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Weighted Inequalities Involving Conformable Integrals and Its Applications for Random Variable and Numerical Integration

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Abstract. We establish new weighted inequalities for conformable integrals and derivatives. Then, some results involving $\alpha-$fractional moments are presented. Next, $\alpha-$fractional Uniform, Gamma and Weibull distribution are defined and exclusive results by using these inequalities developed are obtained. Also, some applications of obtained inequalities in numerical integration are given.

1. Introduction

In 1938, Ostrowski established the integral inequality which is one of the fundemental inequalitıes of mathematic as follows (see, [19]):

Let $f : [a,b] \rightarrow \mathbb{R}$ be a differentiable mapping on (a,b) whose derivative $f' : (a,b) \rightarrow \mathbb{R}$ is bounded on (a,b), i.e., $\left\|f'\right\|_\infty = \sup\limits_{t\in(a,b)} \left|f'(t)\right| < \infty$. Then, the inequality holds:

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)dt \right| \leq \left[\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2} \right] (b-a) \left\|f'\right\|_\infty \tag{1}$$

for all $x \in [a,b]$. The constant $\frac{1}{4}$ is the best possible.

The inequality (1) has wide applications in numerical analysis and the theory of some special means; estimating error bounds for these, some mid-point, trapezoid, Simpson rules and quadrature rules, etc. Some results and generalizations concerning Ostrowski's inequality have been written in various papers, see [5], [10]-[12], [18], [20], [21] and [22] the references therein.

In [10], Cerone et.al. established the following inequalities of Ostrowski type.

Theorem 1.1. *Let $f : [a,b] \rightarrow \mathbb{R}$ be a twice differentiable mapping on (a,b) and $f'' : (a,b) \rightarrow \mathbb{R}$ is bounded, i.e.,*

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$\left\|f''\right\|_\infty = \sup\limits_{t\in(a,b)} \left|f''(t)\right| < \infty$. *Then we have the inequality:*

$$\begin{aligned} & \left| f(x) - \frac{1}{b-a}\int_a^b f(t)dt - \left(x - \frac{a+b}{2}\right) f'(x) \right| \\ \leq & \left[\frac{1}{24}(b-a)^2 + \frac{1}{2}\left(x - \frac{a+b}{2}\right)^2 \right] \left\|f''\right\|_\infty \\ \leq & \frac{(b-a)^2}{6} \left\|f''\right\|_\infty \end{aligned}$$

for all $x \in [a, b]$.

2. Definitions and properties of conformable fractional derivative and integral

Recently, the authors have introduced a new simple well-behaved definition of the fractional derivative called the "conformable fractional derivative" depending just on the basic limit definition of the derivative in [14]. Namely, for given a function $f : [0, \infty) \to \mathbb{R}$ the conformable fractional derivative of order $0 < \alpha \leq 1$ of f at $t > 0$ was defined by

$$D_\alpha (f)(t) = \lim_{\epsilon\to 0} \frac{f\left(t + \epsilon t^{1-\alpha}\right) - f(t)}{\epsilon}$$

If f is $\alpha-$differentiable in some $(0, a)$, $\lim\limits_{t\to 0^+} f^{(\alpha)}(t)$ exist, then define

$$f^{(\alpha)}(0) = \lim_{t\to 0^+} f^{(\alpha)}(t).$$

Also, note that if f is differentiable, then

$$D_\alpha (f)(t) = t^{1-\alpha} f'(t),$$

where

$$f'(t) = \lim_{\epsilon\to 0} \frac{f(t+\epsilon) - f(t)}{\epsilon}.$$

We can write $f^{(\alpha)}(t)$ for $D_\alpha(f)(t)$ to denote the conformable fractional derivatives of f of order α. In addition, if the conformable fractional derivative of f of order α exists, then we simply say f is $\alpha-$differentiable.

The following definitions and theorems related to conformable fractional derivative and integral are referred in [1]-[4], [6], [13] and [14].

Theorem 2.1. *[14] Let* $\alpha \in (0, 1]$ *and* f, g *be* $\alpha-$*differentiable at a point* $t > 0$. *Then*

i. $D_\alpha(af + bg) = aD_\alpha(f) + bD_\alpha(g)$, for all $a, b \in \mathbb{R}$,

ii. $D_\alpha(\lambda) = 0$, for all constant functions $f(t) = \lambda$,

iii. $D_\alpha(fg) = fD_\alpha(g) + gD_\alpha(f)$,

iv. $D_\alpha\left(\frac{f}{g}\right) = \frac{gD_\alpha(f) - fD_\alpha(g)}{g^2}$.

Definition 2.2 (Conformable fractional integral). *[14] Let $\alpha \in (0,1]$ and $0 \leq a < b$. A function $f : [a,b] \to \mathbb{R}$ is α-fractional integrable on $[a,b]$ if the integral*

$$\int_a^b f(x)\, d_\alpha x := \int_a^b f(x)\, x^{\alpha-1} dx$$

exists and is finite.

Remark 2.3. *[14]*

$$I_\alpha^a(f)(t) = I_1^a\left(t^{\alpha-1} f\right) = \int_a^t \frac{f(x)}{x^{1-\alpha}} dx,$$

where the integral is the usual Riemann improper integral, and $\alpha \in (0,1]$.

Theorem 2.4. *[14] Let $f : (a,b) \to \mathbb{R}$ be differentiable and $0 < \alpha \leq 1$. Then, for all $t > a$ we have*

$$I_\alpha^a D_\alpha^a f(t) = f(t) - f(a).$$

Theorem 2.5. *(**Integration by parts**) [1] Let $f, g : [a,b] \to \mathbb{R}$ be two functions such that fg is differentiable. Then*

$$\int_a^b f(x)\, D_\alpha^a(g)(x)\, d_\alpha x = fg\Big|_a^b - \int_a^b g(x)\, D_\alpha^a(f)(x)\, d_\alpha x.$$

Theorem 2.6. *[14] Assume that $f : [a,\infty) \to \mathbb{R}$ such that $f^{(n)}(t)$ is continuous and $\alpha \in (n, n+1]$. Then, for all $t > a$ we have*

$$D_\alpha^a I_\alpha^a f(t) = f(t).$$

Theorem 2.7. *[6] Let $\alpha \in (0,1]$ and $f : [a,b] \to \mathbb{R}$ be a continuous on $[a,b]$ with $0 \leq a < b$. Then,*

$$\left|I_\alpha^a(f)(x)\right| \leq I_\alpha^a \left|f\right|(x).$$

In [6], Anderson proved Ostrowski's α-fractional inequality using a Montgomery identity as follows:

Theorem 2.8. *Let $a, b, s, t \in \mathbb{R}$ with $0 \leq a < b$, and $f : [a,b] \to \mathbb{R}$ be α-fractional differentiable for $\alpha \in (0,1]$. Then,*

$$\left| f(t) - \frac{\alpha}{b^\alpha - a^\alpha} \int_a^b f(t) d_\alpha t \right| \leq \frac{M}{2\alpha\left(b^\alpha - a^\alpha\right)} \left[\left(t^\alpha - a^\alpha\right)^2 + \left(b^\alpha - t^\alpha\right) \right],$$

where

$$M = \sup_{t \in (a,b)} \left| D_\alpha f(t) \right| < \infty.$$

In recent years, many researchers have studied some integral inequalities for some special moments for random variables whose probability density function is defined on a finite interval. For example, it has been given some inequalities for the dispersion of random variables (see, [7]). Additionally, some inequalities for expectation, variance and moments have been obtained in [8], [9], [15], [16] and [21]. Kumar have established some Hermite-Hadamard type inequalities for the moments of random variables in [17].

In this study, we define firstly some definitions for the $\alpha-$fractional moments of random variables using conformable integrals. Then, some weighted inequalities have been established involving $\alpha-$fractional integrals. We specified these inequalities by using some special weights and we obtain exclusive inequalities based on $\alpha-$fractional moments defined in this paper. Next, we calculate $\alpha-$fractional moments for some well known statistical density functions like uniform distribution, weibull distribution, etc. In addition, these inequalities are extended to account for applications in numerical integration. Finally, we apply these $\alpha-$fractional moments to the inequalities obtained previously.

3. Some inequalities for the moments

In this section, some definitions for $\alpha-$fractional moments are firstly given.

Definition 3.1. *Let $\alpha \in (0,1]$ and $w : [a,b] \to [0,\infty)$ be $\alpha-$fractional integrable function, i.e. $\int_a^b w(t)d_\alpha t < \infty$ with $0 \leq a < b$. Then define*

$$m_{k,\alpha}(a,b) = \alpha \int_a^b t^{k\alpha} w(t) d_\alpha t$$

as the $\alpha-$fractional k.th moment of w.

Also, it should be noted that if we take $w : [a,b] \to [0,\infty)$ as $\alpha-$fractional probability density function for random variable X defined on the interval of real numbers I $(a,b \in I, 0 \leq a < b)$. Then we have

$$m_{0,\alpha}(a,b) = 1.$$

In addition, $\alpha-$fractional exceptation value, $\alpha-$fractional r.th central moment and $\alpha-$fractional variance of random variable X are defined as

$$\mu_\alpha(X) = m_{1,\alpha}(a,b), \tag{2}$$

$$M_{r,\alpha}(X) = \alpha \int_a^b (u^\alpha - \mu_\alpha(X))^r w(t) d_\alpha t$$

and

$$\sigma_\alpha^2(X) = M_{2,\alpha}(X) = m_{2,\alpha}(a,b) - \mu_\alpha^2(X), \tag{3}$$

respectively.

It is needed the following equality involving conformable integrals, in order to prove weighted integral inequalities:

Lemma 3.2. *Let $\alpha \in (0,1]$ and $f : [a,b] \to \mathbb{R}$ be a continuous on $[a,b]$ and twice $\alpha-$fractional differentiable function on (a,b) with $0 \leq a < b$. Also, assume that $w : [a,b] \to \mathbb{R}$ be nonnegative and continuous on $[a,b]$. Then we have the equality*

$$\int_a^b K(x,t) D_\alpha^{(2)} f(t) d_\alpha t \tag{4}$$

$$= \left(\int_a^b (u^\alpha - x^\alpha) w(u) d_\alpha u \right) D_\alpha f(x)$$

$$+\alpha \left(\int_a^b w(u) d_\alpha u \right) f(x) - \alpha \int_a^b w(t) f(t) d_\alpha t,$$

where $D_\alpha^{(2)} f(t) = D_\alpha D_\alpha f(t)$ and

$$K_\alpha(x,t) := \begin{cases} \int_a^t (u^\alpha - t^\alpha) w(u) d_\alpha u, & a \leq t < x \\ \\ \int_b^t (u^\alpha - t^\alpha) w(u) d_\alpha u, & x \leq t \leq b. \end{cases} \tag{5}$$

Proof. For the proof of the Theorem, if we apply integration by parts, then we have

$$\begin{aligned}
&\int_a^b K_\alpha(x,t)\, D_\alpha^{(2)} f(t) d_\alpha t \\
&= \int_a^x \left(\int_a^t (u^\alpha - t^\alpha)\, w(u)\, d_\alpha u \right) D_\alpha^{(2)} f(t) d_\alpha t + \int_x^b \left(\int_b^t (u^\alpha - t^\alpha)\, w(u)\, d_\alpha u \right) D_\alpha^{(2)} f(t) d_\alpha t \\
&= \left(\int_a^b (u^\alpha - x^\alpha)\, w(u)\, d_\alpha u \right) D_\alpha f(x) + \alpha \int_a^x \left(\int_a^t w(u)\, d_\alpha u \right) D_\alpha f(t) d_\alpha t \\
&\quad + \alpha \int_x^b \left(\int_b^t w(u)\, d_\alpha u \right) D_\alpha f(t) d_\alpha t.
\end{aligned}$$

It is obtained the required identity in (4) by using the process of integration by parts again. Hence, the proof is completed. □

A new inequality for the functions whose twice $\alpha-$fractional derivatives are bounded is constructed.

Theorem 3.3. *Suppose that all the assumptions of Lemma 3.2 hold. If $D_\alpha^{(2)} f : (a,b) \to \mathbb{R}$ is bounded, i.e., $\left\| D_\alpha^{(2)} f \right\|_\infty = \sup_{t\in(a,b)} \left| D_\alpha^{(2)} f(t) \right| < \infty$, then we have the inequalities*

$$\begin{aligned}
&\left| \left(\int_a^b (u^\alpha - x^\alpha)\, w(u)\, d_\alpha u \right) D_\alpha f(x) \right. \qquad (6) \\
&\left. + \alpha \left(\int_a^b w(u)\, d_\alpha u \right) f(x) - \alpha \int_a^b w(t)\, f(t)\, d_\alpha t \right| \\
&\leq \frac{1}{2\alpha} \left\| D_\alpha^{(2)} f \right\|_{[a,b],\infty} \int_a^b (u^\alpha - x^\alpha)^2\, w(u) d_\alpha u \\
&\leq \frac{1}{2\alpha} \left\| D_\alpha^{(2)} f \right\|_{[a,b],\infty} \left(\frac{b^\alpha - a^\alpha}{2} + \left| \frac{a^\alpha + b^\alpha}{2} - x^\alpha \right| \right)^2 \int_a^b w(u) d_\alpha u
\end{aligned}$$

for all $x \in [a,b]$.

Proof. If we take absolute value of both sides of the equality (4), from the conditions of theorem, we can write

$$\begin{aligned}
&\left|\left(\int_a^b (u^\alpha - x^\alpha)\, w(u)\, d_\alpha u\right) D_\alpha f(x) + \alpha\left(\int_a^b w(u)\, d_\alpha u\right) f(x) - \alpha \int_a^b w(t)\, f(t)\, d_\alpha t\right| \\
\leq\ & \left\|D_\alpha^{(2)} f\right\|_{[a,x],\infty} \int_a^x \left(\int_a^t (t^\alpha - u^\alpha)\, w(u)\, d_\alpha u\right) d_\alpha t \\
&+ \left\|D_\alpha^{(2)} f\right\|_{[x,b],\infty} \int_x^b \left(\int_t^b (u^\alpha - t^\alpha)\, w(u)\, d_\alpha u\right) d_\alpha t.
\end{aligned}$$

By using the change of order of integration, we get

$$\begin{aligned}
&\left|\left(\int_a^b (u^\alpha - x^\alpha)\, w(u)\, d_\alpha u\right) D_\alpha f(x) \right. \qquad (7)\\
&\left. +\alpha\left(\int_a^b w(u)\, d_\alpha u\right) f(x) - \alpha \int_a^b w(t)\, f(t)\, d_\alpha t\right| \\
\leq\ & \frac{\left\|D_\alpha^{(2)} f\right\|_{[a,x],\infty}}{2\alpha} \int_a^x (u^\alpha - x^\alpha)^2\, w(u) d_\alpha u + \frac{\left\|D_\alpha^{2} f\right\|_{[x,b],\infty}}{2\alpha} \int_x^b (u^\alpha - x^\alpha)^2\, w(u) d_\alpha u \\
\leq\ & \frac{\left\|D_\alpha^{(2)} f\right\|_{[a,b],\infty}}{2\alpha} \int_a^b (u^\alpha - x^\alpha)^2\, w(u) d_\alpha u.
\end{aligned}$$

To obtain second inequality of (6) obserwe that

$$\begin{aligned}
\int_a^b (u^\alpha - x^\alpha)^2\, w(u) d_\alpha u &\leq \sup_{u\in[a,b]} (u^\alpha - x^\alpha)^2 \int_a^b w(u) d_\alpha u \qquad (8)\\
&= \max\left\{(u^\alpha - a^\alpha)^2, (u^\alpha - b^\alpha)^2\right\} \int_a^b w(u) d_\alpha u \\
&= \left(\frac{b^\alpha - a^\alpha}{2} + \left|\frac{a^\alpha + b^\alpha}{2} - x^\alpha\right|\right)^2 \int_a^b w(u) d_\alpha u.
\end{aligned}$$

Combining the inequalities (7) and (8), we obtain the desired inequalities (6). The proof is thus completed. □

Corollary 3.4. *Suppose that all the assumptions of Theorem 3.3 hold. Also, we have the weighted inequality*

$$\begin{aligned}
&\left|\left(\int_a^b (u^\alpha - x^\alpha)\, w(u)\, d_\alpha u\right) D_\alpha f(x) + \alpha\left(\int_a^b w(u)\, d_\alpha u\right) f(x) - \alpha \int_a^b w(t)\, f(t)\, d_\alpha t\right| \\
\leq\ & \frac{\left\|D_\alpha^{(2)} f\right\|_{[a,b],\infty} \|w\|_{[a,b],\infty} (b^\alpha - a^\alpha)^3}{2\alpha^2} \left[\frac{1}{12} + \frac{\left(x^\alpha - \frac{a^\alpha + b^\alpha}{2}\right)^2}{(b^\alpha - a^\alpha)^2}\right].
\end{aligned}$$

Corollary 3.5. *If we take $w(u) = 1$ in Corollary 3.4, then we obtain the Ostrowski type α-fractional inequality*

$$\left|\left(\frac{a^\alpha + b^\alpha}{2} - x^\alpha\right) D_\alpha f(x) + \alpha f(x) - \frac{\alpha^2}{b^\alpha - a^\alpha} \int_a^b f(t)\, d_\alpha t\right| \tag{9}$$
$$\leq \frac{(b^\alpha - a^\alpha)^2}{2\alpha}\left[\frac{1}{12} + \frac{\left(x^\alpha - \frac{a^\alpha + b^\alpha}{2}\right)^2}{(b^\alpha - a^\alpha)^2}\right]\left\|D_\alpha^{(2)} f\right\|_{[a,b],\infty}.$$

Now, we give some inequalities related to (6) based on $\alpha-$fractional moments.

Corollary 3.6. *In addition to condition ofTheorem 3.3, if we assume w as a $\alpha-$fractional p.d.f. of a random variable, then we have*

$$\left|\frac{1}{\alpha}\left[\mu_\alpha(X) - x^\alpha\right] D_\alpha f(x) + f(x) - \alpha \int_a^b w(t) f(t)\, d_\alpha t\right| \tag{10}$$
$$\leq \frac{\left\|D_\alpha^{(2)} f\right\|_{[a,b],\infty}}{2\alpha^2}\left[\sigma_\alpha^2(X) + (x^\alpha - \mu_\alpha(X))^2\right]$$
$$\leq \frac{\left\|D_\alpha^{(2)} f\right\|_{[a,b],\infty}}{2\alpha^2}\left(\frac{b^\alpha - a^\alpha}{2} + \left|x^\alpha - \frac{a^\alpha + b^\alpha}{2}\right|\right)^2.$$

Corollary 3.7. *If we choose $x^\alpha = \mu_\alpha(X)$ in Corollary 3.6, then we have the following inequality*

$$\left|f(\mu_\alpha(X)) - \alpha \int_a^b w(t) f(t)\, d_\alpha t\right| \leq \frac{1}{2\alpha^2}\left\|D_\alpha^{(2)} f\right\|_{[a,b],\infty} M_{2,\alpha}(X).$$

It is also examined that weighted inequality for $p-$norm in the following theorem.

Theorem 3.8. *Suppose that all the assumptions of Lemma 3.2 hold. If $D_\alpha^{(2)} \in L_p[a,b]$, $\frac{1}{p} + \frac{1}{q} = 1$ and $p > 1$, then we have the following inequality*

$$\left|\left(\int_a^b (u^\alpha - x^\alpha) w(u)\, d_\alpha u\right) D_\alpha f(x) + \alpha\left(\int_a^b w(u)\, d_\alpha u\right) f(x) - \alpha \int_a^b w(t) f(t)\, d_\alpha t\right|$$
$$\leq \frac{\|w\|_{[a,b],\infty}\left\|D_\alpha^{(2)} f\right\|_p}{2\alpha}\left[\frac{(b^\alpha - x^\alpha)^{2q+1} + (x^\alpha - a^\alpha)^{2q+1}}{(2q+1)\alpha}\right]^{\frac{1}{q}}$$

where

$$\left\|D_\alpha^{(2)} f\right\|_p = \left(\int_a^b \left|D_\alpha^{(2)} f(t)\right|^p d_\alpha t\right)^{\frac{1}{p}}.$$

Proof. Taking modulus of (4) and then using Hölder's inequality, we can write

$$\left|\left(\int_a^b (u^\alpha - x^\alpha)\, w(u)\, d_\alpha u\right) D_\alpha f(x) + \alpha \left(\int_a^b w(u)\, d_\alpha u\right) f(x) - \alpha \int_a^b w(t)\, f(t)\, d_\alpha t\right|$$

$$\leq \left(\int_a^b |K_\alpha(x,t)|^q\, d_\alpha t\right)^{\frac{1}{q}} \left(\int_a^b \left|D_\alpha^{(2)} f(t)\right|^p d_\alpha t\right)^{\frac{1}{p}}$$

$$\leq \|w\|_{[a,b],\infty} \left\|D_\alpha^{(2)} f\right\|_p$$

$$\times \left[\int_a^x \left(\int_a^t (t^\alpha - u^\alpha)\, d_\alpha u\right)^q d_\alpha t + \int_x^b \left(\int_b^t (u^\alpha - t^\alpha)\, d_\alpha u\right)^q d_\alpha t\right]^{\frac{1}{q}}$$

which completes the proof. □

4. Some inequalities for Special Distribution (Density function) Function

In this section, we reconsider the inequalities (10) developed in previous section. The inequalities are illustrated for some well known distributions which have been used frequently in statistical modelling.

4.1. $\alpha-$fractional Uniform distribution

Consider a uniform distribution whose probability density function $f(x) = \frac{1}{b-a}$ with $a < x < b$. Substituting $w(t) = \frac{1}{b^\alpha - a^\alpha}$ into (2) and (3)

$$\mu_\alpha(X) = \frac{\alpha}{b^\alpha - a^\alpha} \int_a^b t^\alpha d_\alpha t = \frac{a^\alpha + b^\alpha}{2}$$

and

$$\sigma_\alpha^2(X) = \frac{\alpha}{b^\alpha - a^\alpha} \int_a^b t^{2\alpha} d_\alpha t - \mu_\alpha^2(X) = \frac{(b^\alpha - a^\alpha)^2}{12}$$

are obtained respectively. It should be also noted that (10) is reduced to (9) if we take the above $\alpha-$fractional mean and $\alpha-$fractional variance of uniform distribution.

4.2. $\alpha-$fractional Gamma distribution

If we choose the $\alpha-$fractional density function $w(t) = \frac{t^{\alpha(k-1)} e^{\frac{-t^\alpha}{\theta^\alpha}}}{\Gamma(k)\theta^{\alpha k}}$ with $k, \theta > 0$, then we have

$$\mu_\alpha(X) = \alpha \int_0^\infty t^\alpha \frac{t^{\alpha(k-1)} e^{\frac{-t^\alpha}{\theta^\alpha}}}{\Gamma(k)\, \theta^{\alpha k}} d_\alpha t = k\theta^\alpha$$

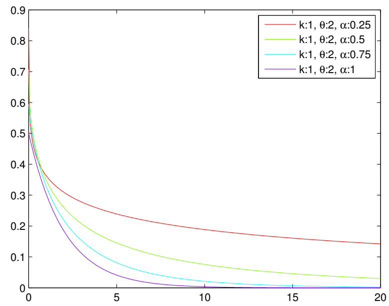
and

$$\sigma_\alpha^2(X) = \alpha \int_0^\infty t^{2\alpha} \frac{t^{\alpha(k-1)} e^{\frac{-t^\alpha}{\theta^\alpha}}}{\Gamma(k)\, \theta^{\alpha k}} d_\alpha t - \mu_\alpha^2(X) = k\theta^{2\alpha}.$$

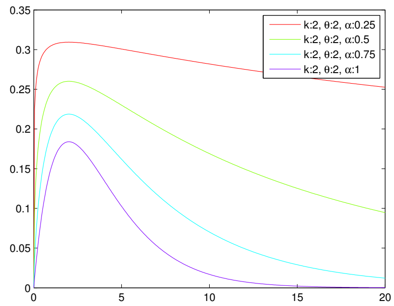
Substituting the above results in (10), the following inequality is hold:

$$\left| \frac{1}{\alpha} \left[k\theta^{\alpha} - x^{\alpha} \right] D_{\alpha} f(x) + f(x) - \alpha \int_{a}^{b} \frac{t^{\alpha(k-1)} e^{\frac{-t^{\alpha}}{\theta^{\alpha}}}}{\Gamma(k)\, \theta^{\alpha k}} f(t)\, d_{\alpha} t \right|$$

$$\leq \frac{\left\| D_{\alpha}^{(2)} f \right\|_{[a,b],\infty}}{2\alpha^2} \left[k\theta^{2\alpha} + \left(x^{\alpha} - k\theta^{\alpha} \right)^2 \right].$$



(a)



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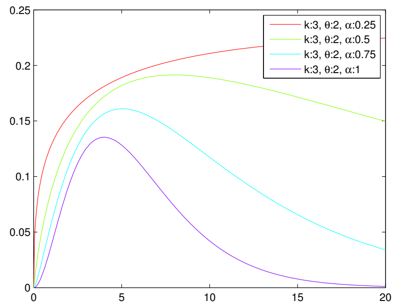

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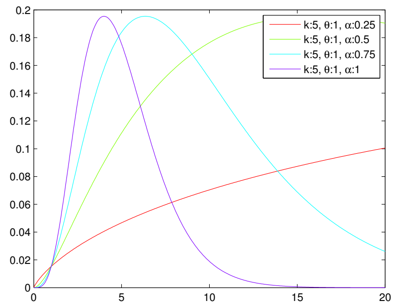

(d)

Figure 1: $\alpha-$fractional Gamma distributions

Figure 1 shows the pdf of gamma distribution for some representative α values. It can be easily seen from the graphs, if $\alpha = 1$, the pdf becomes well-known gamma distribution which have been widely used in statistical analysis especially in failure data and survival analysis. As α becomes smaller, the functions have more skewness and have more expansion.

4.3. $\alpha-$fractional Weibull distribution

Consider the $\alpha-$fractional density function $w(t) = \frac{k}{\lambda^{\alpha}} \left(\frac{t}{\lambda}\right)^{\alpha(k-1)} e^{-\left(\frac{t}{\lambda}\right)^{\alpha k}}$ with $k, \lambda > 0$. The $\alpha-$fractional mean and variance for the w function

$$\mu_{\alpha}(X) = \alpha \int_{0}^{\infty} t^{\alpha} \frac{k}{\lambda^{\alpha}} \left(\frac{t}{\lambda}\right)^{\alpha(k-1)} e^{-\left(\frac{t}{\lambda}\right)^{\alpha k}} d_{\alpha} t = \Gamma\left(1 + \frac{1}{k}\right) \lambda^{\alpha}$$

and

$$\sigma_{\alpha}^{2}(X) = \alpha \int_{0}^{\infty} t^{2\alpha} \frac{k}{\lambda^{\alpha}} \left(\frac{t}{\lambda}\right)^{\alpha(k-1)} e^{-\left(\frac{t}{\lambda}\right)^{\alpha k}} d_{\alpha}t - \mu_{\alpha}^{2}(X)$$
$$= \left[\Gamma\left(1+\frac{2}{k}\right) - \left(\Gamma\left(1+\frac{1}{k}\right)\right)^{2}\right] \lambda^{2\alpha}$$

are calculated respectively. Herewith, the result corresponding to (10)

$$\left| \frac{1}{\alpha}\left[\Gamma\left(1+\frac{1}{k}\right)\lambda^{\alpha} - x^{\alpha}\right] D_{\alpha} f(x) + f(x) - \alpha \int_{a}^{b} \frac{k}{\lambda^{\alpha}} \left(\frac{t}{\lambda}\right)^{\alpha(k-1)} e^{-\left(\frac{t}{\lambda}\right)^{\alpha k}} f(t)\, d_{\alpha}t \right|$$
$$\leq \frac{\left\| D_{\alpha}^{(2)} f \right\|_{[a,b],\infty}}{2\alpha^{2}} \left\{ \left[\Gamma\left(1+\frac{2}{k}\right) - \left(\Gamma\left(1+\frac{1}{k}\right)\right)^{2}\right] \lambda^{2\alpha} + \left(x^{\alpha} - \Gamma\left(1+\frac{1}{k}\right)\lambda^{\alpha}\right)^{2} \right\}.$$

is obtained.

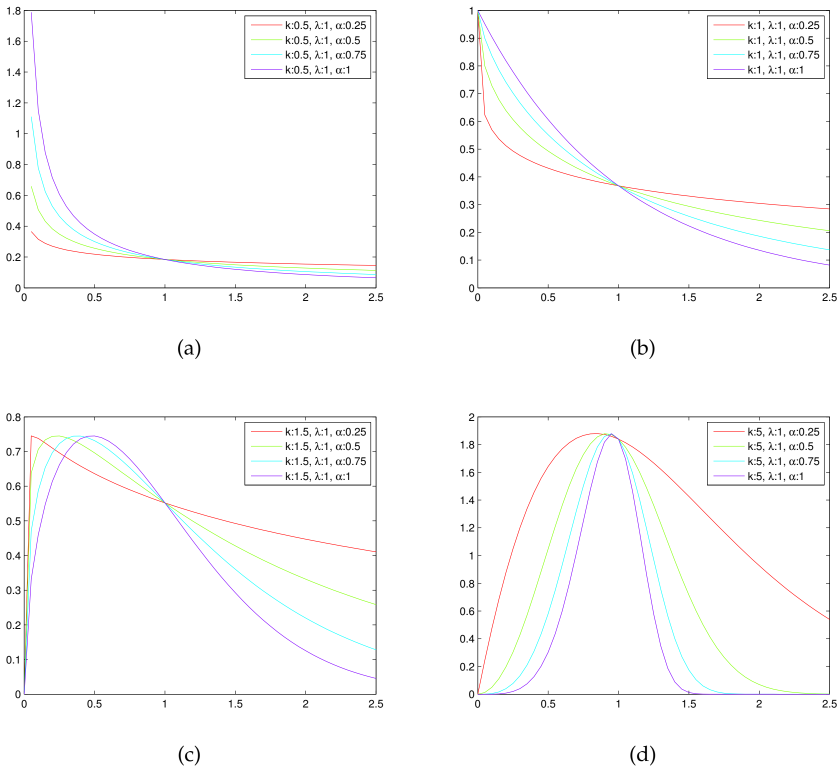

Figure 2: $\alpha-$fractional Weibull distributions

Figure 2 shows the behaviour of pdf of weibull distribution according to the fractional values α. Weibull distribution have been widely used in earth science modellind In the same way, as α becomes smaller, the functions have more deviation and skewness.

5. Applications to Numerical Integration

We now deal with the applications of the integral inequalities involving conformable fractional integral. Also, the new approach to the estimation of the composite quadrature rules is proposed.

Consider a partition $I_n : a = x_0 < x_1 < ... < x_{n-1} < x_n = b$ on the interval $[a, b]$, with $\xi_i \in [x_i, x_{i+1}]$ for $i = 0, ..., n-1$. Define the quadrature

$$S(f, I_n, \xi) := \frac{1}{\alpha^2} \sum_{i=0}^{n-1} \left[\mu_i(X) - \xi_i^\alpha \right] D_\alpha f(\xi_i) + \frac{1}{\alpha} \sum_{i=0}^{n-1} f(\xi_i), \tag{11}$$

where $\mu_i(X) = \mu_\alpha (x_i, x_{i+1})$.

Theorem 5.1. *Let $\alpha \in (0, 1]$ and $f : [a, b] \to \mathbb{R}$ be a continuous on $[a, b]$ and twice α−fractional differentiable function on (a, b) with $0 \le a < b$. If we assume w as a α−fractional p.d.f. of a random variable, then we have the representation*

$$\int_a^b w(t) f(t) d_\alpha t = S(f, I_n, \xi) + R(f, I_n, \xi),$$

where $S(f, I_n, \xi)$ is as defined in (11) and the remainder satisfies the astimation:

$$\begin{aligned} & \left| R(f, I_n, \xi) \right| \\ \le\ & \frac{\left\| D_\alpha^{(2)} f \right\|_{[a,b],\infty}}{2\alpha^3} \sum_{i=0}^{n-1} \left[\sigma_i^2 (X) + \left(\xi_i^\alpha - \mu_i(X) \right)^2 \right] \\ \le\ & \frac{\left\| D_\alpha^{(2)} f \right\|_{[a,b],\infty}}{2\alpha^3} \sum_{i=0}^{n-1} \left(\frac{h_i}{2} + \left| \xi_i^\alpha - \frac{x_i^\alpha + x_{i+1}^\alpha}{2} \right| \right)^2, \end{aligned} \tag{12}$$

where $h_i = (x_{i+1}^\alpha - x_i^\alpha)$, $\mu_i(X) = \mu_\alpha (x_i, x_{i+1})$ and $\sigma_i^2 (X) = \sigma_\alpha^2 (x_i, x_{i+1})$.

Proof. Applying Theorem 3.3 over the interval $[x_i, x_{i+1}]$ for the intermadiate points $x = \xi_i$, we obtain

$$\begin{aligned} & \left| \frac{1}{\alpha^2} \left[\mu_i(X) - \xi_i^\alpha \right] D_\alpha f(\xi_i) + \frac{1}{\alpha} f(\xi_i) - \int_{x_i}^{x_{i+1}} w(t) f(t) d_\alpha t \right| \\ \le\ & \frac{\left\| D_\alpha^{(2)} f \right\|_{[a,b],\infty}}{2\alpha^3} \left[\sigma_i^2 (X) + \left(\xi_i^\alpha - \mu_i(X) \right)^2 \right] \\ \le\ & \frac{\left\| D_\alpha^{(2)} f \right\|_{[a,b],\infty}}{2\alpha^3} \left(\frac{h_i}{2} + \left| \xi_i^\alpha - \frac{x_i^\alpha + x_{i+1}^\alpha}{2} \right| \right)^2 \end{aligned}$$

for all $i = 0, ..., n-1$. The estimation (12) is produced with summing over i from 0 to $n-1$ and using the triangle inequality. □

Now, define the mid-point rule for the weighted integrals as the following:

$$M(f, I_n) := \frac{1}{\alpha} \sum_{i=0}^{n-1} f(\mu_i(X)),$$

where $\mu_i(X) = \mu_\alpha (x_i, x_{i+1})$ for $i = 0, ..., n-1$.

Corollary 5.2. *Under the same assumptions of Theorem 5.1 with $\xi_i^\alpha = \mu_i(X)$. Then, we have the representation*

$$\int_a^b w(t) f(t) d_\alpha t = M(f, I_n) + R(f, I_n),$$

where the remainder is bounded by

$$\left|R(f, I_n)\right| \leq \frac{1}{2\alpha^3} \left\|D_\alpha^{(2)} f\right\|_{[a,b],\infty} \sum_{i=0}^{n-1} M_i(X),$$

where $\mu_i(X) = \mu_\alpha\left(x_i^\alpha, x_{i+1}^\alpha\right)$ and $M_i(X) = M_{2,\alpha}(x_i, x_{i+1})$.

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