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A New Three-Parameter Discrete Distribution With Associated INAR(1) Process and Applications

M. S. ELIWA[1], EMRAH ALTUN[2], M. EL-DAWOODY[3,4], AND M. EL-MORSHEDY[1,3]

[1]Department of Mathematics, Faculty of Science, Mansoura University, Mansoura 35516, Egypt
[2]Department of Mathematics, Bartin University, Bartin 74100, Turkey
[3]Department of Mathematics, College of Sciences and Humanities Studies, Prince Sattam Bin Abdulaziz University, Al-Kharj 11942, Saudi Arabia
[4]Department of Mathematics, Faculty of Science, Aswan University, Aswan 81528, Egypt

Corresponding author: M. S. Eliwa (mseliwa@mans.edu.eg)

ABSTRACT The aim of this article is to propose a new three-parameter discrete Lindley distribution. A wide range of its structural properties are investigated. This includes the shape of the probability mass function, hazard rate function, moments, skewness, kurtosis, index of dispersion, mean residual life, mean past life and stress-strength reliability. These properties are expressed in explicit forms. The maximum likelihood approach is used to estimate the model parameters. A detailed simulation study is carried out to examine the bias and mean square error of the estimators. Using the proposed distribution, a new first-order integer-valued autoregressive process is introduced for the over-dispersed, equi-dispersed and under-dispersed time series of counts. To demonstrate the importance of the proposed distribution, three data sets on coronavirus, length of stay at psychiatric ward and monthly counts of larceny calls are analyzed.

INDEX TERMS Survival discretization method, over-dispersion, INAR(1) process, simulation.

I. INTRODUCTION

Statistical distributions play an important role in data modeling, inference, and forecasting processes. The occurrence times, frequencies and effects of many events in nature are analyzed by statistical modeling techniques. Most of the events in nature or other scientific fields have their own characteristics. Earthquakes, traffic accidents, counts of landslide or number of people dying from the disease are modeled by discrete probability distributions. Researchers have proposed more flexible distributions to reduce estimation errors in the modeling of these data sets. There are two popular methods used to introduce a new discrete distribution. These are mixed-Poisson type discrete distributions and survival discretization method. The recently introduced discrete distribution based on the survival discretization method can be cited as follows: discrete Lindley (DLi) distribution by Gómez-Déniz and Calderín-Ojeda (2011), discrete inverse Weibull (DIW) distribution by Jazi *et al.* (2010), discrete Burr type XII (DB-XII) distribution by Para and Jan (2014), discrete Pareto (DPa) distribution by Krishna and Pundir (2009), generalized geometric by Gómez-Déniz (2010), discrete generalized exponential type II by Nekoukhou *et al.* (2013), discrete Rayleigh (DR) by Roy (2004), two-parameter discrete Lindley by Hussain *et al.* (2016), discrete extended Weibull distribution by Jia *et al.* (2019), discrete Gompertz-G family of distributions by Eliwa *et al.* (2020), exponentiated discrete Lindley by El-Morshedy *et al.* (2020) and discrete Burr-Hatke (DBH) distribution by El-Morshedy *et al.* (2020). Recently, a new one-parameter discrete distribution was introduced by Eliwa and El-Morshedy (2020). The Poisson-Lindley distribution, introduced Sankaran (1970) is one of the popular discrete distributions obtained by assuming that the parameter of the Poisson distribution follows the Lindley distribution. After the work of Sankaran (1970), several generalization of the Poisson-Lindley distribution was proposed such as generalized Poisson-Lindley distribution by Mahmoudi and Zakerzadeh (2010), a new generalized Poisson-Lindley distribution by Bhati *et al.* (2015), a new three-Parameter Poisson-Lindley distribution by Das *et al.* (2018) and Poisson-generalized Lindley distribution by Wongrin and Bodhisuwan (2016). Using the similar approach to Sankaran (1970), several authors have been introduced mixed-Poisson distributions such as Poisson-Bilal

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distribution by Altun (2020a), a new Poisson-weighted exponential distribution by Altun (2020b), Poisson-xgamma distribution Altun *et al.* (2020), Poisson-weighted Lindley distribution by Atikankul *et al.* (2020) and Poisson-transmuted exponential distribution by Bhati *et al.* (2017).

Additionally, the modeling of the time series of counts is an important research area for all applied sciences. For instance, a company in insurance sector cares about predicting the number of claims for next month. Another example is that epidemiologist wants to predict the monthly deaths from a disaster such as coronavirus, bird flu, ebola virus infection. In this case, first-order integer-valued autoregressive process, shortly INAR(1) process, can be used. The INAR(1) process with Poisson innovations was developed by McKenzie (1985) and Al-Osh and Alzaid (1987). After the pioneer works of McKenzie (1985) and Al-Osh and Alzaid (1987), the INAR(1) process have increased its popularity and researchers have focused on the innovation processes of the INAR(1) process to make it more flexible for modeling the over(under)-dispersed time series of counts. The INAR(1) process with geometric innovations (INAR(1)G) by Jazi *et al.* (2012), INAR(1) process with Poisson-Lindley innovations (INAR(1)PL) by Lívio *et al.* (2018) and INAR(1) process with a new Poisson-weighted exponential innovations ((INAR(1)NPWE)) by Altun (2020b) can be given as examples for over-dispersed INAR(1) process.

The goal of this work is to introduce an alternative discrete distribution to model both over-dispersed and under-dispersed count data sets. The over-dispersion and under-dispersion is a widely studied problem of count data modeling. The over-dispersion appears in the case that the empirical variance is greater than empirical mean. The opposite indicates the under-dispersion. In the real-life data modeling, we encounter quite often with these two problems, however, the over-dispersion problem is seen more. We introduce a new three parameter discrete distribution by using the three-parameter Lindley (Li-3P) distribution introduced by Shanker *et al.* (2017) based on the survival discretization method. The proposed distribution is called three parameter discrete-Lindley, shortly (DLi-3P), distribution. The contributions of the presented study to statistics literature can be summarized as follows: (i) an alternative model for the over(under)-dispersed data sets is introduced, (ii) the statistical properties of the DLi-3P distributions is studied in detail, (iii) INAR(1) process with DLi-3P innovations is introduced, (iv) three applications to coronavirus, psychiatric ward and larceny calls data sets are analyzed by the models introduced based on the DLi-3P model.

The article is organized as follows. In Section 2, we introduce the DLi-3P distribution. Different statistical and reliability properties are discussed in Section 3. In Section 4, the model parameters are estimated by using the maximum likelihood estimation (MLE) approach. In Section 5, we discuss the computational complexity of the DLi-3P distribution its limitations. In Section 6, INAR(1) process with DLi-3P innovations is introduced and its statistical properties are derived. Simulation study is presented in Section 7. Three distinctive data sets are analyzed to show the importance of the DLi-3P distribution in Section 8. The detail interpretation of the empirical results are given in Section 9. Finally, Section 10 provides some conclusions.

II. THE DLi-3P DISTRIBUTION

In this section, we derive the discrete analogous of the Li-3P distribution, shortly DLi-3P distribution, by using the survival discretization method. Assume that the random variable X follows a Li-3P distribution whose probability density function (pdf) and the corresponding survival function (sf) are given, respectively, by

$$f(x;\theta,\alpha,\beta)=\frac{\theta^2}{\alpha\theta+\beta}(\alpha+\beta x)\,e^{-\theta x};\quad x>0 \tag{1}$$

and

$$S(x;\theta,\alpha,\beta)=\left(1+\frac{\beta\theta x}{\alpha\theta+\beta}\right)e^{-\theta x};\quad x>0, \tag{2}$$

where $\theta > 0$, $\beta > 0$ and $\alpha\theta+\beta > 0$. The derivation of Li-3P distribution is similar to the Lindley distribution. The Li-3P distribution is obtained as mixture distribution of exponential (θ) and gamma $(2,\theta)$ with a mixing proportion $\frac{\alpha\theta}{\alpha\theta+\beta}$. Using the survival discretization method and survival function of the Li-3P distribution given in (2), the probability mass function (pmf) of the DLi-3P distribution with positive parameter $0<\lambda<1$ can be expressed as

$$\begin{aligned}P_x(x;\lambda,\alpha,\beta)&=\frac{\lambda^x}{\beta-\alpha\ln\lambda}\\&\quad\times\begin{Bmatrix}\beta-\alpha\ln\lambda-\beta x\ln\lambda\\-\lambda\left(\beta-\alpha\ln\lambda-\beta(x+1)\ln\lambda\right)\end{Bmatrix};\ x\in\mathbb{N}_0,\end{aligned} \tag{3}$$

where $\lambda=e^{-\theta}$ and $\mathbb{N}_0=\{0,1,2,3,\ldots,q\}$ for $0<q<\infty$. The corresponding cdf and sf to (3) are given, respectively, by

$$F(x;\lambda,\alpha,\beta)=1-\left(1-\frac{\beta(x+1)\ln\lambda}{\beta-\alpha\ln\lambda}\right)\lambda^{x+1};\quad x\in\mathbb{N}_0 \tag{4}$$

and

$$S(x;\lambda,\alpha,\beta)=\left(1-\frac{\beta(x+1)\ln\lambda}{\beta-\alpha\ln\lambda}\right)\lambda^{x+1};\quad x\in\mathbb{N}_0. \tag{5}$$

The pmf in (3) is log-concave, where $\frac{P_x(x+1;\lambda,\alpha,\beta)}{P_x(x;\lambda,\alpha,\beta)}$ is a decreasing function in x for all values of the model parameters. The several possible shapes of the DLi-3P distribution are displayed in Figure 1. From this figure, we conclude that the DLi-3P distribution could be used to model left-skewed count data sets.

The shapes of the pmf of DLi-3P can be unimodal or decreasing. The hazard rate function (hrf) of the DLi-3P distribution is

$$h(x;\lambda,\alpha,\beta)=\frac{\begin{pmatrix}-\alpha\ln\lambda-\beta x\ln\lambda\\-\lambda\left(\beta-\alpha\ln\lambda-\beta(x+1)\ln\lambda\right)\end{pmatrix}}{\beta-\alpha\ln\lambda-\beta x\ln\lambda}, \tag{6}$$

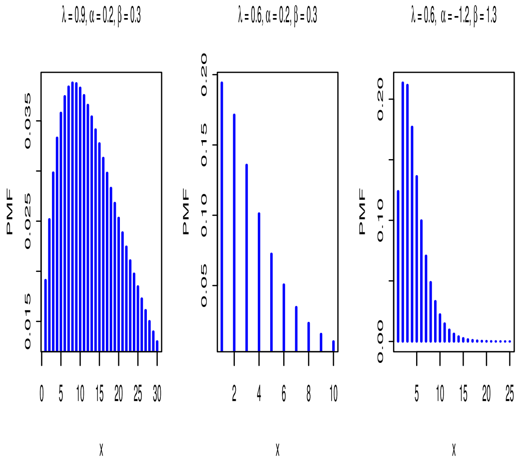

FIGURE 1. The possible pmf shapes of the DLi-3P distribution.

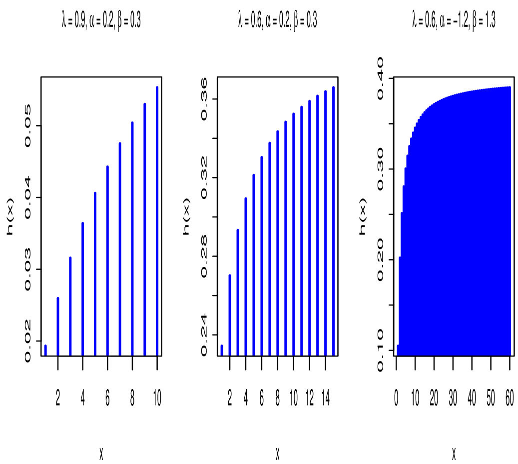

FIGURE 2. The hrf shapes of the DLi-3P distribution for selected parameter values.

where $x \in \mathbb{N}_0$. Figure 2 displays the hrf plots of the DLi-3P distribution for different values of the model parameters. It is observed that the hrf of the DLi-3P distribution has increasing shape.

III. STATISTICAL PROPERTIES

In this section, the statistical properties of the DLi-3P distribution are derived such as mode, raw moments, skewness and kurtosis measures. Additionally, reliability properties of the DLi-3P distribution are derived such as stress-strength, mean residual life (mrl) and mean past life (mpl).

A. MODE

The mode of any discrete distribution shows the value at which the specific discrete distribution takes its maximum value. If X has a DLi-3P distribution, then the mode can be obtained by solving the following non-linear equation

$$\lambda^{x+1}(\beta x+\alpha+\beta)-\lambda^{x}(\beta x+\alpha)=0. \tag{7}$$

Then, the mode of the DLi-3P distribution can be expressed as

$$\mathrm{M}(X)=-\frac{\alpha\lambda+\beta\lambda-\alpha}{\beta(\lambda-1)}. \tag{8}$$

B. MOMENTS, SKEWNESS, KURTOSIS AND DISPERSION INDEX

Assume X be a DLi-3P random variable. Then, the probability generating function (pgf) can be expressed in a closed form as

$$G_X(s)=\frac{\left\{-s\alpha\lambda^2+([1-s]\beta+[1+s]\alpha)\lambda-\alpha\right\}\times\ln\lambda+\beta(\lambda-1)(\lambda s-1)}{(\beta-\alpha\ln\lambda)(\lambda s-1)^2}, \tag{9}$$

where $G_X(s)=\sum_{x=0}^{\infty}s^xP_x(x;\lambda,\alpha,\beta)$. Replacing s by e^s in (9), we get the moment generating function (mgf). The first four derivatives of the mgf, with respect to s at $s=0$, give the first four raw moments. Thus, the first four moments of the DLi-3P model are

$$\mathrm{E}(X)=\lambda\frac{(-\alpha\lambda+\alpha+\beta)\ln\lambda+(\lambda-1)\beta}{(\alpha\ln\lambda-\beta)(\lambda-1)^2}, \tag{10}$$

$$\mathrm{E}(X^2)=\lambda\frac{(-\alpha\lambda^2+3\beta\lambda+\alpha+\beta)\ln\lambda+(\lambda^2-1)\beta}{(\beta-\alpha\ln\lambda)(\lambda-1)^3}, \tag{11}$$

$$\mathrm{E}(X^3)=\lambda\frac{(-\alpha\lambda^3-3\alpha\lambda^2+7\beta\lambda^2+3\alpha\lambda+10\beta\lambda+\alpha+\beta)\times\ln\lambda+(\lambda-1)(\lambda^2+4\lambda+1)\beta}{(\alpha\ln\lambda-\beta)(\lambda-1)^4} \tag{12}$$

and

$$\mathrm{E}(X^4)=\lambda\frac{(-\alpha\lambda^4-10\alpha\lambda^3+15\beta\lambda^3+55\beta\lambda^2+10\alpha\lambda+25\beta\lambda+\alpha+\beta)\ln\lambda+(\lambda^2-1)(\lambda^2+10\lambda+1)\beta}{(\beta-\alpha\ln\lambda)(\lambda-1)^5}. \tag{13}$$

The variance and dispersion index (DI) of the DLi-3P distribution are given, respectively, by (14) and (15), as shown at the bottom of the next page.

The skewness and kurtosis can be derived also in explicit forms by using the below quantities.

$$\mathrm{Skewness}(X)=\frac{\mathrm{E}(X^3)-3\mathrm{E}(X^2)\mathrm{E}(X)+2\,[\mathrm{E}(X)]^3}{[\mathrm{Var}(X)]^{3/2}}$$

and

$$\mathrm{Kurtosis}(X)=\frac{\mathrm{E}(X^4)-4\mathrm{E}(X^2)\mathrm{E}(X)+6\mathrm{E}(X^2)\,[\mathrm{E}(X)]^2-3\,[\mathrm{E}(X)]^4}{[\mathrm{Var}(X)]^2}.$$

The DI can be calculated by dividing the sample variance to sample mean. When the DI is equal to one, it indicates the equi-dispersion. When the DI is greater than one, it indicates the over-dispersion, opposite case indicates the under-dispersion. Table 1 presents some numerical results of the mean, variance, DI, skewness and kurtosis for the DLi-3P distribution for different values of the model parameters.

TABLE 1. Some descriptive statistics for the DLi-3P distribution for $\alpha = 0.01$, $\beta = 0.5$ and various values of λ.

Measure	λ										
	0.01	0.05	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
Mean	0.0531	0.2092	0.3828	0.7372	1.1483	1.6664	2.3673	3.3962	5.0878	8.4430	18.4625
Variance	0.0522	0.2039	0.3817	0.8059	1.4309	2.4443	4.232	7.7392	15.799	40.246	180.248
DI	0.9834	0.9748	0.9970	1.0931	1.2460	1.4668	1.7878	2.2787	3.1053	4.7668	9.7629
Skewness	4.2339	2.1306	1.6674	1.4270	1.3789	1.3761	1.3852	1.3957	1.4042	1.4100	1.4132
Kurtosis	20.3698	7.5370	6.1259	5.7422	5.7709	5.8390	5.8977	5.9408	5.9699	5.9879	5.9972

$$\mathrm{Var}(X) = \frac{\lambda\begin{pmatrix}\alpha^2\lambda^2\log(\lambda)^2 - 2\alpha^2\lambda\log(\lambda)^2 + \alpha^2\log(\lambda)^2 \\ -\alpha\beta\lambda^2\log(\lambda)^2 - 2\alpha\beta\lambda^2\log(\lambda) + 4\alpha\beta\lambda\log(\lambda) \\ +\alpha\beta\log(\lambda)^2 - 2\alpha\beta\log(\lambda) + \beta^2\lambda^2\log(\lambda) + \beta^2\lambda^2 \\ -\beta^2\lambda\log(\lambda)^2 - 2\beta^2\lambda - \beta^2\log(\lambda) + \beta^2\end{pmatrix}}{(\beta - \alpha\log(\lambda))^2(\lambda-1)^4} \tag{14}$$

$$DI = \frac{\begin{pmatrix}\alpha^2\lambda^2\log(\lambda)^2 - 2\alpha^2\lambda\log(\lambda)^2 + \alpha^2\log(\lambda)^2 - \alpha\beta\lambda^2\log(\lambda)^2 \\ -2\alpha\beta\lambda^2\log(\lambda) + 4\alpha\beta\lambda\log(\lambda) + \alpha\beta\log(\lambda)^2 - 2\alpha\beta\log(\lambda) \\ +\beta^2\lambda^2\log(\lambda) + \beta^2\lambda^2 - \beta^2\lambda\log(\lambda)^2 - 2\beta^2\lambda - \beta^2\log(\lambda) + \beta^2\end{pmatrix}}{(\alpha\log(\lambda)-\beta)\ (\lambda-1)^2\ (\beta\lambda - \beta + \alpha\log(\lambda) + \beta\log(\lambda) - \alpha\lambda\log(\lambda))}. \tag{15}$$

From Table 1, the DLi-3P model is appropriate for modeling under(over)-dispersed data sets. Moreover, this model is capable of modeling positively skewed and leptokurtic data sets.

C. MEAN RESIDUAL LIFE AND MEAN PAST LIFE

The mrl and mpl are two commonly used measures to study the ageing behavior of a component or a system of components. The mrl is used to model the burn-in and maintenance of the component. The mrl is defined as

$$\begin{aligned}\Omega_i &= \mathrm{E}(T - i|T \geq i) \\ &= \frac{1}{1 - F(i-1;\lambda,\alpha,\beta)} \\ &\quad \times \sum_{j=i+1}^{q}[1 - F(j-1;\lambda,\alpha,\beta)]; \quad i \in \mathbb{N}_0, \end{aligned}\tag{16}$$

where $\mathbb{N}_0 = \{0, 1, 2, 3, \ldots, q\}$ for $0 < q < \infty$. Let T be a DLi-3P random variable. Then, the mrl, say Ω_i, can be expressed in a closed form as

$$\Omega_i = -\lambda\frac{\begin{pmatrix}\beta\lambda i\ln\lambda + \alpha\lambda\ln\lambda - \beta i\ln\lambda \\ -\alpha\ln\lambda - \beta\ln\lambda - \beta\lambda + \beta\end{pmatrix}}{(\beta i\ln\lambda + \alpha\ln\lambda - \beta)(\lambda-1)^2}. \tag{17}$$

The other reliability measure is mpl which measures the time elapsed since the failure of T given that the system has failed sometime before i. The mpl, Ω_i^*, is given by

$$\begin{aligned}\Omega_i^* = \mathrm{E}(i - T|T < i) &= \frac{1}{F(i-1;\lambda,\alpha,\beta)} \\ &\quad \times \sum_{m=1}^{i} F(m-1;\lambda,\alpha,\beta); \quad i \in \mathbb{N}_0 - \{0\}. \end{aligned}\tag{18}$$

where $\Omega_0^* = 0$. If T be a DLi-3P random variable, then the mpl can be represented in a closed form as

$$\begin{aligned}\Omega_i^* &= \frac{1}{1 - \lambda^i\left(1 - \frac{\beta i\ln\lambda}{\beta - \alpha\ln\lambda}\right)} \\ &\quad \times \left\{i - \frac{\lambda^{i+1} - \lambda}{\lambda - 1} - \frac{\beta\lambda^{i+1}([\lambda-1]i - 1) + \beta\lambda\ln\lambda}{(\alpha\ln\lambda - \beta)(\lambda-1)^2}\right\}.\end{aligned}$$

For $i \in \mathbb{N}_0$, we get $\Omega_i^* \leq i$. The mean of the distribution function can be expressed as

$$\begin{aligned}\text{Mean} &= i - \Omega_i^* F(i-1;\lambda,\alpha,\beta) \\ &\quad + \Omega_i[1 - F(i-1;\lambda,\alpha,\beta)]; \quad i \in \mathbb{N}_0 - \{0\}. \end{aligned}\tag{19}$$

The reversed hrf (rhrf) and the mpl are related as

$$r(i;\lambda,\alpha,\beta) = \frac{1 - \Omega_{i+1}^* + \Omega_i^*}{\Omega_i^*}; \quad i \in \mathbb{N}_0 - \{0\}. \tag{20}$$

If T be a DLi-3P random variable, then the cdf can be recovered by the MPL as

$$\begin{aligned}&F(k;\lambda,\alpha,\beta) \\ &\quad = F(0;\lambda,\alpha,\beta)\prod_{i=1}^{k}\left[\frac{\Omega_i^*}{\Omega_{i+1}^* - 1}\right]; \quad k \in \mathbb{N}_0 - \{0\}, \end{aligned}\tag{21}$$

where $F(0;\lambda,\alpha,\beta) = \left(\prod_{i=1}^{q}\left[\frac{\Omega_i^*}{\Omega_{i+1}^* - 1}\right]\right)^{-1}$ and $0 < q < \infty$.

D. STRESS-STRENGTH ANALYSIS

Stress-strength (Str-Sth) has many applications in different scientific fields. Let X_{Str} be a stress and X_{Sth} be a strength of the system. The expected reliability ($R_{Str-Sth}$) can be calculated by

$$R_{Str-Sth} = \Pr\left[X_{Str} \leq X_{Sth}\right] = \sum_{x=0}^{\infty} f_{X_{Str}}(x) R_{X_{Sth}}(x). \quad (22)$$

Let $X_{Str} \sim$ DLi-3P$(\lambda_1, \alpha_1, \beta_1)$ and $X_{Sth} \sim$ DLi-3P$(\lambda_2, \alpha_2, \beta_2)$. Then, $R_{Str-Sth}$ can be represented in a closed form as

$$R_{\text{Str-Sth}} = \frac{\alpha_1 \lambda_2{}^2 \lambda_1{}^3 (\alpha_2 - \beta_2) - (\beta_2 - \alpha_2)(\beta_1 - \alpha_1) \times \lambda_2 \lambda_2 \lambda_1{}^2 - 2\beta_1 \beta_2 \lambda_2 \lambda_1{}^2 + \alpha_2 \beta_1 \lambda_2 \lambda_1{}^2 - \alpha_1 \beta_2 \lambda_2 \lambda_1{}^2}{\left([\ln(\lambda_2) \ln(\lambda_1)]^{-1} (\beta_1 - \alpha_1 \ln(\lambda_1)) \times (\beta_2 - \alpha_2 \ln(\lambda_2)) (\lambda_2 \lambda_1 - 1)^3 \right)}$$
$$+ \frac{2\alpha_1 \alpha_2 \lambda_2 \lambda_1{}^2 + (-\beta_2 - \alpha_2) \lambda_1 \lambda_2 \beta_1 - \alpha_1 \lambda_1 \lambda_2 \times (\beta_2 - 2\alpha_2) + \lambda_1 \alpha_2 (\beta_1 + \alpha_1) - \alpha_1 \alpha_2}{\left([\ln(\lambda_2) \ln(\lambda_1)]^{-1} (\beta_1 - \alpha_1 \ln(\lambda_1)) \times (\beta_2 - \alpha_2 \ln(\lambda_2)) (\lambda_2 \lambda_1 - 1)^3 \right)}$$
$$- \frac{\beta_2 \begin{bmatrix} \alpha_1 \lambda_2 \lambda_1{}^2 + \beta_1 \lambda_1 \lambda_2 - \alpha_1 \lambda_1 \lambda_2 \\ -\beta_1 \lambda_1 - \alpha_1 \lambda_1 + \alpha_1 \end{bmatrix} \ln(\lambda_1)}{\left((\beta_1 - \alpha_1 \ln(\lambda_1)) (\beta_2 - \alpha_2 \ln(\lambda_2)) \times (\lambda_2 \lambda_1 - 1)^2 \right)}$$
$$+ \frac{(\lambda_1 - 1) \beta_1 \begin{bmatrix} (\lambda_2 \lambda_1 \beta_2 - \lambda_1 \alpha_2 + \alpha_2) \ln(\lambda_2) \\ +\lambda_2 \lambda_1 \beta_2 - \beta_2 \end{bmatrix}}{(\beta_1 - \alpha_1 \ln(\lambda_1)) \times (\beta_2 - \alpha_2 \ln(\lambda_2)) (\lambda_2 \lambda_1 - 1)^2}. \quad (23)$$

From (23), the value of $R_{Str-Sth}$ depends only on the model parameters. Some numerical results of $R_{Str-Sth}$ are reported in Table 2 by using the DLi-3P distribution for the parameters $\alpha_1 = \alpha_2 = 0.01$ and $\beta_1 = \beta_2 = 0.5$.

It is clear that $R_{Str-Sth}$ increases (decreases) with $\lambda_2 \longrightarrow 1$ ($\lambda_1 \longrightarrow 1$) for fixed value of the other parameters.

E. GENERATING RANDOM VARIABLES FROM DLi-3P DISTRIBUTION

We introduce an algorithm to generate random variables from the DLi-3P distribution. The below algorithm could be used for this purpose.

1) Set the parameter values $\lambda = exp(-\theta)$, α β
2) Generate random variable, u, from the standard uniform distribution, U (0, 1).
3) Compute

$$Z = -\frac{\alpha\theta + \beta + \beta W_{-}\left(\begin{matrix} \exp\left(-(\alpha\theta)/\beta - 1\right) \\ \times (u-1)(\beta + \alpha\theta)\beta^{-1} \end{matrix} \right)}{\beta\theta} \quad (24)$$

4) $X = \lfloor Z \rfloor$

To generate random sample of size n from the DLi-3P distribution, the steps 1-4 should be repeated n times. The function $W_{-}(\cdot)$ represents the negative branch of the Lambert-W function.

IV. MAXIMUM LIKELIHOOD ESTIMATION

Assume that the random sample $x_1, x_2, \ldots, x_n$ come from the DLi-3P distribution with unknown parameters λ, α and β. The log-likelihood function of the DLi-3P is

$$\ell(x; \lambda, \alpha, \beta) = \sum_{i=1}^{n} \ln \left\{ \begin{matrix} \beta(\lambda(\log(\lambda) - 1) + 1) \\ + (\lambda - 1)\log(\lambda)(\alpha + \beta x_i) \end{matrix} \right\} + \ln(\lambda) \sum_{i=1}^{n} x_i - n \ln(\beta - \alpha \log(\lambda)) \quad (25)$$

By differentiating (25) with respecto the unknown parameters, we have, $\frac{\partial \ell}{\partial \lambda}$, $\frac{\partial \ell}{\partial \alpha}$, and $\frac{\partial \ell}{\partial \beta}$, as shown at the bottom of this page.

The simultaneous solutions of these likelihood equations give the MLEs of the model parameters. However, these equations cannot be solved analytically; therefore, an iterative procedure like Newton-Raphson is required to solve it numerically. Here, we use the **constrOptim** function of R software to maximize the log-likelihood function of the DLi-3P distribution given in (25). The standard errors of the estimated parameters are obtained by means of the squared root of the inverse of the hessian matrix evaluated at estimated model parameters. The **fdHess** function of R software is used to obtain hessian matrix.

V. THE COMPUTATIONAL COMPLEXITY AND LIMITATIONS OF THE DLi-3P DISTRIBUTION

Proposing a new distribution with adding one or more additional shape parameters increases the model complexity. The proposed distribution, DLi-3P, contains three parameter which requires a good initial parameter vector in estimation step. More importantly, the domain of the parameter λ

$$\frac{\partial \ell}{\partial \lambda} = \frac{1}{\lambda} \sum_{i=1}^{n} x_i - \sum_{i=1}^{n} \frac{\left[\alpha\beta^{-1} - (\ln\lambda)^{-1}\right]^2 + (x_i + 1)\left[\alpha\beta^{-1} - (\ln\lambda)^{-1}\right] + (x_i - \lambda(x_i + 1))(\ln\lambda)^{-2}}{(1-\lambda)\left[\alpha\beta^{-1} - (\ln\lambda)^{-1}\right]^2 + (x_i - \lambda(x_i+1))\left[\alpha\beta^{-1} - (\ln\lambda)^{-1}\right]},$$

$$\frac{\partial \ell}{\partial \alpha} = \sum_{i=1}^{n} \frac{-(x_i - \lambda(x_i+1))\beta^{-1}}{(1-\lambda)\left[\alpha\beta^{-1} - (\ln\lambda)^{-1}\right]^2 + (x_i - \lambda(x_i+1))\left[\alpha\beta^{-1} - (\ln\lambda)^{-1}\right]}$$

$$\frac{\partial \ell}{\partial \beta} = \sum_{i=1}^{n} \frac{(x_i - \lambda(x_i+1))\alpha\beta^{-2}}{(1-\lambda)\left[\alpha\beta^{-1} - (\ln\lambda)^{-1}\right]^2 + (x_i - \lambda(x_i+1))\left[\alpha\beta^{-1} - (\ln\lambda)^{-1}\right]}.$$

TABLE 2. Some numerical results of $R_{Str-Sth}$ for various values of the parameters λ_1 and λ_2.

$\lambda_1 \downarrow \lambda_2 \longrightarrow$	**0.01**	**0.1**	**0.3**	**0.5**	**0.7**	**0.9**	**0.99**
0.01	0.9505	0.9642	0.9815	0.9915	0.9971	0.9997	0.9999
0.1	0.6938	0.7676	0.8684	0.9341	0.9762	0.9973	0.9999
0.3	0.3657	0.4699	0.6414	0.7867	0.9081	0.9874	0.9998
0.5	0.1714	0.2507	0.4085	0.5839	0.7811	0.9617	0.9995
0.7	0.0580	0.0956	0.1866	0.3226	0.5441	0.8813	0.9980
0.9	0.0061	0.0113	0.0267	0.0590	0.14715	0.5131	0.9792
0.99	0.0000	0.0001	0.0003	0.0007	0.0024	0.0221	0.5012

is (0, 1). The estimation of the parameter λ may yield a value which is outside of its domain. To overcome these problems one should take into consideration the points given below.

✓ The initial parameter vector should be correctly determined. The generalized simulated annealing method is implemented to obtain a resonable initial vector. The **GenSA** package of **R** software is used for this purpose.

✓ Since the domain of the parameter λ is (0, 1), the constrained optimization algorithm should be used to obtain estimated of the parameter λ. The **constrOptim** function of **R** software is used for this purpose.

VI. INAR(1)DLi-3P PROCESS

The INAR(1) process is widely used to model the time series of counts in different applied sciences such actuarial, finance, medical sciences. The INAR(1) process differs from the first-order autoregressive, shortly AR(1), process by applying the binomial thinning operator. The INAR(1) process is given by

$$X_t = p \circ X_{t-1} + \varepsilon_t, \quad t \in \mathbb{Z}, \tag{26}$$

where $p \in (0, 1)$ and $\{\varepsilon_t\}$ is an innovation process with mean $E(\varepsilon_t) = \mu_\varepsilon$ and variance $\mathrm{Var}(\varepsilon_t) = \sigma_\varepsilon^2$. The first INAR(1) process was introduced by McKenzie (1985) based on the Poisson innovations. This model is called as INAR(1)P. The symbol, $\circ$, represents the binomial thinning which is defined by $p \circ X_{t-1} := \sum_{j=1}^{X_{t-1}} W_j$ where W_j is the Bernoulli random variable with $\Pr(W_j = 1) = 1 - \Pr(W_j = 0) = p$. The one-step transition probability of INAR(1) process is

$$\begin{aligned} &\Pr(X_t = k | X_{t-1} = l) \\ &\quad = \sum_{i=1}^{\min(k,l)} \times \Pr\left(B_l^\alpha = i\right) \Pr(\varepsilon_t = k - i), \\ &\qquad k, l \geq 0, \end{aligned} \tag{27}$$

where $B_n^\alpha \sim \mathrm{Binomial}(\alpha, n)$ and $\alpha \in [0, 1)$. The mean, variance and DI of the X_t process are given, respectively, by (Weiß, 2018)

$$\mathrm{E}(X_t) = \frac{\mu_\varepsilon}{1-p} \tag{28}$$

$$\mathrm{Var}(X_t) = \frac{p\mu_\varepsilon + \sigma_\varepsilon^2}{1-p^2} \tag{29}$$

$$\mathrm{DI}_{X_t} = \frac{DI_\varepsilon + p}{1+p} \tag{30}$$

where μ_ε, σ_ε^2 and DI_ε are the mean, variance and DI of the innovation distribution.

Following the results of McKenzie (1985) and Al-Osh and Alzaid (1987), we propose a new INAR(1) process with DLi-3P innovations by assuming that the $\{\varepsilon_t\}_{t\in\mathbb{Z}}$ innovations follow a DLi-3P distribution, given in (3). The proposed INAR(1) process opens an opportunity to model both over-dispersed and under-dispersed time series of counts data sets. The one-step transition probability of INAR(1)DLi-3P process is, (31) as shown at the bottom of this page.

Hereafter, (31) is called as the INAR(1)DLi-3P process. The mean, variance and DI of the INAR(1)DLi-3P process can be easily computed by replacing μ_ε, σ_ε^2 and DI_ε in (28), (29) and (30) with (10), (14) and (15), respectively. Since the DLi-3P distribution has an ability to model under-dispersion, equi-dispersion and over-dispersion simultaneously, the INAR(1)DLi-3P will be good candidate to model all kind of dispersed time series of counts. The conditional expectation and variance of INAR(1)DLi-3P process are given, respectively, by (see Weiß, 2018, and Alzaid and Al-Osh, 1988)

$$\mathrm{E}(X_t | X_{t-1}) = \alpha X_{t-1} + \mu_\varepsilon \tag{32}$$

and

$$\mathrm{Var}(X_t | X_{t-1}) = \alpha(1-\alpha) X_{t-1} + \sigma_\varepsilon^2, \tag{33}$$

where μ_ε and σ_ε^2 are given in (10) and (14)

A. ESTIMATION OF INAR(1)DLi-3P PROCESS

The three estimation methods are generally used to estimate the unknown parameters of INAR(1) process.

$$\Pr(X_t = k | X_{t-1} = l) = \sum_{i=1}^{\min(k,l)} \binom{l}{i} p^i (1-p)^{l-i} \times \frac{\lambda^{k-i} \{\beta(\lambda(\log(\lambda) - 1) + 1) + (\lambda - 1)\log(\lambda)(\alpha + \beta(k-i))\}}{\beta - \alpha\log(\lambda)}. \tag{31}$$

These estimation methods are Yule-Walker (YL), conditional least squares (CLS) and conditional maximum likelihood estimation (CMLE). The relative efficiencies of these estimation methods have been discussed in several researches based on the simulation studies (see Bourguignon *et al.*, 2019, and Lívio *et al.*, 2018). According to these simulation studies, CMLE method performs better than other two estimation methods for both small and large sample sizes. Based on these facts, we prefer the CMLE method to obtain the unknown parameters of INAR(1)DLi-3P process. The conditional log-likelihood function of the INAR(1)DLi-3P process is

$$\begin{aligned}\ell(\Theta) &= \sum_{t=2}^{T} \ln\left[\Pr\left(X_t = k \mid X_{t-1} = l\right)\right] \\ &= \sum_{t=2}^{T} \ln\left[\begin{array}{l}\displaystyle\sum_{i=1}^{\min(X_t,X_{t-1})} \binom{X_{t-1}}{i} p^i (1-p)^{X_{t-1}-i} \\ \times \dfrac{\lambda^{X_t-i}\left\{\begin{array}{l}\beta\left(\lambda\left(\log(\lambda)-1\right)+1\right)+(\lambda-1)\\ \times\log(\lambda)\left(\alpha+\beta\left(X_t-i\right)\right)\end{array}\right\}}{\beta-\alpha\log(\lambda)}\end{array}\right]\end{aligned} \tag{34}$$

where $\Theta = (p, \lambda, \alpha, \beta)$ represents the parameter vector to be estimated. It is not possible to obtain the explicit formulations of the CMLE of the parameters of the INAR(1)DLi-3P process. Therefore, (34) has to be maximized by using the statistical software such as R, Matlab, S-Plus or Python. Here, we use the **constrOptim** function of the R software to minimize the minus of the log-likelihood function given in (34). The standard errors of the estimated parameters are obtained by means of the squared roots of the diagonal elements of the Hessian matrix whose elements are numerically calculated by using the **fdHess** function of the R software. The initial parameter vector of the INAR(1)DLi-3P process is obtained by **GenSA** package of the **R** software.

VII. SIMULATION

The finite-sample performance of the MLEs of the parameters of the DLi-3P distribution is investigated by a simulation study. The below simulation procedure is implemented for this purpose.

1) Set the simulation replication number is 1000.
2) Set the parameters of DLi-3P distribution $\lambda = 0.11$, $\alpha = -0.12$ and $\beta = 0.30$.
3) Using the given parameter values, generate random variables with sample size $n = 5, 10, 15, \ldots, 40$ from the DLi-3P by repeating N times.
4) For each generated sample size, obtain the $\widehat{\lambda}_j$, $\widehat{\alpha}_j$ and $\widehat{\beta}_j, j = 1, 2, \ldots, N$.
5) Compute the estimated biases and mean-squared errors (MSEs). The required equations can be found in Altun (2020a).

The simulation results are summarized graphically in Figure 3. From this figure, we conclude that the estimated biases approach to the desired value, zero, for large sample sizes. Also, the estimated MSEs are near the zero for both small and large sample sizes which confirms the consistency property of the MLE.

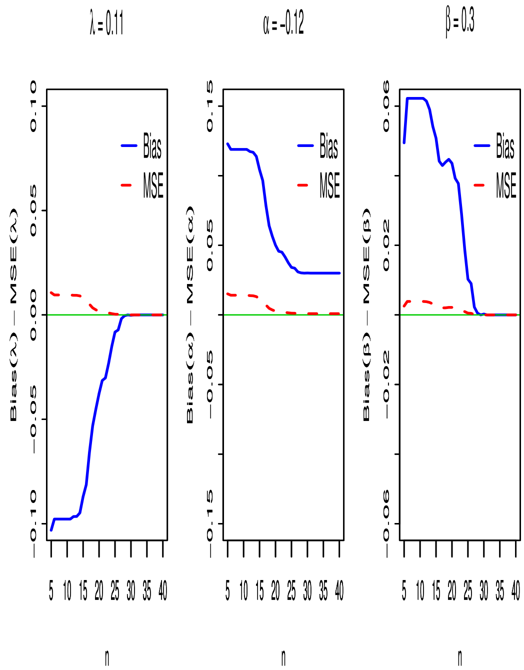

FIGURE 3. The bias and MSE of $\widehat{\lambda}$, $\widehat{\alpha}$ and $\widehat{\beta}$ versus for the DLi-3P model.

VIII. APPLICATIONS

In this section, we analyze three real data sets by using developed models in the previous sections of the presented study. In the first application, the suitable probability distribution for the numbers of daily deaths from the coronavirus in Iran is investigated. In the second application, the length of stay in a psychiatric ward is analyzed. In the third application, the monthly counts of the larceny calls in Pittsburgh are predicted by INAR(1)DLi-3P process. The fitted models are compared utilizing some criteria, namely, the negative maximized log-likelihood ($-\ell$), Akaike information criterion (AIC), corrected Akaike information criterion (CAIC), Bayesian information criterion (BIC), Hannan-Quinn information criterion (HQIC), Chi-square (χ^2) test with its corresponding p-value, and Kolmogorov-Smirnov (K-S) test with its corresponding p-value. The below strategy is used to decide best fitted model.

- ✓ The AIC, CAIC, BIC, HQIC, χ^2, $-\ell$ and KS test with its p-value are computed for all competitive models as well as DLi-3P distribution
- ✓ The models with p-values greater than 0.05 are identified as potential models.
- ✓ Among the potential models, the best model is determined as the model with the smallest values of the AIC, CAIC, BIC, HQIC, chi^2, $-\,ell$.

The computational results are carried out in **R** software. The used computer features are: Intel Core i5-826U CPU

TABLE 3. The estimated parameters of the fitted models for the coronavirus data set.

Model	λ			α			β		
	MLE	SE	C. I	MLE	SE	C. I	MLE	SE	C. I
DLi-3P	0.919	0.008	[0.905, 0.934]	1.418	0.202	[1.022, 1.814]	0.001	0.002	[0, 0.004]
DLi	0.857	0.019	[0.819, 0.894]	–	–	–	–	–	–
DR	0.997	–	–	–	–	–	–	–	–
DIW	0.178	0.077	[0.027, 0.328]	0.662	0.125	[0.417, 0.907]]	–	–	–
DB-XII	0.650	0.102	[0.451, 0.849]	1.239	0.426	[0.405, 2.073]	–	–	–
DPa	0.596	0.062	[0.475, 0.717]	–	–	–	–	–	–
DBH	0.997	0.011	[0.975, 1]	–	–	–	–	–	–
Poi	11.560	0.680	[10.227, 12.893]	–	–	–	–	–	–

TABLE 4. The goodness of fit statistics of the fitted models for the coronavirus data set.

Statistic	Models							
	DLi-3P	DLi	DR	DIW	DB-XII	DPa	DBH	Poi
$-\ell$	87.241	94.731	115.818	89.312	89.917	90.122	96.905	243.850
AIC	180.481	191.462	233.636	182.625	183.834	182.244	195.811	489.701
CAIC	181.624	191.636	233.809	183.170	184.379	182.418	195.984	489.875
BIC	184.138	192.681	234.855	185.062	186.272	183.463	197.029	490.920
HQIC	181.496	191.8	233.974	183.301	184.510	182.582	196.149	490.039
K-S	0.160	0.251	0.742	0.194	0.258	0.301	0.512	0.942
P-value	0.541	0.087	< 0.0001	0.303	0.072	0.021	< 0.0001	< 0.0001

1.80 GHz, 8GB RAM, 2GB graphic card. The execution time of the DLi-3P model is measured with **tictoc** package of **R** software.

A. CORONAVIRUS

The firs data set is reported in https://www.worldometers.info/coronavirus/country/iran/ and represents the daily new deaths in Iran from 15 February to 10 March, 2020. We compare the fits of the DLi-3P model with some competitive models such as DLi, DR, DIW, DB-XII, DPa, DBH and Poisson (Poi) models. The all used competitive models, except the Poi distribution, enables to model over-dispersion. To be fair in comparison, we choose the over-dispersed models. The MLEs with their corresponding standard error (SE), confidence interval (C. I) for the parameter(s) and goodness of fit test for the coronavirus data set are listed in Tables 3 and 4, respectively. As seen from the reported values in Table 4, DLi-3P distribution has the lowest values of the goodness-of-fit statistics which is evidence to conclude that the DLi-3P distribution is more suitable probability distribution than other competitive models for the data used. The execution time of the DLi-3P model for the coronavirus data set is 0.65 seconds.

Figure 4 shows the profile log-likelihood functions of the DLi-3P distribution. It is clear that the estimated parameters are maximizers of the log-likelihood function.

Figures 5 and 6 display the estimated cdfs and P-P plots of the fitted distributions for the data used. From these figures, we conclude that the DLi-3P distribution provides the best fits among others.

Table 5 lists the mean, variance, DI, skewness and kurtosis values of the fitted DLi-3P distribution. As seen from these results, the fitted DLi-3P distribution right skewed and leptokurtic.

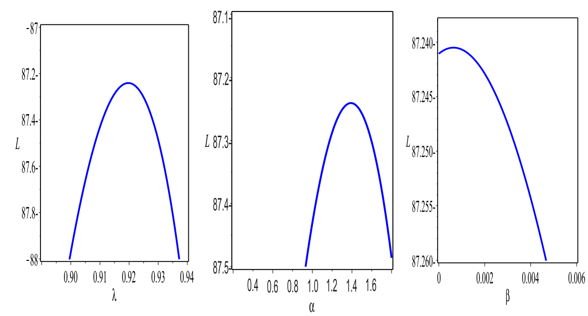

FIGURE 4. The profile log-likelihood functions of the DLi-3P distribution for the coronavirus data set.

TABLE 5. The mean, variance, DI, skewness and kurtosis values of the DLi-3P distribution for the coronavirus data.

Mean	Variance	DI	Skewness	Kurtosis
11.443	142.381	12.441	2.001	9.003

B. PSYCHIATRIC WARD

The data presented herein give the length of stay on a psychiatric ward for 67 Male patients (see Chakraborty and Gupta, 2015). The MLEs with their corresponding SE, C. I for the parameter(s) of the fitted distributions and the goodness of fit test results for the data used are listed in Tables 6 and 7, respectively. Since the DLi-3P distribution has the lowest values of the goodness of fit statistics with highest p-value, it could be selected as a best model among others. The execution time of the DLi-3P model for the length of stay data set is 0.05 seconds.

In the Figure 7, the profile log-likelihood functions of the DLi-3P distribution for the fitted data set are plotted to demonstrate that the estimated parameters of the DLi-3P

TABLE 6. The estimated parameters of the fitted models for the length of stay data set.

Model	λ			α			β		
	MLE	SE	C. I	MLE	SE	C. I	MLE	SE	C. I
DLi-3P	0.484	0.035	$[0.416, 0.552]$	-0.318	0.022	$[-0.361, -0.275]$	1.119	0.347	$[0.439, 1.799]$
DLi	0.585	0.028	$[0.529, 0.639]$	–	–	–	–	–	–
DR	0.927	0.009	$[0.909, 0.944]$	–	–	–	–	–	–
DIW	0.043	0.022	$[0.001, 0.086$	1.749	0.177	$[1.403, 2.096]$	–	–	–
DB-XII	0.843	0.055	$[0.734, 0.951]$	6.059	2.265	$[1.621, 10.498]$	–	–	–
DPa	0.459	0.044	$[0.374, 0.546]$	–	–	–	–	–	–
DBH	0.933	0.035	$[0.864, 1]$	–	–	–	–	–	–
Poi	2.597	0.197	$[2.211, 2.983]$	–	–	–	–	–	–

TABLE 7. The goodness of fit statistics of the fitted models for the length of stay data set.

No. X	Observed Frequency	Expected Frequency							
		DLi-3P	DLi	DR	DIW	DB-XII	DPa	DBH	Poi
0	5	5.01	14.14	4.92	2.89	7.50	27.89	35.74	4.99
1	16	17.70	13.95	12.69	23.42	26.96	10.57	11.81	12.96
2	20	15.97	11.48	15.65	15.99	11.13	5.72	5.84	16.83
3	10	11.30	8.66	13.95	8.43	5.53	3.63	3.45	14.57
4	5	7.20	6.19	9.83	4.77	3.28	2.53	2.26	9.46
5	3	4.32	4.29	5.66	2.93	2.17	1.88	1.58	4.91
6	3	2.5	2.89	2.71	1.92	1.54	1.46	1.16	2.13
7	5	3	5.4	1.59	6.65	8.89	13.32	5.16	1.15
Total	67	67	67	67	67	67	67	67	67
$-\ell$		129.392	135.969	133.987	134.774	143.369	171.155	171.403	134.627
AIC		264.784	273.939	269.974	273.548	290.739	344.311	344.806	271.254
CAIC		265.165	274.001	270.036	273.736	290.926	344.372	344.867	271.316
BIC		271.398	276.144	272.179	277.958	295.148	346.516	347.010	273.459
HQIC		267.401	274.812	270.846	275.293	292.483	345.183	345.678	272.126
χ^2		2.143	13.192	5.462	2.997	19.974	71.835	78.590	5.616
d.f		2	5	3	2	3	2	3	3
P.value		0.342	0.022	0.141	0.223	0.0002	< 0.0001	< 0.0001	0.132

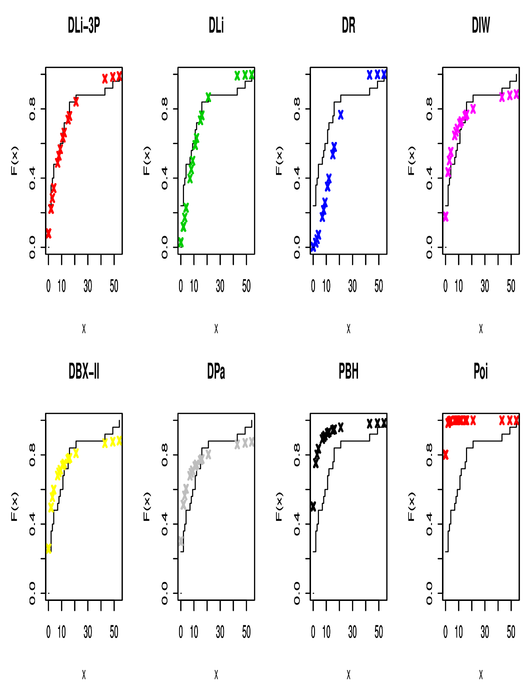

FIGURE 5. The estimated CDFs for the coronavirus data set.

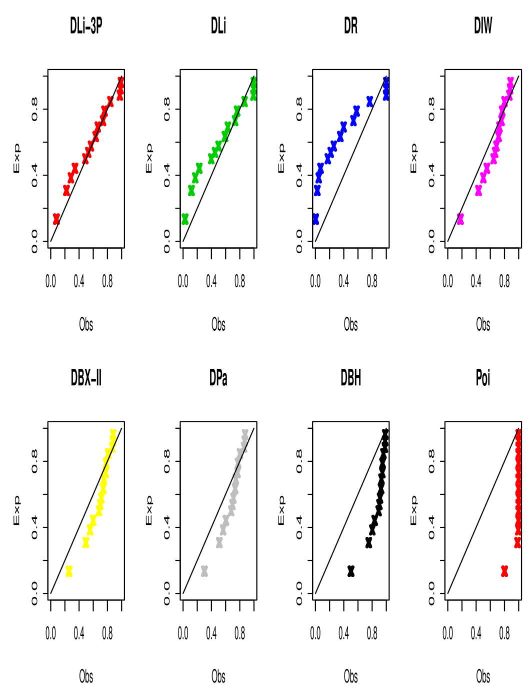

FIGURE 6. The P-P plots for the coronavirus data set.

distribution are the maximizers of the log-likelihood function. As seen from the Figure 7, the estimated parameters of the DLi-3P distribution are the maximizers of the log-likelihood functions of the DLi-3P distribution.

Figures 8 and 9 show the estimated pmfs and P-P plots for the fitted data. These figures reveal that the DLi-3P distributions are the best choice among others for the fitted data.

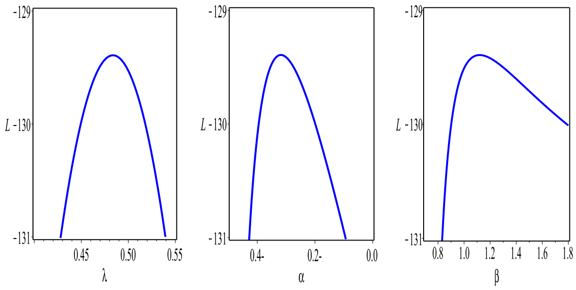

FIGURE 7. The profile log-likelihood functions of the DLi-3P distribution for the length of stay data set.

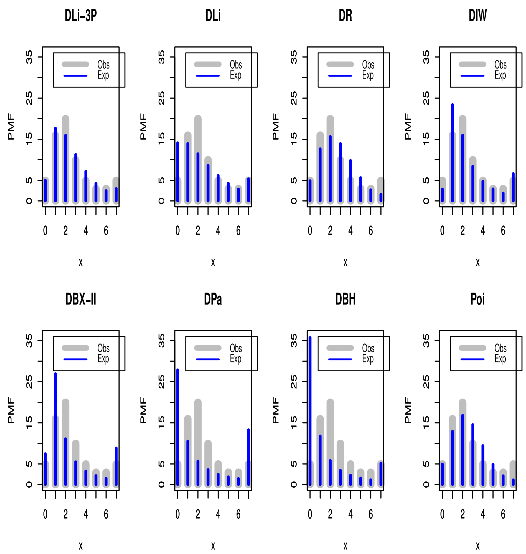

FIGURE 8. The estimated PMFs for the length of stay data set.

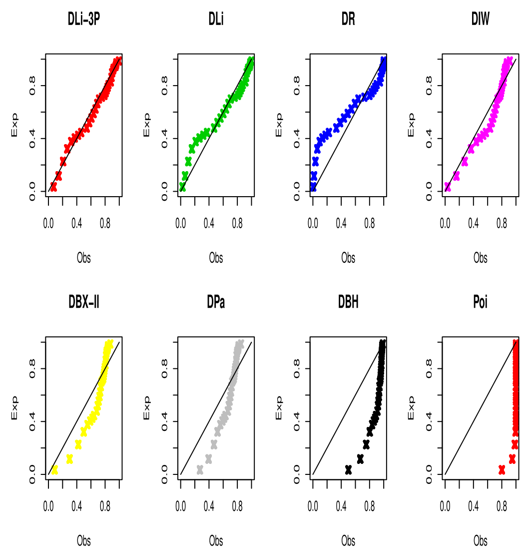

FIGURE 9. The estimated P-P plots for the length of stay data set.

Table 8 lists the mean, variance, DI, skewness and kurtosis values of the fitted DLi-3P distribution. As in first application, the fitted DLi-3P distribution is right-skewed and leptokurtic.

TABLE 8. The mean, variance, DI, skewness and kurtosis values of the DLi-3P distribution for the length of stay data set.

Mean	Variance	IxD	Skewness	Kurtosis
2.599	3.835	1.475	1.380	5.939

C. LARCENY CALLS

Here, the importance of INAR(1)DLi-3P process is compared with INAR(1)NPWE, INAR(1)P, INAR(1)G and INAR(1)PL processes. The required formulation for the translation probabilities of these competitive models can be found in Altun (2020b). The best fitted model is selected based on the information criteria, AIC and BIC statistics. We analyze the crime data set on the monthly counts of 911 larceny calls which contains 144 observations between Jan. 1990 and Dec. 2001. The data is was available in http://www.forecastingprinciples.com/index.php/crimedata. Firstly, we investigate whether the data used displays over-dispersion problem. To do this, we calculate the mean, variance and DI of the data set. The following results are obtained, respectively, 19.951, 39.613 and 1.985. Then, the hypothesis test for over-dispersion, proposed by Schweer and Weiß (2014), is applied to decide the whether the observed over-dispersion is statistically significant. The obtained test statistic is 15.936 and its p-value is less than 0.001 which reveal that the data display significant over-dispersion.

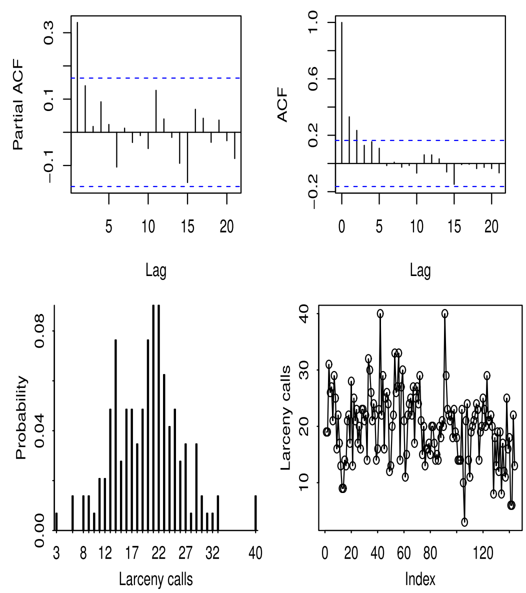

FIGURE 10. The ACF, PACF, histogram and time series plots of the larceny calls.

The fundamental plots of the data used such as autocorrelation function (ACF), partial ACF (PACF), histogram and time series plots are displayed in Figure 10. From the Figure 10 we conclude that the INAR(1) process could be a possible model for this data set since the only first lag is significant in PACF plot.

TABLE 9. The estimated parameters of the fitted models for the 911 larceny calls.

Model	Parameters	Estimate	SE	$-\ell$	AIC	BIC
INAR(1)DLi-3P	p	0.5195	0.0364	468.2365	944.4730	956.3523
	λ	0.8144	0.0138			
	α	-0.3922	0.4595			
	β	1.2776	0.1414			
INAR(1)NPWE	p	0.6149	0.0236	481.8458	969.6916	978.6010
	α	0.1184	0.6481			
	θ	0.1024	0.0345			
INAR(1)G	p	0.6148	0.0236	481.8458	967.6916	973.6312
	θ	0.1154	0.0108			
INAR(1)PL	p	0.5655	0.0283	473.3644	950.7288	956.6684
	θ	0.2108	0.0183			
INAR(1)P	p	0.2356	0.0476	478.7958	961.5916	967.5312
	λ	15.2447	0.9929			

The estimated parameters of the competitive models as well as INAR(1)DLi-3P model and corresponding standard errors, also AIC and BIC statistics, are listed in Table 9. As seen from the results given in Table 9, the INAR(1)DLi-3P model has the lowest values of the AIC and BIC statistics which are evidence to conclude that the INAR(1)DLi-3P model provides better fits than other competitive models for the data used. The execution time of the INAR(1)DLi-3P model for the larceny data set is 1.97 seconds. To check the accuracy of the fitted INAR(1)DLi-3P process, the residual analysis is conducted. We calculate the Pearson residuals of the INAR(1)DLi-3P process by using the following equation

$$r_t = \frac{X_t - E\left(X_t | X_{t-1}\right)}{Var\left(X_t | X_{t-1}\right)^{1/2}} \tag{35}$$

where $E\left(X_t | X_{t-1}\right)$ and $\mathrm{Var}\left(X_t | X_{t-1}\right)$ are given in (32) and (33), respectively. As reported in Harvey and Fernandes (1989), when the fitted INAR(1) process is statistically valid, the Pearson residual has zero mean and unit variance. Additionally, the residuals should be uncorrelated. The mean and variance of the Pearson residuals of the INAR(1)DLi-3P process are obtained 0.002 and 0.989, respectively. These values are very near the desired values of the Pearson residuals for the INAR(1) process. Following the results of Jazi *et al.* (2012), the obtained INAR(1)DLi-3P process for the data used can be given as follows

$$X_t = 0.5195 \circ X_{t-1} + \varepsilon_t.$$

where the innovation process is

$$\varepsilon_t \sim \text{DLi-3P}\,(0.8134, -3922, 1.2776)\,.$$

The predicted values of the larceny calls obtained by the INAR(1)DLi-3P process and the ACF plot of the Pearson residuals are displayed in 11. The ACF plot of the Pearson residuals confirms that the residuals are uncorrelated.

IX. ANALYSIS OF RESULTS

In this section, we interpret the empirical results more efficiently. In previous section, we analyze three data set to convince the readers in favour of the DLi-3P distribution. The all used data sets are over-dispersed. The competitive models, DLi, DR, DIW, DB-XII, DPa and DBH distributions, except the Poisson distribution, have good properties to model the over-dispersion. However, a few of them achieve to demonstrate acceptable fit to used data sets such as DLi, DIW and DB-XII distributions. The parameter λ of the DLi-3P

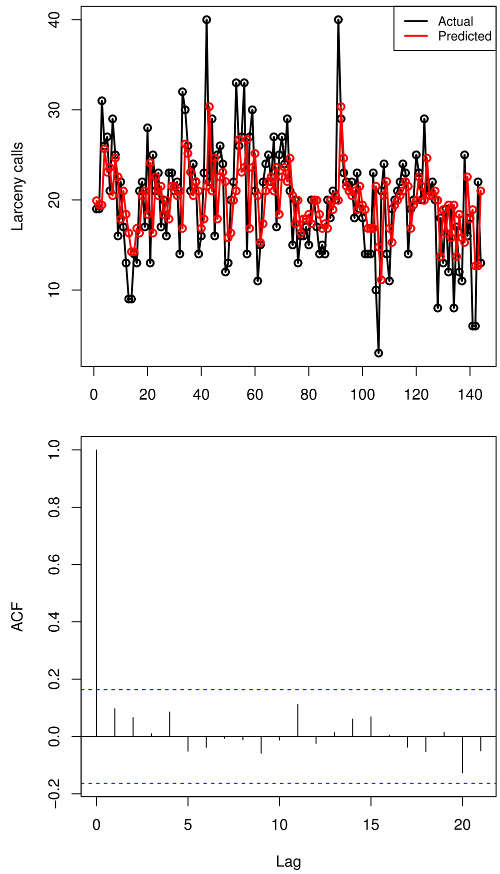

FIGURE 11. The predicted and actual values of the larceny calls (right) and the ACF plot of the residuals (left).

distribution controls the shape of the distribution and thanks to the parameter λ, the DLi-3P distributions gains much more flexibility than the other competitive models. More importantly, the skewness, kurtosis and DI measures of the DLi-3P distribution has wider range than those of competitive models. Additionally, the profile log-likelihood plots of the DLi-3P distribution reveal the righteousness of the used strategy in estimating the unknown parameter vector of the DLi-3P distribution.

X. CONCLUSIONS

This paper introduces a new three-parameter discrete distribution, shortly DLi-3P distribution. The statistical properties of the DLi-3P distribution are derived in great detail. The maximum likelihood estimation method is used to obtain unknown parameters of the proposed distribution and a brief simulation study is given to discuss the performance of the maximum likelihood estimators of the DLi-3P distribution for both small and large sample sizes. More importantly, a new INAR(1) process with DLi-3P innovations are introduced and studied. The three real data sets are analyzed to convince the readers in favour of the DLi-3P distribution against the other competitive models. We believe that the DLi-3P will increase its popularity and find a wider range of application area in different scientific fields.

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M. S. ELIWA is currently an Assistant Professor of mathematical statistics with the Mathematics Department, Faculty of Science, Mansoura University, Egypt. His current research interests include generalized univariate and bivariate classes of distributions and their special models, compounded, cure rate models, applied probability, censored data, survival analysis, and estimation theory. He is a Referee and an Editorial Board Member of international journals.

EMRAH ALTUN is currently an Associate Professor of statistics with the Department of Mathematics, Bartin University, Bartin, Turkey. He has more than 70 publications in international journals on time series, financial risk forecasting, distribution theory, survival models, and count regression models. He is a Referee and an Editorial Board Member of international journals.

M. EL-DAWOODY received the M.Sc. degree in mathematical statistics from the Faculty of Science, Mansoura University, Egypt. He is currently a Lecturer of mathematical statistics with the Mathematics Department, College of Sciences and Humanities Studies, Prince Sattam Bin Abdulaziz University, Al-Kharj, Saudi Arabia. His current research interest includes probability distributions with its applications.

M. EL-MORSHEDY is currently an Assistant Professor of mathematical statistics with the Mathematics Department, College of Sciences and Humanities Studies, Prince Sattam Bin Abdulaziz University, Saudi Arabia. He is also an Assistant Professor with the Faculty of Science, Mansoura University, Egypt. His current research interests include generalized univariate and bivariate classes of distributions and their special models, cure rate models, applied probability, reliability theory, and estimation theory. He is a Referee and an Editorial Board Member of international journals.