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On Fejér Type Inclusions for Products of Interval-Valued Convex Functions

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Abstract. We first get some new Fejér type inclusions for products of interval-valued convex mappings. The most important feature of our work is that it contains Fejér type inclusions for both interval-valued integrals and interval-valued fractional integrals.

1. Introduction

Integral inequalities cover an important area of study in pure and applied mathematics. Especially Hermite-Hadamard inequalities attract the attention of researchers today. C. Hermite and J. Hadamard discovered these inequalities, such that if $\varpi : \mathrm{I} \to \mathbb{R}$ is a convex function on the interval I of real numbers and $\kappa_1, \kappa_2 \in \mathrm{I}$ with $\kappa_1 < \kappa_2$, then

$$\varpi\left(\frac{\kappa_1+\kappa_2}{2}\right) \le \frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2} \varpi(\xi)d\xi \le \frac{\varpi(\kappa_1)+\varpi(\kappa_2)}{2}. \tag{1}$$

If ϖ concave, the two inequalities are in opposite directions. Excellent results associated with this midpoint inequalities and trapezoidal inequalities often used in Special means and estimation errors (see [14] and [25]). Later, many contributors obtained new results for these inequalities under different mapping positions. And then many mathematicians have evaluated generalizations, refinements and counterparts and generalizations of the inequalites (1). For instance, weighted version of the inequalities (1), which is also named Hermite-Hadamard-Fejér inequalities, was established by Fejér in [16] as follow:

Theorem 1.1. *Suppose that $\varpi : [\kappa_1, \kappa_2] \to \mathbb{R}$ is a convex function, and let $\phi : [\kappa_1, \kappa_2] \to \mathbb{R}$ be non-negative, integrable, and symmetric about $\xi = \frac{\kappa_1+\kappa_2}{2}$ (i.e. $\phi(\xi) = \phi(\kappa_1+\kappa_2-\xi)$). Then, we have the inequality*

$$\varpi\left(\frac{\kappa_1+\kappa_2}{2}\right)\int_{\kappa_1}^{\kappa_2} \phi(\xi)d\xi \le \int_{\kappa_1}^{\kappa_2} \varpi(\xi)\phi(\xi)d\xi \le \frac{\varpi(\kappa_1)+\varpi(\kappa_2)}{2}\int_{\kappa_1}^{\kappa_2} \phi(\xi)d\xi. \tag{2}$$

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Many mathematicians derived some generalizations and new results involving fractional integrals regarding to the inequalities (2) to obtain new bounds for left and right sides of the inequalities (2) (for example, [15], [46] and [47]). In addition to all these generalizations, a good many authors have worked on Hermite-Hadamard type inequalities for products two convex functions in recent years. Moreover, some of them attained Hermite-Hadamard type results contains fractional integrals in their works. For instance, Pachpatte provided novel inequalities for products of two non-negative and convex mappings in [40]. After that, some authors examined how the results were obtained by multiplying two mappings selected from various convex function classes in the references [4, 8–10, 22, 26, 48–50]. What is more, some inequalities involving product of two co-ordinated convex mapping were observed by Latif and Alomari in [27]. Thereafter, Ozdemir et al. deduced more general versions of the inequalities presented in [27] by considering the product of two co-ordinated s-convex and product of two co-ordinated h-convex mappings in [38] and [39], respectively. In [3], by using products of two co-ordinated convex mappings, new Hermite-Hadamard type results including fractional integrals were proved by Budak and Sarıkaya.

On the other hand, interval analysis which is considered as one of the means of resolving interval uncertainty is an important factor used in mathematical and computer models. Although this theory has a long history that cannot be traced much study has not been published in this field until the 1950's. The first reference [34] relevant interval analysis was pressed by Ramon E. Moore known as the guide of interval calculus in 1966. After that, many researchers began to investigate the theories and applications of interval analysis. Latterly, many scholars paid attention on integral inequalities obtained by using interval-valued functions. For example, Sadowska [44] has established a Hermite-Hadamard inequality for set-valued functions that is more general version of interval-valued mappings as follows:

Theorem 1.2. *[44] Suppose that $\varphi : [\kappa_1, \kappa_2] \to \mathcal{P}^+$ is interval–valued convex function such that $\varphi(\tau) = \left[\underline{\varphi}(\tau), \overline{\varphi}(\tau)\right]$, Then, we have the inclusions:*

$$\varphi\left(\frac{\kappa_1 + \kappa_2}{2}\right) \supseteq \frac{1}{\kappa_2 - \kappa_1}(IR)\int_{\kappa_1}^{\kappa_2} \varphi(\xi)d\xi \supseteq \frac{\varphi(\kappa_1) + \varphi(\kappa_2)}{2}. \tag{3}$$

In addition, Ostrowski, Minkowski and Beckenbach inequalities and their some applications were provided by evaluated interval-valued functions in [6, 7, 17, 18]. And then, some inclusions involving interval-valued Riemann-Liouville fractional integrals were obtained by Budak et al. in [5]. In [28], Liu et al. gave the definition of interval-valued harmonically convex functions, and so they some Hermite-Hadamard type inclusions including interval fractional integrals. If you want to see more, check out these [11], [12], [19], [20], [33], [35], [36], [37], [52], [53].

The general structure of this paper consist of four main sections including introduction. In this part, we gave some necessary definitons and the concept of interval analysis, and we also mentioned some related works in the literature. In section 2, some basic informations connected interval calculus which forms the basis of this work were presented. In section 3, we provide Fejér type inclusions for products of interval-valued convex functions, and we examine the relation between our results and inclusions presented in the earlier works. Finally, we establish interval-valued Fejer type inequalities including fractional integrals by appling the inequalities given in section 3 to interval-valued fractional integrals in section 4. Briefly, the most important property of this study is that it contains interval-valued Fejer type inclusions for classical and fractional integrals. We note that the opinion and technique of this work may inspire new research in this area.

2. Interval Calculus

In this part, we give some information related to interval analysis which forms the basis of this paper. For this, we denote by $\mathcal{P}$ the space of all closed intervals of $\mathbb{R}$. Suppose that $\mathcal{U}$ is element of $\mathcal{P}$ and bounded, then we have the notation

$$\mathcal{U} = \left[\underline{\mathcal{U}}, \overline{\mathcal{U}}\right] = \left\{\tau \in \mathbb{R} : \underline{\mathcal{U}} \le \tau \le \overline{\mathcal{U}}\right\},$$

where $\underline{\mathcal{U}}, \overline{\mathcal{U}} \in \mathbb{R}$ and $\underline{\mathcal{U}} \leq \overline{\mathcal{U}}$. The numbers $\underline{\mathcal{U}}$ and $\overline{\mathcal{U}}$ are named the left and the right endpoints of interval $\mathcal{U}$, respectively. We say that $\mathcal{U}$ is degenerate if $\underline{\mathcal{U}} = \overline{\mathcal{U}}$, and this situation is denoted by $\mathcal{U} = \mathcal{U} = [\mathcal{U}, \mathcal{U}]$. What's more, $\mathcal{U}$ is said to be positive when $\underline{\mathcal{U}} > 0$, or $\mathcal{U}$ is said to be negative when $\overline{\mathcal{U}} < 0$. The sets of all closed positive intervals of $\mathbb{R}$ and the sets of all closed negative intervals of $\mathbb{R}$ are denoted by $\mathcal{P}^+$ and $\mathcal{P}^-$, respectively. The Hausdorff-Pompeiu distance between the intervals $\mathcal{U}$ and $\mathcal{V}$ is defined by

$$d(\mathcal{U}, \mathcal{V}) = d\left(\left[\underline{\mathcal{U}}, \overline{\mathcal{U}}\right], \left[\underline{\mathcal{V}}, \overline{\mathcal{V}}\right]\right) = \max\left\{\left|\underline{\mathcal{U}} - \underline{\mathcal{V}}\right|, \left|\overline{\mathcal{U}} - \overline{\mathcal{V}}\right|\right\}.$$

Also, $(\mathcal{P}, d)$ is known to be a complete metric space [1].

We now give the properties of fundamental interval analysis operations for the intervals $\mathcal{U}$ and $\mathcal{V}$ as follows:

$$\begin{aligned} \mathcal{U} + \mathcal{V} &= \left[\underline{\mathcal{U}} + \underline{\mathcal{V}}, \overline{\mathcal{U}} + \overline{\mathcal{V}}\right], \\ \mathcal{U} - \mathcal{V} &= \left[\underline{\mathcal{U}} - \overline{\mathcal{V}}, \overline{\mathcal{U}} - \underline{\mathcal{V}}\right], \\ \mathcal{U}.\mathcal{V} &= [\min \Lambda, \max \Lambda] \text{ where } \Lambda = \left\{\underline{\mathcal{U}}\,\underline{\mathcal{V}}, \underline{\mathcal{U}}\,\overline{\mathcal{V}}, \overline{\mathcal{U}}\underline{\mathcal{V}}, \overline{\mathcal{U}}\,\overline{\mathcal{V}}\right\}, \\ \mathcal{U}/\mathcal{V} &= [\min \Delta, \max \Delta] \text{ where } \Delta = \left\{\underline{\mathcal{U}}/\underline{\mathcal{V}}, \underline{\mathcal{U}}/\overline{\mathcal{V}}, \overline{\mathcal{U}}/\underline{\mathcal{V}}, \overline{\mathcal{U}}/\overline{\mathcal{V}}\right\} \text{ and } 0 \notin \mathcal{V}. \end{aligned}$$

Scalar multiplication of the interval $\mathcal{U}$ is indicated by

$$\theta\mathcal{U} = \theta\left[\underline{\mathcal{U}}, \overline{\mathcal{U}}\right] = \begin{cases} \left[\theta\underline{\mathcal{U}}, \theta\overline{\mathcal{U}}\right], & \theta > 0 \\ \{0\}, & \theta = 0 \\ \left[\theta\overline{\mathcal{U}}, \theta\underline{\mathcal{U}}\right], & \theta < 0, \end{cases}$$

where $\theta \in \mathbb{R}$.

The opposite of the interval $\mathcal{U}$ is

$$-\mathcal{U} := (-1)\mathcal{U} = [-\overline{\mathcal{U}}, -\underline{\mathcal{U}}]$$

for $\delta = -1$.

The subtraction is given by

$$\mathcal{U} - \mathcal{V} = \mathcal{U} + (-\mathcal{V}) = [\underline{\mathcal{U}} - \overline{\mathcal{V}}, \overline{\mathcal{U}} - \underline{\mathcal{V}}].$$

In general, $-\mathcal{U}$ is not additive inverse for $\mathcal{U}$ i.e $\mathcal{U} - \mathcal{U} \neq 0$.

The definitions of operations cause a large number of algebraical features that allows $\mathcal{P}$ to be quasilinear space [30] . These features can be written as follows [29],[30],[34],[35]:

(1) (Associativity of addition) $(\mathcal{U} + \mathcal{V}) + \mathcal{W} = \mathcal{U} + (\mathcal{V} + \mathcal{W})$ for all $\mathcal{U}, \mathcal{V}, \mathcal{W} \in \mathcal{P}$,
(2) (Additive element) $\mathcal{U} + 0 = 0 + \mathcal{U} = \mathcal{U}$ for all $\mathcal{U} \in \mathcal{P}$,
(3) (Commutativity of addition) $\mathcal{U} + \mathcal{V} = \mathcal{V} + \mathcal{U}$ for all $\mathcal{U}, \mathcal{V} \in \mathcal{P}$,
(4) (Cancellation law) $\mathcal{U} + \mathcal{W} = \mathcal{V} + \mathcal{W} \Longrightarrow \mathcal{U} = \mathcal{V}$ for all $\mathcal{U}, \mathcal{V}, \mathcal{W} \in \mathcal{P}$,
(5) (Associativity of multiplication) $(\mathcal{U}.\mathcal{V}).\mathcal{W} = \mathcal{U}.(\mathcal{V}.\mathcal{W})$ for all $\mathcal{U}, \mathcal{V}, \mathcal{W} \in \mathcal{P}$,
(6) (Commutativity of multiplication) $\mathcal{U}.\mathcal{V} = \mathcal{V}.\mathcal{U}$ for all $\mathcal{U}, \mathcal{V} \in \mathcal{P}$,
(7) (Unit element) $\mathcal{U}.1 = 1.\mathcal{U}$ for all $\mathcal{U} \in \mathcal{P}$,
(8) (Associate law) $\theta(\gamma U) = (\theta\gamma)\mathcal{U}$ for all $\mathcal{U} \in \mathcal{P}$ and all $\theta, \gamma \in \mathbb{R}$,
(9) (First distributive law) $\theta(\mathcal{U} + \mathcal{V}) = \theta\mathcal{U} + \theta\mathcal{V}$ for all $\mathcal{U}, \mathcal{V} \in \mathcal{P}$ and all $\theta \in \mathbb{R}$,
(10) (Second distributive law) $(\theta + \gamma)\mathcal{U} = \theta\mathcal{U} + \gamma\mathcal{U}$ for all $\mathcal{U} \in \mathcal{P}$ and all $\theta, \gamma \in \mathbb{R}$.

In addition, one of the set features is the inclusion "$\subseteq$" that is defined by

$$\mathcal{U} \subseteq \mathcal{V} \Longleftrightarrow \underline{\mathcal{V}} \leq \underline{\mathcal{U}} \text{ and } \overline{\mathcal{U}} \leq \overline{\mathcal{V}}.$$

If we consider together with arithmetic calculations and inclusion, then we have the following feature that is called inclusion isotony of interval operations:

Assuming that $\odot$ is the addition, multiplication, subtraction or division. If $\mathcal{U}, \mathcal{V}, \mathcal{W}$ and $\mathcal{Y}$ are intervals supplying the conditions

$$\mathcal{U} \subseteq \mathcal{V} \quad \text{and } \mathcal{W} \subseteq \mathcal{Y},$$

then the following connection is true

$$\mathcal{U} \odot \mathcal{W} \subseteq \mathcal{V} \odot \mathcal{Y}.$$

2.1. Integral of Interval-Valued Functions

In this subsection, the notion of integral of the interval-valued mappings is mentioned. Before we can understand the definition of interval integrals, we need to give some concepts in the following.

A function φ is said to be an interval-valued function of τ on $[\kappa_1, \kappa_2]$ if it assigns a nonempty interval to each $\tau \in [\kappa_1, \kappa_2]$

$$\varphi(\tau) = \left[\underline{\varphi}(\tau), \overline{\varphi}(\tau)\right].$$

A partition of $[\kappa_1, \kappa_2]$ is any finite ordered subset D having the form

$$D : \kappa_1 = \tau_0 < \tau_1 < ... < \tau_n = \kappa_2.$$

The mesh of a partition D is indicated by

$$mesh(D) = \max\{\tau_i - \tau_{i-1} : i = 1, 2, ..., n\}.$$

We denote by $D([\kappa_1, \kappa_2])$ the set of all partition of $[\kappa_1, \kappa_2]$. Suppose that $D(\delta, [\kappa_1, \kappa_2])$ is the set of all $D \in D([\kappa_1, \kappa_2])$ such that $mesh(D) < \delta$. We take an arbitrary point ξ_i in interval $[\tau_{i-1}, \tau_i]$, $i = 1, 2, ..., n$, and we define the sum

$$S(\varphi, D, \delta) = \sum_{i=1}^{n} \varphi(\xi_i)\left[\tau_i - \tau_{i-1}\right]$$

where $\varphi : [\kappa_1, \kappa_2] \to \mathcal{P}$. The sum $S(\varphi, D, \delta)$ is said to be a Riemann sum of φ corresponding to $D \in D(\delta, [\kappa_1, \kappa_2])$.

Definition 2.1. *[41],[42],[13] $\varphi : [\kappa_1, \kappa_2] \to \mathcal{P}$ is said to be an interval Riemann integrable function (IR-integrable) on $[\kappa_1, \kappa_2]$ if there exist $A \in \mathcal{P}$ and $\delta > 0$, for each $\varepsilon > 0$, such that*

$$\kappa_4\left(S(\varphi, D, \delta), A\right) < \varepsilon$$

for every Riemann sum S of φ corresponding to each $D \in D(\delta, [\kappa_1, \kappa_2])$ and independent of choice of $\xi_i \in [\tau_{i-1}, \tau_i]$, $1 \leq i \leq n$. In this case, A is called as the IR-integral of φ on $[\kappa_1, \kappa_2]$ and is denoted by

$$A = (IR)\int_{\kappa_1}^{\kappa_2} \varphi(\tau)d\tau.$$

The collection of all functions that are IR-integrable on $[\kappa_1, \kappa_2]$ will be denote by $\mathcal{IR}_{([\kappa_1,\kappa_2])}$.

The next theorem explains connection between IR-integrable and Riemann integrable (R-integrable):

Theorem 2.2. *Assume that $\varphi : [\kappa_1, \kappa_2] \to \mathcal{P}$ is an interval-valued function such that $\varphi(\tau) = \left[\underline{\varphi}(\tau), \overline{\varphi}(\tau)\right]$. $\varphi \in \mathcal{IR}_{([\kappa_1,\kappa_2])}$ if and only if $\underline{\varphi}(\tau)$, $\overline{\varphi}(\tau) \in \mathcal{R}_{([\kappa_1,\kappa_2])}$ and*

$$(IR)\int_{\kappa_1}^{\kappa_2} \varphi(\tau)d\tau = \left[(R)\int_{\kappa_1}^{\kappa_2} \underline{\varphi}(\tau)d\tau, (R)\int_{\kappa_1}^{\kappa_2} \overline{\varphi}(\tau)d\tau\right],$$

where $\mathcal{R}_{([\kappa_1,\kappa_2])}$ denotes the all R-integrable function.

It is easy to see that if $\varphi(\tau) \subseteq \psi(\tau)$ for all $\tau \in [\kappa_1, \kappa_2]$, then $(IR)\int_{\kappa_1}^{\kappa_2} \varphi(\tau)d\tau \subseteq (IR)\int_{\kappa_1}^{\kappa_2} \psi(\tau)d\tau$.

Presented by Zhao et al. in [51], $h-$convex interval-valued function are given in the following definition.

Definition 2.3. *Suppose that $h : [\kappa_3, \kappa_4] \to \mathbb{R}$ is a non-negative function, $(0, 1) \subseteq [\kappa_3, \kappa_4]$ and $h \neq 0$. We say that $\varphi : [\kappa_1, \kappa_2] \to \mathcal{P}^+$ is a $h-$convex interval-valued function if for all $\xi, \eta \in [\kappa_1, \kappa_2]$ and $\tau \in (0, 1)$, we have*

$$h(\tau)\varphi(\xi) + h(1-\tau)\varphi(\eta) \subseteq \varphi(\tau\xi + (1-\tau)\eta). \tag{4}$$

$SX(h, [\kappa_1, \kappa_2], \mathcal{P}^+)$ will show the set of all h-convex interval-valued functions.

If we choose $h(\tau) = \tau$ in the above definition, then we reach the usual notion of convex interval-valued function in [44]. Also, if $h(\tau) = \tau^s$ in (4), then Definition 2.3 gives another convex interval-valued function defined by Breckner [2].

Zhao et al. [51] established the following Hermite-Hadamard inequality for interval-valued functions by using the definition of h-convexity.

Theorem 2.4. *Supposing that $\varphi : [\kappa_1, \kappa_2] \to \mathcal{P}^+$ is an interval-valued mapping such that $\varphi(\tau) = [\underline{\varphi}(\tau), \overline{\varphi}(\tau)]$ and $\varphi \in \mathcal{IR}_{([\kappa_1,\kappa_2])}$, $h : [0, 1] \to \mathbb{R}$ is a non-negative function with $h\left(\frac{1}{2}\right) \neq 0$. If $\varphi \in SX(h, [\kappa_1, \kappa_2], \mathcal{P}^+)$, then*

$$\frac{1}{2h\left(\frac{1}{2}\right)}\varphi\left(\frac{\kappa_1+\kappa_2}{2}\right) \supseteq \frac{1}{\kappa_2-\kappa_1}(IR)\int_{\kappa_1}^{\kappa_2} \varphi(\xi)d\xi \supseteq [\varphi(\kappa_1) + \varphi(\kappa_2)](R)\int_0^1 h(\tau)d\tau. \tag{5}$$

Remark 2.5. *(i) If $h(\tau) = \tau$ in (5), then (5) reduces the result*

$$\varphi\left(\frac{\kappa_1+\kappa_2}{2}\right) \supseteq \frac{1}{\kappa_2-\kappa_1}(IR)\int_{\kappa_1}^{\kappa_2} \varphi(\xi)d\xi \supseteq \frac{\varphi(\kappa_1)+\varphi(\kappa_2)}{2}$$

which is presented by Sadowska in [44].

(ii) If $h(\tau) = \tau^s$ in (5), (5) reduces the result

$$2^{s-1}\varphi\left(\frac{\kappa_1+\kappa_2}{2}\right) \supseteq \frac{1}{\kappa_2-\kappa_1}(IR)\int_{\kappa_1}^{\kappa_2} \varphi(\xi)d\xi \supseteq \frac{1}{s+1}[\varphi(\kappa_1) + \varphi(\kappa_2)]$$

which is given by Osuna-Gómez in [20].

We recall the Riemann-Lioville fractional integrals as follows [24]:

Definition 2.6. *Let $\varpi \in L_1[\kappa_1, \kappa_2]$. The Riemann-Liouville integrals $J^{\alpha}_{\kappa_1+}\varpi$ and $J^{\alpha}_{\kappa_2-}\varpi$ of order $\alpha > 0$ with $\kappa_1 \geq 0$ are defined by*

$$\mathrm{I}^{\alpha}_{\kappa_1+}\varpi(\xi) = \frac{1}{\Gamma(\alpha)} \int_{\kappa_1}^{\xi} (\xi - \tau)^{\alpha-1} \varpi(\tau) d\tau, \quad \xi > \kappa_1$$

and

$$\mathrm{I}^{\alpha}_{\kappa_2-}\varpi(\xi) = \frac{1}{\Gamma(\alpha)} \int_{\xi}^{\kappa_2} (\tau - \xi)^{\alpha-1} \varpi(\tau) d\tau, \quad \xi < \kappa_2,$$

respectively. Here, $\Gamma(\alpha)$ is the Gamma function and $J^{0}_{\kappa_1+}\varpi(\xi) = J^{0}_{\kappa_2-}\varpi(\xi) = \varpi(\xi)$.

In [45], Sarikaya et al. gave the Hermite-Hadamard inequality by using fractional integral as follows:

Theorem 2.7. *Let $\varpi : [\kappa_1, \kappa_2] \to \mathbb{R}$ be a positive function with $0 \leq \kappa_1 < \kappa_2$ and $\varpi \in L_1[\kappa_1, \kappa_2]$. If ϖ is a convex function on $[\kappa_1, \kappa_2]$, then the following inequalities given for fractional integrals hold:*

$$\varpi\left(\frac{\kappa_1 + \kappa_2}{2}\right) \leq \frac{\Gamma(1+\alpha)}{2(\kappa_2 - \kappa_1)^{\alpha}} \left[\mathrm{I}^{\alpha}_{\kappa_1+}\varpi(\kappa_2) + \mathrm{I}^{\alpha}_{\kappa_2-}\varpi(\kappa_1)\right] \leq \frac{\varpi(\kappa_1) + \varpi(\kappa_2)}{2}$$

for $\alpha > 0$.

On the other side, Iscan [23] established following Lemma, and he proved the following Fejer type inequalities for Riemann-Liouville fractional integrals by using this Lemma.

Lemma 2.8. *If $w : [\kappa_1, \kappa_2] \to \mathbb{R}$ is integrable and symmetric to $(\kappa_1 + \kappa_2)/2$ with $\kappa_1 < \kappa_2$, then we have*

$$\mathrm{I}^{\alpha}_{\kappa_1+}w(\kappa_2) = \mathrm{I}^{\alpha}_{\kappa_2-}w(\kappa_1) = \frac{1}{2}\left[\mathrm{I}^{\alpha}_{\kappa_1+}w(\kappa_2) + \mathrm{I}^{\alpha}_{\kappa_2-}w(\kappa_1)\right]$$

for $\alpha > 0$.

Theorem 2.9. *Assuming that $\varpi : [\kappa_1, \kappa_2] \to \mathbb{R}$ is a convex function with $0 \leq \kappa_1 < \kappa_2$ and $\varpi \in L_1[\kappa_1, \kappa_2]$. If $w : [\kappa_1, \kappa_2] \to \mathbb{R}$ is non-negative, integrable and symmetric to $(\kappa_1 + \kappa_2)/2$, then the following fractional integral inequalities hold*

$$\begin{aligned} \varpi\left(\frac{\kappa_1 + \kappa_2}{2}\right)\left[\mathrm{I}^{\alpha}_{\kappa_1+}w(\kappa_2) + \mathrm{I}^{\alpha}_{\kappa_2-}w(\kappa_1)\right] &\leq \left[\mathrm{I}^{\alpha}_{\kappa_1+}(fw)(\kappa_2) + \mathrm{I}^{\alpha}_{\kappa_2-}(fw)(\kappa_1)\right] \\ &\leq \frac{\varpi(\kappa_1) + \varpi(\kappa_2)}{2}\left[\mathrm{I}^{\alpha}_{\kappa_1+}w(\kappa_2) + \mathrm{I}^{\alpha}_{\kappa_2-}w(\kappa_1)\right] \end{aligned} \tag{6}$$

for $\alpha > 0$.

For more information about Riemann-Liouville integrals, you can look over [21],[24],[32],[43].

By considering Riemann-Liouville integral for real valued functions, Lupulescu defined the following interval-valued left-sided Riemann–Liouville fractional integral in [29].

Definition 2.10. *Suppose that $\varphi : [\kappa_1, \kappa_2] \to \mathcal{P}$ is an interval-valued function such that $\varphi(\tau) = \left[\underline{\varphi}(\tau), \overline{\varphi}(\tau)\right]$, and let $\alpha > 0$. The interval-valued left-sided Riemann–Liouville fractional integral of function φ is defined by*

$$\mathcal{J}^{\alpha}_{\kappa_1+}\varphi(\xi) = \frac{1}{\Gamma(\alpha)}(IR)\int_{\kappa_1}^{\xi} (\xi - s)^{\alpha-1} \varphi(\tau) d\tau, \quad \xi > \kappa_1, \tag{7}$$

where Γ is Euler-Gamma function.

Based on the definition of Lupulescu, Budak et al. [5] defined the corresponding interval-valued right-sided Riemann–Liouville fractional integral of function φ by

$$\mathcal{J}_{\kappa_2-}^{\alpha}\varphi(\xi)=\frac{1}{\Gamma(\alpha)}(IR)\int_{\xi}^{\kappa_2}(s-\xi)^{\alpha-1}\varphi(\tau)d\tau,\quad \xi<\kappa_2 \tag{8}$$

where Γ is Euler-Gamma function.

Budak et al. also presented the following results.

Theorem 2.11. *[5] If $\varphi:[\kappa_1,\kappa_2]\to\mathcal{P}$ is an interval-valued function such that $\varphi(\tau)=\left[\underline{\varphi}(\tau),\overline{\varphi}(\tau)\right]$, then one has*

$$\mathcal{J}_{\kappa_1+}^{\alpha}\varphi(\xi)=\left[\mathrm{I}_{\kappa_1+}^{\alpha}\underline{\varphi}(\xi),\mathrm{I}_{\kappa_1+}^{\alpha}\overline{\varphi}(\xi)\right]$$

and

$$\mathcal{J}_{\kappa_2-}^{\alpha}\varphi(\xi)=\left[\mathrm{I}_{\kappa_2-}^{\alpha}\underline{\varphi}(\xi),\mathrm{I}_{\kappa_2-}^{\alpha}\overline{\varphi}(\xi)\right].$$

Theorem 2.12. *[5] Let $\varphi:[\kappa_1,\kappa_2]\to\mathcal{P}^+$ is a convex interval-valued function such that $\varphi(\tau)=\left[\underline{\varphi}(\tau),\overline{\varphi}(\tau)\right]$ and $\alpha>0$, then we have*

$$\varphi\left(\frac{\kappa_1+\kappa_2}{2}\right)\supseteq\frac{\Gamma(1+\alpha)}{2(\kappa_2-\kappa_1)^{\alpha}}\left[\mathcal{J}_{\kappa_1+}^{\alpha}\varphi(\kappa_2)+\mathcal{J}_{\kappa_2-}^{\alpha}\varphi(\kappa_1)\right]\supseteq\frac{\varphi(\kappa_1)+\varphi(\kappa_2)}{2}. \tag{9}$$

In addition to all these results, Liu et al. refined Hermite-Hadamard type inclusions for interval-valued mappings in [28].

Theorem 2.13. *[28] Suppose that $w:[\kappa_1,\kappa_2]\to\mathbb{R}$ is non-negative, integrable, and symmetric about $\xi=\frac{\kappa_1+\kappa_2}{2}$ (i.e. $w(\xi)=w(\kappa_1+\kappa_2-\xi)$). If $\varphi:[\kappa_1,\kappa_2]\to\mathcal{P}^+$ is a convex interval-valued function such that $\varphi=\left[\underline{\varphi}(\tau),\overline{\varphi}(\tau)\right]$, then we possess*

$$\begin{aligned}
&\varphi\left(\frac{\kappa_1+\kappa_2}{2}\right)\left[\mathrm{I}_{\kappa_1+}^{\alpha}w(\kappa_2)+\mathrm{I}_{\kappa_2-}^{\alpha}w(\kappa_1)\right]\\
\supseteq\;&\left[\mathcal{J}_{\kappa_1+}^{\alpha}\varphi(\kappa_2)w(\kappa_2)+\mathcal{J}_{\kappa_2-}^{\alpha}\varphi(\kappa_1)w(\kappa_1)\right]\\
\supseteq\;&\frac{\varphi(\kappa_1)+\varphi(\kappa_2)}{2}\left[\mathrm{I}_{\kappa_1+}^{\alpha}w(\kappa_2)+\mathrm{I}_{\kappa_2-}^{\alpha}w(\kappa_1)\right]
\end{aligned}$$

for $\alpha>0$.

3. Fejér Type Inclusions for Products of Interval-Valued Convex Functions

We present some Fejér type inclusions for products of interval-valued convex functions in this section.

Theorem 3.1. *Suppose that $w:[\kappa_1,\kappa_2]\to\mathbb{R}$ is non-negative, integrable, and symmetric about $\xi=\frac{\kappa_1+\kappa_2}{2}$ (i.e. $w(\xi)=w(\kappa_1+\kappa_2-\xi)$). If φ_1, $\varphi_2:[\kappa_1,\kappa_2]\to\mathcal{P}^+$ are two convex interval-valued functions such that $\varphi_1(\tau)=\left[\underline{\varphi_1}(\tau),\overline{\varphi_1}(\tau)\right]$ and $\varphi_2(\tau)=\left[\underline{\varphi_2}(\tau),\overline{\varphi_2}(\tau)\right]$, then the following interval inclusion holds:*

$$\begin{aligned}
(IR)\int_{\kappa_1}^{\kappa_2}\varphi_1(\xi)\varphi_2(\xi)w(\xi)d\xi\quad\supseteq\quad&\frac{\mathcal{M}(\kappa_1,\kappa_2)}{(\kappa_2-\kappa_1)^2}(R)\int_{\kappa_1}^{\kappa_2}(\kappa_2-\xi)^2\,w(\xi)d\xi\\
&+\frac{\mathcal{N}(\kappa_1,\kappa_2)}{(\kappa_2-\kappa_1)^2}(R)\int_{\kappa_1}^{\kappa_2}(\kappa_2-\xi)\,(\xi-\kappa_1)\,w(\xi)d\xi,
\end{aligned} \tag{10}$$

where

$$\mathcal{M}(\kappa_1,\kappa_2)=\varphi_1(\kappa_1)\varphi_2(\kappa_1)+\varphi_1(\kappa_2)\varphi_2(\kappa_2) \text{ and } \mathcal{N}(\kappa_1,\kappa_2)=\varphi_1(\kappa_1)\varphi_2(\kappa_2)+\varphi_1(\kappa_2)\varphi_2(\kappa_1).$$

Proof. Since φ_1 and φ_2 are interval-valued convex functions on $[\kappa_1,\kappa_2]$, one has

$$\varphi_1\left((1-\tau)\kappa_1+\tau\kappa_2\right)\supseteq(1-\tau)\varphi_1(\kappa_1)+\tau\varphi_1(\kappa_2) \tag{11}$$

and

$$\varphi_2\left((1-\tau)\kappa_1+\tau\kappa_2\right)\supseteq(1-\tau)\varphi_2(\kappa_1)+\tau\varphi_2(\kappa_2). \tag{12}$$

By (11) and (12), we have

$$\begin{aligned} &\varphi_1\left((1-\tau)\kappa_1+\tau\kappa_2\right)\varphi_2\left((1-\tau)\kappa_1+\tau\kappa_2\right) \qquad (13)\\ \supseteq\ &(1-\tau)^2\varphi_1(\kappa_1)\varphi_2(\kappa_1)+\tau^2\varphi_1(\kappa_2)\varphi_2(\kappa_2)+\tau(1-\tau)\left[\varphi_1(\kappa_1)\varphi_2(\kappa_2)+\varphi_1(\kappa_2)\varphi_2(\kappa_1)\right]. \end{aligned}$$

Integrating the resulting inclusion with respect to τ from 0 to 1 after multiplying both sides of (13) by $w\left((1-\tau)\kappa_1+\tau\kappa_2\right)$, we obtain

$$\begin{aligned} &(IR)\int_0^1\varphi_1\left((1-\tau)\kappa_1+\tau\kappa_2\right)\varphi_2\left((1-\tau)\kappa_1+\tau\kappa_2\right)w\left((1-\tau)\kappa_1+\tau\kappa_2\right)d\tau \qquad (14)\\ \supseteq\ &(IR)\int_0^1\varphi_1(\kappa_1)\varphi_2(\kappa_1)(1-\tau)^2w\left((1-\tau)\kappa_1+\tau\kappa_2\right)d\tau\\ &+(IR)\int_0^1\varphi_1(\kappa_2)\varphi_2(\kappa_2)\tau^2w\left((1-\tau)\kappa_1+\tau\kappa_2\right)d\tau\\ &+(IR)\int_0^1\left[\varphi_1(\kappa_1)\varphi_2(\kappa_2)+\varphi_1(\kappa_2)\varphi_2(\kappa_1)\right]\tau(1-\tau)w\left((1-\tau)\kappa_1+\tau\kappa_2\right)d\tau\\ =\ &\varphi_1(\kappa_1)\varphi_2(\kappa_1)(R)\int_0^1(1-\tau)^2w\left((1-\tau)\kappa_1+\tau\kappa_2\right)d\tau\\ &+\varphi_1(\kappa_2)\varphi_2(\kappa_2)(R)\int_0^1\tau^2w\left((1-\tau)\kappa_1+\tau\kappa_2\right)d\tau\\ &+\left[\varphi_1(\kappa_1)\varphi_2(\kappa_2)+\varphi_1(\kappa_2)\varphi_2(\kappa_1)\right](R)\int_0^1\tau(1-\tau)w\left((1-\tau)\kappa_1+\tau\kappa_2\right)d\tau. \end{aligned}$$

By change of the variable $\xi = (1-\tau)\kappa_1 + \tau\kappa_2$, it is found that

$$(IR)\int_0^1 \varphi_1((1-\tau)\kappa_1+\tau\kappa_2)\varphi_2((1-\tau)\kappa_1+\tau\kappa_2)w((1-\tau)\kappa_1+\tau\kappa_2)d\tau \tag{15}$$

$$\begin{aligned}
&= \left[(R)\int_0^1 \underline{\varphi_1}((1-\tau)\kappa_1+\tau\kappa_2)\underline{\varphi_2}((1-\tau)\kappa_1+\tau\kappa_2)w((1-\tau)\kappa_1+\tau\kappa_2)d\tau,\right.\\
&\qquad \left.(R)\int_0^1 \overline{\varphi_1}((1-\tau)\kappa_1+\tau\kappa_2)\overline{\varphi_2}((1-\tau)\kappa_1+\tau\kappa_2)w((1-\tau)\kappa_1+\tau\kappa_2)d\tau\right]\\
&= \left[\frac{1}{\kappa_2-\kappa_1}(R)\int_{\kappa_1}^{\kappa_2}\underline{\varphi_1}(\xi)\underline{\varphi_2}(\xi)w(\xi)d\xi, \frac{1}{\kappa_2-\kappa_1}(R)\int_{\kappa_1}^{\kappa_2}\overline{\varphi_1}(\xi)\overline{\varphi_2}(\xi)w(\xi)d\xi\right]\\
&= \frac{1}{\kappa_2-\kappa_1}(IR)\int_{\kappa_1}^{\kappa_2}\varphi_1(\xi)\varphi_2(\xi)w(\xi)d\xi.
\end{aligned}$$

Moreover, since w is symmetric about $\frac{\kappa_1+\kappa_2}{2}$, it is easily observed that

$$\begin{aligned}
(R)\int_0^1 \tau^2 w((1-\tau)\kappa_1+\tau\kappa_2)d\tau &= \frac{1}{(\kappa_2-\kappa_1)^3}(R)\int_{\kappa_1}^{\kappa_2}(\xi-\kappa_1)^2 w(\xi)d\xi \\
&= \frac{1}{(\kappa_2-\kappa_1)^3}(R)\int_{\kappa_1}^{\kappa_2}(\kappa_2-\xi)^2 w(\xi)d\xi
\end{aligned} \tag{16}$$

and

$$(R)\int_0^1 (1-\tau)^2 w((1-\tau)\kappa_1+\tau\kappa_2)d\tau = \frac{1}{(\kappa_2-\kappa_1)^3}(R)\int_{\kappa_1}^{\kappa_2}(\kappa_2-\xi)^2 w(\xi)d\xi. \tag{17}$$

We also have

$$(R)\int_0^1 \tau(1-\tau) w((1-\tau)\kappa_1+\tau\kappa_2)d\tau = \frac{1}{(\kappa_2-\kappa_1)^3}(R)\int_{\kappa_1}^{\kappa_2}(\kappa_2-\xi)(\xi-\kappa_1) w(\xi)d\xi. \tag{18}$$

By substituting the equalities (15)-(18) in (14), then we possess

$$\frac{1}{\kappa_2-\kappa_1}(IR)\int_{\kappa_1}^{\kappa_2}\varphi_1(\xi)\varphi_2(\xi)w(\xi)d\xi \tag{19}$$

$$\begin{aligned}
&\supseteq \frac{\varphi_1(\kappa_1)\varphi_2(\kappa_1)+\varphi_1(\kappa_2)\varphi_2(\kappa_2)}{(\kappa_2-\kappa_1)^3}(R)\int_{\kappa_1}^{\kappa_2}(\kappa_2-\xi)^2 w(\xi)d\xi \\
&\quad + \frac{\varphi_1(\kappa_1)\varphi_2(\kappa_2)+\varphi_1(\kappa_2)\varphi_2(\kappa_1)}{(\kappa_2-\kappa_1)^3}(R)\int_{\kappa_1}^{\kappa_2}(\kappa_2-\xi)(\xi-\kappa_1) w(\xi)d\xi.
\end{aligned}$$

If we multiply both sides of (19) by $(\kappa_2-\kappa_1)$, then we reach the desired result. $\square$

Remark 3.2. *If we choose $w(\xi)=1$, in Theorem 3.1, then for all $\xi\in[\kappa_1,\kappa_2]$, we have*

$$\frac{1}{\kappa_2-\kappa_1}(IR)\int_{\kappa_1}^{\kappa_2}\varphi_1(\xi)\varphi_2(\xi)d\xi\supseteq\frac{1}{3}\mathcal{M}(\kappa_1,\kappa_2)+\frac{1}{6}\mathcal{N}(\kappa_1,\kappa_2)$$

which is proved by Zhao et al. in [51].

Corollary 3.3. *If we choose $\varphi_2(\xi)=[1,1]$ in Theorem 3.1, then, for all $\xi\in[\kappa_1,\kappa_2]$, we have*

$$(IR)\int_{\kappa_1}^{\kappa_2}\varphi_1(\xi)w(\xi)d\xi\supseteq\frac{\varphi_1(\kappa_1)+\varphi_1(\kappa_2)}{2}(R)\int_{\kappa_1}^{\kappa_2}w(\xi)d\xi.$$

Proof. If we consider the case when $\varphi_2(\xi)=[1,1]$ in (10), then we possess

$$\begin{aligned}(IR)\int_{\kappa_1}^{\kappa_2}\varphi_1(\xi)w(\xi)d\xi &\supseteq \frac{\varphi_1(\kappa_1)+\varphi_1(\kappa_2)}{(\kappa_2-\kappa_1)^2}\left[(R)\int_{\kappa_1}^{\kappa_2}(\kappa_2-\xi)^2w(\xi)d\xi\right.\\ &\quad\left.+(R)\int_{\kappa_1}^{\kappa_2}(\kappa_2-\xi)(\xi-\kappa_1)w(\xi)d\xi\right]\\ &=\frac{\varphi_1(\kappa_1)+\varphi_1(\kappa_2)}{(\kappa_2-\kappa_1)}(R)\int_{\kappa_1}^{\kappa_2}(\kappa_2-\xi)w(\xi)d\xi\end{aligned}\tag{20}$$

for all $\xi\in[\kappa_1,\kappa_2]$.
On the grounds that w is symmetric about $\xi=\frac{\kappa_1+\kappa_2}{2}$, it follows that

$$\begin{aligned}(R)\int_{\kappa_1}^{\kappa_2}(\kappa_2-\xi)w(\xi)d\xi &= (R)\int_{\kappa_1}^{\frac{\kappa_1+\kappa_2}{2}}(\kappa_2-\xi)w(\xi)d\xi+(R)\int_{\frac{\kappa_1+\kappa_2}{2}}^{\kappa_2}(\kappa_2-\xi)w(\xi)d\xi\\ &= (R)\int_{\kappa_1}^{\frac{\kappa_1+\kappa_2}{2}}(\kappa_2-\xi)w(\xi)d\xi+(R)\int_{\kappa_1}^{\frac{\kappa_1+\kappa_2}{2}}(\xi-\kappa_1)w(\kappa_1+\kappa_2-\xi)d\xi\\ &= (\kappa_2-\kappa_1)(R)\int_{\kappa_1}^{\frac{\kappa_1+\kappa_2}{2}}w(\xi)d\xi\\ &= \frac{(\kappa_2-\kappa_1)}{2}(R)\int_{\kappa_1}^{\kappa_2}w(\xi)d\xi.\end{aligned}\tag{21}$$

Putting the equality (21) in (20), we obtain the required result. $\square$

Theorem 3.4. *Suppose that all the conditions of Theorem 3.1 hold, then we have the inclusion*

$$
\begin{aligned}
& 2\varphi_1\left(\frac{\kappa_1+\kappa_2}{2}\right)\varphi_2\left(\frac{\kappa_1+\kappa_2}{2}\right)(R)\int_{\kappa_1}^{\kappa_2} w(\xi)d\xi \\
\supseteq\ & (IR)\int_{\kappa_1}^{\kappa_2} \varphi_1(\xi)\varphi_2(\xi)w(\xi)d\xi \\
& +\frac{\mathcal{M}(\kappa_1,\kappa_2)}{(\kappa_2-\kappa_1)^2}(R)\int_{\kappa_1}^{\kappa_2}(\kappa_2-\xi)(\xi-\kappa_1)\,w(\xi)\,d\xi+\frac{\mathcal{N}(\kappa_1,\kappa_2)}{(\kappa_2-\kappa_1)^2}(R)\int_{\kappa_1}^{\kappa_2}(\kappa_2-\xi)^2\,w(\xi)\,d\xi
\end{aligned}
\tag{22}
$$

where $\mathcal{M}(\kappa_1,\kappa_2)$ and $\mathcal{N}(\kappa_1,\kappa_2)$ are defined as in Theorem 3.1.

Proof. For $\tau\in[0,1]$, we can write

$$
\frac{\kappa_1+\kappa_2}{2}=\frac{(1-\tau)\kappa_1+\tau\kappa_2}{2}+\frac{\tau\kappa_1+(1-\tau)\kappa_2}{2}.
$$

Considering that φ_1 and φ_2 are interval-valued convex mappings on $[\kappa_1,\kappa_2]$, we find that

$$
\begin{aligned}
& \varphi_1\left(\frac{\kappa_1+\kappa_2}{2}\right)\varphi_2\left(\frac{\kappa_1+\kappa_2}{2}\right) \\
=\ & \varphi_1\left(\frac{(1-\tau)\kappa_1+\tau\kappa_2}{2}+\frac{\tau\kappa_1+(1-\tau)\kappa_2}{2}\right) \\
& \times\varphi_2\left(\frac{(1-\tau)\kappa_1+\tau\kappa_2}{2}+\frac{\tau\kappa_1+(1-\tau)\kappa_2}{2}\right) \\
\supseteq\ & \frac{1}{4}\left[\varphi_1((1-\tau)\kappa_1+\tau\kappa_2)+\varphi_1(\tau\kappa_1+(1-\tau)\kappa_2)\right] \\
& \times\left[\varphi_2((1-\tau)\kappa_1+\tau\kappa_2)+\varphi_2(\tau\kappa_1+(1-\tau)\kappa_2)\right] \\
=\ & \frac{1}{4}\left[\varphi_1((1-\tau)\kappa_1+\tau\kappa_2)\varphi_2((1-\tau)\kappa_1+\tau\kappa_2)\right. \\
& \left.+\ \varphi_1(\tau\kappa_1+(1-\tau)\kappa_2)\varphi_2(\tau\kappa_1+(1-\tau)\kappa_2)\right] \\
& +\frac{1}{4}\left[\varphi_1((1-\tau)\kappa_1+\tau\kappa_2)\varphi_2(\tau\kappa_1+(1-\tau)\kappa_2)\right. \\
& \left.+\ \varphi_1(\tau\kappa_1+(1-\tau)\kappa_2)\varphi_2((1-\tau)\kappa_1+\tau\kappa_2)\right].
\end{aligned}
$$

For the second expression in the last equality, by using again the convexity of φ_1 and φ_2, it follows that

$$\begin{aligned}
&\varphi_1\left(\frac{\kappa_1+\kappa_2}{2}\right)\varphi_2\left(\frac{\kappa_1+\kappa_2}{2}\right) \\
\supseteq\ &\frac{1}{4}\left[\varphi_1((1-\tau)\kappa_1+\tau\kappa_2)\varphi_2((1-\tau)\kappa_1+\tau\kappa_2)\right. \\
&\left.+\ \varphi_1(\tau\kappa_1+(1-\tau)\kappa_2)\varphi_2(\tau\kappa_1+(1-\tau)\kappa_2)\right] \\
&+\frac{1}{2}\tau(1-\tau)\left[\varphi_1(\kappa_1)\varphi_2(\kappa_1)+\varphi_1(\kappa_2)\varphi_2(\kappa_2)\right] \\
&+\frac{1}{4}\left[\tau^2+(1-\tau)^2\right]\left[\varphi_1(\kappa_1)\varphi_2(\kappa_2)+\varphi_1(\kappa_2)\varphi_2(\kappa_1)\right].
\end{aligned} \tag{23}$$

Multiplying both sides of (23) by $w((1-\tau)\kappa_1+\tau\kappa_2)$, then integrating the resulting inclusion with respect to τ from 0 to 1, one has

$$\begin{aligned}
&\varphi_1\left(\frac{\kappa_1+\kappa_2}{2}\right)\varphi_2\left(\frac{\kappa_1+\kappa_2}{2}\right)(R)\int_0^1 w((1-\tau)\kappa_1+\tau\kappa_2)\,d\tau \\
\supseteq\ &\frac{1}{4}(IR)\int_0^1 \left[\varphi_1((1-\tau)\kappa_1+\tau\kappa_2)\varphi_2((1-\tau)\kappa_1+\tau\kappa_2)\right. \\
&\left.+\ \varphi_1(\tau\kappa_1+(1-\tau)\kappa_2)\varphi_2(\tau\kappa_1+(1-\tau)\kappa_2)w((1-\tau)\kappa_1+\tau\kappa_2)\right]d\tau \\
&+\frac{\mathcal{M}(\kappa_1,\kappa_2)}{2}(R)\int_0^1 \tau(1-\tau)\,w((1-\tau)\kappa_1+\tau\kappa_2)\,d\tau \\
&+\frac{\mathcal{N}(\kappa_1,\kappa_2)}{4}(R)\int_0^1 \left[\tau^2+(1-\tau)^2\right]w((1-\tau)\kappa_1+\tau\kappa_2)\,d\tau.
\end{aligned} \tag{24}$$

If we substitute the identities (15)-(18) in (24), then we obtain

$$\begin{aligned}
&\varphi_1\left(\frac{\kappa_1+\kappa_2}{2}\right)\varphi_2\left(\frac{\kappa_1+\kappa_2}{2}\right)\frac{1}{\kappa_2-\kappa_1}(R)\int_{\kappa_1}^{\kappa_2} w(\xi)d\xi \\
\supseteq\ &\frac{1}{4}\left[\frac{1}{\kappa_2-\kappa_1}(IR)\int_{\kappa_1}^{\kappa_2}\varphi_1(\xi)\varphi_2(\xi)w(\xi)d\xi+\frac{1}{\kappa_2-\kappa_1}(IR)\int_{\kappa_1}^{\kappa_2}\varphi_1(\xi)\varphi_2(\xi)w(\kappa_1+\kappa_2-\xi)d\xi\right] \\
&+\frac{\mathcal{M}(\kappa_1,\kappa_2)}{2(\kappa_2-\kappa_1)^3}(R)\int_{\kappa_1}^{\kappa_2}(\kappa_2-\xi)(\xi-\kappa_1)\,w(\xi)\,d\xi+\frac{\mathcal{N}(\kappa_1,\kappa_2)}{2(\kappa_2-\kappa_1)^3}(R)\int_{\kappa_1}^{\kappa_2}(\kappa_2-\xi)^2\,w(\xi)\,d\xi.
\end{aligned} \tag{25}$$

Multiplying the both sides of (25) by $2(\kappa_2-\kappa_1)$, then the desired result (22) can be readily attained. □

Remark 3.5. *If we choose $w(\xi) = 1$ for all $\xi \in [\kappa_1, \kappa_2]$ in Theorem 3.4, then we have*

$$2\varphi_1\left(\frac{\kappa_1+\kappa_2}{2}\right)\varphi_2\left(\frac{\kappa_1+\kappa_2}{2}\right) \supseteq \frac{1}{\kappa_2-\kappa_1}(IR)\int_{\kappa_1}^{\kappa_2}\varphi_1(\xi)\varphi_2(\xi)d\xi + \frac{1}{6}\mathcal{M}(\kappa_1,\kappa_2) + \frac{1}{3}\mathcal{N}(\kappa_1,\kappa_2)$$

which is proved by Zhao et al. in [51].

Corollary 3.6. *If we choose $\varphi_2(\xi) = [1, 1]$ in Theorem 3.4, then we have*

$$\begin{aligned} 2\varphi_1\left(\frac{\kappa_1+\kappa_2}{2}\right)(R)\int_{\kappa_1}^{\kappa_2} w(\xi)d\xi \quad \supseteq \quad & (IR)\int_{\kappa_1}^{\kappa_2}\varphi_1(\xi)w(\xi)d\xi \\ & + \frac{\varphi_1(\kappa_1)+\varphi_1(\kappa_2)}{\kappa_2-\kappa_1}(R)\int_{\kappa_1}^{\kappa_2}(\kappa_2-\xi)\,w(\xi)\,d\xi \end{aligned}$$

for all $\xi \in [\kappa_1, \kappa_2]$.

Proof. If we consider again the inclusion (22) for $\varphi_2(\xi) = [1, 1]$, then we get

$$\begin{aligned} & 2\varphi_1\left(\frac{\kappa_1+\kappa_2}{2}\right)(IR)\int_{\kappa_1}^{\kappa_2} w(\xi)d\xi \\ \supseteq \quad & (IR)\int_{\kappa_1}^{\kappa_2}\varphi_1(\xi)w(\xi)d\xi + \frac{\varphi_1(\kappa_1)+\varphi_1(\kappa_2)}{(\kappa_2-\kappa_1)^2}\left[(R)\int_{\kappa_1}^{\kappa_2}(\kappa_2-\xi)(\xi-\kappa_1)\,w(\xi)\,d\xi \right. \\ & \left. + (R)\int_{\kappa_1}^{\kappa_2}(\kappa_2-\xi)^2\,w(\xi)\,d\xi\right] \\ = \quad & (IR)\int_{\kappa_1}^{\kappa_2}\varphi_1(\xi)w(\xi)d\xi + \frac{\varphi_1(\kappa_1)+\varphi_1(\kappa_2)}{\kappa_2-\kappa_1}(R)\int_{\kappa_1}^{\kappa_2}(\kappa_2-\xi)\,w(\xi)\,d\xi. \end{aligned}$$

for all $\xi \in [\kappa_1, \kappa_2]$. By the equality (21), we deduce the desired result. □

4. Some Results For Interval-Valued Fractional Integrals

In this part, we apply the results provided in Section 3 to interval valued fractional integrals. Thus, we establish some Fejér type inclusions involving interval-valued fractional integrals.

Theorem 4.1. *Suppose that $w : [\kappa_1, \kappa_2] \to \mathbb{R}$ is non-negative, integrable, and symmetric about $\xi = \frac{\kappa_1+\kappa_2}{2}$ (i.e. $w(\xi) = w(\kappa_1 + \kappa_2 - \xi)$). If φ_1, $\varphi_2 : [\kappa_1, \kappa_2] \to \mathcal{P}^+$ are two convex interval-valued functions such that $\varphi_1(\tau) = \left[\underline{\varphi_1}(\tau), \overline{\varphi_1}(\tau)\right]$ and $\varphi_2(\tau) = \left[\underline{\varphi_1}(\tau), \overline{\varphi_2}(\tau)\right]$, then we have*

$$\begin{aligned} & \mathcal{J}^{\alpha}_{\kappa_1+}\varphi_1(\kappa_2)\varphi_2(\kappa_2)w(\kappa_2) + \mathcal{J}^{\alpha}_{\kappa_2-}\varphi_1(\kappa_1)\varphi_2(\kappa_1)w(\kappa_1) \qquad (26) \\ \supseteq \quad & \frac{\mathcal{M}(\kappa_1,\kappa_2)}{(\kappa_2-\kappa_1)^2\,\Gamma(\alpha)}(R)\int_{\kappa_1}^{\kappa_2}(\kappa_2-\xi)^{\alpha-1}\left[(\kappa_2-\xi)^2+(\xi-\kappa_1)^2\right]w(\xi)d\xi \\ & + \frac{2\mathcal{N}(\kappa_1,\kappa_2)}{(\kappa_2-\kappa_1)^2\,\Gamma(\alpha)}(R)\int_{\kappa_1}^{\kappa_2}(\xi-\kappa_1)(\kappa_2-\xi)^{\alpha}\,w(\xi)d\xi \end{aligned}$$

for $\alpha > 0$. Here, Γ is the Gamma function.

Proof. Since w is non-negative, integrable and symmetric about $\xi = \frac{\kappa_1+\kappa_2}{2}$, it is obvious that $h(\xi) = \frac{1}{\Gamma(\alpha)}\left[(\kappa_2-\xi)^{\alpha-1} + (\xi-\kappa_1)^{\alpha-1}\right] w(\xi)$ is non-negative, integrable and symmetric about $\xi = \frac{\kappa_1+\kappa_2}{2}$. Thus, by using Theorem 3.1, we can write the inclusion,

$$
\begin{aligned}
(IR)\int_{\kappa_1}^{\kappa_2} \varphi_1(\xi)\varphi_2(\xi)h(\xi)d\xi \quad \supseteq \quad & \frac{\mathcal{M}(\kappa_1,\kappa_2)}{(\kappa_2-\kappa_1)^2}(R)\int_{\kappa_1}^{\kappa_2} (\kappa_2-\xi)^2 h(\xi)d\xi \\
&+\frac{\mathcal{N}(\kappa_1,\kappa_2)}{(\kappa_2-\kappa_1)^2}(R)\int_{\kappa_1}^{\kappa_2} (\kappa_2-\xi)(\xi-\kappa_1) h(\xi)d\xi,
\end{aligned}
$$

that is,

$$
\begin{aligned}
& \frac{1}{\Gamma(\alpha)}(IR)\int_{\kappa_1}^{\kappa_2} (\kappa_2-\xi)^{\alpha-1} \varphi_1(\xi)\varphi_2(\xi)w(\xi)d\xi \qquad (27)\\
& +\frac{1}{\Gamma(\alpha)}(IR)\int_{\kappa_1}^{\kappa_2} (\xi-\kappa_1)^{\alpha-1} \varphi_1(\xi)\varphi_2(\xi)w(\xi)d\xi \\
\supseteq \quad & \frac{\mathcal{M}(\kappa_1,\kappa_2)}{(\kappa_2-\kappa_1)^2\Gamma(\alpha)}(R)\int_{\kappa_1}^{\kappa_2} (\kappa_2-\xi)^2 \left[(\kappa_2-\xi)^{\alpha-1} + (\xi-\kappa_1)^{\alpha-1}\right] w(\xi)d\xi \\
& +\frac{\mathcal{N}(\kappa_1,\kappa_2)}{(\kappa_2-\kappa_1)^2\Gamma(\alpha)}(R)\int_{\kappa_1}^{\kappa_2} (\kappa_2-\xi)(\xi-\kappa_1)\left[(\kappa_2-\xi)^{\alpha-1} + (\xi-\kappa_1)^{\alpha-1}\right] w(\xi)d\xi.
\end{aligned}
$$

Then, from the (7) and (8), it is easy to see that

$$
\begin{aligned}
& \frac{1}{\Gamma(\alpha)}(IR)\int_{\kappa_1}^{\kappa_2} (\kappa_2-\xi)^{\alpha-1} \varphi_1(\xi)\varphi_2(\xi)w(\xi)d\xi \qquad (28)\\
& +\frac{1}{\Gamma(\alpha)}(IR)\int_{\kappa_1}^{\kappa_2} (\xi-\kappa_1)^{\alpha-1} \varphi_1(\xi)\varphi_2(\xi)w(\xi)d\xi \\
= \quad & \mathcal{J}^{\alpha}_{\kappa_1+}\varphi_1(\kappa_2)\varphi_2(\kappa_2)w(\kappa_2) + \mathcal{J}^{\alpha}_{\kappa_2-}\varphi_1(\kappa_1)\varphi_2(\kappa_1)w(\kappa_1).
\end{aligned}
$$

Moreover, since w is symmetric about $\xi = \frac{\kappa_1+\kappa_2}{2}$, we conclude that

$$(R)\int_{\kappa_1}^{\kappa_2} (\kappa_2 - \xi)^2 \left[(\kappa_2 - \xi)^{\alpha-1} + (\xi - \kappa_1)^{\alpha-1}\right] w(\xi)d\xi \tag{29}$$

$$= (R)\int_{\kappa_1}^{\kappa_2} (\kappa_2 - \xi)^{\alpha+1} w(\xi)d\xi + (R)\int_{\kappa_1}^{\kappa_2} (\kappa_2 - \xi)^2 (\xi - \kappa_1)^{\alpha-1} w(\xi)d\xi$$

$$= (R)\int_{\kappa_1}^{\kappa_2} (\kappa_2 - \xi)^{\alpha+1} w(\xi)d\xi + (R)\int_{\kappa_1}^{\kappa_2} (\xi - \kappa_1)^2 (\kappa_2 - \xi)^{\alpha-1} w(\xi)d\xi$$

$$= (R)\int_{\kappa_1}^{\kappa_2} (\kappa_2 - \xi)^{\alpha-1} \left[(\kappa_2 - \xi)^2 + (\xi - \kappa_1)^2\right] w(\xi)d\xi$$

and

$$(R)\int_{\kappa_1}^{\kappa_2} (\kappa_2 - \xi)(\xi - \kappa_1) \left[(\kappa_2 - \xi)^{\alpha-1} + (\xi - \kappa_1)^{\alpha-1}\right] w(\xi)d\xi \tag{30}$$

$$= (R)\int_{\kappa_1}^{\kappa_2} (\xi - \kappa_1)(\kappa_2 - \xi)^{\alpha} w(\xi)d\xi + (R)\int_{\kappa_1}^{\kappa_2} (\kappa_2 - \xi)(\xi - \kappa_1)^{\alpha} w(\xi)d\xi$$

$$= (R)\int_{\kappa_1}^{\kappa_2} (\xi - \kappa_1)(\kappa_2 - \xi)^{\alpha} w(\xi)d\xi + (R)\int_{\kappa_1}^{\kappa_2} (\xi - \kappa_1)(\kappa_2 - \xi)^{\alpha} w(\xi)d\xi$$

$$= 2(R)\int_{\kappa_1}^{\kappa_2} (\xi - \kappa_1)(\kappa_2 - \xi)^{\alpha} w(\xi)d\xi.$$

If we substitute the equalities (28)-(30) in (27), then we obtain the desired inclusion (26) which completes the proof. □

Remark 4.2. *If we choose $w(\xi) = 1$ in Theorem 4.1, then, for all $\xi \in [\kappa_1, \kappa_2]$, we have*

$$\frac{\Gamma(\alpha+1)}{2(\kappa_2 - \kappa_1)^{\alpha}} \left[\mathcal{J}^{\alpha}_{\kappa_1+}\varphi_1(\kappa_2)\varphi_2(\kappa_2) + \mathcal{J}^{\alpha}_{\kappa_2-}\varphi_1(\kappa_1)\varphi_2(\kappa_1)\right]$$

$$\supseteq \left(\frac{1}{2} - \frac{\alpha}{(\alpha+1)(\alpha+2)}\right)\mathcal{M}(\kappa_1, \kappa_2) + \frac{\alpha}{(\alpha+1)(\alpha+2)}\mathcal{N}(\kappa_1, \kappa_2)$$

which is proved by Budak et al. in [5].

Remark 4.3. *If we choose $\alpha = 1$ in Theorem 4.1, then Theorem 4.1 reduces to Theorem 3.1.*

Corollary 4.4. *If we choose $\varphi_2(\xi) = [1, 1]$ in Theorem 4.1, then, for all $\xi \in [\kappa_1, \kappa_2]$, we have*

$$\mathcal{J}^{\alpha}_{\kappa_1+}\varphi_1(\kappa_2)w(\kappa_2) + \mathcal{J}^{\alpha}_{\kappa_2-}\varphi_1(\kappa_1)w(\kappa_1) \supseteq \frac{\varphi_1(\kappa_1) + \varphi_1(\kappa_2)}{2}\left[\mathrm{I}^{\alpha}_{\kappa_1+}w(\kappa_2) + \mathrm{I}^{\alpha}_{\kappa_2-}w(\kappa_1)\right]$$

which is the same as the second inclusion in Theorem 2.13.

Proof. Using the Lemma 2.8 after taking $\varphi_2(\xi) = [1,1]$ in the inclusion (26), for all $\xi \in [\kappa_1, \kappa_2]$, we find that

$$\begin{aligned}
& \mathcal{J}^{\alpha}_{\kappa_1+}\varphi_1(\kappa_2)w(\kappa_2) + \mathcal{J}^{\alpha}_{\kappa_2-}\varphi_1(\kappa_1)w(\kappa_1) \qquad (31)\\
\supseteq\ & \frac{\varphi_1(\kappa_1)+\varphi_1(\kappa_2)}{(\kappa_2-\kappa_1)^2\,\Gamma(\alpha)}(R)\int_{\kappa_1}^{\kappa_2}(\kappa_2-\xi)^{\alpha-1}\left[(\kappa_2-\xi)^2+(\xi-\kappa_1)^2\right]w(\xi)d\xi \\
& +\frac{2\varphi_1(\kappa_1)+\varphi_1(\kappa_2)}{(\kappa_2-\kappa_1)^2\,\Gamma(\alpha)}(R)\int_{\kappa_1}^{\kappa_2}(\xi-\kappa_1)(\kappa_2-\xi)^{\alpha}\,w(\xi)d\xi \\
=\ & \frac{\varphi_1(\kappa_1)+\varphi_1(\kappa_2)}{(\kappa_2-\kappa_1)^2\,\Gamma(\alpha)} \\
& \times\left[(R)\int_{\kappa_1}^{\kappa_2}(\kappa_2-\xi)^{\alpha-1}\left[(\kappa_2-\xi)^2+(\xi-\kappa_1)^2+2(\xi-\kappa_1)(\kappa_2-\xi)\right]w(\xi)d\xi\right] \\
=\ & \frac{\varphi_1(\kappa_1)+\varphi_1(\kappa_2)}{\Gamma(\alpha)}\int_{\kappa_1}^{\kappa_2}(\kappa_2-\xi)^{\alpha-1}\,w(\xi)d\xi \\
=\ & \frac{\varphi_1(\kappa_1)+\varphi_1(\kappa_2)}{2}\left[\mathrm{I}^{\alpha}_{\kappa_1+}w(\kappa_2)+\mathrm{I}^{\alpha}_{\kappa_2-}w(\kappa_1)\right]
\end{aligned}$$

which completes the proof. □

Theorem 4.5. *Suppose that all the conditions of Theorem 4.1 hold. Then, we have the inclusion*

$$\begin{aligned}
& 2\varphi_1\left(\frac{\kappa_1+\kappa_2}{2}\right)\varphi_2\left(\frac{\kappa_1+\kappa_2}{2}\right)\left[\mathrm{I}^{\alpha}_{\kappa_1+}w(\kappa_2)+\mathrm{I}^{\alpha}_{\kappa_2-}w(\kappa_1)\right] \qquad (32)\\
\supseteq\ & \mathcal{J}^{\alpha}_{\kappa_1+}\varphi_1(\kappa_2)\varphi_2(\kappa_2)w(\kappa_2)+\mathcal{J}^{\alpha}_{\kappa_2-}\varphi_1(\kappa_1)\varphi_2(\kappa_1)w(\kappa_1) \\
& +\frac{2\mathcal{M}(\kappa_1,\kappa_2)}{(\kappa_2-\kappa_1)^2\,\Gamma(\alpha)}(R)\int_{\kappa_1}^{\kappa_2}(\xi-\kappa_1)(\kappa_2-\xi)^{\alpha}\,w(\xi)d\xi \\
& +\frac{\mathcal{N}(\kappa_1,\kappa_2)}{(\kappa_2-\kappa_1)^2\,\Gamma(\alpha)}(R)\int_{\kappa_1}^{\kappa_2}(\kappa_2-\xi)^{\alpha-1}\left[(\kappa_2-\xi)^2+(\xi-\kappa_1)^2\right]w(\xi)d\xi.
\end{aligned}$$

Proof. Inasmuch as w is non-negative, integrable and symmetric about $\xi = \frac{\kappa_1+\kappa_2}{2}$, it is clear that $h(\xi) = \frac{1}{\Gamma(\alpha)}\left[(\kappa_2-\xi)^{\alpha-1}+(\xi-\kappa_1)^{\alpha-1}\right]w(\xi)$ is non-negative, integrable and symmetric about $\xi = \frac{\kappa_1+\kappa_2}{2}$. Thus, by using Theorem 3.4, we can write the inclusion

$$
\begin{aligned}
& 2\varphi_1\left(\frac{\kappa_1+\kappa_2}{2}\right)\varphi_2\left(\frac{\kappa_1+\kappa_2}{2}\right)(R)\int_{\kappa_1}^{\kappa_2} h(\xi)d\xi \\
\supseteq\ & (IR)\int_{\kappa_1}^{\kappa_2} \varphi_1(\xi)\varphi_2(\xi)h(\xi)d\xi \\
& +\frac{\mathcal{M}(\kappa_1,\kappa_2)}{(\kappa_2-\kappa_1)^2}(R)\int_{\kappa_1}^{\kappa_2}(\kappa_2-\xi)(\xi-\kappa_1)h(\xi)d\xi+\frac{\mathcal{N}(\kappa_1,\kappa_2)}{(\kappa_2-\kappa_1)^2}(R)\int_{\kappa_1}^{\kappa_2}(\kappa_2-\xi)^2h(\xi)d\xi.
\end{aligned}
$$

That is, we have

$$
\begin{aligned}
& 2\varphi_1\left(\frac{\kappa_1+\kappa_2}{2}\right)\varphi_2\left(\frac{\kappa_1+\kappa_2}{2}\right)\frac{1}{\Gamma(\alpha)}(R)\int_{\kappa_1}^{\kappa_2}\left[(\kappa_2-\xi)^{\alpha-1}+(\xi-\kappa_1)^{\alpha-1}\right]w(\xi) \\
\supseteq\ & \frac{1}{\Gamma(\alpha)}(IR)\int_{\kappa_1}^{\kappa_2}\varphi_1(\xi)\varphi_2(\xi)\left[(\kappa_2-\xi)^{\alpha-1}+(\xi-\kappa_1)^{\alpha-1}\right]w(\xi)d\xi \\
& +\frac{\mathcal{M}(\kappa_1,\kappa_2)}{(\kappa_2-\kappa_1)^2\Gamma(\alpha)}(R)\int_{\kappa_1}^{\kappa_2}(\kappa_2-\xi)(\xi-\kappa_1)\left[(\kappa_2-\xi)^{\alpha-1}+(\xi-\kappa_1)^{\alpha-1}\right]w(\xi)d\xi \\
& +\frac{\mathcal{N}(\kappa_1,\kappa_2)}{(\kappa_2-\kappa_1)^2\Gamma(\alpha)}(R)\int_{\kappa_1}^{\kappa_2}(\kappa_2-\xi)^2\left[(\kappa_2-\xi)^{\alpha-1}+(\xi-\kappa_1)^{\alpha-1}\right]w(\xi)d\xi.
\end{aligned}
$$

Using the identities (28)-(30), we reach the desired result (32). □

Remark 4.6. *If we choose $\alpha=1$ in Theorem 4.5, then the inclusion (32) reduces to the inclusion (22).*

Remark 4.7. *If we choose $w(\xi)=1$ in Theorem 4.5, then we have*

$$
\begin{aligned}
& 2\varphi_1\left(\frac{\kappa_1+\kappa_2}{2}\right)\varphi_2\left(\frac{\kappa_1+\kappa_2}{2}\right) \\
\supseteq\ & \frac{\Gamma(\alpha+1)}{2(\kappa_2-\kappa_1)^\alpha}\left[\mathcal{J}^\alpha_{\kappa_1+}\varphi_1(\kappa_2)\varphi_2(\kappa_2)+\mathcal{J}^\alpha_{\kappa_2-}\varphi_1(\kappa_1)\varphi_2(\kappa_1)\right] \\
& +\frac{\alpha}{(\alpha+1)(\alpha+2)}\mathcal{M}(\kappa_1,\kappa_2)+\left(\frac{1}{2}-\frac{\alpha}{(\alpha+1)(\alpha+2)}\right)\mathcal{N}(\kappa_1,\kappa_2)
\end{aligned}
$$

which is proved by Budak et al. in [5].

Corollary 4.8. *If we take $\varphi_2(\xi)=[1,1]$ in Theorem 4.5, then, for all $\xi\in[\kappa_1,\kappa_2]$, we have the result*

$$
\begin{aligned}
& 2\varphi_1\left(\frac{\kappa_1+\kappa_2}{2}\right)\left[\mathrm{I}^\alpha_{\kappa_1+}w(\kappa_2)+\mathrm{I}^\alpha_{\kappa_2-}w(\kappa_1)\right] \\
\supseteq\ & \mathcal{J}^\alpha_{\kappa_1+}\varphi_1(\kappa_2)w(\kappa_2)+\mathcal{J}^\alpha_{\kappa_2-}\varphi_1(\kappa_1)w(\kappa_1) \\
& +\frac{\varphi_1(\kappa_1)+\varphi_1(\kappa_2)}{2}\left[\mathrm{I}^\alpha_{\kappa_1+}w(\kappa_2)+\mathrm{I}^\alpha_{\kappa_2-}w(\kappa_1)\right].
\end{aligned}
$$

Proof. The proof is obvious from the inclusion (31). □

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