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# The Lomax regression model with residual analysis: an application to insurance data

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## ABSTRACT

In this paper, we introduce a new regression model, called *Lomax regression model*, as an alternative to the gamma regression model. The maximum-likelihood method is used to estimate the unknown parameters of the proposed model, and the finite sample performance of the maximum-likelihood estimation method is evaluated by means of the Monte-Carlo simulation study. The randomized quantile residuals are used to check the adequacy of the fitted model. The insurance data are analyzed to demonstrate the usefulness of the proposed regression model against the gamma regression model.

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## 1. Introduction

The heavy-tailed distributions play an important role in modeling the finance, insurance and actuarial data sets. The normal distribution is inadequate in many insurance applications since the empirical distribution of insurance loss is asymmetric and has thick tails. Additionally, the empirical distribution of losses exhibits right skewness and excess kurtosis as well as non-negative values which means that the normal distribution cannot be used in this case. In statistic literature, many statistical distributions are available to model the insurance data (see [10]). The modeling of the tail distribution of insurance losses is an important issue in actuarial risk modeling, because the losses are located at the tail of the underlying loss distribution. Therefore, misspecification of the loss distribution yields the over-estimation or under-estimation of the real insurance loss. The predictions related to the rare events are only possible with a flexible distribution which enables us to model the empirical tail distribution of the losses accurately. The commonly used distributions in modeling the insurance loss data are the Weibull, log-gamma, Burr and the generalized Pareto (GP) distributions. Ahn *et al.* [1] used the log phase-type (logPH) distribution to model the Danish fire insurance data and compared the logPH distribution with GP distribution.

The gamma regression is a widely used model in modeling the right-skewed response variable (see [8]). Assuming that the response variable follows a gamma distribution, Cepeda-Cuervo [6] introduced the gamma regression model with joint modeling of

the mean and shape parameters. Cepeda-Cuervo *et al.* [5] introduced a new diagnostic tool to analyze the residuals of the gamma regression model. The weighted-exponential (WE) regression model, an alternative model to the gamma regression, was proposed by Altun [2]. The empirical results of [2] have shown that the WE regression model has worked better than the gamma regression model for the cabinet durations data set. In the literature, there are no many alternative regression models to the gamma regression model in modeling the right-skewed response variable.

The purpose of the presented study is to develop a new statistical model to model the insurance loss data with much more accuracy than the gamma regression model. To achieve this purpose, a new regression model for the right-skewed response variable is introduced through the Lomax distribution based on the generalized linear model framework. Lomax [12] introduced the Lomax distribution, known as Pareto Type II distribution, which is widely used in economics, actuarial science and queueing theory. In recent years, the Lomax distribution has increased its popularity. The researchers have shown great interest to introduce new extensions of Lomax distribution such as McDonald Lomax [11], power Lomax [13] and double Lomax distributions [3]. These studies aim to increase the flexibility of Lomax distribution by adding extra (one or more) parameters. The proposed Lomax regression model gives an opportunity to simultaneously modeling the mean and shape parameters, as in the gamma regression model. The parameter estimation problem of the Lomax regression model is addressed with the maximum likelihood estimation (MLE) method. The performance of the MLEs of the Lomax regression model is discussed with a simulation study. The importance of the Lomax regression model in insurance loss modeling is illustrated with an application on a real data set. Two types of residuals are used to check the accuracy of the Lomax regression model.

The remaining parts of the presented paper are organized as follows: in Section 2, re-parametrized Lomax distribution and its regression model is presented. The residual analysis and parameter estimation of the Lomax regression model are also addressed in this section and an extensive simulation study. Section 3 is devoted to the empirical results of the study. The concluding remarks are given in Section 4.

## 2. The Lomax distribution and its regression model

Assume that  $Y$  is a random variable following a Lomax distribution. The probability density function (pdf) of  $Y$  is

$$f(y; \alpha, \lambda) = \frac{\alpha}{\lambda} \left[ 1 + \frac{y}{\lambda} \right]^{-(\alpha+1)}, \quad y \geq 0, \quad (1)$$

where  $\alpha > 0$  is the shape parameter and  $\lambda > 0$  is the scale parameter. The corresponding cumulative distribution function (cdf) to (1) is

$$F(y; \alpha, \lambda) = 1 - \left[ 1 + \frac{y}{\lambda} \right]^{-\alpha}. \quad (2)$$

The mean and variance of the Lomax distribution are given, respectively, by

$$E(X) = \frac{\lambda}{\alpha - 1}, \quad (3)$$

$$\text{Var}(X) = \frac{\lambda^2 \alpha}{(\alpha - 1)^2 (\alpha - 2)}. \tag{4}$$

Note that the mean and variance are only valid for  $\alpha > 1$  and  $\alpha > 2$ , respectively. The suitable re-parametrization of the Lomax distribution is important to introduce the Lomax regression model.

**Proposition 2.1.** *Let  $\lambda = \mu(\alpha - 1)$ , then the pdf of  $Y$  is*

$$f(y; \mu, \alpha) = \frac{\alpha}{\mu(\alpha - 1)} \left[ 1 + \frac{y}{\mu(\alpha - 1)} \right]^{-(\alpha+1)}, \quad y \geq 0, \tag{5}$$

where  $\mu > 0$  and  $\alpha > 1$ . The parameter  $\mu$  controls the location of the distribution and  $\alpha$  treats as a shape parameter. Hereafter, (5) is denoted as  $Y \sim L(\mu, \alpha)$ . In this representation, the mean and variance of  $Y$  are

$$E(Y|\mu, \alpha) = \mu \tag{6}$$

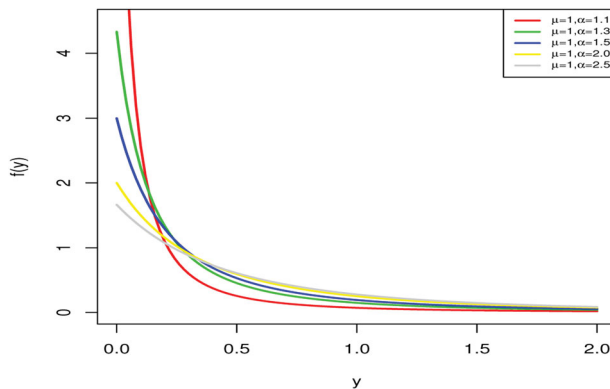
and

$$\text{Var}(Y) = \alpha \mu^2 / (\alpha - 2). \tag{7}$$

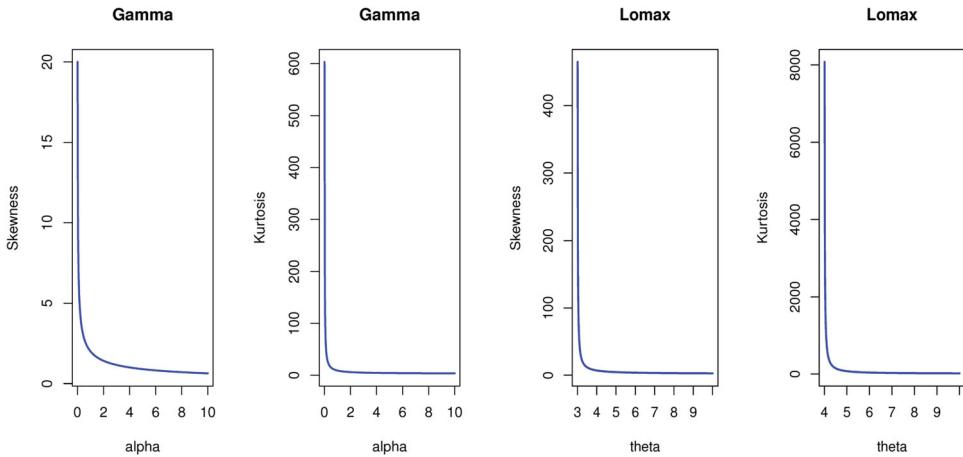
As seen from (7), the variance of the Lomax distribution is valid only if  $\alpha > 2$ . The variance of  $Y$  increases once the parameter  $\alpha$  decreases. Here, we refer the inverse of the parameter  $\alpha$ , say  $\alpha^{-1}$ , as a dispersion parameter.

The pdf plots of the re-parametrized Lomax distribution are displayed in Figure 1. As seen from these figures, the re-parametrized Lomax distribution is extremely right-skewed and the skewness decreases when the parameter  $\alpha$  increase for the fixed values of  $\mu$ .

The flexibility of the Lomax distribution is compared with the gamma distribution based on the skewness and kurtosis measures. Figure 2 displays the skewness and kurtosis values of the re-parametrized Lomax and re-parametrized gamma distribution (see [5] for re-parametrized gamma distribution). As seen from Figure 2, the ranges of the skewness and kurtosis measures of the Lomax distribution are wider than those of the gamma distribution. Therefore, the Lomax distribution is more preferable than the gamma distribution for



**Figure 1.** The pdf plots of the re-parametrized Lomax distribution.



**Figure 2.** The skewness and kurtosis plots of the Gamma (left) and Lomax (right) distributions.

the extremely right-skewed and leptokurtic data sets. Note that the skewness and kurtosis measures of the Lomax distribution are valid only if  $\alpha > 3$  and  $\alpha > 4$ , respectively.

Assume that we have a random sample,  $y_1, y_2, \dots, y_n$ , from  $L(\mu_i, \alpha)$  by unknown parameters  $\mu_i$  and  $\alpha$ . The Lomax regression model is introduced by using the appropriate link function, given by

$$g(\mu_i) = \mathbf{x}_i^T \boldsymbol{\beta}, \tag{8}$$

where  $\mu_i, i = 1, 2, 3, \dots, n$ , is the mean of the Lomax distribution,  $\boldsymbol{\beta} = (\beta_0, \beta_1, \dots, \beta_k)^T$  is the regression parameters,  $k$  is the number of the independent variables and  $\mathbf{x}_i = (x_{i1}, x_{i2}, \dots, x_{ik})^T$  represents the covariates. In (8),  $g(\cdot) : (0, \infty) \mapsto \Re$  is a link function which an increasing and twice differentiable function. The appropriate choice of link function is based on the domain of the random variable  $Y$ . Here, since the random variable  $Y$  is defined on  $\Re^+$ , the log-link function,  $\log(\mu_i) = \mathbf{x}_i^T \boldsymbol{\beta}$ , is suitable for this regression model. As seen in (7), the Lomax regression model has a heteroscedastic structure since its variance is a function of its mean. The variance of the response variable,  $y_i$ , is

$$\text{Var}(y_i) = \frac{\alpha \mu_i^2}{(\alpha - 2)} = \frac{\alpha g^{-1}(\mathbf{x}_i^T \boldsymbol{\beta})^2}{(\alpha - 2)}. \tag{9}$$

Using the re-parametrized pdf of the Lomax, given in (5), the log-likelihood function of the Lomax regression model is

$$\ell(\boldsymbol{\tau}) = \sum_{i=1}^n \ln\left(\frac{\alpha}{\mu_i(\alpha - 1)}\right) - (\alpha + 1) \sum_{i=1}^n \ln\left(1 + \frac{y_i}{\mu_i(\alpha - 1)}\right), \tag{10}$$

where  $\mu_i = \exp(\mathbf{x}_i^T \boldsymbol{\beta})$ . The dispersion parameter may not be constant for all observations. In this case, the shape parameter of the Lomax regression model can be modeled with an appropriate link function. In this case, this model is called as variable dispersion Lomax

regression model which is defined as

$$\begin{aligned} g_1(\mu_i) &= \mathbf{x}_i^T \boldsymbol{\beta}, \\ g_2(\alpha_i) &= \mathbf{z}_i^T \boldsymbol{\gamma}, \end{aligned} \tag{11}$$

where  $\boldsymbol{\beta} = (\beta_0, \beta_1, \dots, \beta_k)^T$  and  $\boldsymbol{\gamma} = (\gamma_0, \gamma_1, \dots, \gamma_h)^T$  are the vectors of regression parameters and  $\mathbf{x}_i = (x_{i1}, x_{i2}, \dots, x_{ik})^T$  and  $\mathbf{z}_i = (z_{i1}, z_{i2}, \dots, z_{ih})^T$  are the vector of covariates. The log-likelihood function of the above regression structure can be simply written by replacing the  $\alpha$  with  $\alpha_i$  in (12).

**2.1. Estimation**

The MLE method is used to estimate the dispersion parameter,  $\alpha$ , of the Lomax regression model as well as regression parameters,  $\boldsymbol{\beta} = (\beta_0, \beta_1, \dots, \beta_k)^T$ . Assume that the random sample  $y_1, y_2, \dots, y_n$  comes from the Lomax distribution with unknown parameters,  $\mu_i$  and  $\alpha_i$ . Using this random sample, the variable dispersion Lomax regression model has the following log-likelihood function

$$\ell(\boldsymbol{\tau}) = \sum_{i=1}^n \ln \left( \frac{\alpha_i}{\mu_i (\alpha_i - 1)} \right) - (\alpha_i + 1) \sum_{i=1}^n \ln \left( 1 + \frac{y_i}{\mu_i (\alpha_i - 1)} \right), \tag{12}$$

where  $\boldsymbol{\tau} = (\boldsymbol{\beta}, \boldsymbol{\gamma})$  and  $\mu_i$  and  $\alpha_i$  are in (11). The partial derivatives of (12) according to the unknown regression parameters, we have the following likelihood equations

$$\frac{\partial \ell}{\partial \boldsymbol{\beta}} = \sum_{i=1}^n \left( \frac{\mathbf{x}_i y_i \exp(-\mathbf{x}_i \boldsymbol{\beta})}{1 + (y_i \exp(-\mathbf{x}_i \boldsymbol{\beta})) / (\exp(\mathbf{z}_i \boldsymbol{\gamma}) - 1)} - \mathbf{x}_i \right), \tag{13}$$

$$\begin{aligned} \frac{\partial \ell}{\partial \boldsymbol{\gamma}} &= \sum_{i=1}^n \frac{\mathbf{z}_i y_i \exp(\mathbf{z}_i \boldsymbol{\gamma} - \mathbf{x}_i \boldsymbol{\beta})}{[1 + (y_i \exp(-\mathbf{x}_i \boldsymbol{\beta})) / (\exp(\mathbf{z}_i \boldsymbol{\gamma}) - 1)] (\exp(\mathbf{z}_i \boldsymbol{\gamma}) - 1)} \\ &\quad - \sum_{i=1}^n \mathbf{z}_i \exp(\mathbf{z}_i \boldsymbol{\gamma}) \ln \left( 1 + \frac{y_i \exp(-\mathbf{x}_i \boldsymbol{\beta})}{\exp(\mathbf{z}_i \boldsymbol{\gamma}) - 1} \right) \\ &\quad + \sum_{i=1}^n \exp(\mathbf{z}_i \boldsymbol{\gamma} - 1) \exp(\mathbf{x}_i \boldsymbol{\beta} - y_i \mathbf{z}_i) \\ &\quad \times \left( \frac{\mathbf{z}_i \exp(\mathbf{z}_i \boldsymbol{\gamma} - \mathbf{x}_i \boldsymbol{\beta})}{\exp(\mathbf{z}_i \boldsymbol{\gamma}) - 1} - \frac{\mathbf{z}_i \exp(2\mathbf{z}_i y_i - \mathbf{x}_i \boldsymbol{\beta})}{(\exp(\mathbf{z}_i \boldsymbol{\gamma}) - 1)^2} \right). \end{aligned} \tag{14}$$

The MLE of the unknown parameter vector,  $\boldsymbol{\tau}$ , can be obtained by simultaneous solution of the likelihood equations for zero,  $\frac{\partial \ell}{\partial \boldsymbol{\beta}} = 0, \frac{\partial \ell}{\partial \boldsymbol{\gamma}} = 0$ . However, it is not possible since the likelihood equations contain non-linear functions. In this case, the direct maximization of the log-likelihood function should be done by means of statistical software such as **R**, **Matlab** or **S-PLUS**. Here, we use the **optim** function of the **R** software to implement the Nelder–Mead optimization algorithm. Considering the regularity conditions of the MLE method, the asymptotic distribution of  $(\widehat{\boldsymbol{\tau}} - \boldsymbol{\tau})$  follows a  $(k + h)$ -variate normal distribution with zero mean and variance–covariance matrix  $K(\boldsymbol{\tau})^{-1}$ . The expected information

matrix can be replaced with  $(k + h) \times (k + h)$  observed information matrix which is used to obtain the asymptotic standard errors of the MLEs evaluated at  $\hat{\tau}$ . The `fdHess` function of R software is used to compute the observed information matrix.

## 2.2. Simulation

Now, we discuss the performance of the MLE method for estimating the parameters of the Lomax regression. We follow the results in [5] for the simulation study. The simulation replication is determined as  $N = 1000$ . Four different sample sizes are used,  $n = 20, 50, 250, 500$  and  $1000$ . The below regression structures are used:

$$\mu_i = \beta_0 + \beta_1 x_{i1}$$

and

$$\alpha_i = \exp(\gamma_0 + \gamma_1 z_{i1}),$$

where the covariates are generated from the standard uniform distribution. The true values of the regression parameters are determined as  $\beta_0 = 2, \beta_1 = 2, \gamma_0 = 0.5$  and  $\gamma_1 = 0.5$ . The response variable  $y_i$  is generated by using the generated  $\mu_i$  and  $\alpha_i$  parameters. The `optim` function is used to implement the Nelder–Mead optimization algorithm to obtain the parameter estimates of the Lomax regression model for each generated data set.

The simulation results are reported in Table 1. The simulation results are interpreted based on the estimated biases, averages of the estimates (AEs) and mean square errors (MSEs). The biases and MSEs should be near the zero, and the AEs should be near the true parameter values for large sample sizes. When the results given in Table 1 are examined, we conclude that the estimated biases and MSEs are near the zero for all sample sizes. As expected, the AEs are also stable for all sample sizes and near the true parameter values. These results verify that the MLEs of the parameters of the Lomax regression model are asymptotically unbiased and consistent.

**Table 1.** The simulation results of the Lomax regression model.

Sample size	Parameters	$\beta_0$	$\beta_1$	$\gamma_0$	$\gamma_1$
$n = 20$	AE	2.0145	1.8798	0.8511	0.7249
	Bias	0.0145	-0.1202	0.3511	0.2249
	MSE	0.4431	0.8655	0.5751	0.8121
$n = 50$	AE	2.0498	2.0717	0.7409	0.4147
	Bias	0.0498	0.0717	0.2409	-0.0853
	MSE	0.2755	0.6525	0.5842	0.7308
$n = 250$	AE	1.9787	2.0521	0.6289	0.4106
	Bias	-0.0213	0.0521	0.1289	-0.0894
	MSE	0.1056	0.2888	0.1779	0.5747
$n = 500$	AE	1.9859	2.0332	0.5639	0.4510
	Bias	-0.0141	0.0332	0.0639	-0.0490
	MSE	0.0623	0.1425	0.0715	0.2233
$n = 1000$	AE	1.9948	2.0149	0.5370	0.4624
	Bias	-0.0052	0.0149	0.0370	-0.0376
	MSE	0.0295	0.0745	0.0329	0.1213

### 2.3. Residuals analysis

Here, we use two residuals to check the accuracy of the fitted Lomax regression model. These are randomized quantile residuals by Dunn and Smyth [9] and Cox–Snell residuals by Cox and Snell [7].

#### 2.3.1. Cox–Snell residual

Cox–Snell residuals are given by (see [7])

$$\hat{e}_i = -\ln \left[ 1 - F \left( y_i | \hat{\beta}_i \hat{\gamma}_i \right) \right], \quad i = 1, 2, \dots, n, \tag{15}$$

where  $F(\cdot)$  is the cdf of the re-parametrized Lomax distribution. The Cox–Snell residuals follow an exponential distribution one scale parameter once the fitted model is valid for the data used.

#### 2.3.2. Randomized quantile residual

The randomized quantile residuals are given by (see [9])

$$\hat{r}_i = \Phi^{-1} \left( \hat{u}_i \right), \tag{16}$$

where  $\hat{u}_i = F(y_i | \hat{\beta}_i \hat{\gamma}_i)$ . Assume that the fitted model is valid for data used. In this case, the distribution of the randomized quantile residuals should be  $N(0, 1)$ . The normality of the randomized quantile residuals can be checked by graphically using the quantile-quantile or probability-probability (PP) plots.

## 3. Empirical study

In this section, we use the data set reported in **CASdatasets** package of R software. The data set consists of 12,513 individuals for which we have the driver age, the vehicle age, and the exposure. Exposure represents the level of risk for an individual. The insurance companies calculate the insurance premiums based on the individual’s exposure value. Therefore, the accurate modeling of the exposure is important to compute the more realistic insurance premium. The goal is to model the exposure ( $y$ ) with vehicle age (VA) ( $x_1$ ) and driver age (DA) ( $x_2$ ) variables. The gamma, WE and Lomax regression models are used to model this data set. The **Gammareg** package of [4] is used to obtain the unknown parameters of the gamma regression model. After obtaining the estimated values of the parameters of the gamma regression, we use these estimates as an initial parameter vector for the Lomax and WE regression models in estimation steps. The log-link function is used for both location and dispersion parameters. The below regression model is fitted by the gamma, WE and Lomax regression models for the used data:

$$\begin{aligned} \log(\mu_i) &= \beta_0 + \beta_1 VA_i + \beta_2 DA_i, \\ \log(\alpha_i) &= \gamma_1 DA_i. \end{aligned} \tag{17}$$

The estimated parameters and its standard errors, corresponding  $p$ -values as well as goodness-of-fit statistics are reported in Table 2. All the estimated parameters are found

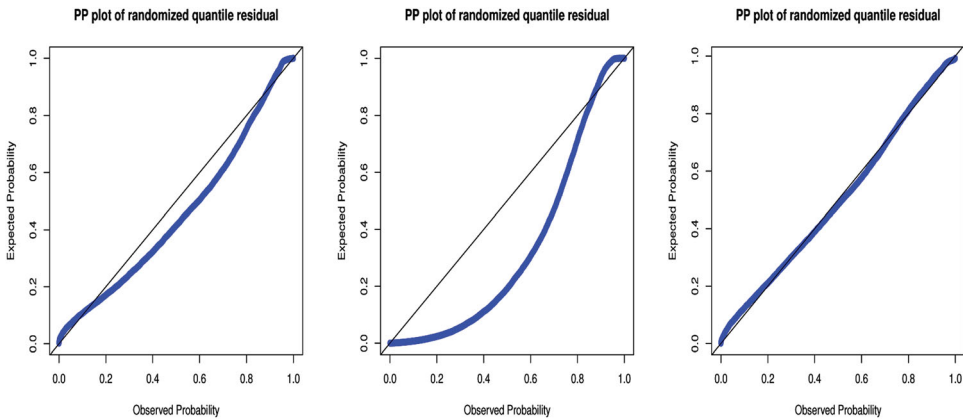


**Table 2.** The estimated parameters and corresponding standard errors (SE) of the fitted regression models.

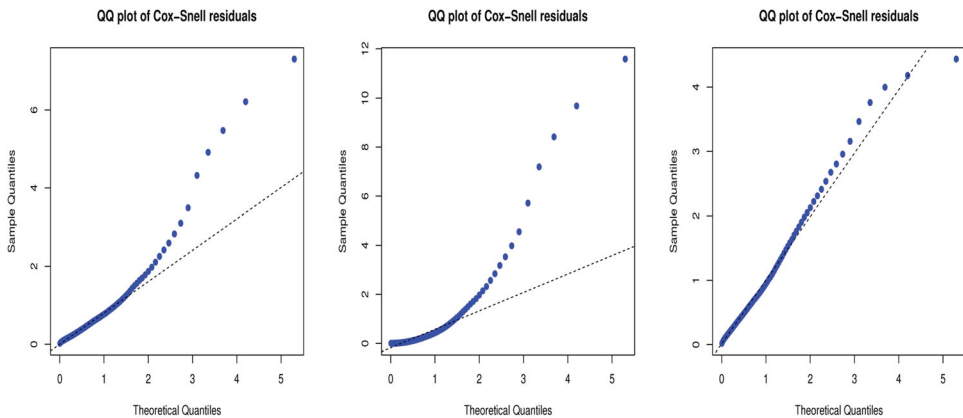
Parameters	Gamma			WE			Lomax		
	Estimate	SE	<i>p</i> -Value	Estimate	SE	<i>p</i> -Value	Estimate	SE	<i>p</i> -Value
$\beta_0$	6.103	0.044	< 0.001	6.154	0.059	< 0.001	5.805	0.059	< 0.001
$\beta_1$	-0.083	0.002	< 0.001	-0.081	0.002	< 0.001	-0.062	0.002	< 0.001
$\beta_2$	-0.018	0.001	< 0.001	-0.019	0.001	< 0.001	-0.014	0.001	< 0.001
$\gamma_1$	-0.007	< 0.001	< 0.001	0.631	< 0.001	< 0.001	0.012	0.000	< 0.001
$-\ell$	67,347.130			67,958.000			65,866.000		
AIC	134,702.260			135,924.000			131,740.000		
BIC	134,731.998			135,953.738			135,934.869		

statistically significant at any significance level for all fitted regression models. According to the estimated values of the parameters, we conclude that when the vehicle age and driver age increase, the exposure of the individual decreases. Moreover, when the driver’s age increases, the variance of the exposure decreases.

To decide the best-fitted model for the used data set, we calculate the  $-\ell$  AIC and BIC statistics. The Lomax regression model has lower values of these statistics than those of the gamma and WE regression models. So, we conclude that the Lomax regression model provides better modeling ability than the gamma and WE regression models for the used data. To check the accuracy of the fitted Lomax regression model, we compute the randomized quantile and Cox–Snell residuals. As mentioned before, when the fitted regression model is suitable for the data set, the randomized quantile residuals should be normally distributed with zero mean and unit variance. Also, Cox–Snell residuals should be distributed as exponential distributions with one scale parameter. The PP plots of the randomized quantile residuals are displayed in Figure 3 for all fitted regression models. From these figures, it is clear that the Lomax regression model is more appropriate than the gamma and WE regression models for used data since the plotted points are far from the diagonal line for the gamma and WE regression models. The quantile-quantile (QQ) plots of the Cox–Snell



**Figure 3.** The PP plots of randomized quantile residuals: gamma (left), WE (middle) and Lomax regression models (right).



**Figure 4.** The QQ plots of Cox–Snell residuals: gamma (left), WE (middle) and Lomax regression models (right).

residuals are displayed in Figure 4. From these figures, it is clear that the Cox–Snell residuals of the Lomax regression model are more near the diagonal line than those of the gamma and WE regression models.

One can prefer to fit the WE regression model without using the estimated parameters of the gamma regression model as an initial parameter vector. In this case, we suggest using a generalized simulated annealing optimization algorithm which can be implemented by the **GenSA** package of R software. This package is very useful to find a suitable initial parameter vector for iterative optimization algorithms. The obtained initial parameter vector by the **GenSA** package is  $\tau = (4.509, -0.041, -0.031, 0.015)$ . Using it as an initial parameter vector in the Nelder–Mead algorithm, the estimated parameter values of the Lomax regression model are the same with the values reported in Table 2.

## 4. Conclusion

In this paper, a new regression model, called the *Lomax regression model*, is introduced as an alternative to gamma WE regression models. The maximum-likelihood method is discussed to estimate the unknown parameters of the proposed model via the Monte-Carlo simulation study. The randomized quantile and Cox–Snell residuals are used to check the accuracy of the fitted models. The insurance data set is analyzed to prove the applicability and importance of the Lomax regression model over the gamma and WE regression models. Empirical findings reveal that the Lomax regression model could be very attractive in modeling the extremely right-skewed response variable. We believe that the Lomax regression model will attract attention in other fields of science.

## Disclosure statement

No potential conflict of interest was reported by the author(s).

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