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Research Article

New Results on $\mathcal{I}_2$-Statistically Limit Points and $\mathcal{I}_2$-Statistically Cluster Points of Sequences of Fuzzy Numbers

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In this paper, some existing theories on convergence of fuzzy number sequences are extended to $\mathcal{I}_2$-statistical convergence of fuzzy number sequence. Also, we broaden the notions of $\mathcal{I}$-statistical limit points and $\mathcal{I}$-statistical cluster points of a sequence of fuzzy numbers to $\mathcal{I}_2$-statistical limit points and $\mathcal{I}_2$-statistical cluster points of a double sequence of fuzzy numbers. Also, the researchers focus on important fundamental features of the set of all $\mathcal{I}_2$-statistical cluster points and the set of all $\mathcal{I}_2$-statistical limit points of a double sequence of fuzzy numbers and examine the relationship between them.

1. Introduction

The theory of statistical convergence reverts to the first edition of monograph of Zygmund [1]. Statistical convergence of number sequences was given by Fast [2] and then was reissued by Schoenberg [3] independently for real and complex sequences. This conception was studied for the double sequences by Mursaleen and Edely [4]. Fridy [5] considered statistical limit points and statistical cluster points of real number sequences. When we focus on the statistical convergence in the literature, we meet Fridy [6], Temizsu and Mikail [7], Braha et al. [8], Nuray and Ruckle [9], Das et al. [10], and so many other researchers (see [11–13]).

The concept of ideal convergence was given by Kostyrko et al. [14] which generalizes and combines different concepts of convergence of sequences containing usual convergence and statistical convergence. Das et al. [15] presented the concept of $\mathcal{I}$-convergence of double sequences in a metric space. In [16], Savas and Das extended the conception of ideal convergence as studied by Kostyrko et al. [14] to $\mathcal{I}$-statistical convergence and examined remarkable basic features of it. For different studies on these topics, we refer to [17–23].

The theory of fuzzy sets was firstly given by Zadeh [24]. Matloka [25] identified the convergence of a sequence of fuzzy numbers. Nanda [26] worked on the sequences of fuzzy numbers and displayed that the set of all convergent sequences of fuzzy numbers generates a complete metric space. Nuray and Savas [27] generalized ordinary convergence and defined statistically Cauchy and statistical convergent sequences of fuzzy number. Later on, it was studied and advanced by Aytar and Pehlivan [28] and many others. Aytar [29] worked on the conception of statistical limit points and cluster points for sequences of fuzzy number. Kumar et al. [30, 31] worked $\mathcal{I}$-convergence, $\mathcal{I}$-limit points, and $\mathcal{I}$-cluster point for sequence of fuzzy numbers. The concepts of $\mathcal{I}$-statistically convergence for sequences of fuzzy numbers were established by Debnath and Debnath [32]. Later on, $\mathcal{I}$-statistically limit points and $\mathcal{I}$-statistically cluster points of sequences of fuzzy numbers were studied by Tripathy et al. [33].

In this paper, we examine some essential features of $\mathcal{I}_2$-statistically convergent sequence of fuzzy numbers and describe $\mathcal{I}_2$-statistical limit point and $\mathcal{I}_2$-statistical cluster point for fuzzy number sequences.

2. Preliminaries

First, we emphasize some properties of double sequences which are not satisfied by a (single) sequence. This provides a proper motivation for studying double sequences.

The essential deficiency of this kind of convergence is that a convergent sequence does not require to be bounded. Hardy [34] defined the concept of regular sense, which does not have this shortcoming, for double sequence. In regular convergence, both the row-index and the column-index of the double sequence need to be convergent besides the convergent in Pringsheim's sense.

The notion of Cesàro summable double sequences was described by [35]. Note that if a bounded sequence (x_{mn}) is statistically convergent then it is also Cesàro summable but not contrariwise.

Let $(x_{mn}) = (-1)^m, \forall n$; then, $\lim_{p,r}\sum_{m=1}^{p}\sum_{n=1}^{r}x_{mn} = 0$, but apparently x is not statistically convergent.

The convergence of double sequences plays a significant part not only in pure mathematics but also in other subjects of science including computer science, biological science, and dynamical systems, as well. Also, the double sequence can be used in convergence of double trigonometric series and in the opening series of double functions and in the making differential solution.

Now, we remember some notions and fundamental definitions required in this study.

We signify by $\mathcal{D}$ the set of all bounded and closed intervals on $\mathbb{R}$, i.e.,

$$\mathcal{D} = \left\{M \subset \mathbb{R} : M = \left[\underline{M}, \bar{M}\right]\right\}. \tag{1}$$

For $M, N \in \mathcal{D}$, we describe $M \leq N$ iff $\underline{M} \leq \underline{N}$ and $\bar{M} \leq \bar{N}$ and $\bar{d} = \max\{|\underline{M} - \underline{N}|, |\bar{M} - \bar{N}|\}$. $(\mathcal{D}, \bar{d})$ forms a complete metric space.

Definition 1. A fuzzy number is a function X from $\mathbb{R}$ to $[0, 1]$, which satisfies the subsequent conditions:

(i) X is normal

(ii) X is fuzzy convex

(iii) X is upper semicontinuous

(iv) The closure of the set $\{x \in \mathbb{R} : X(x) > 0\}$ is compact

The features (i)-(iv) give that for each $\alpha \in [0, 1]$, the α-level set,

$$X^\alpha = \{x \in \mathbb{R} : X(x) \geq \alpha\} = \left[\underline{X}^\alpha, \bar{X}^\alpha\right], \tag{2}$$

is a nonempty compact convex subset of $\mathbb{R}$. The 0-level set is the class of the strong 0-cut, i.e., $cl(\{x \in \mathbb{R} : X(x) \geq 0\})$. The set of all fuzzy numbers is indicated by $L(\mathbb{R})$. Consider a map $\bar{d}(X, Y) = \sup_{\alpha\in[0,1]} d(X^\alpha, Y^\alpha)$. $(L(\mathbb{R}), \bar{d})$ also forms a complete metric space [36].

Definition 2 (see [25]). A sequence (X_k) of fuzzy numbers is named to be convergent to a fuzzy number X_0 if for each $\varepsilon > 0$ there is $m > 0$ such that $\bar{d}(X_k, X_0) < \varepsilon$ for every $k \geq m$. We write $\lim_{k\longrightarrow\infty} X_k = X_0$.

Definition 3 (see [25]). A fuzzy number X_0 is known as a limit point of a sequence of fuzzy number (X_k) on condition that there is a subsequence of (X_k) that converges to X_0.

Definition 4 (see [27]). A sequence (X_k) of fuzzy numbers is named to be statistically convergent to a fuzzy number X_0 if for each $\varepsilon > 0$ the set

$$A(\varepsilon) = \left\{k \in \mathbb{N} : \bar{d}(X_k, X_0) \geq \varepsilon\right\}, \tag{3}$$

has natural density zero. We write $St - \lim_{k\longrightarrow\infty} X_k = X_0$.

Definition 5 (see [30]). Take $\mathcal{I}$ as a nontrivial ideal. A sequence (X_k) of fuzzy numbers is known as $\mathcal{I}$-convergent to a fuzzy number X_0 provided that each $\xi > 0$

$$A(\xi) = \left\{k \in \mathbb{N} : \bar{d}(X_k, X_0) \geq \xi\right\} \in \mathcal{I}. \tag{4}$$

We write $\mathcal{I} - \lim_{k\longrightarrow\infty} X_k = X_0$.

Definition 6 (see [31]). A fuzzy number X_0 is known as ideal limit point of a sequence of fuzzy number (X_k) provided that there is a subset $M = \{k_1 < k_2 < \cdots\} \subset \mathbb{N}$ such that $M \notin \mathcal{I}$ and $\lim X_{k_n} = X_0$.

Definition 7 (see [31]). A fuzzy number X_0 is known as ideal cluster point of a sequence of fuzzy number (X_k) provided that for each $\xi > 0$ the set $\{k \in \mathbb{N} : \bar{d}(X_k, X_0) < \xi\} \notin \mathcal{I}$.

The set of all $\mathcal{I}$-limit points and $\mathcal{I}$-cluster points of the sequence X is shown by $\mathcal{I}(\Lambda_X)$ and $\mathcal{I}(\Gamma_X)$, respectively.

Natural density of a subset K of $\mathbb{N} \times \mathbb{N}$ is demonstrated by

$$d(K) = \lim_{m,n\longrightarrow\infty} \frac{K(m, n)}{m.n}, \tag{5}$$

where $K(m, n) = |\{(j, k) \in \mathbb{N} \times \mathbb{N} : j \leq m, k \leq n\}|$.

A nontrivial ideal $\mathcal{I}_2$ of $\mathbb{N} \times \mathbb{N}$ is known as strongly admissible if $\{i\} \times \mathbb{N}$ and $\mathbb{N} \times \{i\}$ belong to $\mathcal{I}_2$ for each $i \in \mathbb{N}$.

It is the proof that a strongly admissible ideal is admissible also.

Throughout the paper, we consider $\mathcal{I}_2$ as a strongly admissible ideal in $\mathbb{N} \times \mathbb{N}$.

3. Main Results

In this study, the researchers focus on remarkable features of the set of all $\mathcal{I}_2$-statistical cluster points and the set of all $\mathcal{I}_2$-statistical limit points of fuzzy number sequences. We examine interrelationship between them.

Theorem 8. *If (X_{kl}) be a double sequence of fuzzy numbers such that $\mathcal{I}_2 - st\lim X_{kl} = X_0$, then, X_0 identified uniquely.*

Proof. Presume that $\mathcal{I}_2 - st\lim X_{kl} = X_0$ and $\mathcal{I}_2 - st\lim X_{kl} = Y_0$, where $X_0 \neq Y_0$. For any $\varepsilon, \delta > 0$, we get

$$K_1 = \left\{ (s, w) \in \mathbb{N} \times \mathbb{N} : \frac{1}{sw} \left| \left\{ k \leq s, l \leq w : \bar{d}(X_{kl}, X_0) \geq \varepsilon \right\} \right| < \delta \right\} \in \mathcal{F}(\mathcal{I}_2), \tag{6}$$

$$K_2 = \left\{ (s, w) \in \mathbb{N} \times \mathbb{N} : \frac{1}{sw} \left| \left\{ k \leq s, l \leq w : \bar{d}(X_{kl}, Y_0) \geq \varepsilon \right\} \right| < \delta \right\} \in \mathcal{F}(\mathcal{I}_2). \tag{7}$$

Therefore, $K_1 \cap K_2 \neq \varnothing$, since $K_1 \cap K_2 \in \mathcal{F}(\mathcal{I}_2)$. Let $(i, j) \in K_1 \cap K_2$ and take $\varepsilon := \bar{d}(X_0, Y_0)/3 > 0$ such that we have

$$\frac{1}{ij} \left| \left\{ k \leq i, l \leq j : \bar{d}(X_{kl}, X_0) \geq \varepsilon \right\} \right| < \delta, \tag{8}$$

and it goes along with

$$\frac{1}{ij} \left| \left\{ k \leq i, l \leq j : \bar{d}(X_{kl}, Y_0) \geq \varepsilon \right\} \right| < \delta, \tag{9}$$

i.e., for maximum $k \leq i$, $l \leq j$, we have $\bar{d}(X_{kl}, X_0) < \varepsilon$ and $\bar{d}(X_{kl}, Y_0) < \varepsilon$ for a very small $\delta > 0$. Thus, we have to acquire

$$\left\{ k \leq i, l \leq j : \bar{d}(X_{kl}, X_0) < \varepsilon \right\} \cap \left\{ k \leq i, l \leq j : \bar{d}(X_{kl}, Y_0) < \varepsilon \right\} \neq \varnothing, \tag{10}$$

a contradiction, as the nbd of X_0 and Y_0 are disjoint. Hence, X_0 is uniquely identified. □

Theorem 9. *Let (X_{kl}) be a fuzzy numbers sequence then $st\lim X_{kl} = X_0$ implies $\mathcal{I}_2 - st\lim X_{kl} = X_0$.*

Proof. Let $st\lim X_{kl} = X_0$. Then, for each $\varepsilon > 0$, the set

$$K(\varepsilon) = \left\{ k \leq s, l \leq w : \bar{d}(X_{kl}, X_0) \geq \varepsilon \right\}, \tag{11}$$

has natural density zero, i.e.,

$$\lim_{s,w \longrightarrow \infty} \frac{1}{sw} \left| \left\{ k \leq s, l \leq w : \bar{d}(X_{kl}, X_0) \geq \varepsilon \right\} \right| = 0. \tag{12}$$

Therefore, for every $\varepsilon > 0$ and $\delta > 0$,

$$T(\varepsilon, \delta) = \left\{ (s, w) \in \mathbb{N} \times \mathbb{N} : \frac{1}{sw} \left| \left\{ k \leq s, l \leq w : \bar{d}(X_{kl}, X_0) \geq \varepsilon \right\} \right| \geq \delta \right\}, \tag{13}$$

is a finite set and so $T(\varepsilon, \delta) \in \mathcal{I}_2$, where $\mathcal{I}_2$ is an admissible ideal. Hence, we get $\mathcal{I}_2 - st\lim X_{kl} = X_0$. □

Theorem 10. *Take (X_{kl}) as a sequence of fuzzy numbers. $\mathcal{I}_2 - \lim X_{kl} = X_0$ implies $\mathcal{I}_2 - st\lim X_{kl} = X_0$.*

Proof. The proof of this theorem is clear. □

But the reverse is not true. For instance, take for $\mathcal{I}_2$ the class $\mathcal{I}_2^f$ of all finite subsets of $\mathbb{N} \times \mathbb{N}$, the fuzzy number (X_{kl}), where

$$X_{kl}(p) := \begin{cases} \dfrac{n+m+p}{n+m}, & -n-m \leq p \leq 0, \\ \dfrac{n+m-p}{n+m} & 0 \leq p \leq n+m, \end{cases} \tag{14}$$

for $k = n^2, l = m^2,\ n, m \in \mathbb{N}$, and

$$X_{kl}(p) := \begin{cases} 1 + pnm, & -\dfrac{1}{nm} \leq p \leq 0, \\ 1 - pnm & 0 \leq p \leq \dfrac{1}{nm}, \end{cases} \tag{15}$$

for $k \neq n^2, l \neq m^2,\ n, m \in \mathbb{N}$. Then, (X_{kl}) is $\mathcal{I}_2$-statistically convergent, but not $\mathcal{I}_2$-convergent.

Theorem 11. *Let (X_{kl}) and (Y_{kl}) be two fuzzy numbers sequence. Then,*

(i) $\mathcal{I}_2 - st\lim X_{kl} = X_0$, $c \in \mathbb{R}$ implies $\mathcal{I}_2 - st\lim cX_{kl} = cX_0$

(ii) $\mathcal{I}_2 - st\lim X_{kl} = X_0$, $\mathcal{I}_2 - st\lim Y_{kl} = Y_0$ implies $\mathcal{I}_2 - st\lim(X_{kl} + Y_{kl}) = X_0 + Y_0$

Proof. (i) For $c = 0$, there is nothing to prove. So, presume that $c \neq 0$. Now

$$\begin{aligned} &\frac{1}{sw} \left| \left\{ k \leq s, l \leq w : \bar{d}(cX_{kl}, cX_0) \geq \varepsilon \right\} \right| \\ &\quad = \frac{1}{sw} \left| \left\{ k \leq s, l \leq w : |c| \bar{d}(X_{kl}, X_0) \geq \varepsilon \right\} \right| \\ &\quad \leq \frac{1}{sw} \left| \left\{ k \leq s, l \leq w : \bar{d}(X_{kl}, X_0) \geq \frac{\varepsilon}{|c|} \right\} \right| < \delta. \end{aligned} \tag{16}$$

Therefore, we obtain

$$\left\{ (s, w) \in \mathbb{N} \times \mathbb{N} : \frac{1}{sw} \left| \left\{ k \leq s, l \leq w : \bar{d}(cX_{kl}, cX_0) \geq \varepsilon \right\} \right| < \delta \right\} \in \mathcal{F}(\mathcal{I}_2), \tag{17}$$

i.e., $\mathcal{I}_2 - st\lim cX_{kl} = cX_0$.

(ii) We have

$$K_1 = \left\{ (s, w) \in \mathbb{N} \times \mathbb{N} : \frac{1}{sw} \left| \left\{ k \leq s, l \leq w : \bar{d}(X_{kl}, X_0) \geq \frac{\varepsilon}{2} \right\} \right| < \frac{\delta}{2} \right\} \in \mathcal{F}(\mathcal{I}_2), \tag{18}$$

$$K_2 = \left\{ (s, w) \in \mathbb{N} \times \mathbb{N} : \frac{1}{sw} \left| \left\{ k \leq s, l \leq w : \bar{d}(Y_{kl}, Y_0) \geq \frac{\varepsilon}{2} \right\} \right| < \frac{\delta}{2} \right\} \in \mathcal{F}(\mathcal{I}_2). \tag{19}$$

Since, $K_1 \cap K_2 \neq \emptyset$, therefore, for all $(s, w) \in K_1 \cap K_2$, we get

$$\begin{aligned} &\frac{1}{sw}\left|\left\{k \le s, l \le w : \bar{d}(X_{kl} + Y_{kl}, X_0 + Y_0) \ge \varepsilon\right\}\right| \\ &\quad \le \frac{1}{sw}\left|\left\{k \le s, l \le w : \bar{d}(X_{kl}, X_0) \ge \frac{\varepsilon}{2}\right\}\right| \\ &\qquad + \frac{1}{sw}\left|\left\{k \le s, l \le w : \bar{d}(Y_{kl}, Y_0) \ge \frac{\varepsilon}{2}\right\}\right| < \delta, \end{aligned} \tag{20}$$

i.e.,

$$\left\{(s, w) \in \mathbb{N} \times \mathbb{N} : \frac{1}{sw}\left|\left\{k \le s, l \le w : \bar{d}(X_{kl} + Y_{kl}, X_0 + Y_0) \ge \varepsilon\right\}\right| < \delta\right\} \in \mathcal{F}(\mathcal{I}_2). \tag{21}$$

Hence, we have $\mathcal{I}_2 - st\lim(X_{kl} + Y_{kl}) = X_0 + Y_0$. □

Definition 12. An element $X_0 \in L(\mathbb{N})$ is called to be an $\mathcal{I}_2$-statistical limit point of a fuzzy number sequence $X = (X_{kl})$ provided that for each $\varepsilon > 0$ there is a set

$$M = \left\{(k_1, l_1) < (k_2, l_2) < \cdots < (k_r, l_s) < \cdots\right\} \subset \mathbb{N} \times \mathbb{N}, \tag{22}$$

such that $M \notin \mathcal{I}_2$ and $st - \lim X_{k_r, l_s} = X_0$.

$\mathcal{I}_2 - S(\Lambda_X)$ indicates the set of all $\mathcal{I}_2$-statistical limit point of a fuzzy number sequence (X_{kl}).

Theorem 13. *Take (X_{kl}) as a sequence of fuzzy numbers. If $\mathcal{I}_2 - st\lim X_{kl} = X_0$, then $\mathcal{I}_2 - S(\Lambda_X) = \{X_0\}$.*

Proof. Since $\mathcal{I}_2 - st\lim X_{kl} = X_0$, for each $\varepsilon, \delta > 0$, the set

$$K = \left\{(s, w) \in \mathbb{N} \times \mathbb{N} : \frac{1}{sw}\left|\left\{k \le s, l \le w : \bar{d}(X_{kl}, X_0) \ge \varepsilon\right\}\right| \ge \delta\right\} \in \mathcal{I}_2, \tag{23}$$

where $\mathcal{I}_2$ is an admissible ideal.

Assume that $\mathcal{I}_2 - S(\Lambda_X)$ involves Y_0 different from X_0, i.e., $Y_0 \in \mathcal{I}_2 - S(\Lambda_X)$. So, there is a $M \subset \mathbb{N} \times \mathbb{N}$ such that $M \notin \mathcal{I}_2$ and $st - \lim X_{k_r, l_s} = Y_0$.

Let

$$P = \left\{(s, w) \in M : \frac{1}{sw}\left|\left\{k \le s, l \le w : \bar{d}(X_{kl}, Y_0) \ge \varepsilon\right\}\right| \ge \delta\right\}. \tag{24}$$

So P is a finite set and therefore $P \in \mathcal{I}_2$. So

$$P^c = \left\{(s, w) \in M : \frac{1}{sw}\left|\left\{k \le s, l \le w : \bar{d}(X_{kl}, Y_0) \ge \varepsilon\right\}\right| < \delta\right\} \in \mathcal{F}(\mathcal{I}_2). \tag{25}$$

Again let

$$K_1 = \left\{(s, w) \in M : \frac{1}{sw}\left|\left\{k \le s, l \le w : \bar{d}(X_{kl}, X_0) \ge \varepsilon\right\}\right| \ge \delta\right\}. \tag{26}$$

So $K_1 \subset K \in \mathcal{I}_2$, i.e., $K_1^c \in \mathcal{F}(\mathcal{I}_2)$. Therefore, $K_1^c \cap P^c \neq \emptyset$, since $K_1^c \cap P^c \in \mathcal{F}(\mathcal{I}_2)$.

Let $(i, j) \in K_1^c \cap P^c$ and take $\varepsilon := \bar{d}(X_0, Y_0)/3 > 0$, so

$$\frac{1}{ij}\left|\left\{k \le i, l \le j : \bar{d}(X_{kl}, X_0) \ge \varepsilon\right\}\right| < \delta \quad \text{and,} \tag{27}$$

$$\frac{1}{ij}\left|\left\{k \le i, l \le j : \bar{d}(X_{kl}, Y_0) \ge \varepsilon\right\}\right| < \delta, \tag{28}$$

i.e., for maximum $k \le i, l \le j$ will satisfy $\bar{d}(X_{kl}, X_0) < \varepsilon$ and $\bar{d}(X_{kl}, Y_0) < \varepsilon$ for a very small $\delta > 0$. Thus, we have to obtain

$$\left\{k \le i, l \le j : \bar{d}(X_{kl}, X_0) < \varepsilon\right\} \cap \left\{k \le i, l \le j : \bar{d}(X_{kl}, Y_0) < \varepsilon\right\} \neq \emptyset, \tag{29}$$

a contradiction, as the nbd of X_0 and Y_0 are disjoint. Hence, $\mathcal{I}_2 - S(\Lambda_X) = \{X_0\}$. □

Definition 14. An element $X_0 \in L(\mathbb{R})$ is known as $\mathcal{I}_2$-statistical cluster point of a fuzzy number sequence $X = (X_{kl})$ if for each $\varepsilon > 0$ and $\delta > 0$, the set

$$\left\{(s, w) \in \mathbb{N} \times \mathbb{N} : \frac{1}{sw}\left|\left\{k \le s, l \le w : \bar{d}(X_{kl}, X_0) \ge \varepsilon\right\}\right| < \delta\right\} \notin \mathcal{I}_2. \tag{30}$$

$\mathcal{I}_2 - S(\Gamma_X)$ demonstrates the set of all $\mathcal{I}_2$-statistical cluster point of a fuzzy number sequence (X_{kl}).

Theorem 15. *For any sequence (X_{kl}) of fuzzy numbers $\mathcal{I}_2 - S(\Gamma_X)$ is closed.*

Proof. Let the fuzzy number Y_0 be a limit point of the set $\mathcal{I}_2 - S(\Gamma_X)$. Then, for any $\varepsilon > 0$,

$$\mathcal{I}_2 - S(\Gamma_X) \cap B(Y_0, \varepsilon) \neq \emptyset, \tag{31}$$

where

$$B(Y_0, \varepsilon) = \left\{W \in L(\mathbb{R}): \bar{d}(W, Y_0) < \varepsilon\right\}. \tag{32}$$

Let $Z_0 \in \mathcal{I}_2 - S(\Gamma_X) \cap B(Y_0, \varepsilon)$ and select $\varepsilon_1 > 0$ such that $B(Z_0, \varepsilon_1) \subseteq B(Y_0, \varepsilon)$. Then, we get

$$\left\{k \le s, l \le w : \bar{d}(X_{kl}, Z_0) \ge \varepsilon_1\right\} \supseteq \left\{k \le s, l \le w : \bar{d}(X_{kl}, Y_0) \ge \varepsilon\right\}, \tag{33}$$

which implies that

$$\frac{1}{sw}\left|\left\{k\le s, l\le w : \bar{d}(X_{kl}, Z_0)\ge\varepsilon_1\right\}\right| \ge \frac{1}{sw}\left|\left\{k\le s, l\le w : \bar{d}(X_{kl}, Y_0)\ge\varepsilon\right\}\right|. \tag{34}$$

Now, for any $\delta>0$, we obtain

$$\left\{(s,w)\in\mathbb{N}\times\mathbb{N} : \frac{1}{sw}\left|\left\{k\le s, l\le w : \bar{d}(X_{kl}, Z_0)\ge\varepsilon_1\right\}\right|<\delta\right\}, \tag{35}$$

$$\subseteq\left\{(s,w)\in\mathbb{N}\times\mathbb{N} : \frac{1}{sw}\left|\left\{k\le s, l\le w : \bar{d}(X_{kl}, Y_0)\ge\varepsilon\right\}\right|<\delta\right\}. \tag{36}$$

Since $Z_0\in\mathcal{I}_2-S(\Gamma_X)$, we have

$$\left\{(s,w)\in\mathbb{N}\times\mathbb{N} : \frac{1}{sw}\left|\left\{k\le s, l\le w : \bar{d}(X_{kl}, Y_0)\ge\varepsilon\right\}\right|<\delta\right\}\notin\mathcal{I}_2, \tag{37}$$

i.e., $Y_0\in\mathcal{I}_2-S(\Gamma_X)$. This concludes the proof. □

Theorem 16. *For any fuzzy number sequence* (X_{kl}),

$$\mathcal{I}_2-S(\Lambda_X)\subseteq\mathcal{I}_2-S(\Gamma_X). \tag{38}$$

Proof. Let $X_0\in\mathcal{I}_2-S(\Lambda_X)$. In that case, there is a set

$$M=\{(k_1,l_1)<(k_2,l_2)<\cdots<(k_r,l_s)<\cdots\}\notin\mathcal{I}_2, \tag{39}$$

such that $st-\lim X_{k_r,l_s}=X_0$. So, we have

$$\lim_{k,l\longrightarrow\infty}\frac{1}{kl}\left|\left\{k_r\le k, l_s\le l : \bar{d}\left(X_{k_r l_s}, X_0\right)\ge\varepsilon\right\}\right|=0. \tag{40}$$

Take $\delta>0$, so there is $n_0\in\mathbb{N}$ such that for $s,w>n_0$, we have

$$\frac{1}{sw}\left|\left\{k_r\le s, l_s\le w : \bar{d}\left(X_{k_r l_s}, X_0\right)\ge\varepsilon\right\}\right|<\delta. \tag{41}$$

Let

$$K=\left\{(s,w)\in\mathbb{N}\times\mathbb{N} : \frac{1}{sw}\left|\left\{k_r\le s, l_s\le w : \bar{d}\left(X_{k_r l_s}, X_0\right)\ge\varepsilon\right\}\right|<\delta\right\}. \tag{42}$$

Also, we have

$$K\supset M\setminus\left\{(k_1,l_1),\left((k_2,l_2),\cdots,\left(k_{n_0},l_{n_0}\right)\right)\right\}. \tag{43}$$

Considering that $\mathcal{I}_2$ is an admissible ideal and $M\notin\mathcal{I}_2$, therefore, $K\notin\mathcal{I}_2$. Hence, according to the definition of $\mathcal{I}_2$-statistical cluster point $X_0\in\mathcal{I}_2-S(\Gamma_X)$, this finalizes the proof. □

Theorem 17. *If* (X_{kl}) *and* (Y_{kl}) *are two sequences of fuzzy numbers such that*

$$\{(k,l)\in\mathbb{N}\times\mathbb{N} : X_{kl}\neq Y_{kl}\}\in\mathcal{I}_2, \tag{44}$$

then

$$\mathcal{I}_2-S(\Lambda_X)=\mathcal{I}_2-S(\Lambda_Y). \tag{45}$$

$$\mathcal{I}_2-S(\Gamma_X)=\mathcal{I}_2-S(\Gamma_Y). \tag{46}$$

Proof. (i) Let $X_0\in\mathcal{I}_2-S(\Lambda_X)$. So, according to the definition, there is a set

$$M=\{(k_1,l_1)<(k_2,l_2)<\cdots<(k_r,l_s)<\cdots\}\subset\mathbb{N}\times\mathbb{N}, \tag{47}$$

such that $M\notin\mathcal{I}_2$ and $st-\lim X_{k_r,l_s}=X_0$. Since

$$\{(k,l)\in M : X_{kl}\neq Y_{kl}\}\subseteq\{(k,l)\in\mathbb{N}\times\mathbb{N} : X_{kl}\neq Y_{kl}\}\in\mathcal{I}_2, \tag{48}$$

$$M'=\{(k,l)\in M : X_{kl}=Y_{kl}\}\notin\mathcal{I}_2 \quad\text{and}\quad M'\subseteq M. \tag{49}$$

So, we have $st-\lim Y_{k'_r,l'_s}=X_0$. This denotes that $X_0\in\mathcal{I}_2-S(\Lambda_Y)$ and therefore $\mathcal{I}_2-S(\Lambda_X)\subseteq\mathcal{I}_2-S(\Lambda_Y)$. By symmetry, $\mathcal{I}_2-S(\Lambda_Y)\subseteq\mathcal{I}_2-S(\Lambda_X)$. Hence, we obtain $\mathcal{I}_2-S(\Lambda_X)=\mathcal{I}_2-S(\Lambda_Y)$.

(ii) Let $X_0\in\mathcal{I}_2-S(\Gamma_X)$. So, by the definition for each $\varepsilon>0$, we have

$$K=\left\{(s,w)\in\mathbb{N}\times\mathbb{N} : \frac{1}{sw}\left|\left\{k\le s, l\le w : \bar{d}(X_{kl}, X_0)\ge\varepsilon\right\}\right|<\delta\right\}\notin\mathcal{I}_2. \tag{50}$$

Let

$$L=\left\{(s,w)\in\mathbb{N}\times\mathbb{N} : \frac{1}{sw}\left|\left\{k\le s, l\le w : \bar{d}(Y_{kl}, X_0)\ge\varepsilon\right\}\right|<\delta\right\}. \tag{51}$$

We have to prove that $L\notin\mathcal{I}_2$. Presume that $L\in\mathcal{I}_2$, So

$$L^c=\left\{(s,w)\in\mathbb{N}\times\mathbb{N} : \frac{1}{sw}\left|\left\{k\le s, l\le w : \bar{d}(Y_{kl}, X_0)\ge\varepsilon\right\}\right|\ge\delta\right\}\in\mathcal{F}(\mathcal{I}_2). \tag{52}$$

By hypothesis,

$$P=\{(k,l)\in\mathbb{N}\times\mathbb{N} : X_{kl}=Y_{kl}\}\in\mathcal{F}(\mathcal{I}_2). \tag{53}$$

Therefore, $L^c\cap P\in\mathcal{F}(\mathcal{I}_2)$. Also, it is clear that $L^c\cap P\subseteq K^c\in\mathcal{F}(\mathcal{I}_2)$, i.e., $K\in\mathcal{I}_2$, this is a contradiction. Therefore, $L\notin\mathcal{I}_2$ and thus the desired result was achieved. □

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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