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Research Article

Some New Observations on Wijsman $\mathcal{I}_2$-Lacunary Statistical Convergence of Double Set Sequences in Intuitionistic Fuzzy Metric Spaces

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In this study, we investigate the notions of the Wijsman $\mathcal{I}_2$-statistical convergence, Wijsman $\mathcal{I}_2$-lacunary statistical convergence, Wijsman strongly $\mathcal{I}_2$-lacunary convergence, and Wijsman strongly $\mathcal{I}_2$-Cesàro convergence of double sequence of sets in the intuitionistic fuzzy metric spaces (briefly, IFMS). Also, we give the notions of Wijsman strongly $\mathcal{I}_2^*$-lacunary convergence, Wijsman strongly $\mathcal{I}_2$-lacunary Cauchy, and Wijsman strongly $\mathcal{I}_2^*$-lacunary Cauchy set sequence in IFMS and establish noteworthy results.

1. Introduction and Background

Statistical convergence was firstly examined by Henry Fast [1]. This notion was redefined for double sequences by Mursaleen and Edely [2]. As a consequence of the study of ideal convergence defined by Kostyrko et al. [3], there has been valuable research to discover summability works of the classical theories. Das et al. [4] rethought $\mathcal{I}$-convergence of double sequences and worked some features of this convergence. Ideal convergence became a noteworthy topic in summability theory after the studies of [5–11].

Fridy and Orhan [12] examined the notion of lacunary statistical convergence by using lacunary sequence. The publication of the paper affected deeply all the scientific fields. Çakan and Altay [13] demonstrated multidimensional similarities of the conclusions given by Fridy and Orhan [12]. Some works in lacunary statistical convergence can be found in [13–17].

Theory of fuzzy sets (FSs) was firstly given by Zadeh [18]. This work affected deeply all the scientific fields. The theory of FSs has submitted to employ imprecise, vagueness, and inexact data [18]. FSs have been widely implemented in different disciplines and technologies. The theory of FSs cannot always cope with the lack of knowledge of membership degrees. That is why Atanassov [19] investigated the theory of IFS which is the extension of the theory of FSs. Kramosil and Michalek [20] investigated fuzzy metric space (FMS) utilizing the concepts fuzzy and probabilistic metric space. The FMS as a distance between two points to be a nonnegative fuzzy number was examined by Kaleva and Seikkala [21]. George and Veeramani [22] gave some qualifications of FMS. Some basic features of FMS were given, and significant theorems were proved in [23]. Moreover, FMS has used in practical research studies, for example, decision-making, fixed point theory, and medical imaging. Park [24] generalized FMSs and defined IF metric space (IFMS). Park utilized George and Veeramani's [22] opinion of using t-norm and t-conorm to the FMS meantime describing IFMS and investigating its fundamental properties. The concept of IF-normed spaces (IFNS for shortly) was given by Lael and Nourouzi [25]. In order to have a different topology from the topology generated by the F-norm μ, the condition $\mu + \nu \leq 1$ was omitted from Park's definition. Statistical convergence, ideal convergence, and different features of sequences in INFS were examined by several authors [26–29].

Recently, convergence of sequences of sets was studied by several authors. Nuray and Rhoades [30] presented the idea of statistical convergence of set sequences and established some essential theorems. Ulusu and Nuray [31] examined the lacunary statistical convergence of sequence of sets. Convergence for sequences of sets became a notable topic in summability theory after the studies of (see [32–38]).

Lacunary statistical convergence and lacunary strongly convergence for sequence of sets in IFMS were worked by Kisi [39]. Further, Wijsman $\mathscr{I}$-convergence and Wijsman $\mathscr{I}^*$-convergence for sequence of sets in IFMS were investigated by Esi et al. [40].

Throughout this work, we indicate $\mathscr{I}_2$ to be the admissible ideal in $\mathbb{N}\times\mathbb{N}$, $\theta_2 = \{(j_u, k_s)\}$ to be a double lacunary sequence, $(\mathscr{X}, \eta, \nu, *, \diamond)$ to be the IFMS, and F $\{F_{wq}\}$ to be nonempty closed subsets of $\mathscr{X}$.

2. Main Results

Definition 1. A sequence $\{F_{wq}\}$ of nonempty closed subsets of $\mathscr{X}$ is known as Wijsman $\mathscr{I}_2$-statistical convergent to F or $S(\mathscr{I}_{W_2}^{(\eta,\nu)})$-convergent to F with regard to IFM (η,ν), if for every $\xi \in (0,1)$, $\sigma > 0$, for each $x \in \mathscr{X}$, and for all $p > 0$,

$$\left\{(s,t)\in\mathbb{N}\times\mathbb{N}\colon \frac{1}{st}\left|\left\{\begin{array}{c}(w,q)\colon w\le s, q\le t, \left|\eta\left(x,F_{wq},p\right)-\eta(x,F,p)\right|\le 1-\xi\\ \text{or}\left|\nu\left(x,F_{wq},p\right)-\nu(x,F,p)\right|\ge\xi\end{array}\right\}\right|\ge\sigma\right\}\in\mathscr{I}_2. \quad (1)$$

We demonstrate this symbolically by $F_{wq}\, S(\mathscr{I}_{W_2}^{(\eta,\nu)})\, F$ or $F_{wq} \longrightarrow F(S(\mathscr{I}_{W_2}^{(\eta,\nu)}))$. The set of all Wijsman $\mathscr{I}_2$-statistical convergent sequences in IFMS is indicated by $S(\mathscr{I}_{W_2}^{(\eta,\nu)})$.

Example 1. Let $\mathscr{X} = \mathbb{R}^2$ and double sequence $\{F_{wq}\}$ be determined as follows:

$$F_{wq} := \begin{cases}(d,e)\in\mathbb{R}^2\colon\ (d+w)^2+(e+q)^2 = 1, & \text{if } w \text{ and } q \text{ are square integers},\\ \{(1,1)\}, & \text{otherwise}.\end{cases} \quad (2)$$

If $\mathscr{I}_2 = \mathscr{I}_2^\delta$ ($\mathscr{I}_2^\delta$ is the class of $K\subset\mathbb{N}\times\mathbb{N}$ with density of K equal to 0), then the sequence $\{F_{wq}\}$ is Wijsman $\mathscr{I}_2$-statistical convergent to $F = \{(1,1)\}$ with regard to IFM (η,ν).

Definition 2. A sequence $\{F_{wq}\}$ is Wijsman strong $\mathscr{I}_2$-Cesàro summable to F or $C_1[\mathscr{I}_{W_2}^{(\eta,\nu)}]$-summable to F with regard to IFM (η,ν), if for every $\xi\in(0,1)$, for each $x\in\mathscr{X}$, and for all $p>0$,

$$\left\{\begin{array}{c}(s,t)\in\mathbb{N}\times\mathbb{N}\colon \frac{1}{st}\sum_{w,q=1,1}^{s,t}\left|\eta\left(x,F_{wq},p\right)-\eta(x,F,p)\right|\le 1-\xi\\ \text{or}\ \frac{1}{st}\sum_{w,q=1,1}^{s,t}\left|\nu\left(x,F_{wq},p\right)-\nu(x,F,p)\right|\ge\xi\end{array}\right\}\in\mathscr{I}_2. \quad (3)$$

We write $F_{wq} \stackrel{C_1}{\longrightarrow} (\mathscr{I}_{W_2}^{\eta,\nu})\, F$ or $F_{wq} \longrightarrow F(C_1[\mathscr{I}_{W_2}^{(\eta,\nu)}])$.

Example 2. Let $\mathscr{X} = \mathbb{R}^2$ and double sequence $\{F_{wq}\}$ be determined as follows:

$$F_{wq} := \begin{cases}(d,e)\in\mathbb{R}^2\colon\ (d+1)^2+e^2 = \dfrac{1}{wq}; & \text{if } w \text{ and } q \text{ are square integers},\\ \{(0,1)\}; & \text{otherwise}.\end{cases} \quad (4)$$

If $\mathcal{I}_2 = \mathcal{I}_2^f$ ($\mathcal{I}_2^f$ is the class of finite subsets of $\mathbb{N}\times\mathbb{N}$), then sequence $\{F_{wq}\}$ is Wijsman strong $\mathcal{I}_2$-Cesàro summable to $F = \{(0,1)\}$ with regard to IFM (η,ν).

Definition 3. The sequence $\{F_{wq}\}$ is known as Wijsman $\mathcal{I}_2$-lacunary statistically convergent to F or S_{θ_2} $(\mathcal{I}_{W_2}^{(\eta,\nu)})$-convergent to F with regard to IFM (η,ν), if for every $\xi \in (0,1)$, $\sigma > 0$, for each $x \in \mathcal{X}$, and for all $p > 0$,

$$\left\{(u,s)\in\mathbb{N}\times\mathbb{N}: \frac{1}{h_{us}}\left|\left\{\begin{array}{c}(w,q)\in I_{us}: \left|\eta\left(x,F_{wq},p\right)-\eta(x,F,p)\right|\le 1-\xi\\ \text{or}\left|\nu\left(x,F_{wq},p\right)-\nu(x,F,p)\right|\ge\xi\end{array}\right\}\right|\ge\sigma\right\}\in\mathcal{I}_2. \tag{5}$$

In that case, we write $F_{wq} \xrightarrow{S_{\theta_2}} (\mathcal{I}_{W_2}^{\eta,\nu}) F$ or $F_{wq} \longrightarrow F(S_{\theta_2}(\mathcal{I}_{W_2}^{(\eta,\nu)}))$.

Example 3. Let $\mathcal{X} = \mathbb{R}^2$ and double sequence $\{F_{wq}\}$ be determined as follows:

$$F_{wq} := \begin{cases}(d,e)\in\mathbb{R}^2: (d-w)^2+(e+q)^2 = 1, & \text{if } (w,q)\in I_{us}; w \text{ and } q \text{ are square integers},\\ \{(-1,1)\}, & \text{otherwise}.\end{cases} \tag{6}$$

If we take $\mathcal{I}_2 = \mathcal{I}_2^\delta$, then the sequence $\{F_{wq}\}$ is Wijsman $\mathcal{I}_2$-lacunary statistical convergent to $F = \{(-1,1)\}$ with regard to IFM (η,ν).

Definition 4. A double sequence $\{F_{wq}\}$ is Wijsman strong $\mathcal{I}_2$-lacunary summable to F or $N_{\theta_2}[\mathcal{I}_{W_2}^{(\eta,\nu)}]$-summable to F with regard to IFM (η,ν), if for every $\xi \in (0,1)$, for all $p > 0$, and for each $x \in \mathcal{X}$,

$$\left\{\begin{array}{c}(u,s)\in\mathbb{N}\times\mathbb{N}: \dfrac{1}{h_{us}}\displaystyle\sum_{(w,q)\in I_{us}}\left|\eta\left(x,F_{wq},p\right)-\eta(x,F,p)\right|\le 1-\xi\\ \text{or}\ \dfrac{1}{h_{us}}\displaystyle\sum_{(w,q)\in I_{us}}\left|\nu\left(x,F_{wq},p\right)-\nu(x,F,p)\right|\ge\xi\end{array}\right\}\in\mathcal{I}_2. \tag{7}$$

We shall write $F_{wq} \xrightarrow{N_{\theta_2}[\mathcal{I}_{W_2}^{(\eta,\nu)}]} F$ or $F_{wq} \longrightarrow F(N_{\theta_2}[\mathcal{I}_{W_2}^{(\eta,\nu)}])$.

Example 4. Let $\mathcal{X} = \mathbb{R}^2$ and double sequence $\{F_{wq}\}$ be determined as follows:

$$F_{wq} := \begin{cases}(d,e)\in\mathbb{R}^2: d^2+(e-1)^2 = \dfrac{1}{wq}; & \text{if } (w,q)\in I_{us}; w \text{ and } q \text{ are square integers},\\ \{(1,0)\}; & \text{otherwise}.\end{cases} \tag{8}$$

If we take $\mathcal{I}_2 = \mathcal{I}_2^f$, then the sequence $\{F_{wq}\}$ is Wijsman strong $\mathcal{I}_2$-lacunary summable to $F = \{(1,0)\}$ with regard to IFM (η,ν).

Theorem 1. *Let $\theta_2 = \{(j_u,k_s)\}$ be a double lacunary sequence. Then,*

$$F_{wq}\longrightarrow F\left(N_{\theta_2}\left[\mathcal{I}_{W_2}^{(\eta,\nu)}\right]\right)\Longrightarrow F_{wq}\longrightarrow F\left(S_{\theta_2}\left(\mathcal{I}_{W_2}^{(\eta,\nu)}\right)\right). \tag{9}$$

Proof. Let $\xi \in (0,1)$ and $F_{wq} \longrightarrow F(N_{\theta_2}[\mathcal{I}_{W_2}^{(\eta,\nu)}])$. At that time, for every $x \in \mathcal{X}$, we acquire

$$\begin{aligned}
&\sum_{(w,q)\in I_{us}} \left|\eta\left(x,F_{wq},p\right)-\eta(x,F,p)\right| \text{ or } \left|\nu\left(x,F_{wq},p\right)-\nu(x,F,p)\right| \\
&\geq \sum_{\substack{(w,q)\in I_{us} \\ \left|\eta\left(x,F_{wq},p\right)-\eta(x,F,p)\right|\leq 1-\xi \text{ or} \\ \left|\nu\left(x,F_{wq},p\right)-\nu(x,F,p)\right|\geq \xi}} \left|\eta\left(x,F_{wq},p\right)-\eta(x,F,p)\right| \text{ or } \left|\nu\left(x,F_{wq},p\right)-\nu(x,F,p)\right| \\
&\geq \xi.\left|\left\{(w,q)\in I_{us}\colon \left|\eta\left(x,F_{wq},p\right)-\eta(x,F,p)\right|\leq 1-\xi \text{ or } \left|\nu\left(x,F_{wq},p\right)-\nu(x,F,p)\right|\geq \xi\right\}\right|,
\end{aligned} \tag{10}$$

and so

$$\begin{aligned}
&\frac{1}{\xi h_{us}}\sum_{(w,q)\in I_{us}} \left|\eta\left(x,F_{wq},p\right)-\eta(x,F,p)\right| \text{ or } \left|\nu\left(x,F_{wq},p\right)-\nu(x,F,p)\right| \\
&\geq \frac{1}{h_{us}}\left|\left\{(w,q)\in I_{us}\colon \left|\eta\left(x,F_{wq},p\right)-\eta(x,F,p)\right|\leq 1-\xi \text{ or } \left|\nu\left(x,F_{wq},p\right)-\nu(x,F,p)\right|\geq \xi\right\}\right|.
\end{aligned} \tag{11}$$

Then, for any $\sigma>0$ and for every $x\in\mathscr{X}$,

$$\begin{aligned}
&\left\{(u,s)\in\mathbb{N}\times\mathbb{N}\colon \frac{1}{h_{us}}\left|\left\{\begin{array}{c}(w,q)\in I_{us}\colon \left|\eta\left(x,F_{wq},p\right)-\eta(x,F,p)\right|\leq 1-\xi \\ \text{or } \left|\nu\left(x,F_{wq},p\right)-\nu(x,F,p)\right|\geq \xi\end{array}\right\}\right|\geq\sigma\right\} \\
&\subseteq \left\{\begin{array}{c}(u,s)\in\mathbb{N}\times\mathbb{N}\colon \dfrac{1}{h_{us}}\displaystyle\sum_{(w,q)\in I_{us}}\left|\eta\left(x,F_{wq},p\right)-\eta(x,F,p)\right|\leq(1-\xi).\sigma \\ \text{or } \dfrac{1}{h_{us}}\displaystyle\sum_{(w,q)\in I_{us}}\left|\nu\left(x,F_{wq},p\right)-\nu(x,F,p)\right|\geq\xi.\sigma\end{array}\right\}
\end{aligned} \tag{12}$$

This proof is concluded. □

Theorem 2. *Let $\theta_2=\{(j_u,k_s)\}$ be a double lacunary sequence. Then, $\{F_{wq}\}$ is bounded ($\{F_{wq}\}\in L^2_\infty(\mathscr{X})$) and*

$$F_{wq}\longrightarrow F\left(S_{\theta_2}\left(\mathscr{I}_{W_2}^{(\eta,\nu)}\right)\right)\Longrightarrow F_{wq}\longrightarrow F\left(N_{\theta_2}\left[\mathscr{I}_{W_2}^{(\eta,\nu)}\right]\right). \tag{13}$$

The set of all bounded double sequences of sets in IFMS is indicated by $L^2_\infty(\mathscr{X})$.

Proof. Presume that $F_{wq}\longrightarrow F(S_{\theta_2}(\mathscr{I}_{W_2}^{(\eta,\nu)}))$ and $\{F_{wq}\}\in L^2_\infty$. At this point, there is an $H>0$ such that

$$\left|\eta\left(x,F_{wq},p\right)-\eta(x,F,p)\right|\geq 1-H \text{ or } \left|\nu\left(x,F_{wq},p\right)-\nu(x,F,p)\right|\leq H, \tag{14}$$

for every $x\in\mathscr{X}$ and all $w,q\in\mathbb{N}$. Given $\xi\in(0,1)$, we obtain

$$\begin{aligned}
&\frac{1}{h_{us}}\sum_{(w,q)\in I_{us}}\left|\eta\left(x,F_{wq},p\right)-\eta(x,F,p)\right| \text{ or } \left|\nu\left(x,F_{wq},p\right)-\nu(x,F,p)\right| \\
&=\frac{1}{h_{us}}\sum_{\substack{(w,q)\in I_{us}\\ \left|\eta\left(x,F_{wq},p\right)-\eta(x,F,p)\right|\leq 1-(\xi/2) \text{ or}\\ \left|\nu\left(x,F_{wq},p\right)-\nu(x,F,p)\right|\geq(\xi/2)}}\left|\eta\left(x,F_{wq},p\right)-\eta(x,F,p)\right| \text{ or } \left|\nu\left(x,F_{wq},p\right)-\nu(x,F,p)\right| \\
&\quad+\frac{1}{h_{us}}\sum_{\substack{(w,q)\in I_{us}\\ \left|\eta\left(x,F_{wq},p\right)-\eta(x,F,p)\right|>1-(\xi/2) \text{ or}\\ \left|\nu\left(x,F_{wq},p\right)-\nu(x,F,p)\right|<(\xi/2)}}\left|\eta\left(x,F_{wq},p\right)-\eta(x,F,p)\right| \text{ or } \left|\nu\left(x,F_{wq},p\right)-\nu(x,F,p)\right| \\
&\leq\frac{H}{h_{us}}\left|\left\{\begin{array}{c}(w,q)\in I_{us}\colon \left|\eta\left(x,F_{wq},p\right)-\eta(x,F,p)\right|\leq 1-\dfrac{\xi}{2}\\ \text{or } \left|\nu\left(x,F_{wq},p\right)-\nu(x,F,p)\right|\geq\dfrac{\xi}{2}\end{array}\right\}\right|+\frac{\xi}{2}.
\end{aligned}\tag{15}$$

As a consequence, for each $x\in\mathcal{X}$, we get

$$\begin{aligned}
&\left\{\begin{array}{c}(u,s)\in\mathbb{N}\times\mathbb{N}\colon \dfrac{1}{h_{us}}\displaystyle\sum_{(w,q)\in I_{us}}\left|\eta\left(x,F_{wq},p\right)-\eta(x,F,p)\right|\leq 1-\xi\\ \text{or } \dfrac{1}{h_{us}}\displaystyle\sum_{(w,q)\in I_{us}}\left|\nu\left(x,F_{wq},p\right)-\nu(x,F,p)\right|\geq\xi\end{array}\right\} \\
&\subseteq\left\{(u,s)\in\mathbb{N}\times\mathbb{N}\colon \frac{1}{h_{us}}\left|\left\{\begin{array}{c}(w,q)\in I_{us}\colon \left|\eta\left(x,F_{wq},p\right)-\eta(x,F,p)\right|\leq 1-\dfrac{\xi}{2}\\ \text{or } \left|\nu\left(x,F_{wq},p\right)-\nu(x,F,p)\right|\geq\dfrac{\xi}{2}\end{array}\right\}\right|\geq\frac{\xi}{2H}\right\}\in\mathcal{I}_2.
\end{aligned}\tag{16}$$

This proof is concluded. □

Corollary 1. *We have the following result:*

$$\left\{S_{\theta_2}\left(\mathcal{I}_{W_2}^{(\eta,\nu)}\right)\right\}\cap L_\infty^2(\mathcal{X})=\left\{N_{\theta_2}\left[\mathcal{I}_{W_2}^{(\eta,\nu)}\right]\right\}\cap L_\infty^2(\mathcal{X}).\tag{17}$$

Theorem 3. *If* $\liminf_u q_u>1$ *and* $\liminf_s q_s>1$, *then* $F_{wq}\longrightarrow F(S(\mathcal{I}_{W_2}^{(\eta,\nu)}))$ *implies* $F_{wq}\longrightarrow F(S_{\theta_2}(\mathcal{I}_{W_2}^{(\eta,\nu)}))$.

Proof. Presume that $\liminf_u q_u>1$ and $\liminf_s q_s>1$. Then, there are $\zeta>0,\psi>0$ such that

$$q_u\geq 1+\zeta \text{ and } q_s\geq 1+\psi,\tag{18}$$

for sufficiently large u, s which gives that

$$\frac{h_{us}}{j_u k_s}\geq\frac{\zeta\psi}{(1+\zeta)(1+\psi)}.\tag{19}$$

Assume that $F_{wq}\longrightarrow F(S(\mathcal{I}_{W_2}^{(\eta,\nu)}))$. For each $\xi\in(0,1)$, for all $p>0$, and for each $x\in\mathcal{X}$, we have

$$\begin{aligned}
&\frac{1}{j_u k_s}\left|\left\{\begin{array}{c}(w,q)\colon w\le j_u, q\le k_s, \left|\eta\left(x,F_{wq},p\right)-\eta(x,F,p)\right|\le 1-\xi\\ \text{or}\left|\nu\left(x,F_{wq},p\right)-\nu(x,F,p)\right|\ge\xi\end{array}\right\}\right|\\
&\ge\frac{1}{j_u k_s}\left|\left\{\begin{array}{c}(w,q)\in I_{us}\colon \left|\eta\left(x,F_{wq},p\right)-\eta(x,F,p)\right|\le 1-\xi\\ \text{or}\left|\nu\left(x,F_{wq},p\right)-\nu(x,F,p)\right|\ge\xi\end{array}\right\}\right|\\
&=\frac{h_{us}}{j_u k_s}\frac{1}{h_{us}}\left|\left\{\begin{array}{c}(w,q)\in I_{us}\colon \left|\eta\left(x,F_{wq},p\right)-\eta(x,F,p)\right|\le 1-\xi\\ \text{or}\left|\nu\left(x,F_{wq},p\right)-\nu(x,F,p)\right|\ge\xi\end{array}\right\}\right|\\
&\ge\frac{\zeta\psi}{(1+\zeta)(1+\psi)}\frac{1}{h_{us}}\left|\left\{\begin{array}{c}(w,q)\in I_{us}\colon \left|\eta\left(x,F_{wq},p\right)-\eta(x,F,p)\right|\le 1-\xi\\ \text{or}\left|\nu\left(x,F_{wq},p\right)-\nu(x,F,p)\right|\ge\xi\end{array}\right\}\right|.
\end{aligned}\tag{20}$$

Thus, for any $\sigma>0$,

$$\begin{aligned}
&\left\{(u,s)\in\mathbb{N}\times\mathbb{N}\colon \frac{1}{h_{us}}\left|\left\{\begin{array}{c}(w,q)\in I_{us}\colon \left|\eta\left(x,F_{wq},p\right)-\eta(x,F,p)\right|\le 1-\xi\\ \text{or}\left|\nu\left(x,F_{wq},p\right)-\nu(x,F,p)\right|\ge\xi\end{array}\right\}\right|\ge\sigma\right\}\\
&\subseteq\left\{(u,s)\in\mathbb{N}\times\mathbb{N}\colon \frac{1}{j_u k_s}\left|\left\{\begin{array}{c}(w,q)\colon w\le j_u, q\le k_s, \left|\eta\left(x,F_{wq},p\right)-\eta(x,F,p)\right|\le 1-\xi\\ \text{or}\left|\nu\left(x,F_{wq},p\right)-\nu(x,F,p)\right|\ge\xi\end{array}\right\}\right|\ge\frac{\zeta\psi\sigma}{(1+\zeta)(1+\psi)}\right\}.
\end{aligned}\tag{21}$$

Hence, by our supposition, the set on the right side belongs to $\mathcal{I}_2$, and clearly the set on the left side belongs to $\mathcal{I}_2$. As a result, we obtain $F_{wq}\longrightarrow F(S_{\theta_2}(\mathcal{I}_{W_2}^{(\eta,\nu)}))$. ☐

Theorem 4. *If $\limsup_u q_u<\infty$ and $\limsup_s q_s<\infty$, then $F_{wq}\longrightarrow F(S_{\theta_2}(\mathcal{I}_{W_2}^{(\eta,\nu)}))$ implies $F_{wq}\longrightarrow F(S(\mathcal{I}_{W_2}^{(\eta,\nu)}))$.*

Proof. Presume that $\limsup_u q_u<\infty$ and $\limsup_s q_s<\infty$. Then, there are $P,R>0$ such that $q_u<P$ and $q_s<R$ for all u and s. Assume that $F_{wq}\longrightarrow F(S_{\theta_2}(\mathcal{I}_{W_2}^{(\eta,\nu)}))$ and let

$$H_{us}\coloneqq\left|\left\{(w,q)\in I_{us}\colon \left|\eta\left(x,F_{wq},p\right)-\eta(x,F,p)\right|\le 1-\xi\ \text{or}\left|\nu\left(x,F_{wq},p\right)-\nu(x,F,p)\right|\ge\xi\right\}\right|.\tag{22}$$

Since $F_{wq}\longrightarrow F(S_{\theta_2}(\mathcal{I}_{W_2}^{(\eta,\nu)}))$, it holds for each $\xi\in(0,1)$, $\sigma>0$, for every $x\in\mathcal{X}$, and for all $p>0$,

$$\begin{aligned}
&\left\{(u,s)\in\mathbb{N}\times\mathbb{N}\colon \frac{1}{h_{us}}\left|\left\{\begin{array}{c}(w,q)\in I_{us}\colon \left|\eta\left(x,F_{wq},p\right)-\eta(x,F,p)\right|\le 1-\xi\\ \text{or}\left|\nu\left(x,F_{wq},p\right)-\nu(x,F,p)\right|\ge\xi\end{array}\right\}\right|\ge\sigma\right\}\\
&=\left\{(u,s)\in\mathbb{N}\times\mathbb{N}\colon \frac{H_{us}}{h_{us}}\ge\sigma\right\}\in\mathcal{I}_2.
\end{aligned}\tag{23}$$

So, we can select positive integers $u_0, s_0 \in \mathbb{N}$ such that $H_{us}/h_{us} < \sigma$ for all $u \geq u_0, s \geq s_0$. Now, take

$$T := \max\{H_{us}: 1 \leq u \leq u_0, 1 \leq s \leq s_0\}, \tag{24}$$

and let m and n be integers providing $j_{u-1} < m \leq j_u$ and $k_{s-1} < n \leq k_s$. Then, for every $\xi > 0$ and each $x \in \mathscr{X}$, we get

$$\begin{aligned}
&\frac{1}{mn}\left|\left\{(w,q)\colon w \leq m, q \leq n, \left|\eta\left(x, F_{wq}, p\right) - \eta(x,F,p)\right| \leq 1-\xi \text{ or} \left|\nu\left(x, F_{wq}, p\right) - \nu(x,F,p)\right| \geq \xi\right\}\right| \\
&\leq \frac{1}{j_{u-1}k_{s-1}}\left|\left\{(w,q)\colon w \leq j_u, q \leq k_s, \left|\eta\left(x, F_{wq}, p\right) - \eta(x,F,p)\right| \leq 1-\xi \text{ or} \left|\nu\left(x, F_{wq}, p\right) - \nu(x,F,p)\right| \geq \xi\right\}\right| \\
&= \frac{1}{j_{u-1}k_{s-1}}\left\{H_{11} + H_{12} + H_{21} + H_{22} + \cdots + H_{u_0 s_0} + \cdots + H_{us}\right\} \\
&\leq \frac{u_0 s_0}{j_{u-1}k_{s-1}}\left(\max_{\substack{1\leq w\leq u_0\\ 1\leq q\leq s_0}}\left\{H_{wq}\right\}\right) + \frac{1}{j_{u-1}k_{s-1}}\left\{h_{u_0(s_0+1)}\frac{H_{u_0(s_0+1)}}{h_{u_0(s_0+1)}} + h_{(u_0+1)s_0}\frac{H_{(u_0+1)s_0}}{h_{(u_0+1)s_0}} + h_{(u_0+1)(s_0+1)}\frac{H_{(u_0+1)(s_0+1)}}{h_{(u_0+1)(s_0+1)}} + \cdots + h_{us}\frac{H_{us}}{h_{us}}\right\} \\
&\leq \frac{u_0 s_0 T}{j_{u-1}k_{s-1}} + \frac{1}{j_{u-1}k_{s-1}}\left(\sup_{\substack{u>u_0\\ s>s_0}}\frac{H_{us}}{h_{us}}\right)\left(\sum_{\substack{w\geq u_0\\ q\geq s_0}}^{u,s} h_{wq}\right) \\
&\leq \frac{u_0 s_0 T}{j_{u-1}k_{s-1}} + \sigma\frac{\left(j_u - j_{u_0}\right)\left(k_s - k_{s_0}\right)}{j_{u-1}k_{s-1}} \leq \frac{u_0 s_0 T}{j_{u-1}k_{s-1}} + \sigma q_u q_s \leq \frac{u_0 s_0 T}{j_{u-1}k_{s-1}} + \sigma P R.
\end{aligned} \tag{25}$$

Since $j_{u-1}k_{s-1} \longrightarrow \infty$ as $m, n \longrightarrow \infty$, it concludes that for each $x \in \mathscr{X}$,

$$\frac{1}{mn}\left|\left\{(w,q)\colon w \leq m, q \leq n, \left|\eta\left(x, F_{wq}, p\right) - \eta(x,F,p)\right| \leq 1-\xi \text{ or} \left|\nu\left(x, F_{wq}, p\right) - \nu(x,F,p)\right| \geq \xi\right\}\right| \longrightarrow 0, \tag{26}$$

and as a result for any $\sigma_1 > 0$, the set

$$\left\{(m,n) \in \mathbb{N}\times\mathbb{N}\colon \frac{1}{mn}\left|\left\{\begin{array}{c}(w,q)\colon w \leq m, q \leq n, \left|\eta\left(x, F_{wq}, p\right) - \eta(x,F,p)\right| \leq 1-\xi \\ \text{or} \left|\nu\left(x, F_{wq}, p\right) - \nu(x,F,p)\right| \geq \xi\end{array}\right\}\right| \geq \sigma_1\right\} \in \mathscr{I}_2. \tag{27}$$

It gives that $F_{wq} \longrightarrow F(S(\mathscr{I}_{W_2}^{(\eta,\nu)}))$. □

Theorem 5. *Let θ_2 be a double lacunary sequence. If*

$$1 < \lim\inf_u q_u < \lim\sup_u uq < \infty, \text{ and } 1 < \lim\inf_s q_s < \lim\sup_s sq < \infty, \tag{28}$$

then $F_{wq} \longrightarrow F(S_{\theta_2}(\mathscr{I}_{W_2}^{(\eta,\nu)}))$ if $F_{wq} \longrightarrow F(S(\mathscr{I}_{W_2}^{(\eta,\nu)}))$.

Proof. It obvious from Theorem 3 and Theorem 4. □

Theorem 6. *Let $\mathcal{I}_2$ be a strongly admissible ideal providing feature (AP_2), $\theta_2 \in \mathcal{F}(\mathcal{I}_2)$. If $\{F_{wq}\} \in S\ (\mathcal{I}_{W_2}^{(\eta,\nu)}) \cap S_{\theta_2}\ (\mathcal{I}_{W_2}^{(\eta,\nu)})$, then*

$$S\left(\mathcal{I}_{W_2}^{(\eta,\nu)}\right) - \lim_{w,q\longrightarrow\infty} F_{wq} = S_{\theta_2}\left(\mathcal{I}_{W_2}^{(\eta,\nu)}\right) - \lim_{w,q\longrightarrow\infty} F_{wq}. \quad (29)$$

Proof. Presume that $S(\mathcal{I}_{W_2}^{(\eta,\nu)}) - \lim_{w,q\longrightarrow\infty} F_{wq} = A$ and $S_{\theta_2}(\mathcal{I}_{W_2}^{(\eta,\nu)}) - \lim_{w,q\longrightarrow\infty} F_{wq} = B$ and $A \neq B$. Let

$$0 < \xi < \frac{1}{2}|\eta(x, A, p) - \eta(x, B, p)|, \quad \text{and } 0 < \xi < \frac{1}{2}|\nu(x, A, p) - \nu(x, B, p)|, \quad (30)$$

for every $x \in \mathcal{X}$. Since $\mathcal{I}_2$ provides the feature (AP_2), then there is $M \in \mathcal{F}(\mathcal{I}_2)$ (i.e., $\mathbb{N} \times \mathbb{N} \setminus M \in \mathcal{I}_2$) such that for every $x \in \mathcal{X}$ and for $(m, n) \in M$,

$$\lim_{m,n\longrightarrow\infty} \frac{1}{mn}\left|\left\{\begin{array}{c} w \leq m, q \leq n: \left|\eta\left(x, F_{wq}, p\right) - \eta(x, A, p)\right| \leq 1 - \xi \\ \text{or } \left|\nu\left(x, F_{wq}, p\right) - \nu(x, A, p)\right| \geq \xi \end{array}\right\}\right| = 0. \quad (31)$$

Let

$$\begin{aligned} T &= \left\{w \leq m, q \leq n: \left|\eta\left(x, F_{wq}, p\right) - \eta(x, A, p)\right| \leq 1 - \xi, \text{ or } \left|\nu\left(x, F_{wq}, p\right) - \nu(x, A, p)\right| \geq \xi\right\}, \\ S &= \left\{w \leq m, q \leq n: \left|\eta\left(x, F_{wq}, p\right) - \eta(x, B, p)\right| \leq 1 - \xi, \text{ or } \left|\nu\left(x, F_{wq}, p\right) - \nu(x, B, p)\right| \geq \xi\right\}. \end{aligned} \quad (32)$$

Then, $mn = |T \cup S| \leq |T| + |S|$. This gives that $1 \leq (|T|/mn) + (|S|/mn)$. Since $|S|/mn \leq 1$ and $\lim_{m,n\longrightarrow\infty} |T|/mn = 0$, we have to get $\lim_{m,n\longrightarrow\infty}|S|/mn = 1$.

Let $M^* = M \cap \theta_2 \in \mathcal{F}(\mathcal{I}_2)$. Then, for every $x \in \mathcal{X}$ and for $(w_k, q_j) \in M^*$, the $w_k q_j$th term of the statistical limit expression

$$\frac{1}{mn}\left|\left\{w \leq m, q \leq n: \left|\eta\left(x, F_{wq}, p\right) - \eta(x, B, p)\right| \leq 1 - \xi, \text{ or } \left|\nu\left(x, F_{wq}, p\right) - \nu(x, B, p)\right| \geq \xi\right\}\right|, \quad (33)$$

is

$$\begin{aligned} &\frac{1}{w_k q_j}\left|\left\{(w, q) \in \bigcup_{u,s=1,1}^{k,j} I_{us}: \left|\eta\left(x, F_{wq}, p\right) - \eta(x, B, p)\right| \leq 1 - \xi, \text{ or } \left|\nu\left(x, F_{wq}, p\right) - \nu(x, B, p)\right| \geq \xi\right\}\right| \\ &= \frac{1}{\bigcup_{u,s=1,1}^{k,j} h_{us}} \bigcup_{u,s=1,1}^{k,j} v_{us} h_{us}, \end{aligned} \quad (34)$$

where

$$v_{us} = \frac{1}{h_{us}}\left|\left\{(w, q) \in I_{us}: \left|\eta\left(x, F_{wq}, p\right) - \eta(x, B, p)\right| \leq 1 - \xi, \text{ or } \left|\nu\left(x, F_{wq}, p\right) - \nu(x, B, p)\right| \geq \xi\right\}\right| \xrightarrow{\mathcal{I}_2} 0, \quad (35)$$

because $S_{\theta_2}(\mathcal{I}_{W_2}^{(\eta,\nu)}) - \lim_{w,q\longrightarrow\infty} F_{wq} = B$. Since θ_2 is a lacunary sequence, (34) is a regular weighted mean transform of v_{us}'s, and as a result, it is $\mathcal{I}_2$-convergent to 0 as $k, j \longrightarrow \infty$, and also it has a subsequence which is convergent to 0 since $\mathcal{I}_2$ provides the feature (AP_2). However, since this a subsequence of

$$\left\{\frac{1}{mn}\left|\left\{w \le m, q \le n\colon \left|\eta\left(x, F_{wq}, p\right) - \eta(x, B, p)\right| \le 1 - \xi, \text{ or} \left|\nu\left(x, F_{wq}, p\right) - \nu(x, B, p)\right| \ge \xi\right\}\right|\right\}_{(m,n)\in M}, \tag{36}$$

we conclude that

$$\left\{\frac{1}{mn}\left|\left\{w \le m, q \le n\colon \left|\eta\left(x, F_{wq}, p\right) - \eta(x, B, p)\right| \le 1 - \xi, \text{ or} \left|\nu\left(x, F_{wq}, p\right) - \nu(x, B, p)\right| \ge \xi\right\}\right|\right\}_{(m,n)\in M}, \tag{37}$$

which is not convergent to 1. This contradiction indicates that we cannot have $A \neq B$. □

Theorem 7. *If* $\liminf_u q_u > 1$ *and* $\liminf_s q_s > 1$, *then* $C_1[\mathcal{I}_{W_2}^{(\eta,\nu)}] \subseteq N_{\theta_2}[\mathcal{I}_{W_2}^{(\eta,\nu)}]$.

Proof. Let $\liminf_u q_u > 1$ and $\liminf_s q_s > 1$. Then, there are $\zeta, \psi > 0$ such that $q_u \ge 1 + \zeta$ and $q_s \ge 1 + \psi$ for all u and s which gives that

$$\frac{j_u k_s}{h_{us}} \le \frac{(1+\zeta)(1+\psi)}{\zeta\psi}, \quad \text{and } \frac{j_{u-1}k_{s-1}}{h_{us}} \le \frac{1}{\zeta\psi}. \tag{38}$$

Presume that $F_{wq} \longrightarrow F(C_1[\mathcal{I}_{W_2}^{(\eta,\nu)}])$. For each $x \in \mathcal{X}$, we get

$$\begin{aligned}
\frac{1}{h_{us}} \sum_{(w,q)\in I_{us}} \left|\eta\left(x, F_{wq}, p\right) - \eta(x, F, p)\right| - 1 &= \frac{1}{h_{us}} \sum_{w,q=1,1}^{j_u,k_s} \left|\eta\left(x, F_{wq}, p\right) - \eta(x, F, p)\right| - \frac{1}{h_{us}} \sum_{w,q=1,1}^{j_{u-1},k_{s-1}} \left|\eta\left(x, F_{wq}, p\right) - \eta(x, F, p)\right| - 1 \\
&= \frac{j_u k_s}{h_{us}}\left[\frac{1}{j_u k_s} \sum_{w,q=1,1}^{j_u,k_s} \left|\eta\left(x, F_{wq}, p\right) - \eta(x, F, p)\right| - 1\right] - \frac{j_{u-1}k_{s-1}}{h_{us}}\left[\frac{1}{j_{u-1}k_{s-1}} \sum_{w,q=1,1}^{j_{u-1},k_{s-1}} \left|\eta\left(x, F_{wq}, p\right) - \eta(x, F, p)\right| - 1\right].
\end{aligned} \tag{39}$$

Since $F_{wq} \longrightarrow F(C_1[\mathcal{I}_{W_2}^{(\eta,\nu)}])$, then for each

$$\frac{1}{j_u k_s} \sum_{w,q=1,1}^{j_u,k_s} \left|\eta\left(x, F_{wq}, p\right) - \eta(x, F, p)\right| - 1 \xrightarrow{\mathcal{I}_2} 0, \quad \text{and,} \frac{1}{j_{u-1}k_{s-1}} \sum_{w,q=1,1}^{j_{u-1},k_{s-1}} \left|\eta\left(x, F_{wq}, p\right) - \eta(x, F, p)\right| - 1 \xrightarrow{\mathcal{I}_2} 0. \tag{40}$$

Hence, when the above equality is examined, for every $x \in \mathcal{X}$, we have

$$\frac{1}{h_{us}} \sum_{(w,q)\in I_{us}} \left|\eta\left(x, F_{wq}, p\right) - \eta(x, F, p)\right| - 1 \xrightarrow{\mathcal{I}_2} 0. \tag{41}$$

Similarly, we obtain

$$\frac{1}{h_{us}} \sum_{(w,q)\in I_{us}} \left|\nu\left(x, F_{wq}, p\right) - \nu(x, F, p)\right| \xrightarrow{\mathcal{I}_2} 0. \tag{42}$$

That is, $F_{wq} \longrightarrow F(N_{\theta_2}[\mathcal{I}_{W_2}^{(\eta,\nu)}])$. As a result, we obtain $C_1[\mathcal{I}_{W_2}^{(\eta,\nu)}] \subseteq N_{\theta_2}[\mathcal{I}_{W_2}^{(\eta,\nu)}]$. □

Theorem 8. *If* $\liminf_u q_u = 1$ *and* $\liminf_s q_s = 1$, *then* $N_{\theta_2}[\mathcal{I}_{W_2}^{(\eta,\nu)}] \subseteq C_1[\mathcal{I}_{W_2}^{(\eta,\nu)}]$.

Proof. Take $\liminf_u q_u = 1$, $\liminf_s q_s = 1$, and $\{F_{wq}\} \in N_{\theta_2}[\mathcal{I}_{W_2}^{(\eta,\nu)}]$. Then, for every $p > 0$, we acquire

$$H_{us} = \frac{1}{h_{us}} \sum_{(w,q)\in I_{us}} \left|\eta\left(x, F_{wq}, p\right) - \eta(x, F, p)\right| \xrightarrow{\mathcal{I}_2} 1, H'_{us} = \frac{1}{h_{us}} \sum_{(w,q)\in I_{us}} \left|\nu\left(x, F_{wq}, p\right) - \nu(x, F, p)\right| \xrightarrow{\mathcal{I}_2} 0, \tag{43}$$

as $u, s \longrightarrow \infty$. Then, for $\xi > 0$, there are $u_0, s_0 \in \mathbb{N}$ such that $H_{us} < 1 + \xi$ for all $u > u_0$, $s > s_0$. Also, we can find $K > 0$ such that $H_{us} < K$ and $H'_{us} < K$, $u, s = 1, 2, \ldots$. Let m and n be an integer with $j_{u-1} < m \leq j_u$ and $k_{s-1} \leq n \leq k_s$. Then,

$$\begin{aligned}
\frac{1}{mn} \sum_{w,q=1,1}^{m,n} \left|\eta\left(x, F_{wq}, p\right) - \eta(x, F, p)\right| &\leq \frac{1}{j_{u-1}k_{s-1}} \sum_{w,q=1,1}^{j_u,k_s} \left|\eta\left(x, F_{wq}, p\right) - \eta(x, F, p)\right|, \\
&= \frac{1}{j_{u-1}k_{s-1}} \left[\sum_{(w,q)\in I_{11}} \left|\eta\left(x, F_{wq}, p\right) - \eta(x, F, p)\right| + \cdots + \sum_{(w,q)\in I_{us}} \left|\eta\left(x, F_{wq}, p\right) - \eta(x, F, p)\right|\right] \\
&= \sup_{1\leq u\leq u_0, 1\leq s\leq s_0} H_{us} \frac{j_{u_0}k_{s_0}}{j_{u-1}k_{s-1}} + \frac{h_{(u_0+1)(s_0+1)}}{j_{u-1}k_{s-1}} H_{(u_0+1)(s_0+1)} + \cdots + \frac{h_{us}}{j_{u-1}k_{s-1}} H_{us} < K \frac{j_{u_0}k_{s_0}}{j_{u-1}k_{s-1}} + (1+\xi) \frac{j_u k_s - j_{u_0}k_{s_0}}{j_{u-1}k_{s-1}}.
\end{aligned} \tag{44}$$

Since $j_{u-1}k_{s-1} \longrightarrow \infty$ as $m, n \longrightarrow \infty$, it follows that $1/mn\sum_{w,q=1}^{m,n} |\eta(x, F_{wq}, p) - \eta(x, F, p)| \longrightarrow^{\mathcal{I}_2} 1$. Similarly, we can show that $1/mn\sum_{w,q=1}^{m,n} |\nu(x, F_{wq}, p) - \nu(x, F, p)| \longrightarrow^{\mathcal{I}_2} 0$. Hence, $\{F_{wq}\} \in C_1[\mathcal{I}_{W_2}^{(\eta,\nu)}]$. □

Theorem 9. *If* $\{F_{wq}\} \in N_{\theta_2}[\mathcal{I}_{W_2}^{(\eta,\nu)}] \cap C_1[\mathcal{I}_{W_2}^{(\eta,\nu)}]$, *then* $N_{\theta_2}[\mathcal{I}_{W_2}^{(\eta,\nu)}] - limF_{wq} = C_1[\mathcal{I}_{W_2}^{(\eta,\nu)}] - limF_{wq}$.

Proof. Let $F_{wq} \longrightarrow F_1 (N_{\theta_2}[\mathcal{I}_{W_2}^{(\eta,\nu)}])$ and $F_{wq} \longrightarrow F_2 (C_1[\mathcal{I}_{W_2}^{(\eta,\nu)}])$. Assume $r \in \mathbb{N}$ and $\xi > 0$ in such way that $r > 2/\xi$. Then, for any $p > 0$, there are $u_0, s_0 \in \mathbb{N}$ such that

$$\frac{1}{h_{us}} \sum_{(w,q)\in I_{us}} \left|\eta\left(x, F_{wq}, \frac{p}{2}\right) - \eta\left(x, F_1, \frac{p}{2}\right)\right| > 1 - \frac{1}{r}, \quad \text{and} \frac{1}{h_{us}} \sum_{(w,q)\in I_{us}} \left|\nu\left(x, F_{wq}, \frac{p}{2}\right) - \nu\left(x, F_1, \frac{p}{2}\right)\right| < \frac{1}{r}, \tag{45}$$

for all $u > u_0$ and $s > s_0$. Also, there are $m_0, n_0 \in \mathbb{N}$ such that

$$\frac{1}{mn} \sum_{w,q=1,1}^{m,n} \left|\eta\left(x, F_{wq}, \frac{p}{2}\right) - \eta\left(x, F_2, \frac{p}{2}\right)\right| > 1 - \frac{1}{r}, \quad \text{and} \frac{1}{mn} \sum_{w,q=1,1}^{m,n} \left|\nu\left(x, F_{wq}, \frac{p}{2}\right) - \nu\left(x, F_2, \frac{p}{2}\right)\right| < \frac{1}{r}, \tag{46}$$

for all $m > m_0$ and $n > n_0$. Take $r_1 = \max\{u_0, m_0\}$ and $r_2 = \max\{s_0, n_0\}$. Then, we take $k, t \in \mathbb{N}$ such that

$$\begin{aligned}
\left|\eta\left(x, F_{kt}, \frac{p}{2}\right) - \eta\left(x, F_1, \frac{p}{2}\right)\right| &\geq \frac{1}{h_{us}} \sum_{(w,q)\in I_{us}} \left|\eta\left(x, F_{wq}, \frac{p}{2}\right) - \eta\left(x, F_1, \frac{p}{2}\right)\right| > 1 - \frac{1}{r}, \\
\left|\eta\left(x, F_{kt}, \frac{p}{2}\right) - \eta\left(x, F_2, \frac{p}{2}\right)\right| &\geq \frac{1}{mn} \sum_{w,q=1,1}^{m,n} \left|\eta\left(x, F_{wq}, \frac{p}{2}\right) - \eta\left(x, F_2, \frac{p}{2}\right)\right| > 1 - \frac{1}{r}.
\end{aligned} \tag{47}$$

Therefore, we get

$$|\eta(x, F_1, p) - \eta(x, F_2, p)| \leq \left|\eta\left(x, F_{kt}, \frac{p}{2}\right) - \eta\left(x, F_1, \frac{p}{2}\right)\right| + \left|\eta\left(x, F_{kt}, \frac{p}{2}\right) - \eta\left(x, F_2, \frac{p}{2}\right)\right| > \left(1 - \frac{1}{r}\right) + \left(1 - \frac{1}{r}\right) > 1 - \xi. \tag{48}$$

Since $\xi > 0$ is arbitrary, we get $|\eta(x, F_1, p) - \eta(x, F_2, p)| = 1$ for all $p > 0$, which yields $F_1 = F_2$. □

Throughout the following definitions and theorems, we consider $(\mathcal{X}, \eta, \nu, *, \diamond)$ to be a separable IFMS and $\mathcal{I}_2$ to be a strongly admissible ideal.

Definition 5. The sequence $\{F_{wq}\}$ is strongly $\mathcal{I}_2$-lacunary Cauchy sequence (Wijsman sense) if for each $\xi \in (0,1)$, for each $x \in \mathcal{X}$, and for all $p > 0$, there are $s = s(\xi, x), t = t(\xi, x) \in \mathbb{N}$ such that

$$A(\xi, x, p) = \left\{ \begin{array}{c} (u,s) \in \mathbb{N} \times \mathbb{N}: \dfrac{1}{h_{us}} \displaystyle\sum_{(w,q) \in I_{us}} \left|\eta\left(x, F_{wq}, p\right) - \eta(x, F_{st}, p)\right| \leq 1 - \xi \\ \\ \text{or, } \dfrac{1}{h_{us}} \displaystyle\sum_{(w,q) \in I_{us}} \left|\nu\left(x, F_{wq}, p\right) - \nu(x, F_{st}, p)\right| \geq \xi \end{array} \right\} \in \mathcal{I}_2. \tag{49}$$

Theorem 10. *Every Wijsman strongly $\mathcal{I}_2$-lacunary convergent sequence of closed sets $\{F_{wq}\}$ is Wijsman strongly $\mathcal{I}_2$-lacunary Cauchy with regard to IFM (η, ν).*

Proof. Let $F_{wq} \longrightarrow^{N_{\theta_2}} (\mathcal{I}_{W_2}^{\eta,\nu}) F$. At that case, for each $\xi \in (0,1)$, for every $x \in \mathcal{X}$, and for all $p > 0$,

$$A(\xi, x, p) = \left\{ \begin{array}{c} (u,s) \in \mathbb{N} \times \mathbb{N}: \dfrac{1}{h_{us}} \displaystyle\sum_{(w,q) \in I_{us}} \left|\eta\left(x, F_{wq}, p\right) - \eta(x, F, p)\right| \leq 1 - \xi \\ \\ \text{or, } \dfrac{1}{h_{us}} \displaystyle\sum_{(w,q) \in I_{us}} \left|\nu\left(x, F_{wq}, p\right) - \nu(x, F, p)\right| \geq \xi \end{array} \right\} \in \mathcal{I}_2. \tag{50}$$

Since $\mathcal{I}_2$ is a strongly admissible ideal, the set

$$A^c(\xi, x, p) = \left\{ \begin{array}{c} (u,s) \in \mathbb{N} \times \mathbb{N}: \dfrac{1}{h_{us}} \displaystyle\sum_{(w,q) \in I_{us}} \left|\eta\left(x, F_{wq}, p\right) - \eta(x, F, p)\right| > 1 - \xi \\ \\ \text{and } \dfrac{1}{h_{us}} \displaystyle\sum_{(w,q) \in I_{us}} \left|\nu\left(x, F_{wq}, p\right) - \nu(x, F, p)\right| < \xi \end{array} \right\}, \tag{51}$$

is nonempty and belongs to $\mathcal{F}(\mathcal{I}_2)$. So, we select positive integers u and s such that $(u,s) \notin A(\xi, x, p)$, and we get

$$\frac{1}{h_{us}} \sum_{(w_0,q_0) \in I_{us}} \left|\eta\left(x, F_{w_0q_0}, p\right) - \eta(x, F, p)\right| > 1 - \xi, \text{ and } \frac{1}{h_{us}} \sum_{(w_0,q_0) \in I_{us}} \left|\nu\left(x, F_{w_0q_0}, p\right) - \nu(x, F, p)\right| < \xi. \tag{52}$$

Now, presume that

$$B(\xi, x, p) = \left\{ \begin{array}{c} (u,s) \in \mathbb{N}\times\mathbb{N}: \dfrac{1}{h_{us}} \sum_{(w,q),\left(w_0,q_0\right)\in I_{us}} \left|\eta\left(x, F_{wq}, p\right) - \eta\left(x, F_{w_0q_0}, p\right)\right| \leq 1 - 2\xi \\ \\ \text{or } \dfrac{1}{h_{us}} \sum_{(w,q),\left(w_0,q_0\right)\in I_{us}} \left|\nu\left(x, F_{wq}, p\right) - \nu\left(x, F_{w_0q_0}, p\right)\right| \geq 2\xi \end{array} \right\}. \tag{53}$$

Consider the inequality

$$\begin{aligned} &\frac{1}{h_{us}} \sum_{(w,q),\left(w_0,q_0\right)\in I_{us}} \left|\eta\left(x, F_{wq}, p\right) - \eta\left(x, F_{w_0q_0}, p\right)\right| \leq \frac{1}{h_{us}} \sum_{(w,q)\in I_{us}} \left|\eta\left(x, F_{wq}, p\right) - \eta(x, F, p)\right| + \frac{1}{h_{us}} \sum_{\left(w_0,q_0\right)\in I_{us}} \left|\eta\left(x, F_{w_0q_0}, p\right) - \eta(x, F, p)\right|, \\ &\frac{1}{h_{us}} \sum_{(w,q),\left(w_0,q_0\right)\in I_{us}} \left|\nu\left(x, F_{wq}, p\right) - \nu\left(x, F_{w_0q_0}, p\right)\right| \leq \frac{1}{h_{us}} \sum_{(w,q)\in I_{us}} \left|\nu\left(x, F_{wq}, p\right) - \nu(x, F, p)\right| + \frac{1}{h_{us}} \sum_{\left(w_0,q_0\right)\in I_{us}} \left|\nu\left(x, F_{w_0q_0}, p\right) - \nu(x, F, p)\right|. \end{aligned} \tag{54}$$

Notice that if $(u, s) \in B(\xi, x, p)$, therefore,

$$\begin{aligned} &\frac{1}{h_{us}} \sum_{(w,q)\in I_{us}} \left|\eta\left(x, F_{wq}, p\right) - \eta(x, F, p)\right| + \frac{1}{h_{us}} \sum_{\left(w_0,q_0\right)\in I_{us}} \left|\eta\left(x, F_{w_0q_0}, p\right) - \eta(x, F, p)\right| \leq 1 - 2\xi, \\ &\frac{1}{h_{us}} \sum_{(w,q)\in I_{us}} \left|\nu\left(x, F_{wq}, p\right) - \nu(x, F, p)\right| + \frac{1}{h_{us}} \sum_{\left(w_0,q_0\right)\in I_{us}} \left|\nu\left(x, F_{w_0q_0}, p\right) - \nu(x, F, p)\right| \geq 2\xi. \end{aligned} \tag{55}$$

From another point of view, since $(u, s) \notin A(\xi, x, p)$, we get

$$\frac{1}{h_{us}} \sum_{\left(w_0,q_0\right)\in I_{us}} \left|\eta\left(x, F_{w_0q_0}, p\right) - \eta(x, F, p)\right| > 1 - \xi, \text{ and } \frac{1}{h_{us}} \sum_{\left(w_0,q_0\right)\in I_{us}} \left|\nu\left(x, F_{w_0q_0}, p\right) - \nu(x, F, p)\right| < \xi. \tag{56}$$

We reach that

$$\frac{1}{h_{us}} \sum_{(w,q)\in I_{us}} \left|\eta\left(x, F_{wq}, p\right) - \eta(x, F, p)\right| \leq 1 - \xi, \text{ or } \frac{1}{h_{us}} \sum_{(w,q)\in I_{us}} \left|\nu\left(x, F_{wq}, p\right) - \nu(x, F, p)\right| \geq \xi. \tag{57}$$

Hence, $(u, s) \in A(\xi, x, p)$. This gives that $B(\xi, x, p) \subset A(\xi, x, p) \in \mathcal{I}_2$ for every $\xi \in (0, 1)$ and for each $x \in \mathcal{X}$. Therefore, $B(\xi, x, p) \in \mathcal{I}_2$, so the sequence is Wijsman strongly $\mathcal{I}_2$-lacunary sequence. □

Definition 6. The sequence $\{F_{wq}\}$ is Wijsman strongly $\mathcal{I}_2^*$-lacunary convergent to F iff there is a set $M = \{(w, q) \in \mathbb{N}\times\mathbb{N}\}$ such that $M' = \{(u, s) \in \mathbb{N}\times\mathbb{N}: (w, q) \in I_{us}\} \in \mathcal{F}(\mathcal{I}_2)$ for each $x \in \mathcal{X}$,

$$\lim_{u,s\longrightarrow\infty}\frac{1}{h_{us}}\sum_{(w,q)\in I_{us}}\left|\eta\left(x,F_{wq},p\right)-\eta(x,F,p)\right|=1,$$
$$\lim_{u,s\longrightarrow\infty}\frac{1}{h_{us}}\sum_{(w,q)\in I_{us}}\left|\nu\left(x,F_{wq},p\right)-\nu(x,F,p)\right|=0. \tag{58}$$

In this case, we write $F_{wq}\longrightarrow F(N_{\theta_2}[\mathcal{I}^{*\,(\eta,\nu)}_{W_2}])$.

Theorem 11. *If the sequence $\{F_{wq}\}$ is Wijsman strongly $\mathcal{I}_2^*$-lacunary convergent to F, then $\{F_{wq}\}$ is Wijsman strongly $\mathcal{I}_2$-lacunary convergent to F.*

Proof. Presume that $F_{wq}\longrightarrow F(N_{\theta_2}[\mathcal{I}^{*\,(\eta,\nu)}_{W_2}])$. Then, there is a set $M=\{(w,q)\in\mathbb{N}\times\mathbb{N}\}$ such that

$$M'=\{(u,s)\in\mathbb{N}\times\mathbb{N}\colon\ (w,q)\in I_{us}\}\in\mathcal{F}(\mathcal{I}_2), \tag{59}$$

for each $x\in\mathcal{X}$,

$$\frac{1}{h_{us}}\sum_{(w,q)\in I_{us}}\left|\eta\left(x,F_{wq},p\right)-\eta(x,F,p)\right|>1-\xi,$$
$$\frac{1}{h_{us}}\sum_{(w,q)\in I_{us}}\left|\nu\left(x,F_{wq},p\right)-\nu(x,F,p)\right|<\xi. \tag{60}$$

for every $\xi>0$ and for all $w,q\geq k_0=k_0(\xi,x)\in\mathbb{N}$. Hereby, for each $\xi>0$ and $x\in\mathcal{X}$, we get

$$P(\xi,x,p)=\left\{\begin{array}{c}(u,s)\in\mathbb{N}\times\mathbb{N}\colon\frac{1}{h_{us}}\sum_{(w,q)\in I_{us}}\left|\eta\left(x,F_{wq},p\right)-\eta(x,F,p)\right|\leq 1-\xi\\ \\ \text{or,}\ \frac{1}{h_{us}}\sum_{(w,q)\in I_{us}}\left|\nu\left(x,F_{wq},p\right)-\nu(x,F,p)\right|\geq\xi\end{array}\right\},$$
$$\subset H\cup\left(M'\cap\left(\left(\{1,2,\ldots,(k_0-1)\}\times\mathbb{N}\right)\cup\left(\mathbb{N}\times\{1,2,\ldots,(k_0-1)\}\right)\right)\right), \tag{61}$$

for $\mathbb{N}\times\mathbb{N}\setminus M'=H\in\mathcal{I}_2$. Since $\mathcal{I}_2$ is an admissible ideal, we obtain

$$H\cup\left(M'\cap\left(\left(\{1,2,\ldots,(k_0-1)\}\times\mathbb{N}\right)\cup\left(\mathbb{N}\times\{1,2,\ldots,(k_0-1)\}\right)\right)\right)\in\mathcal{I}_2, \tag{62}$$

and so $P(\xi,x,p)\in\mathcal{I}_2$. Hence, $F_{wq}\longrightarrow F(N_{\theta_2}[\mathcal{I}^{(\eta,\nu)}_{W_2}])$. ☐

Theorem 12. *Let $\mathcal{I}_2$ be a strongly admissible ideal involving feature (AP_2). Then, $F_{wq}\longrightarrow F(N_{\theta_2}[\mathcal{I}^{(\eta,\nu)}_{W_2}])$ implies $F_{wq}\longrightarrow F(N_{\theta_2}[\mathcal{I}^{*\,(\eta,\nu)}_{W_2}])$.*

Definition 7. The sequence $\{F_{wq}\}$ is known as Wijsman strongly $\mathcal{I}_2^*$-lacunary Cauchy sequence if for each $\xi\in(0,1)$, for all $x\in\mathcal{X}$, and for all $p>0$, and there is a set $M=\{(w,q)\in\mathbb{N}\times\mathbb{N}\}$ such that $M'=\{(u,s)\in\mathbb{N}\times\mathbb{N}:(w,q)\in I_{us}\}\in\mathcal{F}(\mathcal{I}_2)$ and $N=N(\varepsilon,x)\in\mathbb{N}$ such that

$$\frac{1}{h_{us}}\sum_{(w,q),(s,t)\in I_{us}}\left|\eta\left(x,F_{wq},p\right)-\eta(x,F_{st},p)\right|>1-\xi,\quad\text{and}\ \frac{1}{h_{us}}\sum_{(w,q),(s,t)\in I_{us}}\left|\nu\left(x,F_{wq},p\right)-\nu(x,F_{st},p)\right|<\xi, \tag{63}$$

for every $w,q,s,t\geq N$.

Theorem 13. *Every Wijsman strongly $\mathcal{I}_2^*$-lacunary Cauchy sequence of closed sets is Wijsman strongly $\mathcal{I}_2$-lacunary Cauchy in IFMS with regard to (η,ν).*

Proof. If the hypothesis is provided, then for each $\xi\in(0,1)$, for each $x\in\mathcal{X}$, and for all $p>0$, there is a set $M=\{(w,q)\in\mathbb{N}\times\mathbb{N}\}$ such that

$$M'=\{(u,s)\in\mathbb{N}\times\mathbb{N}\colon\ (w,q)\in I_{us}\}\in\mathcal{F}(\mathcal{I}_2), \tag{64}$$

and $N=N(\varepsilon,x)\in\mathbb{N}$ such that

$$\frac{1}{h_{us}}\sum_{(w,q),(s,t)\in I_{us}}\left|\eta\left(x,F_{wq},p\right)-\eta(x,F_{st},p)\right|>1-\xi,\ \text{and}\ \frac{1}{h_{us}}\sum_{(w,q),(s,t)\in I_{us}}\left|\nu\left(x,F_{wq},p\right)-\nu(x,F_{st},p)\right|<\xi, \tag{65}$$

for each $w, q, s, t \geq N$. Let $H = \mathbb{N} \times \mathbb{N} \setminus M'$. It is clear that $H \in \mathcal{I}_2$ and

$$P(\xi, x, p) = \left\{ \begin{array}{c} (u,s) \in \mathbb{N} \times \mathbb{N}\colon \dfrac{1}{h_{us}} \displaystyle\sum_{(w,q),(s,t) \in I_{us}} \left|\eta\left(x, F_{wq}, p\right) - \eta\left(x, F_{st}, p\right)\right| > 1 - \xi, \\ \text{or } \dfrac{1}{h_{us}} \displaystyle\sum_{(w,q),(s,t) \in I_{us}} \left|\nu\left(x, F_{wq}, p\right) - \nu\left(x, F_{st}, p\right)\right| < \xi, \end{array} \right\}, \tag{66}$$
$$\subset H \cup \left(M' \cap \left((\{1, 2, \ldots, (N-1)\} \times \mathbb{N}) \cup (\mathbb{N} \times \{1, 2, \ldots, (N-1)\})\right)\right).$$

As $\mathcal{I}_2$ be a strongly admissible ideal, then

$$H \cup \left(M' \cap \left((\{1, 2, \ldots, (N-1)\} \times \mathbb{N}) \cup (\mathbb{N} \times \{1, 2, \ldots, (N-1)\})\right)\right) \in \mathcal{I}_2. \tag{67}$$

Therefore, we obtain $P(\xi, x, p) \in \mathcal{I}_2$; that is, $\{F_{wq}\}$ is strongly $\mathcal{I}_2$-lacunary Cauchy sequence (Wijsman sense) with regard to (η, ν). □

Theorem 14. *Let $\mathcal{I}_2$ be an admissible ideal involving property (AP_2). Then, the concept of Wijsman strongly $\mathcal{I}_2$-lacunary Cauchy sequence of sets coincides with Wijsman strongly $\mathcal{I}_2^*$-lacunary Cauchy sequence of sets.*

Proof. If a set sequence is strongly $\mathcal{I}_2^*$-lacunary Cauchy sequence, then it is strongly $\mathcal{I}_2$-lacunary Cauchy sequence according to Theorem 13, where $\mathcal{I}_2$ need not to have the feature (AP_2).

Now, it is adequate to demonstrate that a sequence $\{F_{wq}\}$ in $\mathcal{X}$ is a strongly $\mathcal{I}_2^*$-lacunary Cauchy sequence under assumption that it is a strongly $\mathcal{I}_2$-lacunary Cauchy sequence. Let $\{F_{wq}\}$ be a Wijsman strongly lacunary Cauchy sequence. In this case, for each $\xi \in (0,1)$, for all $x \in \mathcal{X}$, there is a number $s = s(\varepsilon, x), t = t(\varepsilon, x) \in \mathbb{N}$ such that

$$A(\xi, x, p) = \left\{ \begin{array}{c} (u,s) \in \mathbb{N} \times \mathbb{N}\colon \dfrac{1}{h_{us}} \displaystyle\sum_{(w,q) \in I_{us}} \left|\eta\left(x, F_{wq}, p\right) - \eta\left(x, F_{st}, p\right)\right| \leq 1 - \xi \\ \text{or } \dfrac{1}{h_{us}} \displaystyle\sum_{(w,q) \in I_{us}} \left|\nu\left(x, F_{wq}, p\right) - \nu\left(x, F_{st}, p\right)\right| \geq \xi \end{array} \right\} \in \mathcal{I}_2. \tag{68}$$

Let

$$P_j(\xi, x, p) = \left\{ \begin{array}{c} (u,s) \in \mathbb{N} \times \mathbb{N}\colon \dfrac{1}{h_{us}} \displaystyle\sum_{(w,q) \in I_{us}} \left|\eta\left(x, F_{wq}, p\right) - \eta\left(x, F_{s_j t_j}, p\right)\right| > 1 - \dfrac{1}{j} \\ \text{or } \dfrac{1}{h_{us}} \displaystyle\sum_{(w,q) \in I_{us}} \left|\nu\left(x, F_{wq}, p\right) - \nu\left(x, F_{s_j t_j}, p\right)\right| < \dfrac{1}{j} \end{array} \right\}, \tag{69}$$

where $s(j) = s(1/j)$ and $t(j) = t(1/j)$, $j = 1, 2, \ldots$. Clearly, for $j = 1, 2, \ldots$, $P_j(\xi, x, p) \in \mathcal{F}(\mathcal{I}_2)$. Since $\mathcal{I}_2$ has the property (AP_2), then by Theorem 3.3 in [9], there is $P \subset \mathbb{N} \times \mathbb{N}$ so that $P \in \mathcal{F}(\mathcal{I}_2)$ and $P \setminus P_j$ is finite for all j. Now, we demonstrate that

$$\lim_{w,q,s,t \longrightarrow \infty} \frac{1}{h_{us}} \sum_{(w,q) \in I_{us}, (s,t) \in I_{us}} \left|\eta\left(x, F_{wq}, p\right) - \eta\left(x, F_{st}, p\right)\right| = 1, \tag{70}$$

$\mathcal{I}_2$ and

$$\lim_{w,q,s,t\longrightarrow\infty}\frac{1}{h_{us}}\sum_{(w,q)\in I_{us},(s,t)\in I_{us}}\left|\nu\left(x,F_{wq},p\right)-\nu\left(x,F_{st},p\right)\right|=0, \tag{71}$$

for each $x\in\mathcal{X}$ and $(w,q),(s,t)\in P$. To show these, let $\xi\in(0,1)$ and $r\in\mathbb{N}$ such that $r>2/\xi$. If $(w,q),(s,t)\in P$, then $P\setminus P_r$ is a finite set; therefore, there is $u=u(r)$ so that

$$\begin{aligned}
&\frac{1}{h_{us}}\sum_{(w,q)\in I_{us}}\left|\eta\left(x,F_{wq},p\right)-\eta\left(x,F_{s_rt_r},p\right)\right|>1-\frac{1}{r},\\
&\frac{1}{h_{us}}\sum_{(s,t)\in I_{us}}\left|\eta\left(x,F_{st},p\right)-\eta\left(x,F_{s_rt_r},p\right)\right|>1-\frac{1}{r},\\
&\frac{1}{h_{us}}\sum_{(w,q)\in I_{us}}\left|\nu\left(x,F_{wq},p\right)-\nu\left(x,F_{s_rt_r},p\right)\right|<\frac{1}{r},\\
&\frac{1}{h_{us}}\sum_{(s,t)\in I_{us}}\left|\nu\left(x,F_{st},p\right)-\nu\left(x,F_{s_rt_r},p\right)\right|<\frac{1}{r},
\end{aligned} \tag{72}$$

for all $w,q,s,t>u(r)$. So, it follows that

$$\begin{aligned}
&\frac{1}{h_{us}}\sum_{(w,q)\in I_{us}}\left|\eta\left(x,F_{wq},p\right)-\eta\left(x,F_{st},p\right)\right|\leq\frac{1}{h_{us}}\sum_{(w,q)\in I_{us}}\left|\eta\left(x,F_{wq},p\right)-\eta\left(x,F_{s_rt_r},p\right)\right|\\
&+\frac{1}{h_{us}}\sum_{(s,t)\in I_{us}}\left|\eta\left(x,F_{st},p\right)-\eta\left(x,F_{s_rt_r},p\right)\right|>\left(1-\frac{1}{r}\right)+\left(1-\frac{1}{r}\right)>1-\xi,\\
&\frac{1}{h_{us}}\sum_{(w,q)\in I_{us}}\left|\nu\left(x,F_{wq},p\right)-\nu\left(x,F_{st},p\right)\right|\leq\frac{1}{h_{us}}\sum_{(w,q)\in I_{us}}\left|\nu\left(x,F_{wq},p\right)-\nu\left(x,F_{s_rt_r},p\right)\right|\\
&+\frac{1}{h_{us}}\sum_{(s,t)\in I_{us}}\left|\nu\left(x,F_{st},p\right)-\nu\left(x,F_{s_rt_r},p\right)\right|<\frac{1}{r}+\frac{1}{r}<\xi.
\end{aligned} \tag{73}$$

Therefore, for each $\xi\in(0,1)$, $\exists u=u(\xi)$, and $(w,q),(s,t)\in P\in\mathcal{F}(\mathcal{I}_2)$, we get

$$\left\{\begin{aligned}
&(u,s)\in\mathbb{N}\times\mathbb{N}:\frac{1}{h_{us}}\sum_{(w,q)\in I_{us}}\left|\eta\left(x,F_{wq},p\right)-\eta\left(x,F_{st},p\right)\right|\leq1-\xi\\
&\text{or }\frac{1}{h_{us}}\sum_{(w,q)\in I_{us}}\left|\nu\left(x,F_{wq},p\right)-\nu\left(x,F_{st},p\right)\right|\geq\xi
\end{aligned}\right\}\in\mathcal{I}_2. \tag{74}$$

This implies that $\{F_{wq}\}$ is Wijsman strongly $\mathcal{I}_2^*$-lacunary Cauchy. □

Definition 8. A sequence $\{F_{wq}\}$ in IFMS is called to be Wijsman lacunary convergent to F with regard to IFM (η,ν) if, for every $p>0$ and $\xi\in(0,1)$, there is $m_0,n_0\in\mathbb{N}$ such that

$$\frac{1}{h_{us}}\sum_{(w,q)\in I_{us}}\left|\eta\left(x,F_{wq},p\right)-\eta\left(x,F,p\right)\right|>1-\xi,\quad\text{and }\frac{1}{h_{us}}\sum_{(w,q)\in I_{us}}\left|\nu\left(x,F_{wq},p\right)-\nu\left(x,F,p\right)\right|<\xi, \tag{75}$$

for all $u \geq m_0$ and $s \geq n_0$. We write $(\mu, \nu)^{\theta_2} - \lim F_{wq} = F$.

Definition 9. Take $(\mathcal{X}, \eta, \nu, *, \diamond)$ as a separable IFMS and take $\{F_{wq}\} \in \mathcal{X}$.

(a) $F \in \mathcal{X}$ is known as Wijsman lacunary $\mathcal{I}_2$-limit point of $\{F_{wq}\}$ if there is set $M = \{(w_1, q_1) < (w_2, q_2) < \cdots < (w_u, q_s) < \cdots\} \subset \mathbb{N} \times \mathbb{N}$ such that the set

$$M' = \{(u, s) \in \mathbb{N} \times \mathbb{N}: (w_u, q_s) \in I_{us}\} \notin \mathcal{I}_2, \tag{76}$$

and $(\eta, \nu)^{\theta_2} - \lim F_{w_u q_s} = F$.

(b) $F \in \mathcal{X}$ is known as Wijsman lacunary $\mathcal{I}_2$-cluster point of $\{F_{wq}\}$ if, for every $p > 0$ and $\varepsilon \in (0, 1)$, we get

$$\left\{ \begin{array}{c} (u, s) \in \mathbb{N} \times \mathbb{N}: \dfrac{1}{h_{us}} \sum_{(w,q) \in I_{us}} \left| \eta\left(x, F_{wq}, p\right) - \eta(x, F, p) \right| > 1 - \xi \\ \text{and, } \dfrac{1}{h_{us}} \sum_{(w,q) \in I_{us}} \left| \nu\left(x, F_{wq}, p\right) - \nu(x, F, p) \right| < \xi \end{array} \right\} \notin \mathcal{I}_2. \tag{77}$$

Here, $\Lambda_{(\eta,\nu)}^{\mathcal{I}_{\theta_2}}(F_{wq})$ denotes the set of all Wijsman lacunary $\mathcal{I}_2$-limit points and $\Gamma_{(\eta,\nu)}^{\mathcal{I}_{\theta_2}}(F_{wq})$ indicates the set of all Wijsman lacunary $\mathcal{I}_2$-cluster points in IFMS.

Theorem 15. *For each sequence* $\{F_{wq}\}$ *in IFMS, we have* $\Lambda_{(\eta,\nu)}^{\mathcal{I}_{\theta_2}}(F_{wq}) \subseteq \Gamma_{(\eta,\nu)}^{\mathcal{I}_{\theta_2}}(F_{wq})$.

Proof. Let $F \in \Lambda_{(\eta,\nu)}^{\mathcal{I}_{\theta_2}}(F_{wq})$. So, there is a set $M \subset \mathbb{N} \times \mathbb{N}$ such that $M' \notin \mathcal{I}_2$, where M and M' are as in Definition 9, satisfying $(\eta, \nu)^{\theta_2} - \lim F_{w_u q_s} = F$. Hence, for every $p > 0$ and $\xi \in (0, 1)$, there are $m_0, n_0 \in \mathbb{N}$ such that

$$\frac{1}{h_{us}} \sum_{(w,q) \in I_{us}} \left| \eta\left(x, F_{w_u q_s}, p\right) - \eta(x, F, p) \right| > 1 - \xi, \quad \text{and } \frac{1}{h_{us}} \sum_{(w,q) \in I_{us}} \left| \nu\left(x, F_{w_u q_s}, p\right) - \nu(x, F, p) \right| < \xi, \tag{78}$$

for all $u \geq m_0$ and $s \geq n_0$. Therefore,

$$\begin{aligned} B = &\left\{ \begin{array}{c} (u, s) \in \mathbb{N} \times \mathbb{N}: \dfrac{1}{h_{us}} \sum_{(w,q) \in I_{us}} \left| \eta\left(x, F_{wq}, p\right) - \eta(x, F, p) \right| > 1 - \xi \\ \text{and } \dfrac{1}{h_{us}} \sum_{(w,q) \in I_{us}} \left| \nu\left(x, F_{wq}, p\right) - \nu(x, F, p) \right| < \xi \end{array} \right\}, \\ &\supseteq M' \backslash \left\{(w_1, q_1), (w_2, q_2), \ldots, \left(w_{m_0}, q_{n_0}\right)\right\}. \end{aligned} \tag{79}$$

Now, with $\mathcal{I}_2$ being admissible, we must have $M' \backslash \left\{(w_1, q_1), (w_2, q_2), \ldots, (w_{m_0}, q_{n_0})\right\} \notin \mathcal{I}_2$ and as such $B \notin \mathcal{I}_2$. Hence, $F \in \Gamma_{(\eta,\nu)}^{\mathcal{I}_{\theta_2}}(F_{wq})$. □

3. Conclusion

In this study, we examined a version of ideal convergence, named Wijsman lacunary ideal convergence of double set sequences, in IFMS. We investigated new convergence concepts for double set sequences in IFMS and obtained some meaningful results. In addition, Wijsman lacunary $\mathcal{I}_2$-limit points and Wijsman lacunary $\mathcal{I}_2$-cluster points of double set sequences in IFMS were defined. Some of the results presented in this article are analogous to the research studies in the relevant topic, but in most situations, the proofs follow a different approach.

Data Availability

No data were used to support this study.

Conflicts of Interest

The author declares that there are no conflicts of interest.

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