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Some New Simpson's and Newton's Formulas Type Inequalities for Convex Functions in Quantum Calculus

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Abstract: In this paper, using the notions of q^{κ_2}-quantum integral and q^{κ_2}-quantum derivative, we present some new identities that enable us to obtain new quantum Simpson's and quantum Newton's type inequalities for quantum differentiable convex functions. This paper, in particular, generalizes and expands previous findings in the field of quantum and classical integral inequalities obtained by various authors.

Keywords: Simpson's inequalities; Newton's inequalities; quantum calculus; convex functions

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1. Introduction

Thomas Simpson developed crucial techniques for numerical integration and estimation of definite integrals, which became known as Simpson's law during his lifetime (1710–1761). However, J. Kepler used a similar approximation nearly a century before, so it is also known as Kepler's law. Since Simpson's rule includes the three-point Newton–Cotes quadrature rule, estimations based entirely on a three-step quadratic kernel are often referred to as Newton-type results.

(1) Simpson's quadrature formula (Simpson's 1/3 rule):

$$\int_{\kappa_1}^{\kappa_2} \mathcal{F}(x)\mathrm{d}x \approx \frac{\kappa_2-\kappa_1}{6}\left[\mathcal{F}(\kappa_1)+4\mathcal{F}\left(\frac{\kappa_1+\kappa_2}{2}\right)+\mathcal{F}(\kappa_2)\right].$$

(2) Simpson's second formula or Newton–Cotes quadrature formula (Simpson's 3/8 rule):

$$\int_{\kappa_1}^{\kappa_2} \mathcal{F}(x)\mathrm{d}x \approx \frac{\kappa_2-\kappa_1}{8}\left[\mathcal{F}(\kappa_1)+3\mathcal{F}\left(\frac{2\kappa_1+\kappa_2}{3}\right)+3\mathcal{F}\left(\frac{\kappa_1+2\kappa_2}{3}\right)+\mathcal{F}(\kappa_2)\right].$$

Within the literature, there are a plethora of estimations correlated with certain quadrature laws, one of which is the following estimation known as Simpson's inequality:

Theorem 1. *Let a mapping* $\mathcal{F} : [\kappa_1, \kappa_2] \to \mathbb{R}$ *be four times continuously differentiable on* (κ_1, κ_2), *and let* $\left\|\mathcal{F}^{(4)}\right\|_\infty = \sup_{x \in (\kappa_1, \kappa_2)} \left|\mathcal{F}^{(4)}(x)\right| < \infty$. *Then, one has the inequality*

$$\left|\frac{1}{3}\left[\frac{\mathcal{F}(\kappa_1)+\mathcal{F}(\kappa_2)}{2}+2\mathcal{F}\left(\frac{\kappa_1+\kappa_2}{2}\right)\right]-\frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2}\mathcal{F}(x)dx\right| \leq \frac{1}{2880}\left\|\mathcal{F}^{(4)}\right\|_\infty(\kappa_2-\kappa_1)^4.$$

Many authors have concentrated on the Simpson's type inequality in different categories of mappings in recent years. Since convexity theory is an easy and efficient way to solve a large number of problems from various branches of pure and applied mathematics; some mathematicians have worked on the results of Simpson's and Newton's type in obtaining a convex map. Dragomir et al. [1], for example, introduced the recent Simpson's inequalities and their applications in numerical integration quadrature formulas. In addition, Alomari et al. in [2] determined several inequalities of Simpson's type for s-convex functions. The variance of Simpson's type inequality dependent on convexity was then noted by Sarikaya et al. in [3]. The authors presented Newton's inequality for harmonic convex and p-harmonic convex mappings in [4,5]. Iftikhar et al., in [6], described a new Newton-type inequality for functions with the local fractional derivative, which is generalized and convex.

On the other hand, several works in the field of q-analysis are being carried out, beginning with Euler, in order to achieve mastery in the mathematics that drives quantum computing. The link between physics and mathematics is referred to as q-calculus. It has a wide range of applications in mathematics, including number theory, combinatorics, orthogonal polynomials, basic hypergeometric functions, and other disciplines, as well as mechanics, relativity theory, and quantum theory [7,8]. Quantum calculus is also useful in quantum information theory, which is an interdisciplinary field that includes computer science, information theory, philosophy, and cryptography, among other things [9,10]. This important branch of mathematics is thought to have been invented by Euler. The q-parameter was used by Newton in his work on infinite series. Jackson [11] was the first to develop the q-calculus, which knew no limit calculus, in a systematic way. Al-Salam [12] introduced a q-analogue of the q-fractional integral and a q-Riemann–Liouville fractional in 1966. Since then, the amount of research in this area has steadily increased. Specifically, Tariboon and Ntouyas [13] demonstrated the ${}_{\kappa_1}D_q$-difference operator and q_{κ_1}-integral in 2013. The notions of the ${}^{\kappa_2}D_q$ difference operator and q^{κ_2}-integral were given by Bermudo et al. [14] in 2020. In [15], Sadjang generalized to quantum calculus and introduced the notions of post-quantum calculus or (p,q)-calculus. Soontharanon et al. [16] introduced the concepts of fractional (p,q)-calculus later on. In [17], Tunç and Göv gave the post-quantum variant of the ${}_{\kappa_1}D_q$-difference operator and q_{κ_1}-integral. Recently, in 2021, Chu et al. introduced the notions of the ${}^{\kappa_2}D_{p,q}$ derivative and $(p,q)^{\kappa_2}$-integral in [18].

For various types of functions, quantum and post-quantum integrals have been used to investigate many integral inequalities. The authors used ${}_{\kappa_1}D_q$,${}^{\kappa_2}D_q$-derivatives and $q_{\kappa_1}, q^{\kappa_2}$-integrals to prove HH integral inequalities and their left–right estimates for convex and co-ordinated convex functions in [19–28]. Noor et al. presented a generalized version of quantum integral inequalities in [29]. Nwaeze et al. proved certain parameterized quantum integral inequalities for generalized quasi-convex functions in [30]. In [31], Khan et al. proved quantum HH inequality using the green function. For convex and coordinated convex functions, Budak et al. [32], Ali et al. [33,34], and Vivas-Cortez et al. [35] developed new quantum Simpson's and quantum Newton's type inequalities. For quantum Ostrowski's inequalities for convex and co-ordinated convex functions one can consult [36–38]. Using the ${}_{\kappa_1}D_{p,q}$-difference operator and $(p,q)_{\kappa_1}$-integral, Kunt et al. [39] generalized the results of [21] and proved HH type inequalities and their left estimates. Recently, Latif et al. [40] discovered the right estimates of HH type inequalities, which had previously been proven by Kunt et al. [39]. Chu et al. [18] used the concepts of the ${}^{\kappa_2}D_{p,q}$-difference operator and $(p,q)^{\kappa_2}$-integral to prove Ostrowski's inequalities. Recently, Vivas-Cortez et al. [41]

generalized the results of [14] and proved HH type inequalities and their left estimates using the ${}^{\kappa_2}D_{p,q}$-difference operator and $(p,q)^{\kappa_2}$-integral.

Inspired by the ongoing studies, we use the q-integral to develop some new quantum Simpson's and quantum Newton's formulas type inequalities for q-differentiable convex functions, and these inequalities can be helpful for finding the bounds of Simpson's and Newton's formulas for numerical integration. We also show that the newly developed inequalities are extensions of some previously known inequalities.

The following is the structure of this paper: Section 2 provides a brief overview of the fundamentals of q-calculus as well as other related studies in this field. In Section 3, we establish two crucial identities that play an essential role in developing the main results of this paper. The Simpson's and Newton's type inequalities for q-differentiable functions via q-integrals are described in Section 4. The relationship between the findings reported here and similar findings in the literature are also taken into account. Section 5 concludes with some recommendations for future research.

2. Preliminaries of q-Calculus and Some Inequalities

In this section, we first present the definitions and some properties of quantum derivatives and quantum integrals. We also mention some well-known inequalities for quantum integrals. The following notation will be used frequently in this work (see [8]):

$$[n]_q = \frac{q^n - 1}{q - 1}.$$

Jackson defined the q-Jackson integral from 0 to κ_2 for $0 < q < 1$ as follows:

$$\int_0^{\kappa_2} \mathcal{F}(x)\ d_q x\ = (1-q)\kappa_2 \sum_{k=0}^{\infty} q^k \mathcal{F}\left(\kappa_2 q^k\right) \tag{1}$$

provided the sum converges absolutely [11].

In [11], Jackson also defined the q-Jackson integral on any closed interval $[\kappa_1, \kappa_2]$:

$$\int_{\kappa_1}^{\kappa_2} \mathcal{F}(x)\ d_q x\ = \int_0^{\kappa_2} \mathcal{F}(x)\ d_q x\ - \int_0^{\kappa_1} \mathcal{F}(x)\ d_q x\ . \tag{2}$$

Tariboon and Ntouyas defined the following q_{κ_1}-derivative and q_{κ_1}-integral:

Definition 1 ([13]). *The q_{κ_1}-derivative of the mapping $\mathcal{F} : [\kappa_1, \kappa_2] \to \mathbb{R}$ is defined as:*

$${}_{\kappa_1}D_q\mathcal{F}(x)\ = \frac{\mathcal{F}(x) - \mathcal{F}(qx + (1-q)\kappa_1)}{(1-q)(x-\kappa_1)},\ x \neq \kappa_1. \tag{3}$$

If $x = \kappa_1$, we write ${}_{\kappa_1}D_q f(\kappa_1) = \lim_{x\to\kappa_1}\ {}_{\kappa_1}D_q f(x)$ if it exists and it is finite.

Definition 2 ([13]). *The q_{κ_1}-definite integral of the mapping $\mathcal{F} : [\kappa_1, \kappa_2] \to \mathbb{R}$ on $[\kappa_1, \kappa_2]$ is defined as:*

$$\int_{\kappa_1}^{\kappa_2} \mathcal{F}(x)\ {}_{\kappa_1}d_q x\ = (1-q)(\kappa_2 - \kappa_1) \sum_{k=0}^{\infty} q^k \mathcal{F}\left(q^k\kappa_2 + \left(1 - q^k\right)\kappa_1\right).$$

Alp et al. proved quantum Hermite–Hadamard inequalities for q_{κ_1}-integrals by utilizing the convex functions, as follows:

Theorem 2 ([21]). *For a convex mapping $\mathcal{F}:[\kappa_1,\kappa_2]\to\mathbb{R}$ that is differentiable on $[\kappa_1,\kappa_2]$, the following inequality holds:*

$$\mathcal{F}\left(\frac{q\kappa_1+\kappa_2}{[2]_q}\right)\leq\frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2}\mathcal{F}(x)\ {}_{\kappa_1}d_qx\ \leq\frac{q\mathcal{F}(\kappa_1)+\mathcal{F}(\kappa_2)}{[2]_q},\tag{4}$$

where $q\in(0,1)$.

On the other hand, Bermudo et al. defined a new quantum derivative and a quantum integral, which are called the q^{κ_2}-derivative and q^{κ_2}-integral:

Definition 3 ([14]). *The q^{κ_2}-derivative of mapping $\mathcal{F}:[\kappa_1,\kappa_2]\to\mathbb{R}$ is defined as:*

$${}^{\kappa_2}D_q\mathcal{F}(x)\ =\frac{\mathcal{F}(qx+(1-q)\kappa_2)-\mathcal{F}(x)}{(1-q)(\kappa_2-x)},\ x\neq\kappa_2.$$

If $x=\kappa_2$, we write ${}^{\kappa_2}D_qf(\kappa_2)=\lim_{x\to\kappa_2}\ {}^{\kappa_2}D_qf(x)$ if it exists and it is finite.

Definition 4 ([14]). *The q^{κ_2}-definite integral of mapping $\mathcal{F}:[\kappa_1,\kappa_2]\to\mathbb{R}$ on $[\kappa_1,\kappa_2]$ is defined as:*

$$\int_{\kappa_1}^{\kappa_2}\mathcal{F}(x)\ {}^{\kappa_2}d_qx\ =(1-q)(\kappa_2-\kappa_1)\sum_{k=0}^{\infty}q^k\mathcal{F}\left(q^k\kappa_1+\left(1-q^k\right)\kappa_2\right).$$

Bermudo et al. also proved the corresponding quantum Hermite–Hadamard inequalities for the q^{κ_2}-integral:

Theorem 3 ([14]). *For a convex mapping $\mathcal{F}:[\kappa_1,\kappa_2]\to\mathbb{R}$ that is differentiable on $[\kappa_1,\kappa_2]$, the following inequality holds:*

$$\mathcal{F}\left(\frac{\kappa_1+q\kappa_2}{[2]_q}\right)\leq\frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2}\mathcal{F}(x)\ {}^{\kappa_2}d_qx\ \leq\frac{\mathcal{F}(\kappa_1)+q\mathcal{F}(\kappa_2)}{[2]_q},\tag{5}$$

where $q\in(0,1)$.

Moreover, we will need to use the subsequent Lemma in our key results:

Lemma 1 ([8]). *We have the equality*

$$\int_{\kappa_1}^{\kappa_2}(x-\kappa_1)^a\ {}_{\kappa_1}d_qx\ =\frac{(\kappa_2-\kappa_1)^{a+1}}{[a+1]_q}$$

for $a\in\mathbb{R}\backslash\{-1\}$.

3. Identities

We deal with identities, which is necessary to attain our main estimations in this section. We first establish an identity based on a two steps kernel in the following Lemma.

Lemma 2. *Let $\mathcal{F}:[\kappa_1,\kappa_2]\to\mathbb{R}$ be a q^{κ_2}-differentiable function on (κ_1,κ_2) and $0<q<1$. If ${}^{\kappa_2}D_q\mathcal{F}$ is continuous and integrable on $[\kappa_1,\kappa_2]$, then one has the identity*

$$\frac{1}{(\kappa_2-\kappa_1)}\int_{\kappa_1}^{\kappa_2}\mathcal{F}(s)\ {}^{\kappa_2}d_qs\ -\frac{1}{[6]_q}\left[\mathcal{F}(\kappa_1)+q^2[4]_q\mathcal{F}\left(\frac{\kappa_1+q\kappa_2}{[2]_q}\right)+q\mathcal{F}(\kappa_2)\right]\tag{6}$$

$$= q(\kappa_2 - \kappa_1) \int_0^1 \Lambda(s) \ {}^{\kappa_2}D_q\mathcal{F}(s\kappa_1 + (1-s)\kappa_2) \quad d_q s,$$

where:

$$\Lambda(s) = \begin{cases} s - \frac{1}{[6]_q}, & s \in \left[0, \frac{1}{[2]_q}\right) \\ s - \frac{[5]_q}{[6]_q}, & s \in \left[\frac{1}{[2]_q}, 1\right]. \end{cases}$$

Proof. Using Formula (2), from the definition of the function $\Lambda(s)$, we find that

$$\begin{aligned} & \int_0^1 \Lambda(s) \ {}^{\kappa_2}D_q\mathcal{F}(s\kappa_1 + (1-s)\kappa_2) \quad d_q s \qquad (7) \\ = \ & \frac{[5]_q - 1}{[6]_q} \int_0^{\frac{1}{[2]_q}} {}^{\kappa_2}D_q\mathcal{F}(s\kappa_1 + (1-s)\kappa_2) \quad d_q s \\ & + \int_0^1 \left(s - \frac{[5]_q}{[6]_q}\right) \ {}^{\kappa_2}D_q\mathcal{F}(s\kappa_1 + (1-s)\kappa_2) \quad d_q s. \end{aligned}$$

By Definition 3, one also has

$${}^{\kappa_2}D_q\mathcal{F}(s\kappa_1 + (1-s)\kappa_2) = \frac{\mathcal{F}(qs\kappa_1 + (1-qs)\kappa_2) - \mathcal{F}(s\kappa_1 + (1-s)\kappa_2)}{(1-q)(\kappa_2 - \kappa_1)s}.$$

Now, if we substitute the above Equation into (7), we obtain:

$$\begin{aligned} & \int_0^1 \Lambda(s) \ {}^{\kappa_2}D_q\mathcal{F}(s\kappa_1 + (1-s)\kappa_2) \quad d_q s \qquad (8) \\ = \ & \frac{[5]_q - 1}{[6]_q} \int_0^{\frac{1}{[2]_q}} \frac{\mathcal{F}(qs\kappa_1 + (1-qs)\kappa_2) - \mathcal{F}(s\kappa_1 + (1-s)\kappa_2)}{(1-q)(\kappa_2 - \kappa_1)s} \ d_q s \\ & + \int_0^1 \frac{\mathcal{F}(qs\kappa_1 + (1-qs)\kappa_2) - \mathcal{F}(s\kappa_1 + (1-s)\kappa_2)}{(1-q)(\kappa_2 - \kappa_1)} \ d_q s \\ & - \frac{[5]_q}{[6]_q} \int_0^1 \frac{\mathcal{F}(qs\kappa_1 + (1-qs)\kappa_2) - \mathcal{F}(s\kappa_1 + (1-s)\kappa_2)}{(1-q)(\kappa_2 - \kappa_1)s} \ d_q s. \end{aligned}$$

Calculating the first integral in the right-hand side of (8) by taking into account the case when $\kappa_1 = 0$ of Definition 2, it is found that

$$\begin{aligned} & \int_0^{\frac{1}{[2]_q}} \frac{\mathcal{F}(qs\kappa_1 + (1-qs)\kappa_2) - \mathcal{F}(s\kappa_1 + (1-s)\kappa_2)}{(1-q)(\kappa_2 - \kappa_1)s} \ d_q s \qquad (9) \\ = \ & \frac{1}{(\kappa_2 - \kappa_1)} \frac{1}{[2]_q} \left\{ \sum_{k=0}^{\infty} q^k \frac{\mathcal{F}\left(\frac{q^{k+1}}{[2]_q}\kappa_1 + \left(1 - \frac{q^{k+1}}{[2]_q}\right)\kappa_2\right)}{\frac{q^k}{[2]_q}} \right. \end{aligned}$$

$$
\begin{aligned}
& - \sum_{k=0}^{\infty} q^k \frac{\mathcal{F}\left(\frac{q^k}{[2]_q}\kappa_1 + \left(1 - \frac{q^k}{[2]_q}\right)\kappa_2\right)}{\frac{q^k}{[2]_q}} \Bigg\} \\
= \quad & \frac{1}{(\kappa_2 - \kappa_1)} \left\{ \mathcal{F}(\kappa_2) - \mathcal{F}\left(\frac{\kappa_1 + q\kappa_2}{[2]_q}\right) \right\}.
\end{aligned}
$$

If we similarly observe the other integrals in the right-hand side of (8), from Definition 4, then we obtain

$$
\begin{aligned}
& \int_0^1 \frac{\mathcal{F}(qs\kappa_1 + (1-qs)\kappa_2) - \mathcal{F}(s\kappa_1 + (1-s)\kappa_2)}{(1-q)(\kappa_2 - \kappa_1)} \; d_q s \qquad (10) \\
= \quad & \frac{1}{(\kappa_2 - \kappa_1)} \left\{ \frac{1}{q(\kappa_2 - \kappa_1)} \int_{\kappa_1}^{\kappa_2} \mathcal{F}(s) \; {}^{\kappa_2}d_q s \; - \frac{1}{q}\mathcal{F}(\kappa_1) \right\}
\end{aligned}
$$

and

$$
\begin{aligned}
& \int_0^1 \frac{\varphi(qs\kappa_1 + (1-qs)\kappa_2) - \varphi(s\kappa_1 + (1-s)\kappa_2)}{(1-q)(\kappa_2 - \kappa_1)s} \; d_q s \qquad (11) \\
= \quad & \frac{1}{(\kappa_2 - \kappa_1)} \{\mathcal{F}(\kappa_2) - \mathcal{F}(\kappa_1)\}.
\end{aligned}
$$

Substituting Expressions (9)–(11) into (8), and later multiplying both sides of the resulting identity by $q(\kappa_2 - \kappa_1)$, Equality (6) can be captured. □

We now observe how an equality comes out when we use the kernel mapping with three sections.

Lemma 3. *Let $\mathcal{F} : [\kappa_1, \kappa_2] \to \mathbb{R}$ be a q^{κ_2}-differentiable function on (κ_1, κ_2) and $0 < q < 1$. If ${}^{\kappa_2}D_q\mathcal{F}$ is continuous and integrable on $[\kappa_1, \kappa_2]$, then one has the identity*

$$
\begin{aligned}
& \frac{1}{(\kappa_2 - \kappa_1)} \int_{\kappa_1}^{\kappa_2} \mathcal{F}(s) \; {}^{\kappa_2}d_q s \\
& - \frac{1}{[8]_q} \left[\mathcal{F}(\kappa_1) + \frac{q^3[6]_q}{[2]_q} \mathcal{F}\left(\frac{\kappa_1 + q[2]_q\kappa_2}{[3]_q}\right) + \frac{q^2[6]_q}{[2]_q} \mathcal{F}\left(\frac{[2]_q\kappa_1 + q^2\kappa_2}{[3]_q}\right) + q\mathcal{F}(\kappa_2) \right] \\
& = q(\kappa_2 - \kappa_1) \int_0^1 \Delta(s) \; {}^{\kappa_2}D_q\mathcal{F}(s\kappa_1 + (1-s)\kappa_2) \quad d_q s,
\end{aligned}
$$

where:

$$
\Delta(s) = \begin{cases} s - \frac{1}{[8]_q}, & s \in \left[0, \frac{1}{[3]_q}\right) \\ s - \frac{1}{[2]_q}, & s \in \left[\frac{1}{[3]_q}, \frac{[2]_q}{[3]_q}\right) \\ s - \frac{[7]_q}{[8]_q}, & s \in \left[\frac{[2]_q}{[3]_q}, 1\right]. \end{cases}
$$

Proof. Using Formula (2), from the definition of the function $\Delta(s)$, we find that

$$
\int_0^1 \Delta(s) \; {}^{\kappa_2}D_q\mathcal{F}(s\kappa_1 + (1-s)\kappa_2) \quad d_q s
$$

$$
\begin{aligned}
&= \frac{[8]_q - [2]_q}{[8]_q[2]_q} \int_0^{\frac{1}{[3]_q}} \kappa_2 D_q \mathcal{F}(s\kappa_1 + (1-s)\kappa_2) \quad d_q s \\
&+ \frac{[7]_q[2]_q - [8]_q}{[8]_q[2]_q} \int_0^{\frac{[2]_q}{[3]_q}} \kappa_2 D_q \mathcal{F}(s\kappa_1 + (1-s)\kappa_2) \quad d_q s \\
&+ \int_0^1 \left(s - \frac{[7]_q}{[8]_q} \right) \kappa_2 D_q \mathcal{F}(s\kappa_1 + (1-s)\kappa_2) \quad d_q s.
\end{aligned}
$$

If the same steps in the proof of Lemma 2 are applied for the rest of this proof, the desired result can be obtained. □

4. Main Results

For brevity, we start this section with the following notations, which will be used in the new results:

$$A_1(q) = \frac{2q^2[2]_q^2 + [6]_q^2\left([6]_q - [3]_q\right)}{[2]_q^3[3]_q[6]_q^3}, \tag{12}$$

$$B_1(q) = 2\frac{q[3]_q[6]_q - q^2}{[2]_q[3]_q[6]_q^3} + \frac{1}{[2]_q^3}\left(\frac{q+q^2}{[3]_q} - \frac{q^2+2q}{[6]_q}\right), \tag{13}$$

$$A_2(q) = \frac{2q^2[5]_q^3}{[2]_q[3]_q[6]_q^3} + \frac{[6]_q\left(1+[2]_q^3\right) - [3]_q[5]_q\left(1+[2]_q^2\right)}{[2]_q^3[3]_q[6]_q}, \tag{14}$$

$$
\begin{aligned}
B_2(q) &= 2\frac{q[5]_q^2[6]_q[3]_q - q^2[5]_q^3}{[2]_q[3]_q[6]_q^3} + \frac{q^2}{[2]_q[3]_q} - \frac{q[5]_q}{[2]_q[6]_q} \\
&- \frac{1}{[2]_q^3}\left[\frac{[5]_q(2q+q^2)}{[6]_q} - \frac{q+q^2}{[3]_q}\right],
\end{aligned} \tag{15}
$$

$$A_3(q) = \frac{2q^2[3]_q^3 + [8]_q^2\left([8]_q[2]_q - [3]_q^2\right)}{[8]_q^3[3]_q^4[2]_q}, \tag{16}$$

$$
\begin{aligned}
B_3(q) &= 2\frac{q[8]_q[3]_q - q^2}{[8]_q^3[2]_q[3]_q} + \frac{[3]_q^2 - [2]_q}{[3]_q^4[2]_q} \\
&+ \frac{1 - [3]_q[2]_q}{[8]_q[3]_q^2[2]_q},
\end{aligned} \tag{17}
$$

$$A_4(q) = \frac{2q^2}{[2]_q^4[3]_q} + \frac{[2]_q^2\left(1+[2]_q^3\right) - [3]_q^2\left(1+[2]_q^2\right)}{[3]_q^4[2]_q^2}, \tag{18}$$

$$B_4(q) = \frac{2q}{[2]_q^3} - \frac{q}{[3]_q^2} - \frac{q^2}{[3]_q^2} - A_4(q), \tag{19}$$

$$A_5(q) = \frac{2q^2[7]_q^3}{[8]_q^3[2]_q[3]_q} + \frac{[2]_q[8]_q\left([2]_q^3 + [3]_q^3\right) - [7]_q[3]_q^2\left([2]_q^2 + [3]_q^2\right)}{[3]_q^4[8]_q[2]_q}, \tag{20}$$

and

$$B_5(q) = 2\frac{q[7]_q^2[8]_q[3]_q - q^2[7]_q^3}{[8]_q^3[2]_q[3]_q} + \frac{q^2}{[2]_q[3]_q} - \frac{q[7]_q}{[2]_q[8]_q} + \frac{[2]_q\left([3]_q^2 - [2]_q^2\right)}{[3]_q^4} - \frac{(q+q^2)[7]_q[2]_q}{[3]_q^2[8]_q}. \tag{21}$$

4.1. Simpson's Inequalities for q^{κ_2}-Quantum Integral

In this subsection, we prove some Simpson's type inequalities by using Lemma 2.

Let us start to find some new quantum estimates by using Lemma 2. We first examine a new result for functions whose q^{κ_2}-derivatives in modulus are convex in the following theorem.

Theorem 4. *Let $\mathcal{F} : [\kappa_1, \kappa_2] \to \mathbb{R}$ be a q^{κ_2}-differentiable function on (κ_1, κ_2) such that ${}^{\kappa_2}D_q\mathcal{F}$ is continuous and integrable on $[\kappa_1, \kappa_2]$. If $\left|{}^{\kappa_2}D_q\mathcal{F}\right|$ is convex on $[\kappa_1, \kappa_2]$, then we have following inequality for q^{κ_2}-integrals:*

$$\left|\frac{1}{(\kappa_2-\kappa_1)}\int_{\kappa_1}^{\kappa_2}\mathcal{F}(s)\ {}^{\kappa_2}d_qs - \frac{1}{[6]_q}\left[\mathcal{F}(\kappa_1) + q^2[4]_q\mathcal{F}\left(\frac{\kappa_1+q\kappa_2}{[2]_q}\right) + q\mathcal{F}(\kappa_2)\right]\right| \tag{22}$$
$$\leq q(\kappa_2-\kappa_1)\left\{\left|{}^{\kappa_2}D_q\mathcal{F}(\kappa_1)\right|[A_1(q)+A_2(q)] + \left|{}^{\kappa_2}D_q\mathcal{F}(\kappa_2)\right|[B_1(q)+B_2(q)]\right\},$$

where $0 < q < 1$, and $A_1(q)$, $A_2(q)$, $B_1(q)$, $B_2(q)$ are given as in (12)–(15), respectively.

Proof. On taking the modulus in Lemma 2, because of the properties of modulus, we find that

$$\left|\frac{1}{(\kappa_2-\kappa_1)}\int_{\kappa_1}^{\kappa_2}\mathcal{F}(s)\ {}^{\kappa_2}d_qs - \frac{1}{[6]_q}\left[\mathcal{F}(\kappa_1) + q^2[4]_q\mathcal{F}\left(\frac{\kappa_1+q\kappa_2}{[2]_q}\right) + q\mathcal{F}(\kappa_2)\right]\right| \tag{23}$$
$$\leq q(\kappa_2-\kappa_1)\int_0^{\frac{1}{[2]_q}}\left|s-\frac{1}{[6]_q}\right|\left|{}^{\kappa_2}D_q\mathcal{F}(s\kappa_1+(1-s)\kappa_2)\right|d_qs$$
$$+q(\kappa_2-\kappa_1)\int_{\frac{1}{[2]_q}}^1\left|s-\frac{[5]_q}{[6]_q}\right|\left|{}^{\kappa_2}D_q\mathcal{F}(s\kappa_1+(1-s)\kappa_2)\right|d_qs.$$

To calculate integrals in the right-hand side of (23), using the convexity of $\left|{}^{\kappa_2}D_q\mathcal{F}\right|$, it follows that

$$\int_0^{\frac{1}{[2]_q}}\left|s-\frac{1}{[6]_q}\right|\left|{}^{\kappa_2}D_q\mathcal{F}(s\kappa_1+(1-s)\kappa_2)\right|d_qs$$
$$\leq \left|{}^{\kappa_2}D_q\mathcal{F}(\kappa_1)\right|\int_0^{\frac{1}{[2]_q}}s\left|s-\frac{1}{[6]_q}\right|d_qs + \left|{}^{\kappa_2}D_q\mathcal{F}(\kappa_2)\right|\int_0^{\frac{1}{[2]_q}}(1-s)\left|s-\frac{1}{[6]_q}\right|d_qs.$$

Now, if we apply the concept of Lemma 1 for $\kappa_1 = 0$ to the above quantum integrals, we attain

$$\int_0^{\frac{1}{[2]_q}}s\left|s-\frac{1}{[6]_q}\right|d_qs = \int_0^{\frac{1}{[6]_q}}s\left(\frac{1}{[6]_q}-s\right)d_qs + \int_{\frac{1}{[6]_q}}^{\frac{1}{[2]_q}}s\left(s-\frac{1}{[6]_q}\right)d_qs$$
$$= 2\int_0^{\frac{1}{[6]_q}}s\left(\frac{1}{[6]_q}-s\right)d_qs + \int_0^{\frac{1}{[2]_q}}s\left(s-\frac{1}{[6]_q}\right)d_qs$$

$$= \frac{2q^2[2]_q^2 + [6]_q^2\left([6]_q - [3]_q\right)}{[2]_q^3[3]_q[6]_q^3}$$

and

$$\begin{aligned}
&\int_0^{\frac{1}{[2]_q}} (1-s)\left|s - \frac{1}{[6]_q}\right| d_q s \\
= &\int_0^{\frac{1}{[6]_q}} (1-s)\left(\frac{1}{[6]_q} - s\right) d_q s + \int_{\frac{1}{[6]_q}}^{\frac{1}{[2]_q}} (1-s)\left(s - \frac{1}{[6]_q}\right) d_q s \\
= &\; 2\int_0^{\frac{1}{[6]_q}} (1-s)\left(\frac{1}{[6]_q} - s\right) d_q s + \int_0^{\frac{1}{[2]_q}} (1-s)\left(s - \frac{1}{[6]_q}\right) d_q s \\
= &\; 2\frac{q[3]_q[6]_q - q^2}{[2]_q[3]_q[6]_q^3} + \frac{1}{[2]_q^3}\left(\frac{q+q^2}{[3]_q} - \frac{q^2+2q}{[6]_q}\right).
\end{aligned}$$

Thus, we obtain

$$\begin{aligned}
&\int_0^{\frac{1}{[2]_q}} \left|s - \frac{1}{[6]_q}\right| \left|{}^{\kappa_2}D_q\mathcal{F}(s\kappa_1 + (1-s)\kappa_2)\right| d_q s \qquad (24) \\
\leq &\left|{}^{\kappa_2}D_q\mathcal{F}(\kappa_1)\right| \left(\frac{2q^2[2]_q^2 + [6]_q^2\left([6]_q - [3]_q\right)}{[2]_q^3[3]_q[6]_q^3}\right) \\
&+ \left|{}^{\kappa_2}D_q\mathcal{F}(\kappa_2)\right| \left(2\frac{q[3]_q[6]_q - q^2}{[2]_q[3]_q[6]_q^3} + \frac{1}{[2]_q^3}\left(\frac{q+q^2}{[3]_q} - \frac{q^2+2q}{[6]_q}\right)\right).
\end{aligned}$$

Similarly, using Equation (2) in addition to the convexity of $\left|{}^{\kappa_2}D_q\mathcal{F}\right|$ and Lemma 1, we have

$$\begin{aligned}
&\int_{\frac{1}{[2]_q}}^1 \left|s - \frac{[5]_q}{[6]_q}\right| \left|{}^{\kappa_2}D_q\mathcal{F}(s\kappa_1 + (1-s)\kappa_2)\right| d_q s \qquad (25) \\
\leq &\left|{}^{\kappa_2}D_q\mathcal{F}(\kappa_1)\right| \left(\frac{2q^2[5]_q^3}{[2]_q[3]_q[6]_q^3} + \frac{[6]_q\left(1+[2]_q^3\right) - [3]_q[5]_q\left(1+[2]_q^2\right)}{[2]_q^3[3]_q[6]_q}\right) \\
&+ \left|{}^{\kappa_2}D_q\mathcal{F}(\kappa_2)\right| \left(2\frac{q[5]_q^2[6]_q[3]_q - q^2[5]_q^3}{[2]_q[3]_q[6]_q^3} + \frac{q^2}{[2]_q[3]_q} - \frac{q[5]_q}{[2]_q[6]_q}\right. \\
&\left. - \frac{1}{[2]_q^3}\left[\frac{[5]_q(2q+q^2)}{[6]_q} - \frac{q+q^2}{[3]_q}\right]\right).
\end{aligned}$$

By putting (24) and (25) into (23), we attain Inequality (22), which finishes the proof. □

Corollary 1. *Under the assumptions of Theorem 4 with $q \to 1^-$, we have the following inequality of Simpson's type for the function whose modulus values of first derivative are convex (see [2]):*

$$\begin{aligned}
&\left|\frac{1}{6}\left[\mathcal{F}(\kappa_1) + 4\mathcal{F}\left(\frac{\kappa_1+\kappa_2}{2}\right) + \mathcal{F}(\kappa_2)\right] - \frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2} \mathcal{F}(s)ds\right| \\
\leq &\frac{5(\kappa_2-\kappa_1)}{72}\left[\left|\mathcal{F}'(\kappa_1)\right| + \left|\mathcal{F}'(\kappa_2)\right|\right].
\end{aligned}$$

Now, we observe how the inequalities come out when we use the mappings whose powers of q^{κ_2}-derivatives in absolute value are convex.

Theorem 5. *Let $\mathcal{F} : [\kappa_1, \kappa_2] \to \mathbb{R}$ be a q^{κ_2}-differentiable function on (κ_1, κ_2) such that ${}^{\kappa_2}D_q\mathcal{F}$ is continuous and integrable on $[\kappa_1, \kappa_2]$. If $\left|{}^{\kappa_2}D_q\mathcal{F}\right|^{p_1}$ is convex on $[\kappa_1, \kappa_2]$ for some $p_1 > 1$, then we have following inequality for q^{κ_2}-integrals:*

$$\left|\frac{1}{(\kappa_2-\kappa_1)}\int_{\kappa_1}^{\kappa_2}\mathcal{F}(s)\ {}^{\kappa_2}d_qs\ -\frac{1}{[6]_q}\left[\mathcal{F}(\kappa_1)+q^2[4]_q\mathcal{F}\left(\frac{\kappa_1+q\kappa_2}{[2]_q}\right)+q\mathcal{F}(\kappa_2)\right]\right| \tag{26}$$

$$\begin{aligned}
&\leq q(\kappa_2-\kappa_1)\left[\left(\frac{q^{2r_1}[4]_q^{r_1}}{[2]_q^{r_1+1}[6]_q^{r_1}}\right)^{\frac{1}{r_1}}\right.\\
&\times\left(\frac{1}{[2]_q^3}\left|{}^{\kappa_2}D_q\mathcal{F}(\kappa_1)\right|^{p_1}+\frac{q^2+2q}{[2]_q^3}\left|{}^{\kappa_2}D_q\mathcal{F}(\kappa_2)\right|^{p_1}\right)^{\frac{1}{p_1}}\\
&+\left(\frac{[2]_q^{r_1+1}[5]_q^{r_1}-q^{r_1}[4]_q^{r_1}}{[2]_q^{r_1+1}[6]_q^{r_1}}\right)^{\frac{1}{r_1}}\\
&\left.\times\left(\frac{q^2+2q}{[2]_q^3}\left|{}^{\kappa_2}D_q\mathcal{F}(\kappa_1)\right|^{p_1}+\frac{q^3+q^2-q}{[2]_q^3}\left|{}^{\kappa_2}D_q\mathcal{F}(\kappa_2)\right|^{p_1}\right)^{\frac{1}{p_1}}\right],
\end{aligned}$$

where $0 < q < 1$ and $\frac{1}{p_1}+\frac{1}{r_1}=1$.

Proof. Applying the well-known Hölder's inequality for quantum integrals to the integrals in the right-hand side of (23), it is found that

$$\begin{aligned}
&\left|\frac{1}{(\kappa_2-\kappa_1)}\int_{\kappa_1}^{\kappa_2}\mathcal{F}(s)\ {}^{\kappa_2}d_qs\ -\frac{1}{[6]_q}\left[\mathcal{F}(\kappa_1)+q^2[4]_q\mathcal{F}\left(\frac{\kappa_1+q\kappa_2}{[2]_q}\right)+q\mathcal{F}(\kappa_2)\right]\right|\\
\leq\ & q(\kappa_2-\kappa_1)\left[\left(\int_0^{\frac{1}{[2]_q}}\left|s-\frac{1}{[6]_q}\right|^{r_1}d_qs\right)^{\frac{1}{r_1}}\right.\\
&\left.\times\left(\int_0^{\frac{1}{[2]_q}}\left|{}^{\kappa_2}D_q\mathcal{F}(s\kappa_1+(1-s)\kappa_2)\right|^{p_1}d_qs\right)^{\frac{1}{p_1}}\right]\\
&\left[+q(\kappa_2-\kappa_1)\left(\int_{\frac{1}{[2]_q}}^1\left|s-\frac{[5]_q}{[6]_q}\right|^{r_1}d_qs\right)^{\frac{1}{r_1}}\right.\\
&\left.\times\left(\int_{\frac{1}{[2]_q}}^1\left|{}^{\kappa_2}D_q\mathcal{F}(s\kappa_1+(1-s)\kappa_2)\right|^{p_1}d_qs\right)^{\frac{1}{p_1}}\right].
\end{aligned}$$

By using the convexity of $\left|{}^{\kappa_2}D_q\mathcal{F}\right|^{p_1}$, we obtain

$$\left|\frac{1}{(\kappa_2-\kappa_1)}\int_{\kappa_1}^{\kappa_2}\mathcal{F}(s)\ {}^{\kappa_2}d_qs\ -\frac{1}{[6]_q}\left[\mathcal{F}(\kappa_1)+q^2[4]_q\mathcal{F}\left(\frac{\kappa_1+q\kappa_2}{[2]_q}\right)+q\mathcal{F}(\kappa_2)\right]\right| \tag{27}$$

$$
\begin{aligned}
\leq\ & q(\kappa_2-\kappa_1)\left[\left(\int_0^{\frac{1}{[2]_q}}\left|s-\frac{1}{[6]_q}\right|^{r_1}d_qs\right)^{\frac{1}{r_1}}\right.\\
&\left.\left(\left|{}^{\kappa_2}D_q\mathcal{F}(\kappa_1)\right|^{p_1}\int_0^{\frac{1}{[2]_q}}sd_qs+\left|{}^{\kappa_2}D_q\mathcal{F}(\kappa_2)\right|^{p_1}\int_0^{\frac{1}{[2]_q}}(1-s)d_qs\right)^{\frac{1}{p_1}}\right]\\
&+q(\kappa_2-\kappa_1)\left[\left(\int_{\frac{1}{[2]_q}}^1\left|s-\frac{[5]_q}{[6]_q}\right|^{r_1}d_qs\right)^{\frac{1}{r_1}}\right.\\
&\left.\left(\left|{}^{\kappa_2}D_q\mathcal{F}(\kappa_1)\right|^{p_1}\int_{\frac{1}{[2]_q}}^1sd_qs+\left|{}^{\kappa_2}D_q\mathcal{F}(\kappa_2)\right|^{p_1}\int_{\frac{1}{[2]_q}}^1(1-s)d_qs\right)^{\frac{1}{p_1}}\right].
\end{aligned}
$$

To calculate the integrals in the right-hand side of (27), if we first use Rule (1), then we obtain

$$
\begin{aligned}
\int_0^{\frac{1}{[2]_q}}\left|s-\frac{1}{[6]_q}\right|^{r_1}d_qs &= \frac{1-q}{[2]_q}\sum_{\kappa_2=0}^{\infty}q^{\kappa_2}\left|\frac{q^{\kappa_2}}{[2]_q}-\frac{1}{[6]_q}\right|^{r_1}\\
&\leq \frac{1-q}{[2]_q}\sum_{\kappa_2=0}^{\infty}q^{\kappa_2}\left|\frac{1}{[2]_q}-\frac{1}{[6]_q}\right|^{r_1}\\
&= \left(\frac{1}{[2]_q}-\frac{1}{[6]_q}\right)^{r_1}\frac{1}{[2]_q}\\
&= \frac{q^{2r_1}[4]_q^{r_1}}{[2]_q^{r_1+1}[6]_q^{r_1}}.
\end{aligned}
\tag{28}
$$

Similarly, we have

$$
\int_{\frac{1}{[2]_q}}^1\left|s-\frac{[5]_q}{[6]_q}\right|^{r_1}d_qs\leq\frac{[2]_q^{r_1+1}[5]_q^{r_1}-q^{r_1}[4]_q^{r_1}}{[2]_q^{r_1+1}[6]_q^{r_1}}.
\tag{29}
$$

For the other integrals in the right-hand side of (27), using the case when $\kappa_1=0$ of Lemma 1, we find that

$$
\int_0^{\frac{1}{[2]_q}}sd_qs = \frac{1}{[2]_q^3},
\tag{30}
$$

$$
\int_0^{\frac{1}{[2]_q}}(1-s)d_qs = \frac{q^2+2q}{[2]_q^3}.
\tag{31}
$$

Similarly, we get

$$
\int_{\frac{1}{[2]_q}}^1sd_qs = \frac{q^2+2q}{\left([2]_q\right)^3}
\tag{32}
$$

$$
\int_{\frac{1}{[2]_q}}^1(1-s)d_qs = \frac{q^3+q^2-q}{\left([2]_q\right)^3}.
\tag{33}
$$

By substituting (28)–(33) into (27), we obtain the desired Inequality (26), which completes the proof. □

Theorem 6. *Let $\mathcal{F} : [\kappa_1, \kappa_2] \to \mathbb{R}$ be a q^{κ_2}-differentiable function on (κ_1, κ_2) such that ${}^{\kappa_2}D_q\mathcal{F}$ is continuous and integrable on $[\kappa_1, \kappa_2]$. If $\left|{}^{\kappa_2}D_q\mathcal{F}\right|^{p_1}$ is convex on $[\kappa_1, \kappa_2]$ for some $p_1 \geq 1$, then we have following inequality for q^{κ_2}-integrals:*

$$\begin{aligned}
&\left|\frac{1}{(\kappa_2-\kappa_1)}\int_{\kappa_1}^{\kappa_2}\mathcal{F}(s)\ {}^{\kappa_2}d_qs\ -\frac{1}{[6]_q}\left[\mathcal{F}(\kappa_1)+q^2[4]_q\mathcal{F}\left(\frac{\kappa_1+q\kappa_2}{[2]_q}\right)+q\mathcal{F}(\kappa_2)\right]\right| \\
\leq\ & q(\kappa_2-\kappa_1)\left[\left(\frac{2q}{[2]_q[6]_q^2}+\frac{q^3[3]_q-q}{[6]_q[2]_q^3}\right)^{1-\frac{1}{p_1}}\right. \\
&\times\left(A_1(q)\left|{}^{\kappa_2}D_q\mathcal{F}(\kappa_1)\right|^{p_1}+B_1(q)\left|{}^{\kappa_2}D_q\mathcal{F}(\kappa_2)\right|^{p_1}\right)^{\frac{1}{p_1}} \\
&+\left(2q\frac{[5]_q^2}{[2]_q[6]_q^2}+\frac{1}{[2]_q}-\frac{[5]_q}{[6]_q}-\frac{[5]_q[2]_q^2-[6]_q}{[6]_q[2]_q^3}\right)^{1-\frac{1}{p_1}} \\
&\left.\times\left(A_2(q)\left|{}^{\kappa_2}D_q\mathcal{F}(\kappa_1)\right|^{p_1}+B_2(q)\left|{}^{\kappa_2}D_q\mathcal{F}(\kappa_2)\right|^{p_1}\right)^{\frac{1}{p_1}}\right],
\end{aligned} \tag{34}$$

where $0 < q < 1$, and $A_1(q)$, $A_2(q)$, $B_1(q)$, $B_2(q)$ are given as in (12)–(15), respectively.

Proof. Utilizing from the results in the proof of Theorem 4 after applying the well-known Power mean inequality to the integrals in the right-hand side of (23), owing to the convexity of $\left|{}^{\kappa_2}D_q\mathcal{F}\right|^{p_1}$, we find that

$$\begin{aligned}
&\left|\frac{1}{(\kappa_2-\kappa_1)}\int_{\kappa_1}^{\kappa_2}\mathcal{F}(s)\ {}^{\kappa_2}d_qs\ -\frac{1}{[6]_q}\left[\mathcal{F}(\kappa_1)+q^2[4]_q\mathcal{F}\left(\frac{\kappa_1+q\kappa_2}{[2]_q}\right)+q\mathcal{F}(\kappa_2)\right]\right| \\
\leq\ & q(\kappa_2-\kappa_1)\left[\left(\int_0^{\frac{1}{[2]_q}}\left|s-\frac{1}{[6]_q}\right|d_qs\right)^{1-\frac{1}{p_1}}\right. \\
&\times\left(\left|{}^{\kappa_2}D_q\mathcal{F}(\kappa_1)\right|^{p_1}\int_0^{\frac{1}{[2]_q}}s\left|s-\frac{1}{[6]_q}\right|d_qs\right. \\
&\left.\left.+\left|{}^{\kappa_2}D_q\mathcal{F}(\kappa_2)\right|^{p_1}\int_0^{\frac{1}{[2]_q}}(1-s)\left|s-\frac{1}{[6]_q}\right|d_qs\right)^{\frac{1}{p_1}}\right] \\
&+q(\kappa_2-\kappa_1)\left[\left(\int_{\frac{1}{[2]_q}}^1\left|s-\frac{[5]_q}{[6]_q}\right|d_qs\right)^{1-\frac{1}{p_1}}\right. \\
&\times\left(\left|{}^{\kappa_2}D_q\mathcal{F}(\kappa_1)\right|^{p_1}\int_{\frac{1}{[2]_q}}^1s\left|s-\frac{[5]_q}{[6]_q}\right|d_qs\right. \\
&\left.\left.+\left|{}^{\kappa_2}D_q\mathcal{F}(\kappa_2)\right|^{p_1}\int_{\frac{1}{[2]_q}}^1(1-s)\left|s-\frac{[5]_q}{[6]_q}\right|d_qs\right)^{\frac{1}{p_1}}\right] \\
=\ & q(\kappa_2-\kappa_1)\left[\left(\int_0^{\frac{1}{[2]_q}}\left|s-\frac{1}{[6]_q}\right|d_qs\right)^{1-\frac{1}{p_1}}\right. \\
&\left.\times\left(A_1(q)\left|{}^{\kappa_2}D_q\mathcal{F}(\kappa_1)\right|^{p_1}+B_1(q)\left|{}^{\kappa_2}D_q\mathcal{F}(\kappa_2)\right|^{p_1}\right)^{\frac{1}{p_1}}\right]
\end{aligned}$$

$$+q\left[(\kappa_2-\kappa_1)\left(\int_{\frac{1}{[2]_q}}^{1}\left|s-\frac{[5]_q}{[6]_q}\right|d_qs\right)^{1-\frac{1}{p_1}}\right.$$
$$\left.\times\left(A_2(q)\left|{}^{\kappa_2}D_q\mathcal{F}(\kappa_1)\right|^{p_1}+B_2(q)\left|{}^{\kappa_2}D_q\mathcal{F}(\kappa_2)\right|^{p_1}\right)^{\frac{1}{p_1}}\right]. \tag{35}$$

We also observe that

$$\begin{aligned}\int_0^{\frac{1}{[2]_q}}\left|s-\frac{1}{[6]_q}\right|d_qs &= 2\int_0^{\frac{1}{[6]_q}}\left(\frac{1}{[6]_q}-s\right)d_qs+\int_0^{\frac{1}{[2]_q}}\left(s-\frac{1}{[6]_q}\right)d_qs \\ &= \frac{2q}{[2]_q[6]_q^2}+\frac{q^3[3]_q-q}{[6]_q[2]_q^3},\end{aligned} \tag{36}$$

and by using similar operations, we have

$$\int_{\frac{1}{[2]_q}}^{1}\left|s-\frac{[5]_q}{[6]_q}\right|d_qs=2q\frac{[5]_q^2}{[2]_q[6]_q^2}+\frac{1}{[2]_q}-\frac{[5]_q}{[6]_q}-\frac{[5]_q[2]_q^2-[6]_q}{[6]_q[2]_q^3}. \tag{37}$$

By substituting (36) and (37) into (35), we attain the required Inequality (34). Hence, the proof is completed. □

Corollary 2. *Under the given conditions in Theorem 6 with $q\to 1^-$, we have following inequality given by Alomari et al. (see [2]):*

$$\begin{aligned}&\left|\frac{1}{6}\left[\mathcal{F}(\kappa_1)+4\mathcal{F}\left(\frac{\kappa_1+\kappa_2}{2}\right)+\mathcal{F}(\kappa_2)\right]-\frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2}\mathcal{F}(s)ds\right| \\ \leq\ &\frac{1}{(1296)^{\frac{1}{p_1}}}\left(\frac{5}{72}\right)^{1-\frac{1}{p_1}}(\kappa_2-\kappa_1) \\ &\times\left[61\left|\mathcal{F}'(\kappa_1)\right|^{p_1}+29\left|\mathcal{F}'(\kappa_2)\right|^{p_1}\right]^{\frac{1}{p_1}}+\left[29\left|\mathcal{F}'(\kappa_1)\right|^{p_1}+61\left|\mathcal{F}'(\kappa_2)\right|^{p_1}\right]^{\frac{1}{p_1}}.\end{aligned}$$

4.2. Newton's Inequalities for q^{κ_2}-Quantum Integral

In this subsection, we establish some new Newton's type inequalities by using Lemma 3.

In the next theorems, we will present new quantum estimates by using the convexity of q^{κ_2}-derivatives in modulus and Lemma 3.

Theorem 7. *Let $\mathcal{F}:[\kappa_1,\kappa_2]\to\mathbb{R}$ be a q^{κ_2}-differentiable function on (κ_1,κ_2) such that ${}^{\kappa_2}D_q\mathcal{F}$ is continuous and integrable on $[\kappa_1,\kappa_2]$. If $\left|{}^{\kappa_2}D_q\mathcal{F}\right|$ is convex on $[\kappa_1,\kappa_2]$, then we have the following inequality for q^{κ_2}-integrals:*

$$\begin{aligned}&\left|\frac{1}{(\kappa_2-\kappa_1)}\int_{\kappa_1}^{\kappa_2}\mathcal{F}(s)\ {}^{\kappa_2}d_qs\ -\frac{1}{[8]_q}\left[\mathcal{F}(\kappa_1)+\frac{q^3[6]_q}{[2]_q}\mathcal{F}\left(\frac{\kappa_1+q[2]_q\kappa_2}{[3]_q}\right)\right.\right. \\ &\left.\left.+\frac{q^2[6]_q}{[2]_q}\mathcal{F}\left(\frac{[2]_q\kappa_1+q^2\kappa_2}{[3]_q}\right)+q\mathcal{F}(\kappa_2)\right]\right| \\ &\leq q(\kappa_2-\kappa_1)\left\{\left|{}^{\kappa_2}D_q\mathcal{F}(\kappa_1)\right|[A_3(q)+A_4(q)+A_5(q)]\right. \\ &\left.+\left|{}^{\kappa_2}D_q\mathcal{F}(\kappa_2)\right|[B_3(q)+B_4(q)+B_5(q)]\right\},\end{aligned} \tag{38}$$

where $0<q<1$, and $A_3(q)$, $A_4(q)$, $A_5(q)$, $B_3(q)$, $B_4(q)$,$B_5(q)$ are given as in (16)–(21), respectively.

Proof. On taking the modulus in Lemma 3, we gain

$$\begin{aligned}
&\left|\frac{1}{(\kappa_2-\kappa_1)}\int_{\kappa_1}^{\kappa_2}\mathcal{F}(s)\ {}^{\kappa_2}d_qs\ -\frac{1}{[8]_q}\left[\mathcal{F}(\kappa_1)+\frac{q^3[6]_q}{[2]_q}\mathcal{F}\left(\frac{\kappa_1+q[2]_q\kappa_2}{[3]_q}\right)\right.\right. \qquad (39)\\
&\left.\left.+\frac{q^2[6]_q}{[2]_q}\mathcal{F}\left(\frac{[2]_q\kappa_1+q^2\kappa_2}{[3]_q}\right)+q\mathcal{F}(\kappa_2)\right]\right|\\
\leq\ & q(\kappa_2-\kappa_1)\int_0^{\frac{1}{[3]_q}}\left|s-\frac{1}{[8]_q}\right|\left|{}^{\kappa_2}D_q\mathcal{F}(s\kappa_1+(1-s)\kappa_2)\right|d_qs\\
&+q(\kappa_2-\kappa_1)\int_{\frac{1}{[3]_q}}^{\frac{[2]_q}{[3]_q}}\left|s-\frac{1}{[2]_q}\right|\left|{}^{\kappa_2}D_q\mathcal{F}(s\kappa_1+(1-s)\kappa_2)\right|d_qs\\
&+q(\kappa_2-\kappa_1)\int_{\frac{[2]_q}{[3]_q}}^{1}\left|s-\frac{[7]_q}{[8]_q}\right|\left|{}^{\kappa_2}D_q\mathcal{F}(s\kappa_1+(1-s)\kappa_2)\right|d_qs.
\end{aligned}$$

From now on, it is sufficient to use the same methods as in the proof of Theorem 4 in order to reach desired result (38). □

Remark 1. *Under the given conditions of Theorem 7 with the limit $q\to 1^-$, we have following inequality:*

$$\begin{aligned}
&\left|\frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2}\mathcal{F}(s)ds-\frac{1}{8}\left[\mathcal{F}(\kappa_1)+3\mathcal{F}\left(\frac{2\kappa_1+\kappa_2}{3}\right)+3\mathcal{F}\left(\frac{\kappa_1+2\kappa_2}{3}\right)+\mathcal{F}(\kappa_2)\right]\right|\\
\leq\ & \frac{25}{576}(\kappa_2-\kappa_1)\left[\left|\mathcal{F}'(\kappa_1)\right|+\left|\mathcal{F}'(\kappa_2)\right|\right],
\end{aligned}$$

which is given by Noor et al. in [5].

Theorem 8. *Let $\mathcal{F}:[\kappa_1,\kappa_2]\to\mathbb{R}$ be a q^{κ_2}-differentiable function on (κ_1,κ_2) such that ${}^{\kappa_2}D_q\mathcal{F}$ is continuous and integrable on $[\kappa_1,\kappa_2]$. If $\left|{}^{\kappa_2}D_q\mathcal{F}\right|^{p_1}$ is convex on $[\kappa_1,\kappa_2]$ for some $p_1>1$, then we have the following inequality for q^{κ_2}-integrals:*

$$\begin{aligned}
&\left|\frac{1}{(\kappa_2-\kappa_1)}\int_{\kappa_1}^{\kappa_2}\mathcal{F}(s)\ {}^{\kappa_2}d_qs\ -\frac{1}{[8]_q}\left[\mathcal{F}(\kappa_1)+\frac{q^3[6]_q}{[2]_q}\mathcal{F}\left(\frac{\kappa_1+q[2]_q\kappa_2}{[3]_q}\right)\right.\right. \qquad (40)\\
&\left.\left.+\frac{q^2[6]_q}{[2]_q}\mathcal{F}\left(\frac{[2]_q\kappa_1+q^2\kappa_2}{[3]_q}\right)+q\mathcal{F}(\kappa_2)\right]\right|\\
\leq\ & q(\kappa_2-\kappa_1)\left[\left(\frac{q^{3r_1}[5]_q^{r_1}}{[3]_q^{r_1+1}[8]_q^{r_1}}\right)^{\frac{1}{r_1}}\right.\\
&\times\left(\frac{1}{[3]_q^2[2]_q}\left|{}^{\kappa_2}D_q\mathcal{F}(\kappa_1)\right|^{p_1}+\frac{[3]_q[2]_q-1}{[3]_q^2[2]_q}\left|{}^{\kappa_2}D_q\mathcal{F}(\kappa_2)\right|^{p_1}\right)^{\frac{1}{p_1}}\\
&+\left(\frac{q^{r_1}[2]_q-q^{2r_1}}{[3]_q^{r_1+1}[2]_q^{r_1}}\right)^{\frac{1}{r_1}}\\
&\left.\times\left(\frac{q^2+2}{[3]_q^2[2]_q}\left|{}^{\kappa_2}D_q\mathcal{F}(\kappa_1)\right|^{p_1}+\frac{q[3]_q[2]_q-(q^2+2q)}{[3]_q^2[2]_q}\left|{}^{\kappa_2}D_q\mathcal{F}(\kappa_2)\right|^{p_1}\right)^{\frac{1}{p_1}}\right]
\end{aligned}$$

$$+\left(\frac{q^{7r_1}}{[8]_q^{r_1}}-\frac{[2]_q\left([7]_q[3]_q-[8]_q[2]_q\right)^{r_1}}{[8]_q^{r_1}[3]_q^{r_1+1}}\right)^{\frac{1}{r_1}}$$
$$\times\left(\frac{[3]_q^2-[2]_q^2}{[3]_q^2[2]_q}\left|{}^{\kappa_2}D_q\mathcal{F}(\kappa_1)\right|^{p_1}+\frac{q^2[3]_q[2]_q+[2]_q^2-[3]_q^2}{[3]_q^2[2]_q}\left|{}^{\kappa_2}D_q\mathcal{F}(\kappa_2)\right|^{p_1}\right)^{\frac{1}{p_1}}\Bigg],$$

where $0<q<1$ *and* $\frac{1}{p_1}+\frac{1}{r_1}=1$.

Proof. If we apply Hölder's inequality to the expressions in the right-hand side of (39), then we obtain

$$\begin{aligned}
&\left|\frac{1}{(\kappa_2-\kappa_1)}\int_{\kappa_1}^{\kappa_2}\mathcal{F}(s)\ {}^{\kappa_2}d_qs\ -\frac{1}{[8]_q}\left[\mathcal{F}(\kappa_1)+\frac{q^3[6]_q}{[2]_q}\mathcal{F}\left(\frac{\kappa_1+q[2]_q\kappa_2}{[3]_q}\right)\right.\right.\\
&\left.\left.+\frac{q^2[6]_q}{[2]_q}\mathcal{F}\left(\frac{[2]_q\kappa_1+q^2\kappa_2}{[3]_q}\right)+q\mathcal{F}(\kappa_2)\right]\right|\\
\leq\ & q(\kappa_2-\kappa_1)\left(\int_0^{\frac{1}{[3]_q}}\left|s-\frac{1}{[8]_q}\right|^{r_1}d_qs\right)^{\frac{1}{r_1}}\left(\int_0^{\frac{1}{[3]_q}}\left|{}^{\kappa_2}D_q\mathcal{F}(s\kappa_1+(1-s)\kappa_2)\right|^{p_1}d_qs\right)^{\frac{1}{p_1}}\\
&+q(\kappa_2-\kappa_1)\left(\int_{\frac{1}{[3]_q}}^{\frac{[2]_q}{[3]_q}}\left|s-\frac{1}{[2]_q}\right|^{r_1}d_qs\right)^{\frac{1}{r_1}}\left(\int_{\frac{1}{[3]_q}}^{\frac{[2]_q}{[3]_q}}\left|{}^{\kappa_2}D_q\mathcal{F}(s\kappa_1+(1-s)\kappa_2)\right|^{p_1}d_qs\right)^{\frac{1}{p_1}}\\
&+q(\kappa_2-\kappa_1)\left(\int_{\frac{[2]_q}{[3]_q}}^{1}\left|s-\frac{[7]_q}{[8]_q}\right|^{r_1}d_qs\right)^{\frac{1}{r_1}}\left(\int_{\frac{[2]_q}{[3]_q}}^{1}\left|{}^{\kappa_2}D_q\mathcal{F}(s\kappa_1+(1-s)\kappa_2)\right|^{p_1}d_qs\right)^{\frac{1}{p_1}}.
\end{aligned}$$

For the rest of this proof, if the same procedure that was used in the proof of Theorem 5 is applied to the above inequality, then Result (40) can be captured. □

Theorem 9. *Let* $\mathcal{F}:[\kappa_1,\kappa_2]\to\mathbb{R}$ *be a* q^{κ_2}*-differentiable function on* (κ_1,κ_2) *such that* ${}^{\kappa_2}D_q\mathcal{F}$ *is continuous and integrable on* $[\kappa_1,\kappa_2]$*. If* $\left|{}^{\kappa_2}D_q\mathcal{F}\right|^{p_1}$ *is convex on* $[\kappa_1,\kappa_2]$ *for some* $p_1\geq 1$*, then we have the following inequality for* q^{κ_2}*-integrals:*

$$\begin{aligned}
&\left|\frac{1}{(\kappa_2-\kappa_1)}\int_{\kappa_1}^{\kappa_2}\mathcal{F}(s)\ {}^{\kappa_2}d_qs\ -\frac{1}{[8]_q}\left[\mathcal{F}(\kappa_1)+\frac{q^3[6]_q}{[2]_q}\mathcal{F}\left(\frac{\kappa_1+q[2]_q\kappa_2}{[3]_q}\right)\right.\right.\\
&\left.\left.+\frac{q^2[6]_q}{[2]_q}\mathcal{F}\left(\frac{[2]_q\kappa_1+q^2\kappa_2}{[3]_q}\right)+q\mathcal{F}(\kappa_2)\right]\right|\\
\leq\ & q(\kappa_2-\kappa_1)\left[\left(\frac{2q}{[8]_q^2[2]_q}+\frac{[8]_q-[3]_q[2]_q}{[3]_q^2[2]_q[8]_q}\right)^{1-\frac{1}{p_1}}\right.\\
&\times\left(A_3(q)\left|{}^{\kappa_2}D_q\mathcal{F}(\kappa_1)\right|^{p_1}+B_3(q)\left|{}^{\kappa_2}D_q\mathcal{F}(\kappa_2)\right|^{p_1}\right)^{\frac{1}{p_1}}\\
&+\left(\frac{2q}{[2]_q^3}+\frac{q}{[3]_q^2[2]_q}+\frac{1-[3]_q[2]_q}{[3]_q^2[2]_q}\right)^{1-\frac{1}{p_1}}\\
&\left.\times\left(A_4(q)\left|{}^{\kappa_2}D_q\mathcal{F}(\kappa_1)\right|^{p_1}+B_4(q)\left|{}^{\kappa_2}D_q\mathcal{F}(\kappa_2)\right|^{p_1}\right)^{\frac{1}{p_1}}\right.
\end{aligned}\tag{41}$$

$$+\left(2\frac{q[7]_q^2}{[8]_q^2[2]_q}+\frac{[3]_q^2+[2]_q^2}{[2]_q[3]_q^2}-\frac{[7]_q\left([3]_q+[2]_q\right)}{[8]_q[3]_q}\right)^{1-\frac{1}{p_1}}$$
$$\times\left(A_5(q)\left|{}^{\kappa_2}D_q\mathcal{F}(\kappa_1)\right|^{p_1}+B_5(q)\left|{}^{\kappa_2}D_q\mathcal{F}(\kappa_2)\right|^{p_1}\right)^{\frac{1}{p_1}},$$

where $0<q<1$, *and* $A_3(q)$, $A_4(q)$, $A_5(q)$, $B_3(q)$, $B_4(q)$,$B_5(q)$ *are given as in (16)–(21), respectively.*

Proof. If the strategy that was used in the proof of Theorem 6 is applied by taking into account Lemma 3, the desired Inequality (41) can be attained. □

Remark 2. *Under the conditions of Theorem 9 with the limit* $q\to 1^-$, *we obtain the following inequality:*

$$\begin{aligned}&\left|\frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2}\mathcal{F}(s)ds-\frac{1}{8}\left[\mathcal{F}(\kappa_1)+3\mathcal{F}\left(\frac{2\kappa_1+\kappa_2}{3}\right)+3\mathcal{F}\left(\frac{\kappa_1+2\kappa_2}{3}\right)+\mathcal{F}(\kappa_2)\right]\right|\\ \leq\ &\frac{\kappa_2-\kappa_1}{36}\left\{\left(\frac{17}{16}\right)^{1-\frac{1}{r}}\left(\frac{251}{1152}\left|\mathcal{F}'(\kappa_1)\right|^{p_1}+\frac{973}{1152}\left|\mathcal{F}'(\kappa_2)\right|^{p_1}\right)^{\frac{1}{p_1}}\right.\\ &+\left(\frac{1}{2}\left|\mathcal{F}'(\kappa_1)\right|^{p_1}+\frac{1}{2}\left|\mathcal{F}'(\kappa_2)\right|^{p_1}\right)^{\frac{1}{p_1}}\\ &\left.+\left(\frac{17}{16}\right)^{1-\frac{1}{r}}\left(\frac{973}{1152}\left|\mathcal{F}'(\kappa_1)\right|^{p_1}+\frac{251}{1152}\left|\mathcal{F}'(\kappa_2)\right|^{p_1}\right)^{\frac{1}{p_1}}\right\},\end{aligned}$$

which can be found in [5].

5. Conclusions

We conclude our work by stating that, using the concepts of quantum derivative and quantum integral, we proved some new Simpson's and Newton's type quantum integral inequalities for quantum differentiable convex functions, and these inequalities can be helpful for finding the bounds of Simpson's and Newton's formulas for numerical integration. It is important to note that by considering the limit $q\to 1^-$ in our key results, our results were transformed into some new and well-known results. We believe it is an interesting and creative problem for upcoming researchers who will be able to obtain similar inequalities for various types of convexity and co-ordinated convexity in their future work.

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