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On some new midpoint inequalities for the functions of two variables via quantum calculus

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Abstract

In this paper, first we obtain a new identity for quantum integrals, the result is then used to prove midpoint type inequalities for differentiable coordinated convex mappings. The outcomes provided in this article are an extension of the comparable consequences in the literature on the midpoint inequalities for differentiable coordinated convex mappings.

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1 Introduction

Quantum calculus, which is also named q-calculus, is occasionally mentioned as calculation method without limits. Herewith, one achieves q-analogues of mathematical tools that may be got back as $q \to 1$. There are two techniques in q-addition, one of them is the Nalli–Ward–Al-Salam q-addition (NWA) and the other is Jackson–Hahn–Cigler q-addition (JHC). The first one is commutative and associative, at the same time as the second one is neither. Because of this, there are multiple q-analogs from time to time. These operators constitute the base of the method that combine hypergeometric collection with q-hypergeometric collection and gives many formulations of q-calculus a natural shape. The history of quantum calculus may be traced back to Euler (1707–1783), he first added the expression q in the tracks of Newton's infinite series. Recently, a great number of researchers have shown an eager hobby in studying and investigating quantum calculus and accordingly it emerged as an interdisciplinary subject. The quantum theory has become a cornerstone in theoretical mathematics and applied sciences, due to the fact that quantum analysis is very helpful in several fields and has huge applications in various areas of natural and applied sciences such as computer science and particle physics. Specifically, the theory has been seen as a critical tool for researchers operating with analytic number theory or in theoretical physics. This calculus method is a bridge that provides the connection between mathematics and physics. Owing to a large numbers of applications in quantum group theory, the quantum calculus also has a significant role for physicists. For some recent trends in quantum calculus the reader is referred to [1–6].

In recent decades the idea of convex functions has been drastically studied because of its fantastic significance in numerous fields of pure and applied sciences. Theory of inequalities and concept of convex functions are closely related to each other, thus they resemble inequalities that could be obtained inside the literature which are derived for convex and differentiable convex mappings; see [7–13].

We now consider how the convex functions of two-variables on the coordinates, which may be also called a coordinated convex function, is defined. Dragomir [14] presented the definition of coordinated convexity as follows.

Definition 1 For all $(\varkappa,\zeta),(\eta,\xi)\in\Omega$ and $u,v\in[0,1]$, a mapping $\Psi:\Omega=[\alpha,\beta]\times[\gamma,\delta]\subseteq\mathbb{R}^2\to\mathbb{R}$ is said to be coordinated convex on Ω, if it satisfies the inequality

$$\begin{aligned}&\Psi\bigl(u\varkappa+(1-u)\eta,v\zeta+(1-v)\xi\bigr)\\&\quad\le uv\Psi(\varkappa,\zeta)+u(1-v)\Psi(\varkappa,\xi)+v(1-u)\Psi(\eta,\zeta)+(1-u)(1-v)\Psi(\eta,\xi).\end{aligned}\tag{1.1}$$

The function Ψ is said to be coordinated concave on Ω, if the inequality (1.1) holds in reversed direction for all $u,v\in[0,1]$ and $(\varkappa,\zeta),(\eta,\xi)\in\Omega$.

In [14], Hermite–Hadamard type inequalities for convex function of two-variable on the coordinates are established by Dragomir as follows.

Theorem 1 *If $\Psi:\Omega\to\mathbb{R}$ is coordinated convex, then one has the inequalities*

$$\begin{aligned}\Psi\left(\frac{\alpha+\beta}{2},\frac{\gamma+\delta}{2}\right)&\le\frac{1}{2}\left[\frac{1}{\beta-\alpha}\int_\alpha^\beta\Psi\left(\varkappa,\frac{\gamma+\delta}{2}\right)d\varkappa+\frac{1}{\delta-\gamma}\int_\gamma^\delta\Psi\left(\frac{\alpha+\beta}{2},\eta\right)d\eta\right]\\&\le\frac{1}{(\beta-\alpha)(\delta-\gamma)}\int_\alpha^\beta\int_\gamma^\delta\Psi(\varkappa,\eta)\,d\eta\,d\varkappa\\&\le\frac{1}{4}\left[\frac{1}{\beta-\alpha}\int_\alpha^\beta\Psi(\varkappa,\gamma)\,d\varkappa+\frac{1}{\beta-\alpha}\int_\alpha^\beta\Psi(\varkappa,\delta)\,d\varkappa\right.\\&\quad\left.+\frac{1}{\delta-\gamma}\int_\gamma^\delta\Psi(\alpha,\eta)\,d\eta+\frac{1}{d-\gamma}\int_\gamma^\delta\Psi(\beta,\eta)\,d\eta\right]\\&\le\frac{\Psi(\alpha,\gamma)+\Psi(\alpha,\delta)+\Psi(\beta,\gamma)+\Psi(\beta,\delta)}{4}.\end{aligned}\tag{1.2}$$

The above inequalities are sharp. The inequalities in (1.2) hold in reverse direction if the mapping Ψ is a concave mapping on the coordinates.

For the some papers on Hermite–Hadamard type inequalities for coordinated convex functions, please refer to [15–20].

2 Some important definitions and theorems with regard to quantum calculus

In this section, we review some valuable definitions, notations and inequalities associated to quantum calculus.

Definition 2 ([6]) Suppose that $\Psi : [\alpha,\beta] \to \mathbb{R}$ is a continuous function. Then the q-derivative of Ψ at $\varkappa \in [\alpha,\beta]$ is characterized by the expression

$$_{\alpha}d_q\Psi(\varkappa) = \frac{\Psi(\varkappa) - \Psi(q\varkappa + (1-q)\alpha)}{(1-q)(\varkappa-\alpha)}, \quad \varkappa \neq \alpha. \tag{2.1}$$

Because $\Psi : [\alpha,\beta] \to \mathbb{R}$ is a continuous function, one has the equation $_{\alpha}d_q\Psi(\alpha) = \lim_{\varkappa\to\alpha} {}_{\alpha}d_q\Psi(\varkappa)$. The mapping Ψ is q-differentiable on $[\alpha,\beta]$, if $_{\alpha}d_q\Psi(t)$ exists for all $\varkappa \in [\alpha,\beta]$. If $\alpha = 0$ in (2.1), then the equation $_0d_q\Psi(\varkappa) = d_q\Psi(\varkappa)$ is valid. Here, $d_q\Psi(\varkappa)$ is the familiar q-derivative of Ψ at $\varkappa \in [\alpha,\beta]$ defined by the expression (see [5])

$$d_q\Psi(\varkappa) = \frac{\Psi(\varkappa) - \Psi(q\varkappa)}{(1-q)\varkappa}, \quad \varkappa \neq 0. \tag{2.2}$$

Definition 3 ([6]) Assume that $\Psi : [\alpha,\beta] \to \mathbb{R}$ is a continuous function. Then, for $x \in [\alpha,\beta]$, the q_α-definite integral on $[\alpha,\beta]$ is defined as

$$\int_{\alpha}^{x} \Psi(t)\, {}_{\alpha}d_q t = (1-q)(x-\alpha)\sum_{n=0}^{\infty} q^n \Psi\bigl(q^n x + \bigl(1-q^n\bigr)\alpha\bigr). \tag{2.3}$$

We should note that the notation of the quantum numbers (see [5]) which will be used many times in our main results is defined by

$$[\mu]_q = \frac{q^\mu - 1}{q-1} = 1 + q + \cdots + q^{\mu-1}.$$

Moreover, we need to give the following lemma in order to prove our main results readily.

Lemma 1 ([21]) *One has the identity*

$$\int_{\alpha}^{\beta} (\varkappa-\alpha)^{\mu}\, {}_{\alpha}d_q\varkappa = \frac{(\beta-\alpha)^{\mu+1}}{[\mu+1]_q}$$

for $\mu \in \mathbb{R}\backslash\{-1\}$.

In [7], Alp et al. established the following q_α-Hermite–Hadamard inequalities by using convex functions and quantum integral.

Theorem 2 *If* $\Psi : [\alpha,\beta] \to \mathbb{R}$ *be a convex differentiable function on* $[\alpha,\beta]$ *and* $0 < q < 1$. *Then we have the q-Hermite–Hadamard inequalities*

$$\Psi\left(\frac{q\alpha+\beta}{[2]_q}\right) \leq \frac{1}{\beta-\alpha}\int_{\alpha}^{\beta} \Psi(\varkappa)\, {}_{\alpha}d_q\varkappa \leq \frac{q\Psi(\alpha)+\Psi(\beta)}{[2]_q}. \tag{2.4}$$

On the other side, a new definition of quantum integrals and connected Hermite–Hadamard type inequalities are introduced by Bermudo et al.

Definition 4 ([8]) Suppose that $\Psi : [\alpha,\beta] \to \mathbb{R}$ is a continuous function. Then, for $\varkappa \in [\alpha,\beta]$, the q^β-definite integral on $[\alpha,\beta]$ is defined by

$$\int_{\varkappa}^{\beta} \Psi(t)\, {}^{\beta}d_q t = (1-q)(\beta-\varkappa)\sum_{n=0}^{\infty} q^n \Psi\bigl(q^n\varkappa + \bigl(1-q^n\bigr)\beta\bigr).$$

Theorem 3 ([8]) *If* $\Psi : [\alpha, \beta] \to \mathbb{R}$ *is a convex differentiable mapping on* $[\alpha, \beta]$ *and* $0 < q < 1$. *Then, one has the q-Hermite–Hadamard inequalities*

$$\Psi\left(\frac{\alpha + q\beta}{[2]_q}\right) \le \frac{1}{\beta - \alpha} \int_{\alpha}^{\beta} \Psi(\varkappa)\, {}^{\beta}d_q\varkappa \le \frac{\Psi(\alpha) + q\Psi(\beta)}{[2]_q}. \tag{2.5}$$

Now, we mention some definitions and inequalities related to our main results involving double quantum integrals.

$q_{\alpha\gamma}$-integral and partial q-derivatives for two variables functions are defined by Latif in [22].

Definition 5 Let $\Psi : \Omega \subset \mathbb{R}^2 \to \mathbb{R}$ be a continuous function. Then, for $(\varkappa, \eta) \in \Omega$, the definite $q_{\alpha\gamma}$-integral on Ω is defined by

$$\begin{aligned}\int_{\alpha}^{\varkappa} \int_{\gamma}^{\eta} \Psi(\zeta, \xi)\, {}_{\gamma}d_{q_2}\xi\, {}_{\alpha}d_{q_1}\zeta &= (1 - q_1)(1 - q_2)(\varkappa - \alpha)(\eta - \gamma) \\ &\quad \times \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} q_1^n q_2^m \Psi\bigl(q_1\varkappa + (1 - q_1)\alpha, q_2\eta + (1 - q_2)\gamma\bigr).\end{aligned}$$

Definition 6 ([22]) Assume that $\Psi : \Omega \subseteq \mathbb{R}^2 \to \mathbb{R}$ is a continuous function of two variables. Then the partial q_1-derivatives, q_2-derivatives and q_1q_2-derivatives at $(\varkappa, \eta) \in \Omega$ can be given as follows:

$$\begin{aligned}\frac{{}^{\beta}\partial_{q_1}\Psi(\varkappa, \eta)}{{}^{\beta}\partial_{q_1}\varkappa} &= \frac{\Psi(q_1\varkappa + (1 - q_1)\alpha, \eta) - \Psi(\varkappa, \eta)}{(1 - q_1)(\varkappa - \alpha)}, \quad \varkappa \ne \beta, \\ \frac{{}^{\delta}\partial_{q_1}\Psi(\varkappa, \eta)}{{}^{\beta}\partial_{q_2}\eta} &= \frac{\Psi(\varkappa, q_2\eta + (1 - q_2)\gamma) - \Psi(\varkappa, \eta)}{(1 - q_2)(\eta - \gamma)}, \quad \eta \ne \gamma, \\ \frac{{}_{\alpha,\gamma}\partial^2_{q_1,q_2}\Psi(\varkappa, \eta)}{{}_{\alpha}\partial_{q_1}\varkappa\, {}_{\gamma}\partial_{q_2}\eta} &= \frac{1}{(\varkappa - \alpha)(\eta - \gamma)(1 - q_1)(1 - q_2)}\bigl[\Psi\bigl(q_1\varkappa + (1 - q_1)\alpha, q_2\eta + (1 - q_2)\gamma\bigr) \\ &\quad - \Psi\bigl(q_1\varkappa + (1 - q_1)\alpha, \eta\bigr) - \Psi\bigl(\varkappa, q_2\eta + (1 - q_2)\gamma\bigr) + \Psi(\varkappa, \eta)\bigr], \\ &\quad \varkappa \ne \alpha, \eta \ne \gamma.\end{aligned}$$

For more details related to q-derivatives and integrals for the mappings of two variables, one can refer to [22].

In addition to all these definitions, definitions of q_{α}^{δ}, q_{γ}^{β} and $q^{\beta\delta}$ integrals and related inequalities of Hermite–Hadamard type are presented by Budak et al. in [23].

Definition 7 ([23]) Let $\Psi : \Omega \subset \mathbb{R}^2 \to \mathbb{R}$ be a continuous function. Then, for $(\varkappa, \eta) \in \Omega$, the q_{α}^{δ}, q_{γ}^{β} and $q^{\beta\delta}$ integrals on Ω are defined by

$$\begin{aligned}\int_{\alpha}^{\varkappa} \int_{\eta}^{\delta} \Psi(\zeta, \xi)\, {}^{\delta}d_{q_2}\xi\, {}_{\alpha}d_{q_1}\zeta &= (1 - q_1)(1 - q_2)(\varkappa - \alpha)(\delta - \eta) \\ &\quad \times \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} q_1^n q_2^m \Psi\bigl(q_1\varkappa + (1 - q_1)\alpha, q_2\eta + (1 - q_2)\delta\bigr),\end{aligned} \tag{2.6}$$

$$\int_{\varkappa}^{\beta}\int_{\gamma}^{\eta}\Psi(\zeta,\xi)\,{}_{\gamma}d_{q_2}\xi\,{}^{\beta}d_{q_1}\zeta=(1-q_1)(1-q_2)(\beta-\varkappa)(\eta-\gamma)\times\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}q_1^nq_2^m\Psi\big(q_1\varkappa+(1-q_1)\beta,q_2\eta+(1-q_2)\gamma\big),\tag{2.7}$$

and

$$\int_{\varkappa}^{\beta}\int_{\eta}^{\delta}\Psi(\zeta,\xi)\,{}^{\delta}d_{q_2}\xi\,{}^{\beta}d_{q_1}\zeta=(1-q_1)(1-q_2)(\beta-\varkappa)(\delta-\eta)\times\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}q_1^nq_2^m\Psi\big(q_1\varkappa+(1-q_1)\beta,q_2\eta+(1-q_2)\delta\big),\tag{2.8}$$

respectively.

Theorem 4 ([23]) *Let $\Psi:\Omega\subset\mathbb{R}^2\to\mathbb{R}$ be coordinated on Ω. Then we have the inequalities*

$$\begin{aligned}&\Psi\left(\frac{q_1\alpha+\beta}{[2]_{q_1}},\frac{\gamma+q_2\delta}{[2]_{q_2}}\right)\\&\le\frac{1}{2}\left[\frac{1}{\beta-\alpha}\int_{\alpha}^{\beta}\Psi\left(\varkappa,\frac{\gamma+q_2\delta}{[2]_{q_2}}\right){}_{\alpha}d_{q_1}\varkappa+\frac{1}{\delta-\gamma}\int_{\gamma}^{\delta}\Psi\left(\frac{q_1\alpha+\beta}{[2]_{q_1}},\eta\right){}^{\delta}d_{q_2}\eta\right]\\&\le\frac{1}{(\beta-\alpha)(\delta-\gamma)}\int_{\alpha}^{\beta}\int_{\gamma}^{\delta}\Psi(\varkappa,\eta)\,{}^{\delta}d_{q_2}\eta\,{}_{\alpha}d_{q_1}\varkappa\\&\le\frac{q_1}{2[2]_{q_1}(\delta-\gamma)}\int_{\gamma}^{\delta}\Psi(\alpha,\eta)\,{}^{\delta}d_{q_2}\eta+\frac{1}{2[2]_{q_1}(\delta-\gamma)}\int_{\gamma}^{\delta}\Psi(\beta,\eta)\,{}^{\delta}d_{q_2}\eta\\&\quad+\frac{1}{2[2]_{q_2}(\beta-\alpha)}\int_{\alpha}^{\beta}\Psi(\varkappa,\gamma)\,{}_{\alpha}d_{q_1}\varkappa+\frac{q_2}{2[2]_{q_2}(\beta-\alpha)}\int_{\alpha}^{\beta}\Psi(\varkappa,\delta)\,{}_{\alpha}d_{q_1}\varkappa\\&\le\frac{q_1\Psi(\alpha,\gamma)+q_1q_2\Psi(\alpha,\delta)+\Psi(\beta,\gamma)+q_2\Psi(\beta,\delta)}{[2]_{q_1}[2]_{q_2}}\end{aligned}\tag{2.9}$$

for all $q_1,q_2\in(0,1)$.

Budak et al. gave two similar inequalities in addition to the above result. Also, Latif introduced a quantum version of Hölder's inequality for double integrals in [22].

Theorem 5 (q_1q_2-Hölder's inequality for two variables functions, [22]) *Let $x,y>0$, $0<q_1$, $q_2<1$, $p_1>1$ such that $\frac{1}{p_1}+\frac{1}{r_1}=1$. Then*

$$\begin{aligned}&\int_0^{\varkappa}\int_0^{\eta}\big|\Psi(\varkappa,\eta)\Upsilon(\varkappa,\eta)\big|\,d_{q_1}\varkappa\,d_{q_2}\eta\\&\le\left(\int_0^{\varkappa}\int_0^{\eta}\big|\Psi(\varkappa,\eta)\big|^{p_1}d_{q_1}\varkappa\,d_{q_2}\eta\right)^{\frac{1}{p_1}}\left(\int_0^{\varkappa}\int_0^{\eta}\big|\Upsilon(\varkappa,\eta)\big|^{r_1}d_{q_1}\varkappa\,d_{q_2}\eta\right)^{\frac{1}{r_1}}.\end{aligned}$$

Inspired by these ongoing studies, we establish some new quantum analogues of midpoint type inequalities for q-differentiable coordinated convex functions. Integral inequalities form a crucial branch of analysis and were combined with various types of quantum integrals but we had never seen these before with the integrals that we use here. For this reason, we studied the midpoint type inequalities in quantum calculus.

3 q-Derivatives for the functions of two variables

In this section, we recall partial q-derivatives for mappings of two variables offered by Ali et al. in [24].

Definition 8 Suppose that $\Psi : \Omega \subseteq \mathbb{R}^2 \to \mathbb{R}$ is a continuous function of two variables. Then the partial q_1-derivative, q_2-derivative and q_1q_2-derivatives at $(\varkappa,\eta) \in \Omega$ are defined by

$$\frac{{}^{\beta}\partial_{q_1}\Psi(\varkappa,\eta)}{{}^{\beta}\partial_{q_1}\varkappa} = \frac{\Psi(q_1\varkappa + (1-q_1)\beta,\eta) - \Psi(\varkappa,\eta)}{(1-q_1)(\beta-\varkappa)}, \quad \varkappa \neq \beta,$$

$$\frac{{}^{\delta}\partial_{q_2}\Psi(\varkappa,\eta)}{{}^{\delta}\partial_{q_2}\eta} = \frac{\Psi(\varkappa,q_2\eta + (1-q_2)\delta) - \Psi(\varkappa,\eta)}{(1-q_2)(\delta-\eta)}, \quad \delta \neq \eta,$$

$$\begin{aligned}\frac{{}^{\delta}_{\alpha}\partial^2_{q_1,q_2}\Psi(\varkappa,\eta)}{{}_{\alpha}\partial_{q_1}\varkappa\,{}^{\delta}\partial_{q_2}\eta} &= \frac{1}{(\varkappa-\alpha)(\delta-\eta)(1-q_1)(1-q_2)}\big[\Psi\big(q_1\varkappa + (1-q_1)\alpha, q_2\eta + (1-q_2)\delta\big) \\ &\quad - \Psi\big(q_1\varkappa + (1-q_1)\alpha,\eta\big) - \Psi\big(\varkappa,q_2\eta + (1-q_2)\delta\big) + \Psi(\varkappa,\eta)\big], \\ &\quad \varkappa \neq \alpha, \eta \neq \delta,\end{aligned}$$

$$\begin{aligned}\frac{{}^{\beta}_{\gamma}\partial^2_{q_1,q_2}\Psi(\varkappa,\eta)}{{}^{\beta}\partial_{q_1}\varkappa_{\gamma}\partial_{q_2}\eta} &= \frac{1}{(\beta-\varkappa)(\eta-\gamma)(1-q_1)(1-q_2)}\big[\Psi\big(q_1\varkappa + (1-q_1)\beta, q_2\eta + (1-q_2)\gamma\big) \\ &\quad - \Psi\big(q_1\varkappa + (1-q_1)\beta,\eta\big) - \Psi\big(\varkappa,q_2\eta + (1-q_2)\gamma\big) + \Psi(\varkappa,\eta)\big], \\ &\quad \varkappa \neq \beta, \eta \neq \gamma,\end{aligned}$$

$$\begin{aligned}\frac{{}^{\beta,\delta}\partial^2_{q_1,q_2}\Psi(\varkappa,\eta)}{{}^{\beta}\partial_{q_1}\varkappa\,{}^{\delta}\partial_{q_2}\eta} &= \frac{1}{(\beta-\varkappa)(\delta-\eta)(1-q_1)(1-q_2)}\big[\Psi\big(q_1\varkappa + (1-q_1)\beta, q_2\eta + (1-q_2)\delta\big) \\ &\quad - \Psi\big(q_1\varkappa + (1-q_1)\beta,\eta\big) - \Psi\big(\varkappa,q_2y + (1-q_2)\delta\big) + \Psi(\varkappa,\eta)\big], \\ &\quad \varkappa \neq \beta, \eta \neq \delta,\end{aligned}$$

respectively.

4 Essential lemmas

In this section, we address three new identities, which are necessary to obtain our crucial results.

Let us start with the following lemma.

Lemma 2 *Let $F : \Delta \subseteq \mathbb{R}^2 \to \mathbb{R}$ be a twice partially q_1q_2-differentiable function on Δ°. If partial q_1q_2-derivative $\frac{{}^{b,d}\partial^2_{q_1,q_2}F(t,s)}{{}^{b}\partial_{q_1}t\,{}^{d}\partial_{q_2}s}$ is continuous and integrable on $[a,b] \times [c,d] \subseteq \Delta^\circ$. Then the following identity holds for q_1q_2-integrals*:

$$\begin{aligned}&q_1q_2(b-a)(d-c)\int_0^1\int_0^1 \Lambda(t,s)\frac{{}^{b,d}\partial^2_{q_1,q_2}F(ta+(1-t)b, sc+(1-s)d)}{{}^{b}\partial_{q_1}t\,{}^{d}\partial_{q_2}s}\,{}^{b}d_{q_1}t\,{}^{d}d_{q_2}s \qquad (4.1)\\ &= F\left(\frac{a+q_1b}{[2]_{q_1}}, \frac{c+q_2d}{[2]_{q_2}}\right) - \frac{1}{d-c}\int_c^d F\left(\frac{a+q_1b}{[2]_{q_1}}, y\right)\,{}^{d}d_{q_2}y\end{aligned}$$

$$
\begin{aligned}
&-\frac{1}{b-a}\int_a^b F\left(x,\frac{c+q_2d}{[2]_{q_2}}\right){}^b d_{q_1}x+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d F(x,y)\,{}^b d_{q_1}x\,{}^d d_{q_2}y\\
&={}^{b,d}I_{q_1,q_2}(a,b,c,d)(F),
\end{aligned}
$$

where

$$
\Lambda(t,s)=\begin{cases}
ts, & \text{if } (t,s)\in[0,\frac{1}{[2]_{q_1}}]\times[0,\frac{1}{[2]_{q_2}}],\\
t(s-\frac{1}{q_2}), & \text{if } (t,s)\in[0,\frac{1}{[2]_{q_1}}]\times(\frac{1}{[2]_{q_2}},1],\\
s(t-\frac{1}{q_1}), & \text{if } (t,s)\in(\frac{1}{[2]_{q_1}},1]\times[0,\frac{1}{[2]_{q_2}}],\\
(t-\frac{1}{q_1})(s-\frac{1}{q_2}), & \text{if } (t,s)\in(\frac{1}{[2]_{q_1}},1]\times(\frac{1}{[2]_{q_2}},1],
\end{cases}
$$

and $0<q_1,q_2<1$.

Proof From Definition 8, we have

$$
\begin{aligned}
&\frac{{}^{b,d}\partial^2_{q_1,q_2}F(ta+(1-t)b,sc+(1-s)d)}{{}^b\partial_{q_1}t\,{}^d\partial_{q_2}s}\\
&=\frac{1}{(1-q_1)(1-q_2)(b-a)(d-c)ts}\big[F\big(tq_1a+(1-tq_1)b,sq_2c+(1-sq_2)d\big)\\
&\quad-F\big(tq_1a+(1-tq_1)b,sc+(1-s)d\big)-F\big(ta+(1-t)b,sq_2c+(1-sq_2)d\big)\\
&\quad+F\big(ta+(1-t)b,sc+(1-s)d\big)\big].
\end{aligned}
$$

Also, it is easily observed that

$$
\begin{aligned}
&q_1q_2(b-a)(d-c)\int_0^1\int_0^1\Lambda(t,s)\frac{{}^{b,d}\partial^2_{q_1,q_2}F(ta+(1-t)b,sc+(1-s)d)}{{}^b\partial_{q_1}t\,{}^d\partial_{q_2}s}\,dq_2s\,d_{q_1}t \qquad (4.2)\\
&=q_1q_2(b-a)(d-c)\Bigg[\int_0^{\frac{1}{[2]_{q_1}}}\int_0^{\frac{1}{[2]_{q_2}}}ts\frac{{}^{b,d}\partial^2_{q_1,q_2}F(ta+(1-t)b,sc+(1-s)d)}{{}^b\partial_{q_1}t\,{}^d\partial_{q_2}s}\,d_{q_2}s\,d_{q_1}t\\
&\quad+\int_0^{\frac{1}{[2]_{q_1}}}\int_{\frac{1}{[2]_{q_2}}}^1 t\left(s-\frac{1}{q_2}\right)\frac{{}^{b,d}\partial^2_{q_1,q_2}F(ta+(1-t)b,sc+(1-s)d)}{{}^b\partial_{q_1}t\,{}^d\partial_{q_2}s}\,d_{q_2}s\,d_{q_1}t\\
&\quad+\int_{\frac{1}{[2]_{q_1}}}^1\int_0^{\frac{1}{[2]_{q_2}}} s\left(t-\frac{1}{q_1}\right)\frac{{}^{b,d}\partial^2_{q_1,q_2}F(ta+(1-t)b,sc+(1-s)d)}{{}^b\partial_{q_1}t\,{}^d\partial_{q_2}s}\,d_{q_2}s\,d_{q_1}t\\
&\quad+\int_{\frac{1}{[2]_{q_1}}}^1\int_{\frac{1}{[2]_{q_2}}}^1\left(t-\frac{1}{q_1}\right)\left(s-\frac{1}{q_2}\right)\frac{{}^{b,d}\partial^2_{q_1,q_2}F(ta+(1-t)b,sc+(1-s)d)}{{}^b\partial_{q_1}t\,{}^d\partial_{q_2}s}\,d_{q_2}s\,d_{q_1}t\Bigg]\\
&=q_1q_2(b-a)(d-c)\int_0^1\int_0^1 ts\frac{{}^{b,d}\partial^2_{q_1,q_2}F(ta+(1-t)b,sc+(1-s)d)}{{}^b\partial_{q_1}t\,{}^d\partial_{q_2}s}\,d_{q_2}s\,d_{q_1}t\\
&\quad-q_2(b-a)(d-c)\int_0^1\int_0^1 s\frac{{}^{b,d}\partial^2_{q_1,q_2}F(ta+(1-t)b,sc+(1-s)d)}{{}^b\partial_{q_1}t\,{}^d\partial_{q_2}s}\,d_{q_2}s\,d_{q_1}t\\
&\quad-q_1(b-a)(d-c)\int_0^1\int_0^1 t\frac{{}^{b,d}\partial^2_{q_1,q_2}F(ta+(1-t)b,sc+(1-s)d)}{{}^b\partial_{q_1}t\,{}^d\partial_{q_2}s}\,d_{q_2}s\,d_{q_1}t\\
&\quad+q_2(b-a)(d-c)\int_0^{\frac{1}{[2]_{q_1}}}\int_0^1 s\frac{{}^{b,d}\partial^2_{q_1,q_2}F(ta+(1-t)b,sc+(1-s)d)}{{}^b\partial_{q_1}t\,{}^d\partial_{q_2}s}\,d_{q_2}s\,d_{q_1}t
\end{aligned}
$$

$$
\begin{aligned}
&+ q_1(b-a)(d-c)\int_0^1\int_0^{\frac{1}{[2]_{q_2}}} t\frac{{}^{b,d}\partial^2_{q_1,q_2}F(ta+(1-t)b,sc+(1-s)d)}{{}^{b}\partial_{q_1}t\,{}^{d}\partial_{q_2}s}\,d_{q_2}s\,d_{q_1}t \\
&+ (b-a)(d-c)\int_0^1\int_0^1 \frac{{}^{b,d}\partial^2_{q_1,q_2}F(ta+(1-t)b,sc+(1-s)d)}{{}^{b}\partial_{q_1}t\,{}^{d}\partial_{q_2}s}\,d_{q_2}s\,d_{q_1}t \\
&- (b-a)(d-c)\int_0^{\frac{1}{[2]_{q_1}}}\int_0^1 \frac{{}^{b,d}\partial^2_{q_1,q_2}F(ta+(1-t)b,sc+(1-s)d)}{{}^{b}\partial_{q_1}t\,{}^{d}\partial_{q_2}s}\,d_{q_2}s\,d_{q_1}t \\
&- (b-a)(d-c)\int_0^1\int_0^{\frac{1}{[2]_{q_2}}} \frac{{}^{b,d}\partial^2_{q_1,q_2}F(ta+(1-t)b,sc+(1-s)d)}{{}^{b}\partial_{q_1}t\,{}^{d}\partial_{q_2}s}\,d_{q_2}s\,d_{q_1}t \\
&+ (b-a)(d-c)\int_0^{\frac{1}{[2]_{q_1}}}\int_0^{\frac{1}{[2]_{q_2}}} \frac{{}^{b,d}\partial^2_{q_1,q_2}F(ta+(1-t)b,sc+(1-s)d)}{{}^{b}\partial_{q_1}t\,{}^{d}\partial_{q_2}s}\,d_{q_2}s\,d_{q_1}t \\
&= I_1 - I_2 - I_3 + I_4 + I_5 + I_6 - I_7 - I_8 + I_9.
\end{aligned}
$$

Now by the definition of definite q_1q_2-integrals and properties of q_1q_2-integrals, we obtain

$$
\begin{aligned}
I_1 &= q_1q_2(b-a)(d-c)\int_0^1\int_0^1 \bigl[F\bigl(tq_1a+(1-tq_1)b,sq_2c+(1-sq_2)d\bigr) \\
&\quad - F\bigl(tq_1a+(1-tq_1)b,sc+(1-s)d\bigr) - F\bigl(ta+(1-t)b,sq_2c+(1-sq_2)d\bigr) \\
&\quad + F\bigl(ta+(1-t)b,sc+(1-s)d\bigr)\bigr]\,d_{q_1}t\,d_{q_2}s \\
&= q_1q_2\Biggl[\sum_{n=0}^\infty\sum_{m=0}^\infty \frac{q_1^{n+1}q_2^{m+1}}{q_1q_2}F\bigl(q_1^{n+1}a+\bigl(1-q_1^{n+1}\bigr)b,q_2^{m+1}c+\bigl(1-q_2^{m+1}\bigr)d\bigr) \\
&\quad - \sum_{n=0}^\infty\sum_{m=0}^\infty \frac{q_1^{n+1}q_2^{m}}{q_1}F\bigl(q_1^{n+1}a+\bigl(1-q_1^{n+1}\bigr)b,q_2^{m}c+\bigl(1-q_2^{m}\bigr)d\bigr) \\
&\quad - \sum_{n=0}^\infty\sum_{m=0}^\infty \frac{q_1^{n}q_2^{m+1}}{q_2}F\bigl(q_1^{n}a+\bigl(1-q_1^{n}\bigr)b,q_2^{m+1}c+\bigl(1-q_2^{m+1}\bigr)d\bigr) \\
&\quad + \sum_{n=0}^\infty\sum_{m=0}^\infty q_1^nq_2^mF\bigl(q_1^{n}a+\bigl(1-q_1^{n}\bigr)b,q_2^{m}c+\bigl(1-q_2^{m}\bigr)d\bigr)\Biggr] \\
&= q_1q_2\Biggl[\frac{1}{q_1q_2}\sum_{n=0}^\infty\sum_{m=0}^\infty q_1^nq_2^mF\bigl(q_1^{n}a+\bigl(1-q_1^{n}\bigr)b,q_2^{m}c+\bigl(1-q_2^{m}\bigr)d\bigr) \\
&\quad - \frac{1}{q_1q_2}\sum_{m=0}^\infty q_2^mF\bigl(a,q_2^mc+\bigl(1-q_2^m\bigr)d\bigr) - \frac{1}{q_1q_2}\sum_{n=0}^\infty q_1^nF\bigl(q_1^na+\bigl(1-q_1^n\bigr)b,c\bigr) \\
&\quad + \frac{1}{q_1q_2}F(a,c) - \frac{1}{q_1}\sum_{n=0}^\infty\sum_{m=0}^\infty q_1^nq_2^mF\bigl(q_1^{n}a+\bigl(1-q_1^{n}\bigr)b,q_2^{m}c+\bigl(1-q_2^{m}\bigr)d\bigr) \\
&\quad + \frac{1}{q_1}\sum_{m=0}^\infty q_2^mF\bigl(a,q_2^mc+\bigl(1-q_2^m\bigr)d\bigr) \\
&\quad - \frac{1}{q_2}\sum_{n=0}^\infty\sum_{m=0}^\infty q_1^nq_2^mF\bigl(q_1^{n}a+\bigl(1-q_1^{n}\bigr)b,q_2^{m}c+\bigl(1-q_2^{m}\bigr)d\bigr) \\
&\quad + \frac{1}{q_2}\sum_{n=0}^\infty q_1^nF\bigl(q_1^na+\bigl(1-q_1^n\bigr)b,c\bigr)
\end{aligned}
$$

$$
\begin{aligned}
&\quad + \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} q_1^n q_2^m F\left(q_1^n a + \left(1-q_1^n\right)b, q_2^m c + \left(1-q_2^m\right)d\right)\Bigg] \\
&= (1-q_1)(1-q_2)\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} q_1^n q_2^m F\left(q_1^n a + \left(1-q_1^n\right)b, q_2^m c + \left(1-q_2^m\right)d\right) \\
&\quad - (1-q_2)\sum_{m=0}^{\infty} q_2^m F\left(a, q_2^m c + \left(1-q_2^m\right)d\right) \\
&\quad - (1-q_2)\sum_{n=0}^{\infty} q_1^n F\left(q_1^n a + \left(1-q_1^n\right)b, c\right) - F(a,c) \\
&= \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d F(x,y)\,{}^d d_{q_2}y\,{}^b d_{q_1}x - \frac{1}{d-c}\int_c^d F(a,y)\,{}^d d_{q_2}y \\
&\quad - \frac{1}{b-a}\int_a^b F(x,c)\,{}^b d_{q_1}x + F(a,c).
\end{aligned}
$$

By using the similar operations, one can obtain

$$
\begin{aligned}
I_2 &= -F(b,c) + F(a,c) + \frac{1}{d-c}\int_c^d F(b,y)\,{}^d d_{q_2}y - \frac{1}{d-c}\int_c^d F(a,y)\,{}^d d_{q_2}y, \\
I_3 &= -F(a,d) + F(a,c) + \frac{1}{b-a}\int_a^b F(x,d)\,{}^b d_{q_1}x - \frac{1}{b-a}\int_a^b F(x,c)\,{}^b d_{q_1}x, \\
I_4 &= F\left(\frac{a+q_1 b}{[2]_{q_1}}, c\right) - F(b,c) + \frac{1}{d-c}\int_c^d F(b,y)\,{}^d d_{q_2}y - \frac{1}{d-c}\int_c^d F\left(\frac{a+q_1 b}{[2]_{q_1}}, y\right){}^d d_{q_2}y, \\
I_5 &= F\left(a, \frac{c+q_2 d}{[2]_{q_2}}\right) - F(a,d) + \frac{1}{b-a}\int_a^b F(x,d)\,d_{q_1}x - \frac{1}{b-a}\int_a^b F\left(x, \frac{c+q_2 d}{[2]_{q_2}}\right) d_{q_1}x, \\
I_6 &= F(b,d) - F(a,d) - F(b,c) + F(a,c), \\
I_7 &= F(b,d) - F\left(\frac{a+q_1 b}{[2]_{q_1}}, d\right) - F(b,c) + F\left(\frac{a+q_1 b}{[2]_{q_1}}, c\right), \\
I_8 &= F(b,d) - F(a,d) - F\left(b, \frac{c+q_2 d}{[2]_{q_2}}\right) + F\left(a, \frac{c+q_2 d}{[2]_{q_2}}\right), \\
I_9 &= F(b,d) - F\left(\frac{a+q_1 b}{[2]_{q_1}}, d\right) - F\left(b, \frac{c+q_2 d}{[2]_{q_2}}\right) + F\left(\frac{a+q_1 b}{[2]_{q_1}}, \frac{c+q_2 d}{[2]_{q_2}}\right).
\end{aligned}
$$

Using the calculated integrals (I_1)–(I_9) in (4.2), then we obtain the desired identity (4.1) which ends the proof. □

Remark 1 Under the given conditions of Lemma 2 with $q_1, q_2 \to 1^-$, then we have the following identity:

$$
\begin{aligned}
&(b-a)(d-c)\int_0^1\int_0^1 \Psi(t,s)\frac{\partial^2 F}{\partial t\partial s}\big(ta+(1-t)b, sc+(1-s)d\big)\,ds\,dt \qquad (4.3) \\
&= F\left(\frac{a+b}{2}, \frac{c+d}{2}\right) + \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d F(x,y)\,dy\,dx \\
&\quad - \left[\frac{1}{b-a}\int_a^b F\left(x, \frac{c+d}{2}\right)dx + \frac{1}{d-c}\int_c^d F\left(\frac{a+b}{2}, y\right)dy\right],
\end{aligned}
$$

where

$$\Psi(t,s)=\begin{cases} ts, & \text{if } (t,s)\in[0,\frac{1}{2}]\times[0,\frac{1}{2}],\\ t(s-1), & \text{if } (t,s)\in[0,\frac{1}{2}]\times(\frac{1}{2},1],\\ s(t-1), & \text{if } (t,s)\in(\frac{1}{2},1]\times[0,\frac{1}{2}],\\ (t-1)(s-1), & \text{if } (t,s)\in(\frac{1}{2},1]\times(\frac{1}{2},1],\end{cases}$$

which is proved by Latif and Dragomir in [25, Lemma 1].

Lemma 3 *Let $F:\Delta\subseteq\mathbb{R}^2\to\mathbb{R}$ be a twice partially q_1q_2-differentiable function on Δ°. If the partial q_1q_2-derivative $\frac{{}_a^d\partial^2_{q_1,q_2}F(t,s)}{{}_a\partial_{q_1}t\,{}^d\partial_{q_2}s}$ is continuous and integrable on $[a,b]\times[c,d]\subseteq\Delta^\circ$, then the following identity holds for q_1q_2-integrals:*

$$\begin{aligned} &q_1q_2(b-a)(d-c)\int_0^1\int_0^1\Lambda(t,s)\frac{{}_a^d\partial^2_{q_1,q_2}F(tb+(1-t)a,sc+(1-s)d)}{{}_a\partial_{q_1}t\,{}^d\partial_{q_2}s}\,d_{q_1}t\,d_{q_2}s \qquad (4.4)\\ &=F\left(\frac{q_1a+b}{[2]_{q_1}},\frac{c+q_2d}{[2]_{q_2}}\right)-\frac{1}{d-c}\int_c^d F\left(\frac{q_1a+b}{[2]_{q_1}},y\right){}^dd_{q_2}y\\ &\quad-\frac{1}{b-a}\int_a^b F\left(x,\frac{c+q_2d}{[2]_{q_2}}\right){}_ad_{q_1}x+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d F(x,y)\,{}_ad_{q_1}x\,{}^dd_{q_2}y\\ &={}_a^dI_{q_1,q_2}(a,b,c,d)(F), \end{aligned}$$

where $0<q_1,q_2<1$ and Λ is defined as in Lemma 2.

Proof If the strategy which was used in the proof of Lemma 2 are applied by taking into account the definition of $\frac{{}_a^d\partial^2_{q_1,q_2}F(t,s)}{{}_a\partial_{q_1}t\,{}^d\partial_{q_2}s}$, the desired inequality (4.4) can be obtained. □

Remark 2 If we choose $q_1,q_2\to1^-$ and replace $tb+(1-t)a$ with $ta+(1-t)a$ in Lemma 3, then identity (4.4) reduces to identity (4.3).

Lemma 4 *Let $F:\Delta\subseteq\mathbb{R}^2\to\mathbb{R}$ be a twice partially q_1q_2-differentiable function on Δ°. If the partial q_1q_2-derivative $\frac{{}_c^b\partial^2_{q_1,q_2}F(t,s)}{{}^b\partial_{q_1}t\,{}_c\partial_{q_2}s}$ is continuous and integrable on $[a,b]\times[c,d]\subseteq\Delta^\circ$, then the following identity holds for q_1q_2-integrals:*

$$\begin{aligned} &q_1q_2(b-a)(d-c)\int_0^1\int_0^1\Lambda(t,s)\frac{{}_c^b\partial^2_{q_1,q_2}F(ta+(1-t)b,sd+(1-s)c)}{{}^b\partial_{q_1}t\,{}_c\partial_{q_2}s}\,d_{q_1}t\,d_{q_2}s \qquad (4.5)\\ &=F\left(\frac{a+q_1b}{[2]_{q_1}},\frac{q_2c+d}{[2]_{q_2}}\right)-\frac{1}{d-c}\int_c^d F\left(\frac{a+q_1b}{[2]_{q_1}},y\right){}_cd_{q_2}y\\ &\quad-\frac{1}{b-a}\int_a^b F\left(x,\frac{q_2c+d}{[2]_{q_2}}\right){}^bd_{q_1}x+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d F(x,y)\,{}^bd_{q_1}x\,{}_cd_{q_2}y\\ &={}_c^bI_{q_1,q_2}(a,b,c,d)(F), \end{aligned}$$

where $0<q_1,q_2<1$ and Λ is defined as in Lemma 2.

Proof If the strategy which was used in the proof of Lemma 2 is applied by taking into account the definition of $\frac{{}_c^b\partial^2_{q_1,q_2}F(t,s)}{{}^b\partial_{q_1}t\,{}_c\partial_{q_2}s}$, the desired inequality (4.5) can be obtained. □

Remark 3 If we choose $q_1, q_2 \to 1^-$ and replace $sd + (1-s)c$ with $sc + (1-s)d$ in Lemma 4, then identity (4.5) reduces to identity (4.3).

5 Some new q_1q_2-Hermite–Hadamard like inequalities

For brevity, we give some calculated integrals before giving new estimates:

$$\begin{aligned}
\Upsilon(q_1,q_2) &= \int_0^1 \int_0^1 \Lambda(t,s)\, d_{q_1}t\, d_{q_2}s \\
&= \int_0^{\frac{1}{[2]_{q_1}}} \int_0^{\frac{1}{[2]_{q_2}}} ts\, d_{q_1}t\, d_{q_2}s + \int_0^{\frac{1}{[2]_{q_1}}} \int_{\frac{1}{[2]_{q_2}}}^1 t\left(\frac{1}{q_2} - s\right) d_{q_1}t\, d_{q_2}s \\
&\quad + \int_{\frac{1}{[2]_{q_1}}}^1 \int_0^{\frac{1}{[2]_{q_2}}} s\left(\frac{1}{q_1} - t\right) d_{q_1}t\, d_{q_2}s \\
&\quad + \int_{\frac{1}{[2]_{q_1}}}^1 \int_{\frac{1}{[2]_{q_2}}}^1 \left(\frac{1}{q_1} - t\right)\left(\frac{1}{q_2} - s\right) d_{q_1}t\, d_{q_2}s \\
&= \frac{4 - 2[2]_{q_2}^2 - 2[2]_{q_1}^2 + [2]_{q_1}^2[2]_{q_2}^2}{[2]_{q_1}^3[2]_{q_2}^3} \\
&\quad + \frac{2q_1q_2([2]_{q_2}^2 + [2]_{q_1}^2) - 2q_1q_2[2]_{q_1}^2[2]_{q_2}^2 + q_1[2]_{q_2}^2[2]_{q_1}^2([2]_{q_2} - 1)}{q_1q_2[2]_{q_1}^3[2]_{q_2}^3},
\end{aligned} \tag{5.1}$$

$$\begin{aligned}
A_1(q_1,q_2) &= \int_0^{\frac{1}{[2]_{q_1}}} \int_0^{\frac{1}{[2]_{q_2}}} t^2s^2\, d_{q_1}t\, d_{q_2}s \\
&= \frac{1}{[2]_{q_1}^3[2]_{q_2}^3[3]_{q_1}[3]_{q_2}},
\end{aligned} \tag{5.2}$$

$$\begin{aligned}
A_2(q_1,q_2) &= \int_0^{\frac{1}{[2]_{q_1}}} \int_{\frac{1}{[2]_{q_2}}}^1 t^2s\left(\frac{1}{q_2} - s\right) d_{q_1}t\, d_{q_2}s \\
&= \frac{q_2 + 2}{[2]_{q_1}^3[2]_{q_2}^3[3]_{q_1}} + \frac{1 - [2]_{q_2}^3}{[2]_{q_1}^3[2]_{q_2}^3[3]_{q_1}[3]_{q_2}},
\end{aligned} \tag{5.3}$$

$$\begin{aligned}
A_3(q_1,q_2) &= \int_{\frac{1}{[2]_{q_1}}}^1 \int_0^{\frac{1}{[2]_{q_2}}} ts^2\left(\frac{1}{q_1} - t\right) d_{q_1}t\, d_{q_2}s \\
&= \frac{q_1 + 2}{[2]_{q_1}^3[2]_{q_2}^3[3]_{q_2}} + \frac{1 - [2]_{q_1}^3}{[2]_{q_1}^3[2]_{q_2}^3[3]_{q_1}[3]_{q_2}},
\end{aligned} \tag{5.4}$$

$$\begin{aligned}
&\int_{\frac{1}{[2]_{q_1}}}^1 \int_{\frac{1}{[2]_{q_2}}}^1 ts\left(\frac{1}{q_1} - t\right)\left(\frac{1}{q_2} - s\right) d_{q_1}t\, d_{q_2}s \\
&\quad = A_4(q_1,q_2) \\
&\quad = \frac{([2]_{q_1}^2 - 1)([2]_{q_2}^2 - 1)}{q_1q_2[2]_{q_1}^3[2]_{q_2}^3} + \frac{(1 - [2]_{q_1}^2)([2]_{q_2}^3 - 1)}{q_1[2]_{q_1}^3[2]_{q_2}^3[3]_{q_2}} \\
&\qquad + \frac{(1 - [2]_{q_1}^3)([2]_{q_2}^2 - 1)}{q_2[2]_{q_1}^3[2]_{q_2}^3[3]_{q_1}} + \frac{([2]_{q_1}^3 - 1)([2]_{q_2}^3 - 1)}{[2]_{q_1}^3[2]_{q_2}^3[3]_{q_1}[3]_{q_2}},
\end{aligned} \tag{5.5}$$

$$B_1(q_1,q_2) = \int_0^{\frac{1}{[2]_{q_1}}} \int_0^{\frac{1}{[2]_{q_2}}} t(1-t)s^2\, d_{q_1}t\, d_{q_2}s \tag{5.6}$$

$$= \frac{q_1}{[2]_{q_1}^2[3]_{q_1}[2]_{q_2}^3[3]_{q_2}},$$

$$B_2(q_1,q_2) = \int_0^{\frac{1}{[2]_{q_1}}}\int_{\frac{1}{[2]_{q_2}}}^1 t(1-t)s\left(\frac{1}{q_2}-s\right)d_{q_1}t\,d_{q_2}s \tag{5.7}$$

$$= \frac{q_1(q_2+2)}{[2]_{q_1}^2[2]_{q_2}^3[3]_{q_1}} + \frac{q_1(1-[2]_{q_2}^3)}{[2]_{q_1}^2[2]_{q_2}^3[3]_{q_1}[3]_{q_2}},$$

$$B_3(q_1,q_2) = \int_{\frac{1}{[2]_{q_1}}}^1\int_0^{\frac{1}{[2]_{q_2}}} (1-t)s^2\left(\frac{1}{q_1}-t\right)d_{q_1}t\,d_{q_2}s \tag{5.8}$$

$$= \frac{q_1^2+q_1-1}{[2]_{q_1}^3[2]_{q_2}^3[3]_{q_2}} - \frac{q_1^2+q_1-1}{[2]_{q_1}^2[2]_{q_2}^3[3]_{q_1}[3]_{q_2}},$$

$$B_4(q_1,q_2) = \int_{\frac{1}{[2]_{q_1}}}^1\int_{\frac{1}{[2]_{q_2}}}^1 s(1-t)\left(\frac{1}{q_1}-t\right)\left(\frac{1}{q_2}-s\right)d_{q_1}t\,d_{q_2}s \tag{5.9}$$

$$= \frac{([2]_{q_2}^2-1)([2]_{q_2}^2-(q_1+2))}{q_2[2]_{q_1}^3[2]_{q_2}^3}$$

$$+ \frac{([2]_{q_2}^3-1)((q_1+2)-[2]_{q_1}^2)}{[2]_{q_1}^3[2]_{q_2}^3[3]_{q_2}}$$

$$+ \frac{q_1([2]_{q_2}^3-1)(q_1[2]_{q_1}-1)}{[2]_{q_1}^2[2]_{q_2}^3[3]_{q_1}[3]_{q_2}} + \frac{q_1([2]_{q_2}^2-1)(1-q_1[2]_{q_1})}{[2]_{q_1}^2[2]_{q_2}^3[3]_{q_1}},$$

$$C_1(q_1,q_2) = \int_0^{\frac{1}{[2]_{q_1}}}\int_0^{\frac{1}{[2]_{q_2}}} t^2s(1-s)\,d_{q_1}t\,d_{q_2}s \tag{5.10}$$

$$= \frac{q_2}{[2]_{q_2}^2[3]_{q_2}[2]_{q_1}^3[3]_{q_1}},$$

$$C_2(q_1,q_2) = \int_0^{\frac{1}{[2]_{q_1}}}\int_{\frac{1}{[2]_{q_2}}}^1 t^2(1-s)\left(\frac{1}{q_2}-s\right)d_{q_1}t\,d_{q_2}s \tag{5.11}$$

$$= \frac{q_2(q_1+2)}{[2]_{q_2}^2[2]_{q_1}^3[3]_{q_2}} + \frac{q_2(1-[2]_{q_1}^3)}{[2]_{q_2}^2[2]_{q_1}^3[3]_{q_1}[3]_{q_2}},$$

$$C_3(q_1,q_2) = \int_{\frac{1}{[2]_{q_1}}}^1\int_0^{\frac{1}{[2]_{q_2}}} ts(1-s)\left(\frac{1}{q_1}-t\right)d_{q_1}t\,d_{q_2}s \tag{5.12}$$

$$= \frac{q_2^2+q_2-1}{[2]_{q_2}^3[2]_{q_1}^3[3]_{q_1}} - \frac{q_2^2+q_2-1}{[2]_{q_2}^2[2]_{q_1}^3[3]_{q_1}[3]_{q_2}},$$

$$C_4(q_1,q_2) = \int_{\frac{1}{[2]_{q_1}}}^1\int_{\frac{1}{[2]_{q_2}}}^1 t(1-s)\left(\frac{1}{q_1}-t\right)\left(\frac{1}{q_2}-s\right)d_{q_1}t\,d_{q_2}s \tag{5.13}$$

$$= \frac{([2]_{q_1}^2-1)([2]_{q_1}^2-(q_2+2))}{q_1[2]_{q_2}^3[2]_{q_1}^3}$$

$$+ \frac{([2]_{q_1}^3-1)((q_2+2)-[2]_{q_2}^2)}{[2]_{q_1}^3[2]_{q_2}^3[3]_{q_1}}$$

$$+ \frac{q_2([2]_{q_1}^3-1)(q_2[2]_{q_2}-1)}{[2]_{q_2}^2[2]_{q_1}^3[3]_{q_2}[3]_{q_1}}$$

$$+\frac{q_2([2]_{q_1}^2-1)(1-q_2[2]_{q_2})}{[2]_{q_2}^2[2]_{q_1}^3[3]_{q_2}},$$

$$E_1(q_1,q_2)=\int_0^{\frac{1}{[2]_{q_1}}}\int_0^{\frac{1}{[2]_{q_2}}} t(1-t)s(1-s)\,d_{q_1}t\,d_{q_2}s \tag{5.14}$$
$$=\frac{q_1q_2}{[2]_{q_1}^2[3]_{q_1}[2]_{q_2}^2[3]_{q_2}},$$

$$E_2(q_1,q_2)=\int_0^{\frac{1}{[2]_{q_1}}}\int_{\frac{1}{[2]_{q_2}}}^1 t(1-t)(1-s)\left(\frac{1}{q_2}-s\right)d_{q_1}t\,d_{q_2}s \tag{5.15}$$
$$=q_1\frac{[2]_{q_2}^2-(q_2+2)}{[2]_{q_1}^2[2]_{q_2}^3[3]_{q_1}}+q_1q_2\frac{1-q_2[2]_{q_2}}{[2]_{q_1}^2[2]_{q_2}^2[3]_{q_1}[3]_{q_2}},$$

$$E_3(q_1,q_2)=\int_{\frac{1}{[2]_{q_1}}}^1\int_0^{\frac{1}{[2]_{q_2}}}(1-t)s(1-s)\left(\frac{1}{q_1}-t\right)d_{q_1}t\,d_{q_2}s \tag{5.16}$$
$$=q_2\frac{[2]_{q_1}^2-(q_1+2)}{[2]_{q_1}^3[2]_{q_2}^2[3]_{q_2}}+q_1q_2\frac{1-q_1[2]_{q_1}}{[2]_{q_1}^2[2]_{q_2}^2[3]_{q_1}[3]_{q_2}},$$

$$E_4(q_1,q_2)=\int_{\frac{1}{[2]_{q_1}}}^1\int_{\frac{1}{[2]_{q_2}}}^1(1-t)(1-s)\left(\frac{1}{q_1}-t\right)\left(\frac{1}{q_2}-s\right)d_{q_1}t\,d_{q_2}s \tag{5.17}$$
$$=\frac{([2]_{q_2}^2-(q_2+2))([2]_{q_1}^2-(q_1+2))}{[2]_{q_1}^3[2]_{q_2}^3}$$
$$+\frac{q_2((q_1+2)-[2]_{q_1}^2)(q_2[2]_{q_2}-1)}{[2]_{q_1}^3[2]_{q_2}^2[3]_{q_2}}$$
$$+\frac{q_1(q_1[2]_{q_1}-1)((q_2+2)-[2]_{q_2}^2)}{[2]_{q_1}^2[2]_{q_2}^3[3]_{q_1}}$$
$$+\frac{q_1q_2(q_1[2]_{q_1}-1)(q_2[2]_{q_2}-1)}{[2]_{q_1}^2[2]_{q_2}^2[3]_{q_1}[3]_{q_2}}.$$

Now we give some new quantum estimates by using the identities given in last section.

Let us start to find some new quantum estimates by using Lemma 2. We first examine a new result for functions whose partially q_1q_2-derivatives in modulus are convex in the following theorem.

Theorem 6 *Let $F:\Delta\subseteq\mathbb{R}^2\to\mathbb{R}$ be a twice partially q_1q_2-differentiable function on Δ° such that the partial q_1q_2-derivative $\frac{{}^{b,d}\partial_{q_1,q_2}^2F(t,s)}{{}^{b}\partial_{q_1}t\,{}^{d}\partial_{q_2}s}$ is continuous and integrable on $[a,b]\times[c,d]\subseteq\Delta^\circ$. Then we have the following inequality provided that $|\frac{{}^{b,d}\partial_{q_1,q_2}^2F(t,s)}{{}^{b}\partial_{q_1}t\,{}^{d}\partial_{q_2}s}|$ is convex on $[a,b]\times[c,d]$:*

$$\left|{}^{b,d}I_{q_1,q_2}(a,b,c,d)(F)\right| \tag{5.18}$$
$$\le q_1q_2(b-a)(d-c)\left[A\left|\frac{{}^{b,d}\partial_{q_1,q_2}^2F(a,c)}{{}^{b}\partial_{q_1}t\,{}^{d}\partial_{q_2}s}\right|+B\left|\frac{{}^{b,d}\partial_{q_1,q_2}^2F(b,c)}{{}^{b}\partial_{q_1}t\,{}^{d}\partial_{q_2}s}\right|\right.$$
$$\left.+C\left|\frac{{}^{b,d}\partial_{q_1,q_2}^2F(a,d)}{{}^{b}\partial_{q_1}t\,{}^{d}\partial_{q_2}s}\right|+E\left|\frac{{}^{b,d}\partial_{q_1,q_2}^2F(b,d)}{{}^{b}\partial_{q_1}t\,{}^{d}\partial_{q_2}s}\right|\right],$$

where

$$A=\sum_{i=1}^{4}A_i(q_1,q_2),\qquad B=\sum_{i=1}^{4}B_i(q_1,q_2),$$
$$C=\sum_{i=1}^{4}C_i(q_1,q_2),\qquad E=\sum_{i=1}^{4}E_i(q_1,q_2),$$

and $0<q_1,q_2<1$.

Proof On taking the modulus of the identity of Lemma (4.1), because of the properties of the modulus, we find that

$$\begin{aligned}&\left|{}^{b,d}I_{q_1,q_2}(a,b,c,d)(F)\right| \\ &\quad\le q_1q_2(b-a)(d-c) \\ &\qquad\times\int_0^1\int_0^1\left|\Lambda(t,s)\right|\left|\frac{{}^{b,d}\partial^2_{q_1,q_2}F(ta+(1-t)b,sc+(1-s)d)}{{}^{b}\partial_{q_1}t\,{}^{d}\partial_{q_2}s}\right|{}^{b}d_{q_1}t\,{}^{d}d_{q_2}s.\end{aligned}\tag{5.19}$$

Now using the convexity of $|\frac{{}^{b,d}\partial^2_{q_1,q_2}F(t,s)}{{}^{b}\partial_{q_1}t\,{}^{d}\partial_{q_2}s}|$, then (5.19) becomes

$$\begin{aligned}&\left|{}^{b,d}I_{q_1,q_2}(a,b,c,d)(F)\right| \\ &\quad\le q_1q_2(b-a)(d-c) \\ &\qquad\times\int_0^1\int_0^1\left|\Lambda(t,s)\right|\left[ts\left|\frac{{}^{b,d}\partial^2_{q_1,q_2}F(a,c)}{{}^{b}\partial_{q_1}t\,{}^{d}\partial_{q_2}s}\right|+(1-t)s\left|\frac{{}^{b,d}\partial^2_{q_1,q_2}F(b,c)}{{}^{b}\partial_{q_1}t\,{}^{d}\partial_{q_2}s}\right|\right. \\ &\qquad\left.+t(1-s)\left|\frac{{}^{b,d}\partial^2_{q_1,q_2}F(a,d)}{{}^{b}\partial_{q_1}t\,{}^{d}\partial_{q_2}s}\right|+(1-t)(1-s)\left|\frac{{}^{b,d}\partial^2_{q_1,q_2}F(b,d)}{{}^{b}\partial_{q_1}t\,{}^{d}\partial_{q_2}s}\right|\right]d_{q_1}t\,d_{q_2}s \\ &\quad=q_1q_2(b-a)(d-c)\int_0^{\frac{1}{[2]_{q_1}}}\int_0^{\frac{1}{[2]_{q_2}}}ts\left[ts\left|\frac{{}^{b,d}\partial^2_{q_1,q_2}F(a,c)}{{}^{b}\partial_{q_1}t\,{}^{d}\partial_{q_2}s}\right|+(1-t)s\left|\frac{{}^{b,d}\partial^2_{q_1,q_2}F(b,c)}{{}^{b}\partial_{q_1}t\,{}^{d}\partial_{q_2}s}\right|\right. \\ &\qquad\left.+t(1-s)\left|\frac{{}^{b,d}\partial^2_{q_1,q_2}F(a,d)}{{}^{b}\partial_{q_1}t\,{}^{d}\partial_{q_2}s}\right|+(1-t)(1-s)\left|\frac{{}^{b,d}\partial^2_{q_1,q_2}F(b,d)}{{}^{b}\partial_{q_1}t\,{}^{d}\partial_{q_2}s}\right|\right]d_{q_1}t\,d_{q_2}s \\ &\qquad+\int_0^{\frac{1}{[2]_{q_1}}}\int_{\frac{1}{[2]_{q_2}}}^1 t\left(\frac{1}{q_2}-s\right)\left[ts\left|\frac{{}^{b,d}\partial^2_{q_1,q_2}F(a,c)}{{}^{b}\partial_{q_1}t\,{}^{d}\partial_{q_2}s}\right|+(1-t)s\left|\frac{{}^{b,d}\partial^2_{q_1,q_2}F(b,c)}{{}^{b}\partial_{q_1}t\,{}^{d}\partial_{q_2}s}\right|\right. \\ &\qquad\left.+t(1-s)\left|\frac{{}^{b,d}\partial^2_{q_1,q_2}F(a,d)}{{}^{b}\partial_{q_1}t\,{}^{d}\partial_{q_2}s}\right|+(1-t)(1-s)\left|\frac{{}^{b,d}\partial^2_{q_1,q_2}F(b,d)}{{}^{b}\partial_{q_1}t\,{}^{d}\partial_{q_2}s}\right|\right]d_{q_1}t\,d_{q_2}s \\ &\qquad+\int_{\frac{1}{[2]_{q_1}}}^1\int_0^{\frac{1}{[2]_{q_2}}} s\left(\frac{1}{q_1}-t\right)\left[ts\left|\frac{{}^{b,d}\partial^2_{q_1,q_2}F(a,c)}{{}^{b}\partial_{q_1}t\,{}^{d}\partial_{q_2}s}\right|+(1-t)s\left|\frac{{}^{b,d}\partial^2_{q_1,q_2}F(b,c)}{{}^{b}\partial_{q_1}t\,{}^{d}\partial_{q_2}s}\right|\right. \\ &\qquad\left.+t(1-s)\left|\frac{{}^{b,d}\partial^2_{q_1,q_2}F(a,d)}{{}^{b}\partial_{q_1}t\,{}^{d}\partial_{q_2}s}\right|+(1-t)(1-s)\left|\frac{{}^{b,d}\partial^2_{q_1,q_2}F(b,d)}{{}^{b}\partial_{q_1}t\,{}^{d}\partial_{q_2}s}\right|\right]d_{q_1}t\,d_{q_2}s\end{aligned}\tag{5.20}$$

$$+\int_{\frac{1}{[2]q_1}}^{1}\int_{\frac{1}{[2]q_2}}^{1}\left(\frac{1}{q_1}-t\right)\left(\frac{1}{q_2}-s\right)\left[ts\left|\frac{{}^{b,d}\partial^2_{q_1,q_2}F(a,c)}{{}^{b}\partial_{q_1}t\,{}^{d}\partial_{q_2}s}\right|+(1-t)s\left|\frac{{}^{b,d}\partial^2_{q_1,q_2}F(b,c)}{{}^{b}\partial_{q_1}t\,{}^{d}\partial_{q_2}s}\right|\right.$$

$$\left.+t(1-s)\left|\frac{{}^{b,d}\partial^2_{q_1,q_2}F(a,d)}{{}^{b}\partial_{q_1}t\,{}^{d}\partial_{q_2}s}\right|+(1-t)(1-s)\left|\frac{{}^{b,d}\partial^2_{q_1,q_2}F(b,d)}{{}^{b}\partial_{q_1}t\,{}^{d}\partial_{q_2}s}\right|\right]d_{q_1}t\,d_{q_2}s.$$

This completes the proof. □

Example 1 Define a function $f:[0,1]\times[0,1]\to\mathbb{R}$ by $f(x,y)=x^2y^2$. Then $f(x,y)$ is a convex differentiable function of two variables on $[0,1]\times[0,1]$. For $q_1=q_2=\frac{1}{2}$, we have

$$F\left(\frac{a+q_1b}{[2]_{q_1}},\frac{c+q_2d}{[2]_{q_2}}\right)=\frac{1}{81},$$

$$\frac{1}{d-c}\int_c^d F\left(\frac{a+q_1b}{[2]_{q_1}},y\right){}^{d}d_{q_2}y=\int_0^1 F\left(\frac{1}{3},y\right){}^{1}d_{\frac{1}{2}}y$$
$$=\frac{1}{36},$$

$$\frac{1}{b-a}\int_a^b F\left(x,\frac{c+q_2d}{[2]_{q_2}}\right){}^{b}d_{q_1}x=\int_0^1 F\left(x,\frac{1}{3}\right){}^{1}d_{\frac{1}{2}}x$$
$$=\frac{1}{36},$$

$$\frac{1}{(b-a)(d-c)}\int_c^d\int_a^b F(x,y)\,{}^{b}d_{q_1}x\,{}^{d}d_{q_2}y=\int_0^1\int_0^1 F(x,y)\,{}^{1}d_{\frac{1}{2}}x\,{}^{1}d_{\frac{1}{2}}y$$
$$=\frac{1}{16}.$$

Thus,

$$\left|{}^{b,d}I_{q_1,q_2}(a,b,c,d)\big(F(x,y)\big)\right|=\frac{25}{3^4.2^4}.$$

Now, we can observe that

$$\frac{{}^{1,1}\partial^2_{\frac{1}{2},\frac{1}{2}}F(t,s)}{{}^{1}\partial_{\frac{1}{2}}t\,{}^{1}\partial_{\frac{1}{2}}s}=\frac{4}{(1-t)(1-s)}\left[F\left(\frac{t+1}{2},\frac{s+1}{2}\right)-F\left(\frac{t+1}{2},s\right)-F\left(t,\frac{s+1}{2}\right)+F(t,s)\right],$$

$$\left|\frac{{}^{1,1}\partial^2_{\frac{1}{2},\frac{1}{2}}F(0,0)}{{}^{1}\partial_{\frac{1}{2}}t\,{}^{1}\partial_{\frac{1}{2}}s}\right|=\frac{1}{4},$$

$$\left|\frac{{}^{1,1}\partial^2_{\frac{1}{2},\frac{1}{2}}F(1,0)}{{}^{1}\partial_{\frac{1}{2}}t\,{}^{1}\partial_{\frac{1}{2}}s}\right|=\frac{3}{2},$$

$$\left|\frac{{}^{1,1}\partial^2_{\frac{1}{2},\frac{1}{2}}F(0,1)}{{}^{1}\partial_{\frac{1}{2}}t\,{}^{1}\partial_{\frac{1}{2}}s}\right|=\frac{3}{2},$$

and

$$\left|\frac{{}^{1,1}\partial^2_{\frac{1}{2},\frac{1}{2}}F(1,1)}{{}^{1}\partial_{\frac{1}{2}}t\,{}^{1}\partial_{\frac{1}{2}}s}\right|=\frac{9}{4}.$$

Finally, using the above calculated values in inequality (5.18), we have

$$\frac{25}{1296} < \frac{377}{2640},$$

which shows that the proved inequality is valid for convex functions.

Remark 4 Under the given conditions of Theorem 6 with $q_1, q_2 \to 1^-$, then we obtain the following inequality:

$$\begin{aligned} &\left|\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d F(x,y)\,dy\,dx + F\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right. \\ &\quad\left.-\left[\frac{1}{b-a}\int_a^b F\left(x,\frac{c+d}{2}\right)dx + \frac{1}{d-c}\int_c^d F\left(\frac{a+b}{2},y\right)dy\right]\right| \\ &\quad\leq \frac{(b-a)(d-c)}{16}\left[\frac{|\frac{\partial^2 F}{\partial t\partial s}(a,c)| + |\frac{\partial^2 F}{\partial t\partial s}(a,d)| + |\frac{\partial^2 F}{\partial t\partial s}(b,c)| + |\frac{\partial^2 F}{\partial t\partial s}(b,d)|}{4}\right], \end{aligned} \tag{5.21}$$

which is given by Latif and Dragomir in [25, Theorem 2].

Theorem 7 *Let $F : \Delta \subseteq \mathbb{R}^2 \to \mathbb{R}$ be a twice partially q_1q_2-differentiable function on Δ° such that the partial q_1q_2-derivative $\frac{{}^{b,d}\partial^2_{q_1,q_2}F(t,s)}{{}^b\partial_{q_1}t\,{}^d\partial_{q_2}s}$ is continuous and integrable on $[a,b] \times [c,d] \subseteq \Delta^\circ$. If $|\frac{{}^{b,d}\partial^2_{q_1,q_2}F(t,s)}{{}^b\partial_{q_1}t\,{}^d\partial_{q_2}s}|^{p_1}$ is convex on $[a,b]\times[c,d]$ for some $p_1 > 1$ and $\frac{1}{r_1} + \frac{1}{p_1} = 1$, then we have the following inequality*:

$$\begin{aligned} &\left|{}^{b,d}I_{q_1,q_2}(a,b,c,d)(F)\right| \\ &\quad\leq q_1q_2(b-a)(d-c)\left(\int_0^1\int_0^1 \left|\Lambda(t,s)\right|^{r_1} d_{q_1}t\,d_{q_2}s\right)^{\frac{1}{r_1}} \\ &\quad\quad\times\left[\frac{1}{[2]_{q_1}[2]_{q_2}}\left|\frac{{}^{b,d}\partial^2_{q_1,q_2}F(a,c)}{{}^b\partial_{q_1}t\,{}^d\partial_{q_2}s}\right|^{p_1} + \frac{q_2}{[2]_{q_1}[2]_{q_2}}\left|\frac{{}^{b,d}\partial^2_{q_1,q_2}F(a,d)}{{}^b\partial_{q_1}t\,{}^d\partial_{q_2}s}\right|^{p_1}\right. \\ &\quad\quad\left.+\frac{q_1}{[2]_{q_1}[2]_{q_2}}\left|\frac{{}^{b,d}\partial^2_{q_1,q_2}F(b,c)}{{}^b\partial_{q_1}t\,{}^d\partial_{q_2}s}\right|^{p_1} + \frac{q_1q_2}{[2]_{q_1}[2]_{q_2}}\left|\frac{{}^{b,d}\partial^2_{q_1,q_2}F(b,d)}{{}^b\partial_{q_1}t\,{}^d\partial_{q_2}s}\right|^{p_1}\right]^{\frac{1}{p_1}}, \end{aligned} \tag{5.22}$$

where $0 < q_1, q_2 < 1$.

Proof Applying the well-known Hölder inequality for q_1q_2-integrals to the integrals on the right side of (5.19), it is found that

$$\begin{aligned} &\left|{}^{b,d}I_{q_1,q_2}(a,b,c,d)(F)\right| \\ &\quad\leq q_1q_2(b-a)(d-c)\left[\left(\int_0^1\int_0^1 \left|\Lambda(t,s)\right|^{r_1} d_{q_1}t\,d_{q_2}s\right)^{\frac{1}{r_1}}\right. \\ &\quad\quad\left.\times\left(\int_0^1\int_0^1\left|\frac{{}^{b,d}\partial^2_{q_1,q_2}F(ta+(1-t)b, sc+(1-s)d)}{{}^b\partial_{q_1}t\,{}^d\partial_{q_2}s}\right|^{p_1} d_{q_1}t\,d_{q_2}s\right)^{\frac{1}{p_1}}\right]. \end{aligned} \tag{5.23}$$

By applying convexity of $|\frac{{}^{b,d}\partial^2_{q_1,q_2}F(t,s)}{{}^{b}\partial_{q_1}t\,{}^{d}\partial_{q_2}s}|^{p_1}$, then (5.23) becomes

$$\begin{aligned}
&\left|{}^{b,d}I_{q_1,q_2}(a,b,c,d)(F)\right| \qquad (5.24)\\
&\le q_1q_2(b-a)(d-c)\Bigg[\left(\int_0^1\int_0^1\left|\Lambda(t,s)\right|^{r_1}\,d_{q_1}t\,d_{q_2}s\right)^{\frac{1}{r_1}}\\
&\quad\times\Bigg(\int_0^1\int_0^1\Bigg[ts\left|\frac{{}^{b,d}\partial^2_{q_1,q_2}F(a,c)}{{}^{b}\partial_{q_1}t\,{}^{d}\partial_{q_2}s}\right|^{p_1}+t(1-s)\left|\frac{{}^{b,d}\partial^2_{q_1,q_2}F(a,d)}{{}^{b}\partial_{q_1}t\,{}^{d}\partial_{q_2}s}\right|^{p_1}\\
&\quad+(1-t)s\left|\frac{{}^{b,d}\partial^2_{q_1,q_2}F(b,c)}{{}^{b}\partial_{q_1}t\,{}^{d}\partial_{q_2}s}\right|^{p_1}+(1-t)(1-s)\left|\frac{{}^{b,d}\partial^2_{q_1,q_2}F(b,d)}{{}^{b}\partial_{q_1}t\,{}^{d}\partial_{q_2}s}\right|^{p_1}\Bigg]\,d_{q_1}t\,d_{q_2}s\Bigg)^{\frac{1}{p_1}}\Bigg].
\end{aligned}$$

Now, if we apply the concept of Lemma 1 for $a=0$ to the above quantum integrals, we obtain

$$\begin{aligned}
\int_0^1\int_0^1 ts\,d_{q_1}t\,d_{q_2}s&=\left(\int_0^1 t\,d_{q_1}t\right)\left(\int_0^1 s\,d_{q_2}s\right) \qquad (5.25)\\
&=\frac{1}{[2]_{q_1}[2]_{q_2}},
\end{aligned}$$

$$\int_0^1\int_0^1 t(1-s)\,d_{q_1}t\,d_{q_2}s=\frac{q_2}{[2]_{q_1}[2]_{q_2}}, \qquad (5.26)$$

$$\int_0^1\int_0^1 (1-t)s\,d_{q_1}t\,d_{q_2}s=\frac{q_1}{[2]_{q_1}[2]_{q_2}}, \qquad (5.27)$$

$$\int_0^1\int_0^1 (1-t)(1-s)\,d_{q_1}t\,d_{q_2}s=\frac{q_1q_2}{[2]_{q_1}[2]_{q_2}}. \qquad (5.28)$$

By substituting the calculated integrals (5.25)–(5.28) in (5.24), then we obtain the desired inequality (5.22) which finishes the proof. □

Remark 5 Under the given conditions of Theorem 7 with $q_1,q_2\to 1^-$, then we obtain the following inequality:

$$\begin{aligned}
&\left|\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d F(x,y)\,dy\,dx+F\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right. \qquad (5.29)\\
&\quad\left.-\left[\frac{1}{b-a}\int_a^b F\left(x,\frac{c+d}{2}\right)dx+\frac{1}{d-c}\int_c^d F\left(\frac{a+b}{2},y\right)dy\right]\right|\\
&\le\frac{(b-a)(d-c)}{4(r_1+1)^{\frac{2}{r_1}}}\left[\frac{|\frac{\partial^2F}{\partial t\partial s}(a,c)|^{p_1}+|\frac{\partial^2F}{\partial t\partial s}(a,d)|^{p_1}+|\frac{\partial^2F}{\partial t\partial s}(b,c)|^{p_1}+|\frac{\partial^2F}{\partial t\partial s}(b,d)|^{p_1}}{4}\right]^{\frac{1}{p_1}},
\end{aligned}$$

which is given by Latif and Dragomir in [25, Theorem 3].

Theorem 8 *Let $F:\Delta\subseteq\mathbb{R}^2\to\mathbb{R}$ be a twice partially q_1q_2-differentiable function on Δ° such that the partial q_1q_2-derivative $\frac{{}^{b,d}\partial^2_{q_1,q_2}F(t,s)}{{}^{b}\partial_{q_1}t\,{}^{d}\partial_{q_2}s}$ is continuous and integrable on $[a,b]\times[c,d]\subseteq\Delta^\circ$. If $|\frac{{}^{b,d}\partial^2_{q_1,q_2}F(t,s)}{{}^{b}\partial_{q_1}t\,{}^{d}\partial_{q_2}s}|^{p_1}$ is convex on $[a,b]\times[c,d]$ for some $p_1>1$, then we have the* *following inequality*:

$$\begin{aligned} &\left|{}^{b,d}I_{q_1,q_2}(a,b,c,d)(F)\right| \\ &\leq q_1q_2(b-a)(d-c)\big(\Upsilon(q_1,q_2)\big)^{1-\frac{1}{p_1}} \\ &\quad\times\left[A\left|\frac{{}^{b,d}\partial^2_{q_1,q_2}F(a,c)}{{}^{b}\partial_{q_1}t\,{}^{d}\partial_{q_2}s}\right|^{p_1}+B\left|\frac{{}^{b,d}\partial^2_{q_1,q_2}F(b,c)}{{}^{b}\partial_{q_1}t\,{}^{d}\partial_{q_2}s}\right|^{p_1}\right. \\ &\quad\left.+C\left|\frac{{}^{b,d}\partial^2_{q_1,q_2}F(a,d)}{{}^{b}\partial_{q_1}t\,{}^{d}\partial_{q_2}s}\right|^{p_1}+E\left|\frac{{}^{b,d}\partial^2_{q_1,q_2}F(b,d)}{{}^{b}\partial_{q_1}t\,{}^{d}\partial_{q_2}s}\right|^{p_1}\right]^{\frac{1}{p_1}}, \end{aligned} \tag{5.30}$$

where A, B, C, E *are defined in Theorem* 6 *and* $0 < q_1, q_2 < 1$.

Proof Applying the well-known power mean inequality for q_1q_2-integrals to the integrals on the right side of (5.19), it is found that

$$\begin{aligned} &\left|{}^{b,d}I_{q_1,q_2}(a,b,c,d)(F)\right| \\ &\leq q_1q_2(b-a)(d-c)\left[\left(\int_0^1\int_0^1\left|\Lambda(t,s)\right|d_{q_1}t\,d_{q_2}s\right)^{1-\frac{1}{p_1}}\right. \\ &\quad\times\left(\int_0^1\int_0^1\left|\Lambda(t,s)\right|\right. \\ &\quad\left.\left.\times\left|\frac{{}^{b,d}\partial^2_{q_1,q_2}F(ta+(1-t)b,sc+(1-s)d)}{{}^{b}\partial_{q_1}t\,{}^{d}\partial_{q_2}s}\right|^{p_1}d_{q_1}t\,d_{q_2}s\right)^{\frac{1}{p_1}}\right]. \end{aligned} \tag{5.31}$$

By applying the convexity of $\left|\frac{{}^{b,d}\partial^2_{q_1,q_2}F(t,s)}{{}^{b}\partial_{q_1}t\,{}^{d}\partial_{q_2}s}\right|^{p_1}$, then (5.31) becomes

$$\begin{aligned} &\left|{}^{b,d}I_{q_1,q_2}(a,b,c,d)(F)\right| \\ &\leq q_1q_2(b-a)(d-c)\left(\int_0^1\int_0^1\left|\Lambda(t,s)\right|d_{q_1}t\,d_{q_2}s\right)^{1-\frac{1}{p_1}} \\ &\quad\times\left[A\left|\frac{{}^{b,d}\partial^2_{q_1,q_2}F(a,c)}{{}^{b}\partial_{q_1}t\,{}^{d}\partial_{q_2}s}\right|^{p_1}+B\left|\frac{{}^{b,d}\partial^2_{q_1,q_2}F(b,c)}{{}^{b}\partial_{q_1}t\,{}^{d}\partial_{q_2}s}\right|^{p_1}\right. \\ &\quad\left.+C\left|\frac{{}^{b,d}\partial^2_{q_1,q_2}F(a,d)}{{}^{b}\partial_{q_1}t\,{}^{d}\partial_{q_2}s}\right|^{p_1}+E\left|\frac{{}^{b,d}\partial^2_{q_1,q_2}F(b,d)}{{}^{b}\partial_{q_1}t\,{}^{d}\partial_{q_2}s}\right|^{p_1}\right]^{\frac{1}{p_1}}. \end{aligned} \tag{5.32}$$

By substituting the calculated integral (5.1) in (5.32), then we obtain required inequality (5.30), which ends the proof. □

Remark 6 Under the given conditions of Theorem 8 with $q_1, q_2 \to 1^-$, then the inequality (5.30) reduces to the following one:

$$\begin{aligned} &\left|\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d F(x,y)\,dy\,dx+F\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right. \\ &\quad\left.-\left[\frac{1}{b-a}\int_a^b F\left(x,\frac{c+d}{2}\right)dx+\frac{1}{d-c}\int_c^d F\left(\frac{a+b}{2},y\right)dy\right]\right| \end{aligned} \tag{5.33}$$

$$\leq \frac{(b-a)(d-c)}{16}\left[\frac{|\frac{\partial^2 F}{\partial t\partial s}(a,c)|^{p_1}+|\frac{\partial^2 F}{\partial t\partial s}(a,d)|^{p_1}+|\frac{\partial^2 F}{\partial t\partial s}(b,c)|^{p_1}+|\frac{\partial^2 F}{\partial t\partial s}(b,d)|^{p_1}}{4}\right]^{\frac{1}{p_1}},$$

which is given by Latif and Dragomir in [25, Theorem 4].

Now we use Lemma 3 to find some new quantum estimates. We first examine a new result for functions whose partially q_1q_2-derivatives in modulus are convex in the following theorem.

Theorem 9 *Let $F : \Delta \subseteq \mathbb{R}^2 \to \mathbb{R}$ be a twice partially q_1q_2-differentiable function on Δ° such that the partial q_1q_2-derivative $\frac{{}_a^d\partial^2_{q_1,q_2}F(t,s)}{{}_a\partial_{q_1}t\,{}^d\partial_{q_2}s}$ is continuous and integrable on $[a,b]\times[c,d]\subseteq\Delta^\circ$. Then we have the following inequality provided that $|\frac{{}_a^d\partial^2_{q_1,q_2}F(t,s)}{{}_a\partial_{q_1}t\,{}^d\partial_{q_2}s}|$ is convex on $[a,b]\times[c,d]$*:

$$\begin{aligned}
&\left|{}_a^dI_{q_1,q_2}(a,b,c,d)(F)\right| \qquad (5.34)\\
&\leq q_1q_2(b-a)(d-c)\left[A\left|\frac{{}_a^d\partial^2_{q_1,q_2}F(b,c)}{{}_a\partial_{q_1}t\,{}^d\partial_{q_2}s}\right|+B\left|\frac{{}_a^d\partial^2_{q_1,q_2}F(a,c)}{{}_a\partial_{q_1}t\,{}^d\partial_{q_2}s}\right|\right.\\
&\quad\left.+C\left|\frac{{}_a^d\partial^2_{q_1,q_2}F(b,d)}{{}_a\partial_{q_1}t\,{}^d\partial_{q_2}s}\right|+E\left|\frac{{}_a^d\partial^2_{q_1,q_2}F(a,d)}{{}_a\partial_{q_1}t\,{}^d\partial_{q_2}s}\right|\right],
\end{aligned}$$

where A, B, C, E are defined in Theorem 6 and $0<q_1,q_2<1$.

Proof If the strategy which was used in the proof of Theorem 6 is applied by taking into account Lemma 3, the desired inequality (5.34) can be obtained. □

Remark 7 Under the assumptions of Theorem 9 with $q_1,q_2\to 1^-$, then inequality (5.34) reduces to inequality (5.21).

Theorem 10 *Let $F : \Delta \subseteq \mathbb{R}^2 \to \mathbb{R}$ be a twice partially q_1q_2-differentiable function on Δ° such that the partial q_1q_2-derivative $\frac{{}_a^d\partial^2_{q_1,q_2}F(t,s)}{{}_a\partial_{q_1}t\,{}^d\partial_{q_2}s}$ is continuous and integrable on $[a,b]\times[c,d]\subseteq\Delta^\circ$. If $|\frac{{}_a^d\partial^2_{q_1,q_2}F(t,s)}{{}_a\partial_{q_1}t\,{}^d\partial_{q_2}s}|^{p_1}$ is convex on $[a,b]\times[c,d]$ for some $p_1>1$ and $\frac{1}{r_1}+\frac{1}{p_1}=1$. Then we have the following inequality*:

$$\begin{aligned}
&\left|{}_a^dI_{q_1,q_2}(a,b,c,d)(F)\right| \qquad (5.35)\\
&\leq q_1q_2(b-a)(d-c)\left(\int_0^1\int_0^1\left|\Lambda(t,s)\right|^{r_1}d_{q_1}t\,d_{q_2}s\right)^{\frac{1}{r_1}}\\
&\quad\times\left[\frac{1}{[2]_{q_1}[2]_{q_2}}\left|\frac{{}_a^d\partial^2_{q_1,q_2}F(b,c)}{{}_a\partial_{q_1}t\,{}^d\partial_{q_2}s}\right|^{p_1}+\frac{q_2}{[2]_{q_1}[2]_{q_2}}\left|\frac{{}_a^d\partial^2_{q_1,q_2}F(b,d)}{{}_a\partial_{q_1}t\,{}^d\partial_{q_2}s}\right|^{p_1}\right.\\
&\quad\left.+\frac{q_1}{[2]_{q_1}[2]_{q_2}}\left|\frac{{}_a^d\partial^2_{q_1,q_2}F(a,c)}{{}_a\partial_{q_1}t\,{}^d\partial_{q_2}s}\right|^{p_1}+\frac{q_1q_2}{[2]_{q_1}[2]_{q_2}}\left|\frac{{}_a^d\partial^2_{q_1,q_2}F(a,d)}{{}_a\partial_{q_1}t\,{}^d\partial_{q_2}s}\right|^{p_1}\right]^{\frac{1}{p_1}},
\end{aligned}$$

where $0<q_1,q_2<1$.

Proof If the strategy which was used in the proof of Theorem 7 is applied by taking into account Lemma 3, the desired inequality (5.35) can be obtained. □

Remark 8 Under the assumptions of Theorem 10 with $q_1, q_2 \to 1^-$, then inequality (5.35) reduces to inequality (5.29).

Theorem 11 *Let $F : \Delta \subseteq \mathbb{R}^2 \to \mathbb{R}$ be a twice partially q_1q_2-differentiable function on Δ° such that the partial q_1q_2-derivative $\frac{{}_a^d\partial^2_{q_1,q_2}F(t,s)}{{}_a\partial_{q_1}t\,{}^d\partial_{q_2}s}$ is continuous and integrable on $[a,b] \times [c,d] \subseteq \Delta^\circ$. If $|\frac{{}_a^d\partial^2_{q_1,q_2}F(t,s)}{{}_a\partial_{q_1}t\,{}^d\partial_{q_2}s}|^{p_1}$ is convex on $[a,b] \times [c,d]$ for some $p_1 > 1$, then we have the following inequality*:

$$\begin{aligned}
&\left|{}_a^d I_{q_1,q_2}(a,b,c,d)(F)\right| \\
&\quad\le q_1q_2(b-a)(d-c)\big(\Upsilon(q_1,q_2)\big)^{1-\frac{1}{p_1}} \\
&\qquad\times\left[A\left|\frac{{}_a^d\partial^2_{q_1,q_2}F(b,c)}{{}_a\partial_{q_1}t\,{}^d\partial_{q_2}s}\right|^{p_1}+B\left|\frac{{}_a^d\partial^2_{q_1,q_2}F(a,c)}{{}_a\partial_{q_1}t\,{}^d\partial_{q_2}s}\right|^{p_1}\right. \\
&\qquad\left.+C\left|\frac{{}_a^d\partial^2_{q_1,q_2}F(b,d)}{{}_a\partial_{q_1}t\,{}^d\partial_{q_2}s}\right|^{p_1}+E\left|\frac{{}_a^d\partial^2_{q_1,q_2}F(a,d)}{{}_a\partial_{q_1}t\,{}^d\partial_{q_2}s}\right|^{p_1}\right]^{\frac{1}{p_1}},
\end{aligned}\tag{5.36}$$

where A, B, C, E are defined in Theorem 6 and $0 < q_1, q_2 < 1$.

Proof If the strategy which was used in the proof of Theorem 8 is applied by taking into account Lemma 3, the desired inequality (5.36) can be obtained. □

Remark 9 Under the assumptions of Theorem 11 with $q_1, q_2 \to 1^-$, then inequality (5.36) reduces to inequality (5.33)

Now we use Lemma 4 to find some new quantum estimates. We first examine a new result for functions whose partially q_1q_2-derivatives in modulus are convex in the following theorem.

Theorem 12 *Let $F : \Delta \subseteq \mathbb{R}^2 \to \mathbb{R}$ be a twice partially q_1q_2-differentiable function on Δ° such that the partial q_1q_2-derivative $\frac{{}_c^b\partial^2_{q_1,q_2}F(t,s)}{{}^b\partial_{q_1}t\,{}_c\partial_{q_2}s}$ is continuous and integrable on $[a,b] \times [c,d] \subseteq \Delta^\circ$. Then we have the following inequality provided that $|\frac{{}_c^b\partial^2_{q_1,q_2}F(t,s)}{{}^b\partial_{q_1}t\,{}_c\partial_{q_2}s}|$ is convex on $[a,b] \times [c,d]$*:

$$\begin{aligned}
&\left|{}_c^b I_{q_1,q_2}(a,b,c,d)(F)\right| \\
&\quad\le q_1q_2(b-a)(d-c)\left[A\left|\frac{{}_c^b\partial^2_{q_1,q_2}F(a,d)}{{}^b\partial_{q_1}t\,{}_c\partial_{q_2}s}\right|+B\left|\frac{{}_c^b\partial^2_{q_1,q_2}F(b,d)}{{}^b\partial_{q_1}t\,{}_c\partial_{q_2}s}\right|\right. \\
&\qquad\left.+C\left|\frac{{}_c^b\partial^2_{q_1,q_2}F(a,c)}{{}^b\partial_{q_1}t\,{}_c\partial_{q_2}s}\right|+E\left|\frac{{}_c^b\partial^2_{q_1,q_2}F(b,c)}{{}^b\partial_{q_1}t\,{}_c\partial_{q_2}s}\right|\right],
\end{aligned}\tag{5.37}$$

where A, B, C, E are defined in Theorem 6 and $0 < q_1, q_2 < 1$.

Proof If the strategy which was used in the proof of Theorem 6 are applied by taking into account Lemma 4, the desired inequality (5.37) can be obtained. □

Remark 10 Under the assumptions of Theorem 12 with $q_1, q_2 \to 1^-$, then inequality (5.37) reduces to inequality (5.21).

Theorem 13 *Let $F : \Delta \subseteq \mathbb{R}^2 \to \mathbb{R}$ be a twice partially q_1q_2-differentiable function on Δ° such that the partial q_1q_2-derivative $\frac{{}_c^b\partial^2_{q_1,q_2} F(t,s)}{{}^b\partial_{q_1} t\, {}_c\partial_{q_2} s}$ is continuous and integrable on $[a,b] \times [c,d] \subseteq \Delta^\circ$. If $|\frac{{}_c^b\partial^2_{q_1,q_2} F(t,s)}{{}^b\partial_{q_1} t\, {}_c\partial_{q_2} s}|^{p_1}$ is convex on $[a,b] \times [c,d]$ for some $p_1 > 1$ and $\frac{1}{r_1} + \frac{1}{p_1} = 1$, then we have following inequality*:

$$\begin{aligned} &\left|{}_c^b I_{q_1,q_2}(a,b,c,d)(F)\right| \\ &\leq q_1q_2(b-a)(d-c)\left(\int_0^1\int_0^1 \left|\Lambda(t,s)\right|^{r_1} d_{q_1}t\, d_{q_2}s\right)^{\frac{1}{r_1}} \\ &\quad\times\left[\frac{1}{[2]_{q_1}[2]_{q_2}}\left|\frac{{}_c^b\partial^2_{q_1,q_2} F(a,d)}{{}^b\partial_{q_1} t\, {}_c\partial_{q_2} s}\right|^{p_1} + \frac{q_2}{[2]_{q_1}[2]_{q_2}}\left|\frac{{}_c^b\partial^2_{q_1,q_2} F(a,c)}{{}^b\partial_{q_1} t\, {}_c\partial_{q_2} s}\right|^{p_1}\right. \\ &\quad\left. + \frac{q_1}{[2]_{q_1}[2]_{q_2}}\left|\frac{{}_c^b\partial^2_{q_1,q_2} F(b,d)}{{}^b\partial_{q_1} t\, {}_c\partial_{q_2} s}\right|^{p_1} + \frac{q_1q_2}{[2]_{q_1}[2]_{q_2}}\left|\frac{{}_c^b\partial^2_{q_1,q_2} F(b,c)}{{}^b\partial_{q_1} t\, {}_c\partial_{q_2} s}\right|^{p_1}\right]^{\frac{1}{p_1}}, \end{aligned} \tag{5.38}$$

where $0 < q_1, q_2 < 1$.

Proof If the strategy which was used in the proof of Theorem 7 is applied by taking into account Lemma 4, the desired inequality (5.38) can be obtained. □

Remark 11 Under the assumptions of Theorem 13 with $q_1, q_2 \to 1^-$, then inequality (5.38) reduces to inequality (5.29).

Theorem 14 *Let $F : \Delta \subseteq \mathbb{R}^2 \to \mathbb{R}$ be a twice partially q_1q_2-differentiable function on Δ° such that the partial q_1q_2-derivative $\frac{{}_c^b\partial^2_{q_1,q_2} F(t,s)}{{}^b\partial_{q_1} t\, {}_c\partial_{q_2} s}$ is continuous and integrable on $[a,b] \times [c,d] \subseteq \Delta^\circ$. If $|\frac{{}_c^b\partial^2_{q_1,q_2} F(t,s)}{{}^b\partial_{q_1} t\, {}_c\partial_{q_2} s}|^{p_1}$ is convex on $[a,b] \times [c,d]$ for some $p_1 > 1$, then we have the following inequality*:

$$\begin{aligned} &\left|{}_c^b I_{q_1,q_2}(a,b,c,d)(F)\right| \\ &\leq \frac{q_1q_2(b-a)(d-c)}{[2]_{q_1}[2]_{q_2}}\left(\Upsilon(q_1,q_2)\right)^{1-\frac{1}{p_1}} \\ &\quad\times\left[A\left|\frac{{}_c^b\partial^2_{q_1,q_2} F(a,d)}{{}^b\partial_{q_1} t\, {}_c\partial_{q_2} s}\right|^{p_1} + B\left|\frac{{}_c^b\partial^2_{q_1,q_2} F(b,d)}{{}^b\partial_{q_1} t\, {}_c\partial_{q_2} s}\right|^{p_1}\right. \\ &\quad\left. + C\left|\frac{{}_c^b\partial^2_{q_1,q_2} F(a,c)}{{}^b\partial_{q_1} t\, {}_c\partial_{q_2} s}\right|^{p_1} + D\left|\frac{{}_c^b\partial^2_{q_1,q_2} F(b,c)}{{}^b\partial_{q_1} t\, {}_c\partial_{q_2} s}\right|^{p_1}\right]^{\frac{1}{p_1}}, \end{aligned} \tag{5.39}$$

where A, B, CE are defined in Theorem 6 and $0 < q_1, q_2 < 1$.

Proof If the strategy which was used in the proof of Theorem 8 is applied by taking into account Lemma 4, the desired inequality (5.39) can be obtained. □

Remark 12 Under the assumptions of Theorem 14 with $q_1, q_2 \to 1^-$, then inequality (5.39) reduces to inequality (5.33).

6 Conclusion

In this paper, midpoint type inequalities for coordinated convex functions by applying the notion of q_1q_2-integrals are obtained. It is also shown that the results proved in this paper are the potential generalization of the existing comparable results in the literature. It is an interesting and new problem that the upcoming mathematicians can derive similar inequalities for different kinds of convexities in their future work.

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