{\mathcal{I}}$-statistical convergence in a measure of sequence of uncertain complex variables that for any small numbers $\varepsilon, \delta, \vartheta > 0$,

$$\Big\{n \in \mathbb{N}\colon \frac{1}{\lambda_n}\Big|\Big\{k \in I_n\colon M\big(\|\xi_k - \xi\| \geq \frac{\varepsilon}{\sqrt{2}}\big) \geq \delta\Big\}\Big| \geq \vartheta\Big\} \in \mathcal{I},$$
$$\Big\{n \in \mathbb{N}\colon \frac{1}{\lambda_n}\Big|\Big\{k \in I_n\colon M\big(\|\gamma_k - \gamma\| \geq \frac{\varepsilon}{\sqrt{2}}\big) \geq \delta\Big\}\Big| \geq \vartheta\Big\} \in \mathcal{I}.$$

Note that $\|\zeta_n - \zeta\| = \sqrt{|\xi_n - \xi|^2 + |\gamma_n - \gamma|^2}$. Thus, we have

$$\big\{\|\zeta_n - \zeta\| \geq \varepsilon\big\} \subset \Big\{\|\xi_n - \xi\| \geq \frac{\varepsilon}{\sqrt{2}} \cup \|\gamma_n - \gamma\| \geq \frac{\varepsilon}{\sqrt{2}}\Big\}.$$

Using the subadditivity axiom of an uncertain measure, we obtain

$$\Big\{n \in \mathbb{N}\colon \frac{1}{\lambda_n}\big|\big\{k \in I_n\colon M\big(\|\zeta_k - \zeta\| \geq \varepsilon\big) \geq \delta\big\}\big| \geq \vartheta\Big\} \subseteq$$
$$\subseteq \Big\{n \in \mathbb{N}\colon \frac{1}{\lambda_n}\Big|\Big\{k \in I_n\colon M\Big(\Big\|\xi_k - \xi\Big\| \geq \frac{\varepsilon}{\sqrt{2}}\Big) \geq \delta\Big\}\Big| \geq \vartheta\Big\} \cup$$
$$\cup \Big\{n \in \mathbb{N}\colon \frac{1}{\lambda_n}\Big|\Big\{k \in I_n\colon M\Big(\Big\|\gamma_k - \gamma\Big\| \geq \frac{\varepsilon}{\sqrt{2}}\Big) \geq \delta\Big\}\Big| \geq \vartheta\Big\} \in \mathcal{I}.$$

Hence, we have

$$\Big\{r \in \mathbb{N}\colon \frac{1}{\lambda_n}\Big|\Big\{k \in I_n\colon M\big(\|\zeta_k - \zeta\| \geq \varepsilon\big) \geq \delta\Big\}\Big| \geq \vartheta\Big\} \in \mathcal{I}.$$

Thus, $\{\zeta_n\}$ $\lambda_{\mathcal{I}}$-statistically converges in measure to ζ. □

Proposition 3. *Let $\{\zeta_n\}$ be a sequence of complex uncertain variables with real part $\{\xi_n\}$ and imaginary part $\{\gamma_n\}$, respectively, for $n = 1, 2, \ldots$. If the sequences $\{\xi_n\}$ and $\{\gamma_n\}$ $\lambda_{\mathcal{I}}$-statistically converge in measure to ξ and γ, respectively, then the sequence $\{\zeta_n\}$ $\lambda_{\mathcal{I}}$-statistically converges in distribution to $\zeta = \xi + i\gamma$.*

Proof. Let $c = a + ib$ be a given continuity point of the complex uncertainty distribution Φ. On the other hand, for any $\alpha > a$, $\beta > b$, we have

$$\big\{\xi_n \leq a, \gamma_n \leq b\big\} = \big\{\xi_n \leq a, \gamma_n \leq b, \xi \leq \alpha, \gamma \leq \beta\big\} \cup \big\{\xi_n \leq a, \gamma_n \leq b, \xi > \alpha, \gamma > \beta\big\} \cup$$
$$\cup \big\{\xi_n \leq a, \gamma_n \leq b, \xi \leq \alpha, \gamma > \beta\big\} \cup \big\{\xi_n \leq a, \gamma_n \leq b, \xi > \alpha, \gamma \leq \beta\big\} \subset$$
$$\subset \big\{\xi \leq a, \gamma \leq b\big\} \cup \big\{|\xi_n - \xi| \geq \alpha - a\big\} \cup \big\{|\gamma_n - \gamma| \geq \beta - b\big\}.$$

It follows from the subadditivity axiom that

$$\Phi_n(c) = \Phi_n(a + ib) \leq \Phi(\alpha + i\beta) + M\big\{|\xi_n - \xi| \geq \alpha - a\big\} + M\big\{|\gamma_n - \gamma| \geq \beta - b\big\}.$$

Since $\{\xi_n\}$ and $\{\gamma_n\}$ $\lambda_{\mathcal{I}}$-statistically converge in measure to ξ and γ, respectively, hence, for any small number $\varepsilon, \delta > 0$ we have

$$\Big\{n \in \mathbb{N}\colon \frac{1}{\lambda_n}\big|\big\{k \in I_n\colon M\big(\|\xi_k - \xi\| \geq \alpha - a\big) \geq \varepsilon\big\}\big| \geq \delta\Big\} \in \mathcal{I},$$
$$\Big\{n \in \mathbb{N}\colon \frac{1}{\lambda_n}\big|\big\{k \in I_n\colon M\big(\|\gamma_k - \gamma\| \geq \beta - b\big) \geq \varepsilon\big\}\big| \geq \delta\Big\} \in \mathcal{I}.$$

Thus, we obtain $\mathcal{I}\text{-}\limsup_{n\to\infty} \Phi_n(c) \le \Phi(\alpha+i\beta)$ for any $\alpha > a$, $\beta > b$. Letting $\alpha+i\beta \to a+ib$, we get

$$\mathcal{I}\text{-}\limsup_{n\to\infty} \Phi_n(c) \le \Phi(c). \tag{1}$$

On the other hand, for any $x < a,\ y < b$ we have

$$\begin{aligned}\{\xi \le x, \gamma \le y\} &= \{\xi_n \le a, \gamma_n \le b, \xi \le x, \gamma \le y\} \cup \{\xi_n \le a, \gamma_n \le b, \xi \le x, \gamma \le y\} \cup \\ &\cup \{\xi_n > a, \gamma_n \le b, \xi \le x, \gamma \le y\} \cup \{\xi_n > a, \gamma_n > b, \xi \le x, \gamma \le y\} \subset \\ &\subset \{\xi_n \le a, \gamma_n \le b\} \cup \{|\xi_n - \xi| \ge a - x\} \cup \{|\gamma_n - \gamma| \ge b - y\}.\end{aligned}$$

This implies,

$$\Phi(x+iy) \le \Phi_n(a+ib) + M\{\|\xi_n - \xi\| \ge a - x\} + M\{\|\gamma_n - \gamma\| \ge b - y\}.$$

Since

$$\left\{n \in \mathbb{N} \colon \frac{1}{\lambda_n}\left|\left\{k \in I_n \colon M\left(\|\zeta_k - \zeta\| \ge a - x\right) \ge \varepsilon\right\}\right| \ge \delta\right\} \in \mathcal{I},$$
$$\left\{n \in \mathbb{N} \colon \frac{1}{\lambda_n}\left|\left\{k \in I_n \colon M\left(\|\gamma_k - \gamma\| \ge b - y\right) \ge \varepsilon\right\}\right| \ge \delta\right\} \in \mathcal{I},$$

we obtain

$$\Phi(x+iy) \le \mathcal{I}\text{-}\liminf_{n\to\infty} \Phi_n(a+ib)$$

for any $x < a, y < b$. Taking $x+iy \to a+ib$, we get

$$\Phi(c) \le \mathcal{I}\text{-}\liminf_{n\to\infty} \Phi_n(c) \tag{2}$$

It follows from (1) and (2) that $\Phi_n(c) \to \Phi(c)$ as $n \to \infty$. Hence, the sequence $\{\zeta_n\}$ is $\lambda_{\mathcal{I}}$-statistically convergent in distribution to $\zeta = \xi + i\gamma$. □

Remark 2. Converse of the above theorem is not necessarily true. i.e. $\lambda_{\mathcal{I}}$-statistical convergence in distribution does not imply $\lambda_{\mathcal{I}}$-statistical convergence in measure. The following example illustrates this.

Example 2. Consider the uncertainty space (Γ, L, M) to be $\{\gamma_1, \gamma_2\}$ with $M(\gamma_j) = \frac{1}{2}$, $j = 1, 2$. We define a complex uncertain variable as

$$\zeta(\gamma) = \begin{cases} i, & \text{if } \gamma = \gamma_1, \\ -i, & \text{if } \gamma = \gamma_2. \end{cases}$$

We also define $\zeta_n = -\zeta$ for $n = 1, 2, \ldots$. Then ζ_n and ζ have the same distribution

$$\Phi_n(c) = \Phi_n(a+ib) = \begin{cases} 0, & \text{if } a < 0, -\infty < b < +\infty; \\ 0, & \text{if } a \ge 0, b < -1; \\ \frac{1}{2}, & \text{if } a \ge 0, -1 \le b < 1; \\ 1, & \text{if } a \ge 0, b \ge 1. \end{cases}$$

Then $\{\zeta_n\}$ is $\lambda_{\mathcal{I}}$-statistically convergent in distribution to ζ. However, for a given $\varepsilon > 0$, we have

$$\Big\{n \in \mathbb{N}\colon \frac{1}{\lambda_n}\big|\big\{k \in I_n\colon M\big(\|\xi_k - \xi\| \geq \varepsilon\big) \geq 1\big\}\big| \geq \delta\Big\} =$$
$$= \Big\{n \in \mathbb{N}\colon \frac{1}{\lambda_n}\big|\big\{k \in I_n\colon M\big(\gamma\colon \|\xi_k(\gamma) - \xi(\gamma)\| \geq \varepsilon\big) \geq 1\big\}\big| \geq \delta\Big\} \in \mathcal{I}.$$

Thus, the sequence $\{\zeta_n\}$ does not $\lambda_{\mathcal{I}}$-statistically converge in measure to ζ. By Theorem 4, the real part and imaginary part of $\{\zeta_n\}$ also do not $\lambda_{\mathcal{I}}$-statistically converge in measure. In addition, since $\zeta_n = -\zeta$ for $n = 1, 2, \ldots$, the sequence $\{\zeta_n\}$ does not $\lambda_{\mathcal{I}}$-statistically converge *a.s.* to ζ.

4.3. $\lambda_{\mathcal{I}}$-statistically convergence *a.s.* and $\lambda_{\mathcal{I}}$-statistically converge in measure. $\lambda_{\mathcal{I}}$-statistically convergence *a.s.* does not imply $\lambda_{\mathcal{I}}$-statistically convergence in measure.

Example 3. Consider the uncertainty space (Γ, L, M) to be $\gamma_1, \gamma_2, \ldots$ with

$$M\{\Lambda\} = \begin{cases} \sup\limits_{\gamma_n \in \Lambda} \frac{n}{2n+1}, & \text{if } \sup\limits_{\gamma_n \in \Lambda} \frac{n}{2n+1} < 0.5;\\ 1 - \sup\limits_{\gamma_n \in \Lambda^c} \frac{n}{2n+1}, & \text{if } \sup\limits_{\gamma_n \in \Lambda^c} \frac{n}{2n+1} < 0.5;\\ 0.5, & \text{otherwise} \end{cases}$$

and we define a complex uncertain variable as

$$\zeta_n(\gamma) = \begin{cases} in, & \text{if } \gamma = \gamma_n,\\ 0, & \text{otherwise.} \end{cases}$$

for $n = 1, 2, \ldots$ and $\zeta \equiv 0$. Then the sequence $\{\zeta_n\}$ $\lambda_{\mathcal{I}}$-statistically converges *a.s.* to ζ. However, for some small number $\varepsilon, \delta > 0$, we have

$$\Big\{n \in \mathbb{N}\colon \frac{1}{\lambda_n}\Big|\Big\{k \in I_n\colon M\big(\|\xi_k - \xi\| \geq \varepsilon\big) \geq \frac{1}{2}\Big\}\Big| \geq \delta\Big\} =$$
$$= \Big\{n \in \mathbb{N}\colon \frac{1}{\lambda_n}\Big|\Big\{k \in I_n\colon M\big(\gamma\colon \|\zeta_k(\gamma) - \zeta(\gamma)\| \geq \varepsilon\big) \geq \frac{1}{2}\Big\}\Big| \geq \delta\Big\} =$$
$$= \Big\{n \in \mathbb{N}\colon \frac{1}{\lambda_n}\Big|\Big\{k \in I_n\colon M\{\gamma_n\} \geq \frac{1}{2}\Big\}\Big| \geq \delta\Big\}.$$

Therefore the sequence $\{\zeta_n\}$ does not $\lambda_{\mathcal{I}}$-statistically converge in measure to ζ.

Remark 3. $\lambda_{\mathcal{I}}$-statistically convergence in measure does not imply $\lambda_{\mathcal{I}}$-statistically convergence *a.s.*

Example 4. Consider the uncertainty space (Γ, L, M) to be $[0, 1]$ with the Borel algebra and the Lebesgue measure. For any positive integer n, there is an integer p such that $n = 2^p + k$ where k is an integer between 0 and $2^p - 1$. Then, we define a complex uncertain variable by

$$\zeta_n(\gamma) = \begin{cases} i, & \text{if } k \cdot 2^{-p} \leq \gamma \leq (k+1) \cdot 2^{-p};\\ 0, & \text{otherwise.} \end{cases}$$

for $n = 1, 2, \ldots$ and $\zeta \equiv 0$. For some small number $\varepsilon, \delta, \vartheta > 0$ and $n \geq 2$, we have

$$\Big\{n \in \mathbb{N}\colon \frac{1}{\lambda_n}\big|\big\{k \in I_n\colon M\big(\|\xi_k - \xi\| \geq \varepsilon\big) \geq \delta\big\}\big| \geq \vartheta\Big\} =$$
$$= \Big\{n \in \mathbb{N}\colon \frac{1}{\lambda_n}\big|\big\{k \in I_n\colon M\big(\gamma\colon \|\zeta_k(\gamma) - \zeta(\gamma)\| \geq \varepsilon\big) \geq \delta\big\}\big| \geq \vartheta\Big\} =$$
$$= \Big\{n \in \mathbb{N}\colon \frac{1}{\lambda_n}\big|\big\{k \in I_n\colon M\{\gamma_n\} \geq \delta\big\}\big| \geq \vartheta\Big\}.$$

Thus, the sequence $\{\zeta_n\}$ $\lambda_{\mathcal{I}}$-statistically converges in measure to ζ. In addition for every $\varepsilon, \delta > 0$ we have $\Big\{n \in \mathbb{N}\colon \frac{1}{\lambda_n}\big|\big\{k \in I_n\colon E\big(\|\zeta_k - \zeta\|\big) \geq \varepsilon\big\}\big| \geq \delta\Big\} \in \mathcal{I}$. Hence, the sequence $\{\zeta_n\}$ also $\lambda_{\mathcal{I}}$-statistically converges in mean to ζ. However, for any $\gamma \in [0, 1]$, there is an infinite number of intervals of the form $[k \cdot 2^{-p}, (k + 1) \cdot 2^{-p}]$ containing γ. Thus, $\zeta_n(\gamma)$ does not $\lambda_{\mathcal{I}}$-statistically converge to 0. In other words, the sequence $\{\zeta_n\}$ does not $\lambda_{\mathcal{I}}$-statistically converge *a.s.* to ζ.

4.4. $\lambda_{\mathcal{I}}$-statistically convergence *a.s.* and $\lambda_{\mathcal{I}}$-statistically converge in mean. $\lambda_{\mathcal{I}}$-statistically convergence *a.s.* does not imply $\lambda_{\mathcal{I}}$-statistically convergence in mean.

Example 5. Consider the uncertainty space (Γ, L, M) to be $\{\gamma_1, \gamma_2\}$ with $M\{\Lambda\} = \sum_{\gamma_n \in \Lambda} 3^{-n}$. The complex uncertain variables are defined by

$$\zeta_n(\gamma) = \begin{cases} i3^n, & \text{if } \gamma = \gamma_n; \\ 0, & \text{otherwise.} \end{cases}$$

for $n = 1, 2, \ldots$ and $\zeta \equiv 0$. Then, the sequence $\{\zeta_n\}$ $\lambda_{\mathcal{I}}$-statistically converges *a.s.* to ζ. However, the uncertainty distributions of $\|\zeta_n\|$ are

$$\Phi_n(x) = \begin{cases} 0, & \text{if } x < 0; \\ 1 - \frac{1}{3^n}, & \text{if } 0 \leq x < 3^n; \\ 1, & \text{if } x \geq 3^n, \end{cases}$$

for $n = 1, 2, \ldots$, respectively. Then, we have $\{n \in \mathbb{N}\colon \frac{1}{\lambda_n}\big|\big\{k \in I_n\colon E\big(\|\zeta_k - \zeta\|\big) \geq 1\big\}\big| \geq \delta\} \in \mathcal{I}$. Therefore, the sequence $\{\zeta_n\}$ does not $\lambda_{\mathcal{I}}$-statistically converge in mean to ζ.

From Example 5, we can obtain that $\lambda_{\mathcal{I}}$-statistically convergence in mean does not imply $\lambda_{\mathcal{I}}$-statistically converge *a.s.*

4.5. $\lambda_{\mathcal{I}}$-statistically convergence a.s. and $\lambda_{\mathcal{I}}$-statistically convergence uniformly a.s.

Proposition 4. *Let $\zeta, \zeta_1, \zeta_2, \ldots$ be complex uncertain variables. Then $\{\zeta_n\}$ $\lambda_{\mathcal{I}}$-statistically converges a.s. to ζ if and only if for any $\varepsilon, \delta, \vartheta > 0$, we have*

$$\Big\{n \in \mathbb{N}\colon \frac{1}{\lambda_n}\Big|\Big\{k \in I_n\colon M\Big(\bigcap_{k=1}^{\infty}\bigcup_{n=k}^{\infty}\Big\|\zeta_n - \zeta\Big\| \geq \varepsilon\Big) \geq \delta\Big\}\Big| \geq \vartheta\Big\} \in \mathcal{I}.$$

Proof. By the definition of $\lambda_{\mathcal{I}}$-statistical convergence *a.s.*, we have that there exists an event Λ with $M\Big(\Lambda\Big) = 1$ such that$\{n \in \mathbb{N}\colon \frac{1}{\lambda_n}\big|\big\{k \in I_n\colon \big\|\zeta_k - \zeta\big\| \geq \varepsilon\big\}\big| \geq \delta\} \in \mathcal{I}$ for every $\varepsilon, \delta > 0$. Then for any $\varepsilon, \vartheta > 0$, there exists k such that $\big\|\zeta_n - \zeta\big\| < \varepsilon$ where $n > k$ and for any $\gamma \in \Lambda$, that is equivalent to

$$\Big\{n \in \mathbb{N}\colon \frac{1}{\lambda_n}\Big|\Big\{k \in I_n\colon M\Big(\bigcap_{k=1}^{\infty}\bigcup_{n=k}^{\infty}\big\|\zeta_n - \zeta\big\| < \varepsilon\Big) \geq 1\Big\}\Big| \geq \vartheta\Big\} \in \mathcal{I}.$$

It follows from the duality axiom of an uncertain measure that

$$\Big\{n \in \mathbb{N}\colon \frac{1}{\lambda_n}\Big|\Big\{k \in I_n\colon M\Big(\bigcap_{k=1}^{\infty}\bigcup_{n=k}^{\infty}\big\|\zeta_n - \zeta\big\| \geq \varepsilon\Big) \geq \delta\Big\}\Big| \geq \vartheta\Big\} \in \mathcal{I}.$$

□

Proposition 5. *Let $\zeta, \zeta_1, \zeta_2, \dots$ be complex uncertain variables. Then $\big\{\zeta_n\big\}$ $\lambda_{\mathcal{I}}$-statistically converges uniformly a.s. to ζ if and only if for any $\varepsilon, \delta, \vartheta > 0$, we have*

$$\Big\{n \in \mathbb{N}\colon \frac{1}{\lambda_n}\Big|\Big\{k \in I_n\colon M\Big(\bigcup_{n=k}^{\infty}\big\|\zeta_k - \zeta\big\| \geq \varepsilon\Big) \geq \delta\Big\}\Big| \geq \vartheta\Big\} \in \mathcal{I}.$$

Proof. If $\big\{\zeta_n\big\}$ $\lambda_{\mathcal{I}}$-statistically converges uniformly *a.s.* to ζ, then for any $\vartheta > 0$ there exists K such that $M\big\{K\big\} < \vartheta$ and $\big\{\zeta_n\big\}$ $\lambda_{\mathcal{I}}$-statistically converges to ζ on $\Gamma - K$. Thus, for any $\varepsilon > 0$, there exists $k > 0$ such that $\big\|\zeta_k - \zeta\big\| < \varepsilon$ where $n > k$ and for any $\gamma \in \Gamma - K$. That is

$$\bigcup_{n=k}^{\infty}\{\|\zeta_n - \zeta\| \geq \varepsilon\} \subset K.$$

It follows from the subadditivity axiom that

$$\Big\{n \in \mathbb{N}\colon \frac{1}{\lambda_n}\Big|\Big\{k \in I_n\colon \bigcup_{n=k}^{\infty}\Big\{\big\|\zeta_n - \zeta\big\| \geq \varepsilon\Big\}\Big\}\Big| \geq \delta\Big\} \subseteq \delta^{\mathcal{I}_\theta}M\Big\{K\Big\} \subseteq \vartheta.$$

Then$\{n \in \mathbb{N}\colon \frac{1}{\lambda_n}\big|\big\{k \in I_n\colon M\big(\bigcup_{n=k}^{\infty}\big\|\zeta_n - \zeta\big\| \geq \varepsilon\big) \geq \delta\big\}\big| \geq \vartheta\} \in \mathcal{I}$. On the contrary, if

$$\Big\{n \in \mathbb{N}\colon \frac{1}{\lambda_n}\Big|\Big\{k \in I_n\colon M\Big(\bigcup_{n=k}^{\infty}\big\|\zeta_n - \zeta\big\| \geq \varepsilon\Big) \geq \delta\Big\}\Big| \geq \vartheta\Big\} \in \mathcal{I},$$

for any $\varepsilon, \delta, \vartheta > 0$, then for given $\delta > 0$ and $m \geq 1$, there exists m_k such that

$$\delta\Big(M\Big(\bigcup_{n=m_k}^{\infty}\Big\{\|\zeta_n - \zeta\| \geq \frac{1}{m}\Big\}\Big)\Big) < \frac{\delta}{2^m}.$$

Let $K = \bigcup_{m=1}^{\infty}\bigcup_{n=m_k}^{\infty}\Big\{\big\|\zeta_n - \zeta\big\| \geq \frac{1}{m}\Big\}$. Then

$$\delta\Big(M\{K\}\Big) \leq \sum_{m=1}^{\infty}\delta\Big(M\Big(\bigcup_{n=m_k}^{\infty}\Big\{\big\|\zeta_n - \zeta\big\| \geq \frac{1}{m}\Big\}\Big)\Big) \leq \sum_{m=1}^{\infty}\frac{\delta}{2^m}.$$

In addition, we get $\mathcal{I}\text{-}\sup_{\gamma \in \Gamma - K}\big\|\zeta_n - \zeta\big\| < \frac{1}{m}$ for any $m = 1, 2, \dots$ and $n > m_k$. □

Theorem 1. *If a sequence $\{\zeta_n\}$ of complex uncertain variables $\lambda_{\mathcal{I}}$-statistically converges uniformly a.s. to ζ, then $\{\zeta_n\}$ $\lambda_{\mathcal{I}}$-statistically converges a.s. to ζ.*

Proof. It follows from the above proposition that if $\{\zeta_n\}$ $\lambda_{\mathcal{I}}$-statistically converges uniformly *a.s.* to ζ, then

$$\Big\{n \in \mathbb{N}\colon \frac{1}{\lambda_n}\Big|\Big\{k \in I_n\colon M\Big(\bigcup_{n=k}^{\infty}\{\|\zeta_n - \zeta\| \geq \varepsilon\}\Big) \geq \delta\Big\}\Big| \geq \vartheta\Big\} \in \mathcal{I}.$$

Since

$$\delta\Big(M\Big(\bigcap_{k=1}^{\infty}\bigcup_{n=k}^{\infty}\{\|\zeta_n - \zeta\| \geq \varepsilon\}\Big)\Big) \leq \delta\Big(M\Big(\bigcup_{n=k}^{\infty}\{\|\zeta_n - \zeta\| \geq \varepsilon\}\Big)\Big)$$

taking the limit as $n \to \infty$ on both side of above inequality, we obtain

$$\delta\Big(M\Big(\bigcap_{k=1}^{\infty}\bigcup_{n=k}^{\infty}\{\|\zeta_n - \zeta\| \geq \varepsilon\}\Big)\Big) = 0.$$

By the first proposition, $\{\zeta_n\}$ $\lambda_{\mathcal{I}}$-statistically converges *a.s.* to ζ. □

4.6. $\lambda_{\mathcal{I}}$-statistically convergence uniformly *a.s.* and $\lambda_{\mathcal{I}}$-statistically convergence in measure.

Theorem 2. *If a sequence $\{\zeta_n\}$ of complex uncertain variables $\lambda_{\mathcal{I}}$-statistically converges uniformly a.s. to ζ, then $\{\zeta_n\}$ $\lambda_{\mathcal{I}}$-statistically converges in measure to ζ.*

Proof. If the sequence $\{\zeta_n\}$ complex uncertain variables $\lambda_{\mathcal{I}}$-statistically converges uniformly *a.s.* to ζ, then from the proposition above we have

$$\Big\{r \in \mathbb{N}\colon \frac{1}{\lambda_n}\Big|\Big\{k \in I_r\colon M\Big(\bigcup_{n=k}^{\infty}\{\|\zeta_n - \zeta\| \geq \varepsilon\}\Big) \geq \delta\Big\}\Big| \geq \vartheta\Big\} \in \mathcal{I},$$

$$\delta\Big(M\Big(\{\|\zeta_n - \zeta\|\} \geq \varepsilon\Big)\Big) \leq \delta\Big(M\Big(\bigcup_{n=k}^{\infty}\{\|\zeta_n - \zeta\| \geq \varepsilon\}\Big)\Big).$$

Letting $n \to \infty$, we can obtain $\{\zeta_n\}$ $\lambda_{\mathcal{I}}$-statistically converges in measure to ζ. □

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