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The exponentiated generalized power Lindley distribution: Properties and applications

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Abstract. In this paper, we introduce a new extension of the power Lindley distribution, called the exponentiated generalized power Lindley distribution. Several mathematical properties of the new model such as the shapes of the density and hazard rate functions, the quantile function, moments, mean deviations, Bonferroni and Lorenz curves and order statistics are derived. Moreover, we discuss the parameter estimation of the new distribution using the maximum likelihood and diagonally weighted least squares methods. A simulation study is performed to evaluate the estimators. We use two real data sets to illustrate the applicability of the new model. Empirical findings show that the proposed model provides better fits than some other well-known extensions of Lindley distributions.

§1 Introduction

The Lindley distribution was first introduced by Lindley [18] whose probability density function (pdf) is

$$f_L(t;\lambda) = \frac{\lambda^2}{\lambda+1}(1+t)\mathrm{e}^{-\lambda t}, \quad t>0, \quad \lambda>0. \tag{1}$$

The corresponding cumulative distribution function (cdf) is given by

$$F_L(t;\lambda) = 1 - \frac{\lambda+1+\lambda t}{\lambda+1}\mathrm{e}^{-\lambda t}, \quad t>0, \quad \lambda>0.$$

Ghitany et al. [12] studied the statistical properties of the Lindley distribution. Now, suppose that T has a Lindley distribution with pdf given in (1). Recently, Ghitany et al. [13] introduced a generalization of the Lindley distribution, called the power Lindley distribution, by considering the power transformation $Y = T^{1/\beta}$. The pdf of Y is then given by

$$f_{PL}(y;\lambda,\beta) = \frac{\beta\lambda^2}{\lambda+1}(1+y^\beta)y^{\beta-1}\mathrm{e}^{-\lambda y^\beta}, \quad y>0, \quad \lambda,\beta>0.$$

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Moreover, the cdf of Y is

$$F_{PL}(y;\lambda,\beta)=1-\frac{\lambda+1+\lambda y^{\beta}}{\lambda+1}\mathrm{e}^{-\lambda y^{\beta}},\quad y>0. \tag{2}$$

The power Lindley distribution was generalized by many authors, see for example Alizadeh et al. [1] and Alizadeh et al. [2]. Now, let $G(x)$ and $g(x)$ be the cdf and pdf of a continuous random variable X, respectively. Cordeiro et al. [10] introduced a new interesting generalized family of distributions for random variable X, called the exponentiated generalized (EG) class of distributions, whose cdf is given by

$$F(x;a,b)=\left[1-\bar{G}(x)^{a}\right]^{b}, \tag{3}$$

where $a>0$ and $b>0$ and $\bar{G}(x)=1-G(x)$. The corresponding pdf of (3) is $f(x;a,b)=\frac{\partial}{\partial x}F(x;a,b)$, which is

$$f(x;a,b)=a\,bg(x)\bar{G}(x)^{a-1}\left[1-\bar{G}(x)^{a}\right]^{b-1}.$$

This kind of generalization has become popular and received considerable attention in recent years and many authors have studied EG-compounded lifetime distributions. One advantage of this kind of generalization is that the additional parameters a and b can control the skewness and kurtosis of the distribution and vary the tail weight, allowing for much flexibility of the new distribution in modelling data. Examples of such generalized lifetime distributions include the EG Birnbaum-Saunders distribution (Cordeiro and Lemonte [9]), the EG Dagum distribution (Silva et al. [24]), the EG Weibull distribution (Cordeiro et al. [10] and Oguntunde et al. [21]) and the EG modified Weibull distribution (Aryal and Elbatal [4]). In addition, the EG Gumbel distribution (a useful model for engineering data) was considered by Cordeiro et al. [10] and its properties and applications were discussed in detail by Andrade et al. [3].

In this paper, we introduce a new generalization of the power Lindley distribution by taking $G(x)$ in (3) to be the cdf of the power Lindley distribution. Therefore, the cdf of the new distribution is obtained to be

$$F(x)=\left\{1-\left[(1+\frac{\lambda}{1+\lambda}\,x^{\beta})\mathrm{e}^{-\lambda x^{\beta}}\right]^{a}\right\}^{b}, \tag{4}$$

where $x>0$ and $\lambda,\beta,a,b>0$.

This new distribution, called the exponentiated generalized power Lindley (EG-PL) distribution, contains the power Lindley distribution as a special case having two additional shape parameters, a and b. The additional parameters affect the tail behavior of the distribution and control its skewness and kurtosis so the new generalized distribution becomes more suitable for modelling skewed, leptokurtic, platykurtic data sets that cannot be properly fitted by some existing distributions. Hence, we hope that the new distribution provides a more flexible framework that can be applied in reliability and engineering researches more properly.

Another motivation for the new distribution can be explained as follows: Suppose that a system contains b components such that each component is made up of a subcomponents. Suppose further the system fails when all of its b components have failed and each component fails as soon as at least one of the a subcomponents fails. Let $X_{ij}, i=1,\cdots,a$ and $j=1,\cdots,b$ denote the lifetime of the i-th subcomponent within the j-th component. Now, suppose that $X_{11},\cdots,X_{ab}$ are independent random variables distributed as the power Lindley distribution with cdf (2). If $X_j, j = 1, \cdots, b$ denotes the lifetime of the j-th component and X denotes the lifetime of the whole system, then the probability that failure time of the system, X, becomes less than time x is

$$\begin{aligned} Pr(X \le x) &= Pr(X_1 \le x, \cdots, X_b \le x) = [Pr(X_1 \le x)]^b \\ &= \{1 - Pr(X_1 > x)\}^b = \{1 - Pr(X_{11} > x, \cdots, X_{1a} > x)\}^b \\ &= \{1 - [Pr(X_{11} > x)]^a\}^b = \{1 - [1 - F_{PL}(x; \lambda, \beta)]^a\}^b, \end{aligned}$$

the cdf of the EG-PL distribution.

The rest of the paper is organized as follows. The density and hazard rate functions of the EG-PL distribution are discussed in Section 2. The expansions for the cdf and the pdf are derived in Section 3. In Section 4, we obtain the quantile function. The moments and associated measures are studied in Section 5. The order statistics and their moments from the EG-PL distribution are investigated in Section 6. In Section 7, we find the maximum likelihood (ML) estimators as well as the asymptotic confidence intervals for the unknown parameters. We also present the applications of this new proposed distribution in Section 8. Finally, Section 9 concludes the paper with some concluding remarks.

§2 Density and hazard rate functions

The pdf of the EG-PL distribution is given by

$$\begin{aligned} f(x) &= \frac{a\,b\,\beta\,\lambda^2}{1+\lambda}\, x^{\beta-1}(1+x^\beta)\,\mathrm{e}^{-\lambda x^\beta}\left[(1+\frac{\lambda}{1+\lambda}\,x^\beta)\mathrm{e}^{-\lambda x^\beta}\right]^{a-1} \\ &\times \left\{1-\left[(1+\frac{\lambda}{1+\lambda}\,x^\beta)\mathrm{e}^{-\lambda x^\beta}\right]^a\right\}^{b-1}, \end{aligned} \tag{5}$$

where $x > 0$ and $\lambda, \beta, a, b > 0$. If X is a random variable with pdf (5), then we will use the notation $X \sim \mathrm{EG-PL}(\lambda, \beta, a, b)$.

Special cases of the EG-PL distribution can be categorized as follows

- For $a = 1$, we obtain the exponentiated power Lindley (EPL) distribution, Warahena-Liyanage and Pararai [26] and Ashour and Eltehiwy [5].
- For $\beta = 1$, we obtain the exponentiated generalized Lindley distribution.
- For $a = \beta = 1$, we obtain the generalized Lindley distribution, Nadarajah et al. [19].
- For $a = b = 1$, we obtain the power Lindley distribution, Ghitany et al. [13].
- For $a = b = \beta = 1$, we obtain the Lindley distribution.

Figure 1 displays the plots of the pdfs for selected parameter combinations.

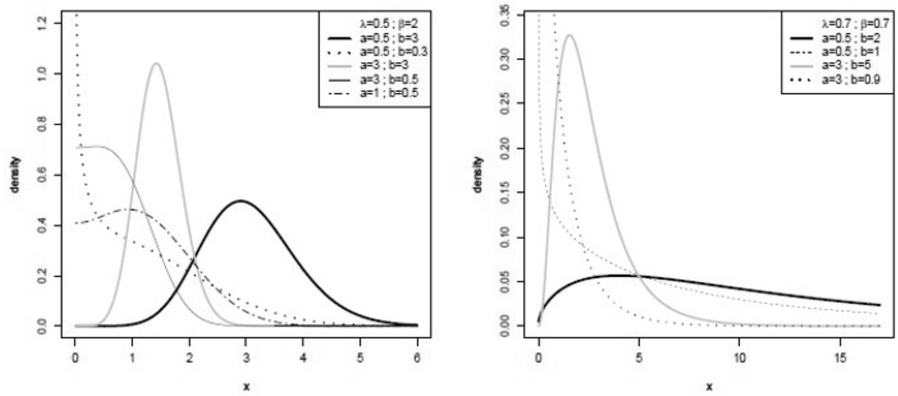

Figure 1: EG-PL densities for selected parameter values.

In addition, the hazard rate function (hrf) of the EG-PL distribution is

$$
\begin{aligned}
h(x;\lambda,\beta,a,b) &= \frac{a\,b\,\beta\,\lambda^2}{1+\lambda}\,x^{\beta-1}(1+x^{\beta})\,\mathrm{e}^{-\lambda x^{\beta}}\left[\left(1+\frac{\lambda}{1+\lambda}\,x^{\beta}\right)\mathrm{e}^{-\lambda x^{\beta}}\right]^{a-1} \\
&\times\ \frac{\left\{1-\left[(1+\frac{\lambda}{1+\lambda}\,x^{\beta})\mathrm{e}^{-\lambda x^{\beta}}\right]^{a}\right\}^{b-1}}{1-\left\{1-\left[(1+\frac{\lambda}{1+\lambda}\,x^{\beta})\mathrm{e}^{-\lambda x^{\beta}}\right]^{a}\right\}^{b}}. \qquad (6)
\end{aligned}
$$

Figure 2 plots of the hazard rate functions for selected parameter combinations. As one can see from Figure 2, the hrf can be monotonically increasing, monotonically decreasing, bathtub-shaped and upside down bathtub shaped and this flexibility makes the EG-PL distribution useful and suitable for many real life data sets that are more likely to be faced. In addition, we plotted a figure related to the regions of the hrf when $\lambda = 0.5$ and $\beta = 0.5$, see Figure 3. Similar figures can be obtained for other different parameter vectors.

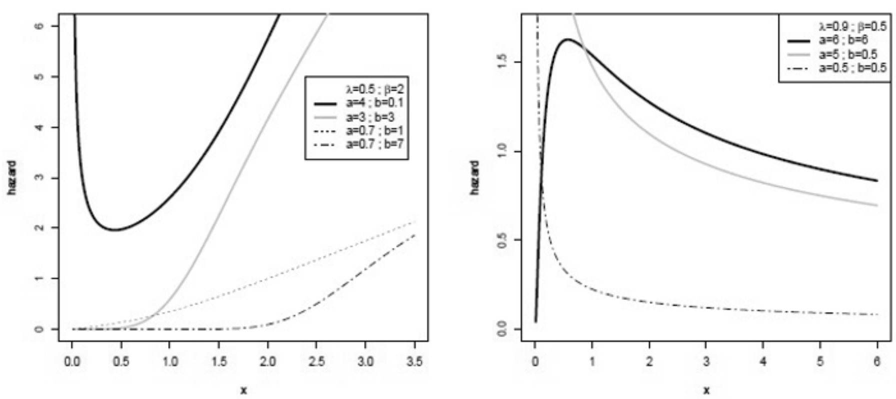

Figure 2: EG-PL hazard functions for selected parameter values.

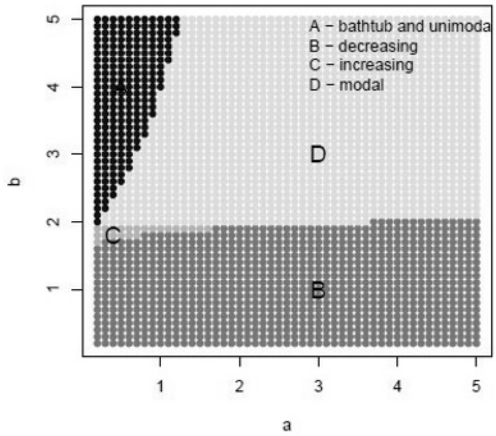

Figure 3: The EG-PL hrf shape as a function of a and b when $\lambda = 0.5$ and $\beta = 0.5$.

§3 Mixture representations for the pdf and cdf

In this section, we show that the EG-PL distribution can be written as a mixture of the EPL distributions. As mentioned before, the EPL distribution, introduced by Warahena-Liyanage and Pararai [26] and Ashour and Eltehiwy [5], is a special case of the EG-PL distribution. The pdf of the EPL distribution with positive parameters λ, β and b is given by

$$f_{EPL}(x;\lambda,\beta,b) = \frac{b\,\beta\,\lambda^2}{\lambda+1}(1+x^\beta)x^{\beta-1}\mathrm{e}^{-\lambda x^\beta}\left[1-(1+\frac{\lambda}{1+\lambda}\,x^\beta)\mathrm{e}^{-\lambda x^\beta}\right]^{b-1}, \quad x>0. \tag{7}$$

We write $X \sim \mathrm{EPL}(\lambda,\beta,b)$ if the pdf of X can be written as (7). Moreover, the cdf of the EPL distribution is

$$F_{EPL}(x;\lambda,\beta,b) = \left[1-(1+\frac{\lambda}{1+\lambda}\,x^\beta)\mathrm{e}^{-\lambda x^\beta}\right]^{b}, \quad x>0.$$

Let us consider the following generalized binomial expansion

$$(1-z)^\eta = \sum_{j=0}^{\infty}\binom{\eta}{j}(-1)^j z^j,$$

where $\eta > 0$. If $|z| \le 1$, then the above expansion converges, see for example Gradshteyn and Ryzhik [14], Section 1.11.

Using the generalized binomial expansion and noting that the cdf of any distribution, like the power Lindley distribution $G(x) = 1-(1+\frac{\lambda}{1+\lambda}\,x^\beta)\mathrm{e}^{-\lambda x^\beta}$ is between 0 and 1, we can expand the cdf of the EG-PL distribution as follows

$$F(x) = \sum_{k=0}^{\infty} c_k\left[1-(1+\frac{\lambda}{1+\lambda}\,x^\beta)\mathrm{e}^{-\lambda x^\beta}\right]^{k},$$

where

$$c_k = c_k(a,b) = \sum_{i=0}^{\infty}\binom{b}{i}\binom{i\,a}{k}(-1)^{i+k}. \tag{8}$$

Or equivalently, we have

$$F(x)=\sum_{k=0}^{\infty} c_k\, F_{EPL}(x;\lambda,\beta,k), \tag{9}$$

where $F_{EPL}(x;\lambda,\beta,k)$ denotes the cdf of the EPL distribution with parameters λ, β and k.

Upon differentiating (9), the pdf of the EG-PL distribution can be written as

$$f(x)=\sum_{k=0}^{\infty} c_{k+1}\, f_{EPL}(x;\lambda,\beta,k+1), \tag{10}$$

where $f_{EPL}(x;\lambda,\beta,k+1)$ denotes the pdf of the EPL distribution with parameters λ, β and $k+1$.

§4 Quantile function

Let $X\sim \mathrm{EG-PL}(\lambda,\beta,a,b)$, then the quantile function, say $Q(p)$, defined by $F(Q(p))=p$ is the root of the equation

$$(1+\lambda+\lambda\, Q(p)^{\beta})\,\mathrm{e}^{-\lambda\, Q(p)^{\beta}}=(1+\lambda)(1-p^{\frac{1}{b}})^{\frac{1}{a}}, \tag{11}$$

for $0<p<1$. Inserting $Z(p)=-1-\lambda-\lambda\, Q(p)^{\beta}$ into (11), we can rewrite Equation (11) as

$$Z(p)\,\mathrm{e}^{Z(p)}=-(1+\lambda)(1-p^{\frac{1}{b}})^{\frac{1}{a}}\,\mathrm{e}^{-1-\lambda}, \tag{12}$$

for $0<p<1$. The solution of $Z(p)$ is

$$Z(p)=W_{-1}\left[-(1+\lambda)\,(1-p^{\frac{1}{b}})^{\frac{1}{a}}\,\mathrm{e}^{-1-\lambda}\right], \tag{13}$$

for $0<p<1$, where $W_{-1}[.]$ is the negative branch of the Lambert function (see Corless et al. [11]). Therefore, from (11), (12) and (13), the quantile function is obtained to be

$$Q(p)=\left\{-1-\frac{1}{\lambda}-\frac{1}{\lambda}\,W_{-1}\left[-(1+\lambda)(1-p^{\frac{1}{b}})^{\frac{1}{a}}\,\mathrm{e}^{-1-\lambda}\right]\right\}^{\frac{1}{\beta}}, \tag{14}$$

for $0<p<1$. The particular case of Equation (14) for $(a=b=\beta=1)$ has been derived recently by Jodrá [16].

§5 Moments and associated measures

We define and compute

$$A(a_1,a_2,a_3;\lambda,\beta)\;=\;\int_0^{\infty} x^{a_1}\,(1+x^{\beta})\,\mathrm{e}^{-a_2\,x^{\beta}}\left[1-(1+\frac{\lambda}{1+\lambda}\,x^{\beta})\mathrm{e}^{-\lambda x^{\beta}}\right]^{a_3}\,\mathrm{d}x,$$

where $a_1>-1$ and $a_i>0$ for $i=2,3$. Using the generalized binomial expansion, one can obtain

$$A(a_1,a_2,a_3;\lambda,\beta)=\sum_{l_1=0}^{\infty}\sum_{l_2=0}^{l_1}\sum_{l_3=0}^{l_2+1}\binom{a_3}{l_1}\binom{l_1}{l_2}\binom{l_2+1}{l_3}\frac{(-1)^{l_1}\,\lambda^{l_2}\,\Gamma(\frac{a_1+1}{\beta}+l_3)}{\beta\,(\lambda+1)^{l_1}\,(a_2+l_1\,\lambda)^{\frac{a_1+1}{\beta}+l_3}}.$$

The above expansion is convergent as the first summation is related to the generalized binomial expansion by noting that $\bar{G}(x)=(1+\frac{\lambda}{1+\lambda}\,x^{\beta})\mathrm{e}^{-\lambda x^{\beta}}$ is between 0 and 1, and the next two summations are ordinary finite binomial expansions.

Therefore, the n-th moment of $X\sim \mathrm{EG-PL}(\lambda,\beta,a,b)$ can be written as

$$E\left[X^n\right]=\frac{\beta\,\lambda^2}{1+\lambda}\sum_{k=0}^{\infty}(k+1)\,c_{k+1}A(n+\beta-1,\lambda,k;\lambda,\beta), \tag{15}$$

where c_k is given in (8).

For integer values of s, let $\mu'_s = E(X^s)$ and $\mu = \mu'_1 = E(X)$, then the s-th central moment of the EG-PL distribution can be found by the following well-known equation

$$\mu_s = E(X-\mu)^s = \sum_{r=0}^{s} \binom{s}{r} \mu'_s (-\mu)^{s-r}. \tag{16}$$

Using (16), the variance, skewness and kurtosis measures can be obtained, respectively, as

$$\begin{aligned} Var(X) &= E(X^2) - [E(X)]^2, \\ Skewness(X) &= \frac{E(X^3) - 3E(X)E(X^2) + 2[E(X)]^3}{[Var(X)]^{\frac{3}{2}}}, \\ Kurtosis(X) &= \frac{E(X^4) - 4E(X)E(X^3) + 6E(X^2)[E(X)]^2 - 3[E(X)]^4}{[Var(X)]^2}. \end{aligned}$$

Figure 4 shows the behaviors of the mean, variance, skewness and kurtosis of the EG-PL distribution with respect to a and b when $\lambda = 1$ and $\beta = 2$.

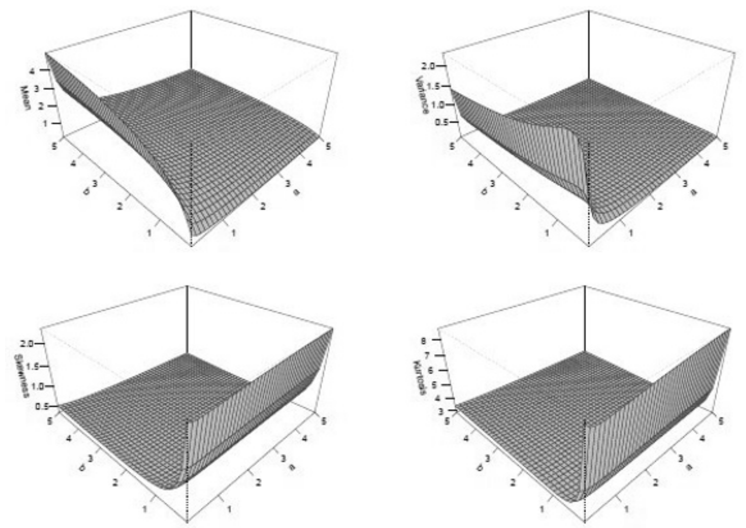

Figure 4: Mean, variance, skewness and kurtosis plots for $\lambda = 1$ and $\beta = 2$.

To find the incomplete moments, we define and compute

$$B(a_1, a_2, a_3; y, \lambda, \beta) = \int_0^y x^{a_1} (1 + x^\beta) \, \mathrm{e}^{-a_2 x^\beta} \left[1 - (1 + \frac{\lambda}{1+\lambda} x^\beta) \mathrm{e}^{-\lambda x^\beta}\right]^{a_3} \mathrm{d}x,$$

where $a_1 > -1$ and $a_i > 0$ for $i = 2, 3$.

Using the generalized binomial expansion, we have

$$B(a_1, a_2, a_3; y, \lambda, \beta) = \sum_{l_1=0}^{\infty} \sum_{l_2=0}^{l_1} \sum_{l_3=0}^{l_2+1} \binom{a_3}{l_1} \binom{l_1}{l_2} \binom{l_2+1}{l_3} \frac{(-1)^{l_1} \lambda^{l_2} \gamma\left(\frac{a_1+1}{\beta} + l_3, (a_2 + l_1 \lambda) y^\beta\right)}{\beta (\lambda+1)^{l_1} (a_2 + l_1 \lambda)^{\frac{a_1+1}{\beta} + l_3}},$$

where $\gamma(\lambda, z) = \int_0^z t^{\lambda-1} \mathrm{e}^{-t} \, \mathrm{d}t$ denotes the incomplete gamma function. So the n-th incomplete moment of the EG-PL distribution is given by

$$m_n(y) = \int_0^y x^n f(x) \, \mathrm{d}x = \frac{\beta \lambda^2}{1+\lambda} \sum_{k=0}^{\infty} (k+1) \, c_{k+1} \, B(n + \beta - 1, \lambda, k; y, \lambda, \beta). \tag{17}$$

5.1 Mean deviations

The amount of scatter in a population may be measured to some extent by deviations from the mean and median. These are known as the mean deviation about the mean and the mean deviation about the median, defined by

$$\delta_1(X) = \int_0^{\infty} |x-\mu|\, f(x)\, \mathrm{d}x,$$

and

$$\delta_2(X) = \int_0^{\infty} |x-M|\, f(x)\, \mathrm{d}x.$$

respectively, where $\mu = E(X)$ and $M = \text{Median}(X) = Q(0.5)$ denotes the median and $Q(p)$ is the quantile function. The measures $\delta_1(X)$ and $\delta_2(X)$ can be calculated using the relations

$$\begin{aligned}\delta_1(X) &= 2\mu F(\mu) - 2\int_0^{\mu} x\, f(x)\, \mathrm{d}x,\\ \delta_2(X) &= \mu - 2\int_0^{M} x\, f(x)\, \mathrm{d}x.\end{aligned}$$

Finally, from (17), we can obtain

$$\delta_1(X) = 2\mu F(\mu) - \frac{2\beta\lambda^2}{1+\lambda}\sum_{k=0}^{\infty}(k+1)\, c_{k+1}\, B(\beta,\lambda,k;\mu,\lambda,\beta),$$

and

$$\delta_2(X) = \mu - \frac{2\beta\lambda^2}{1+\lambda}\sum_{k=0}^{\infty}(k+1)\, c_{k+1}\, B(\beta,\lambda,k;M,\lambda,\beta).$$

5.2 Bonferroni and Lorenz curves

The Bonferroni and Lorenz curves have applications in economics as well as other fields like reliability, medicine and insurance. Let $Y \sim \text{EG}-\text{PL}(\lambda,\beta,a,b)$ and $F(y)$ be the cdf of Y, then the Bonferroni curve of the EG-PL distribution is given by

$$B(F(y)) = \frac{1}{\mu F(y)}\int_0^{y} t f(t)\mathrm{d}t,$$

where $\mu = E(Y)$.

Therefore, from (17), we have

$$B(F(y)) = \frac{\beta\lambda^2}{(1+\lambda)\mu F(y)}\sum_{k=0}^{\infty}(k+1)\, c_{k+1}\, B(\beta,\lambda,k;y,\lambda,\beta).$$

The Lorenz curve of the EG-PL distribution can be obtained using the relation

$$L(F(y)) = F(y)B(F(y)) = \frac{\beta\lambda^2}{(1+\lambda)\mu}\sum_{k=0}^{\infty}(k+1)\, c_{k+1}\, B(\beta,\lambda,k;y,\lambda,\beta).$$

§6 Order statistics

Order statistics are widely applied in many areas of statistical theory and practice, especially when we encounter censoring problems. Suppose $X_1,\ldots,X_n$ are a random sample from an EG-PL distribution. Let $X_{l:n}$ denote the l-th order statistic. The pdf of $X_{l:n}$ can be expressed as

$$\begin{aligned} f_{l:n}(x) &= \frac{n!}{(l-1)!(n-l)!}\, f(x)\, F^{l-1}(x)\, \{1-F(x)\}^{n-l} \\ &= \frac{n!}{(l-1)!(n-l)!} \sum_{s=0}^{n-l} (-1)^s \binom{n-l}{s} f(x)\, F(x)^{l+s-1} \\ &= \frac{n!\, a\, b\, \beta\, \lambda^2\, x^{\beta-1}(1+x^\beta)\mathrm{e}^{-\lambda x^\beta}}{(l-1)!(n-l)!(1+\lambda)} \left[(1+\frac{\lambda}{1+\lambda}\, x^\beta)\mathrm{e}^{-\lambda x^\beta}\right]^{a-1} \\ &\times \sum_{s=0}^{n-l} (-1)^s \binom{n-l}{s} \left\{1-\left[(1+\frac{\lambda}{1+\lambda}\, x^\beta)\mathrm{e}^{-\lambda x^\beta}\right]^a\right\}^{b(s+l)-1} \\ &= \sum_{s=0}^{n-l} c_s^*\, f(x;\lambda,\beta,a,b(s+l)), \end{aligned} \tag{18}$$

where $f(x;\lambda,\beta,a,b(s+l))$ stands for the pdf of the EG-PL distribution with parameters λ, β, a and $b(s+l)$ and

$$c_s^* = c_s^*(l,n) = \frac{n!\,(-1)^s}{(l-1)!\,s!\,(n-l-s)!\,(s+l)}.$$

Therefore, the pdf of $X_{l:n}$ can be written as a finite linear combination of EG-PL densities and we can obtain some mathematical properties of $X_{l:n}$ from those of the EG-PL distribution using this result. For example, we may express $f_{l:n}(x)$ as an infinite linear combination of the EPL densities. From (8), (10) and (18), we can write

$$f_{l:n}(x) = \sum_{k=0}^{\infty} c_{k+1}^{**} f_{EPL}(x;\lambda,\beta,k+1), \tag{19}$$

where $f_{EPL}(x;\lambda,\beta,k+1)$ denotes the pdf of the EPL distribution with parameters λ, β and $k+1$ and

$$c_k^{**} = c_k^{**}(l,n,a,b) = \sum_{s=0}^{n-l}\sum_{i=0}^{\infty} \binom{b(l+s)}{i}\binom{i\,a}{k}(-1)^{i+k}\, c_s^*.$$

In addition, from (15) and (19), the m-th moment of $X_{l:n}$ is obtained to be

$$E\,[X_{l:n}^m] = \frac{\beta\,\lambda^2}{1+\lambda} \sum_{k=0}^{\infty} (k+1)\, c_{k+1}^{**} A(m+\beta-1,\lambda,k;\lambda,\beta).$$

§7 Estimation

Several methods for parameter point estimation have been proposed in the literature but the ML method is still the most commonly employed one. The ML estimators possess asymptotic properties that can be applied to the construction of confidence intervals and regions and also to performing test of hypotheses. Simple approximations, that are based on large sample theory for ML estimators, work rather satisfactorily in finite samples.

The log-likelihood for the parameters of the EG-PL distribution given the random sample

$x_1,\ldots,x_n$ reduces to

$$\begin{aligned}\ell_n &= n\log(\frac{a\,b\beta\,\lambda^2}{1+\lambda})+(\beta-1)\sum_{i=1}^n\log(x_i)+\sum_{i=1}^n\log(1+x_i^\beta)-\lambda\sum_{i=1}^n x_i^\beta\\ &+ (a-1)\sum_{i=1}^n\log(t_i)+(b-1)\sum_{i=1}^n\log(1-t_i^a),\end{aligned}$$

where $t_i=(1+\frac{\lambda}{1+\lambda}\,x_i^\beta)\,\mathrm{e}^{-\lambda\,x_i^\beta}$. Let $\theta=(\lambda,\beta,a,b)$, then the ML estimate of θ may be obtained by solving the following non-linear equations

$$\begin{aligned}\frac{\partial\ell_n}{\partial\lambda} &= \frac{2n}{\lambda}-\frac{n}{1+\lambda}-\sum_{i=1}^n x_i^\beta+(a-1)\sum_{i=1}^n\frac{t_i^{(\lambda)}}{t_i}+a(1-b)\sum_{i=1}^n\frac{t_i^{(\lambda)}t_i^{a-1}}{1-t_i^a}=0,\\ \frac{\partial\ell_n}{\partial\beta} &= \frac{n}{\beta}+\sum_{i=1}^n\log(x_i)+\sum_{i=1}^n\frac{x_i^\beta\,\log(x_i)}{1+x_i^\beta}-\lambda\sum_{i=1}^n x_i^\beta\,\log(x_i)\\ &+(a-1)\sum_{i=1}^n\frac{t_i^{(\beta)}}{t_i}+a(1-b)\sum_{i=1}^n\frac{t_i^{(\beta)}t_i^{a-1}}{1-t_i^a}=0,\\ \frac{\partial\ell_n}{\partial a} &= \frac{n}{a}+\sum_{i=1}^n\log(t_i)+(1-b)\sum_{i=1}^n\frac{t_i^a\,\log(t_i)}{1-t_i^a}=0,\\ \frac{\partial\ell_n}{\partial b} &= \frac{n}{b}+\sum_{i=1}^n\log(1-t_i^a)=0,\end{aligned}$$

where

$$\begin{aligned}t_i^{(\lambda)} &= \frac{\partial t_i}{\partial\lambda}=-x_i^\beta\mathrm{e}^{-\lambda\,x_i^\beta}\left[\frac{-1}{(1+\lambda)^2}+1+\frac{\lambda}{1+\lambda}\,x_i^\beta\right],\\ t_i^{(\beta)} &= \frac{\partial t_i}{\partial\beta}=\frac{-\lambda^2x_i^\beta\,(1+x_i^\beta)\mathrm{e}^{-\lambda\,x_i^\beta}\,\log(x_i)}{1+\lambda}.\end{aligned}$$

Numerical iterative techniques should be used to solve the above equations. One can use the nleqslv function, contained in the nleqslv package (Hasselman [15]) in R (R Core Team [23]) to solve the above equations.

For interval estimation, we utilize the symmetric observed information matrix which is given by

$$\mathbf{I}_F(\theta)=-\begin{pmatrix} I_{\lambda\lambda} & I_{\lambda\beta} & I_{\lambda a} & I_{\lambda b}\\ I_{\beta\lambda} & I_{\beta\beta} & I_{\beta a} & I_{\beta b}\\ I_{a\lambda} & I_{a\beta} & I_{aa} & I_{ab}\\ I_{b\lambda} & I_{b\beta} & I_{ba} & I_{bb}\end{pmatrix}.$$

The elements of the above matrix are obtained by the authors but they are not presented in this paper.

Let $\widehat{\theta}=(\widehat{\lambda},\widehat{\beta},\widehat{a},\widehat{b})$ denote the ML estimator of $\theta=(\lambda,\beta,a,b)$. It is well-known that under the regularity conditions that are fulfilled for the parameters (see for example Lehmann and Casella [17], pp. 461-463), the asymptotic joint distribution of $(\widehat{\lambda},\widehat{\beta},\widehat{a},\widehat{b})$, as $n\to\infty$ is a 4-variate normal distribution with mean (λ,β,a,b) and variance-covariance $\mathbf{I}_F^{-1}(\theta)$. Unknown parameters which may appear in the elements of the matrix $\mathbf{I}_F^{-1}(\theta)$ can be replaced by their corresponding ML estimators. Therefore, the asymptotic equi-tailed $100(1-p)\%$ confidence intervals for the parameters λ, β, a and b, respectively, are given by

$$\widehat{\lambda} \pm z_{p/2}\sqrt{\widehat{Var(\widehat{\lambda})}}, \quad \widehat{\beta} \pm z_{p/2}\sqrt{\widehat{Var(\widehat{\beta})}}, \quad \widehat{a} \pm z_{p/2}\sqrt{\widehat{Var(\widehat{a})}}, \quad \text{and} \quad \widehat{b} \pm z_{p/2}\sqrt{\widehat{Var(\widehat{b})}},$$

where $z_{p/2}$ is the upper $p/2$ quantile of the standard normal distribution.

Next, we use the weighted least squares (WLS) method to estimate the parameters of the EG-PL model. Suppose that $X_{(1)}, \cdots, X_{(n)}$ are the order statistics of a sample of size n coming from the EG-PL distribution with the cdf given in (4). Then, $O_{(i)} = F(X_{(i)})$ is identically distributed as the i-th order statistic extracted from a sample of size n from the standard uniform distribution and therefore the expected value of $O_{(i)}$ is $E_{(i)} = E(O_{(i)}) = \frac{i}{n+1}$. Here, we wish to minimize the distances between the $O_{(i)}$'s and $E_{(i)}$'s. Let $\epsilon = (\epsilon_{(1)}, \cdots, \epsilon_{(n)})^T$ where $\epsilon_{(i)} = O_{(i)} - E_{(i)}$ and $\theta = (\lambda, \beta, a, b)$. Then the WLS estimator of θ will be obtained by minimizing the quantity $S(\theta|\Omega) = \epsilon^T\Omega\epsilon$ where Ω is an $n \times n$ matrix that is related to the procedure of the estimation. One can choose Ω to be the inverse of the variance-covariance matrix of ϵ, however, here we consider the following diagonal weight matrix

$$\Omega = \text{diag}\left\{\frac{1}{Var(\epsilon_{(1)})}, \cdots, \frac{1}{Var(\epsilon_{(n)})}\right\},$$

and obtain the diagonally weighted least squares (DWLS) estimator of θ denoted as $\widetilde{\theta} = (\widetilde{\lambda}, \widetilde{\beta}, \widetilde{a}, \widetilde{b})$. It is clear that $Var(\epsilon_{(i)}) = \frac{i(n-i+1)}{(n+1)^2(n+2)} = \frac{E_{(i)}(1-E_{(i)})}{n+2}, i = 1, \cdots, n$, therefore, we have

$$S(\theta|\Omega) = \sum_{i=1}^{n} \frac{n+2}{E_{(i)}(1-E_{(i)})}\, \epsilon_{(i)}^2. \tag{20}$$

Therefore, the DWLS estimators of the EG-PL parameters will be obtained by minimizing (20) with respect to (λ, β, a, b).

In the sequel, we wish to compare the performances of ML and DWLS estimators of the parameters using a Monte Carlo simulation. To this end, first, we express the procedure of data simulation from the EG-PL distribution.

Here, we propose three different algorithms for generating random data from the EG-PL distribution.

The first algorithm is based on this fact that the Lindley distribution is a mixture of exponential and gamma distributions, see Ghitany et al. [12].

The second algorithm is based on generating random data from the Weibull-generalized gamma (GG) mixture form of the power Lindley distribution, see Ghitany et al. [13].

The third algorithm is based on generating random data using the quantile function of the EG-PL distribution, given in (14).

Algorithm 1. (Mixture Form of the Lindley Distribution).

1. *Generate* $U_i \sim Uniform(0,1), i = 1, \cdots, n$;
2. *Generate* $V_i \sim Exponential(\lambda), i = 1, \cdots, n$;
3. *Generate* $W_i \sim Gamma(2, \lambda), i = 1, \cdots, n$;
4. *If* $(1 - U_i{}^{\frac{1}{b}})^{\frac{1}{a}} \geq \frac{1}{1+\lambda}$ *set* $X_i = V_i^{\frac{1}{\beta}}$, *otherwise, set* $X_i = W_i^{\frac{1}{\beta}}, i = 1, \cdots, n$.

Algorithm 2. (Mixture Form of the Power Lindley Distribution).

1. *Generate* $U_i \sim Uniform(0,1), i = 1, \cdots, n$;
2. *Generate* $Y_i \sim Weibull(\beta, \lambda), i = 1, \cdots, n$;
3. *Generate* $Z_i \sim GG(2, \beta, \lambda), i = 1, \cdots, n$;
4. *If* $(1 - U_i{}^{\frac{1}{b}})^{\frac{1}{a}} \geq \frac{1}{1+\lambda}$ *set* $X_i = Y_i$, *otherwise, set* $X_i = Z_i, i = 1, \cdots, n$.

Algorithm 3. (Quantile function).

1. *Generate* $U_i \sim Uniform(0,1), i = 1, \cdots, n$;
2. *Set*
$$X_i = \left\{-1 - \frac{1}{\lambda} - \frac{1}{\lambda} W\left[-(\lambda+1)(1 - U_i{}^{\frac{1}{b}})^{\frac{1}{a}} \mathrm{e}^{-1-\lambda}\right]\right\}^{\frac{1}{\beta}}, i = 1, \cdots, n.$$

In our simulation, we estimate the biases and the root mean squared errors (RMSEs) of the estimators obtained based on the ML and DWLS methods using the following relations
$$bias_h(n) = \frac{1}{M}\sum_{i=1}^{M}(\hat{h}_i - h),$$
and
$$RMSE_h(n) = \sqrt{\frac{1}{M}\sum_{i=1}^{M}(\hat{h}_i - h)^2},$$
for $h = \lambda, \beta, a$ and b and $\hat{h}_i$ is the corresponding estimate (ML or DWLS estimate) obtained in the i-th iteration and M is the number of the iterations of the simulation. We take the sample sizes $n = 50$ and 100, $\lambda = 0.5, 2$, $\beta = 0.5, 2$, $a = 0.5, 2$, $b = 0.5, 2$ and $M = 10000$ in our simulation. We used Algorithm 1 to generate data. The results are presented in Tables 1 and 2. We note that we used the `optim` function, in R (R Core Team [23]) and we excluded the bad generated samples, for which the solutions were not convergent (or we encountered an error) and/or at least one of the solutions did not get positive, from the simulation. As we can see from these tables, the results are rather stable and the estimates are close to the real values (see the estimated biases) for the most considered cases. In addition, the estimated RMSEs decrease as n increases (a few exceptions exist). We cannot attain any general conclusion that which method of estimation performs better, since in some cases the estimated RMSEs of the ML estimators are less than the corresponding estimated RMSEs of the DWLS estimators and in the other cases, the reverse is true.

§8 Real data application

In this section, we illustrate the fitting performance of the EG-PL distribution using two real data sets. For the purpose of comparison, we fitted the following models to show the fitting

Table 1: The estimated RMSEs (and estimated biases in the parentheses) of the ML estimators.

				$n = 50$			
a	b	λ	β	$\widehat{a}$	$\widehat{b}$	$\widehat{\lambda}$	$\widehat{\beta}$
0.5	0.5	0.5	0.5	0.9165(0.5824)	1.2713(0.7628)	0.6840(0.2113)	0.1635(0.0142)
			2	2.4458(1.9526)	1.3757(0.8549)	0.4067(-0.0627)	0.6017(-0.1120)
		2	0.5	0.4026(-0.0574)	1.2290(0.6900)	2.7057(2.5054)	0.3519(0.1311)
			2	0.3219(-0.1396)	1.0454(0.6191)	3.6430(3.1459)	1.4327(0.5451)
	2	0.5	0.5	2.3699(1.4843)	1.6070(0.1804)	0.7005(0.2853)	0.1555(-0.0197)
			2	3.9759(2.9532)	1.9116(0.4210)	0.4099(-0.0126)	0.5693(-0.2467)
		2	0.5	0.5035(-0.1709)	1.4229(-0.5684)	5.7822(4.8651)	0.3708(0.2039)
			2	0.4271(-0.1881)	1.4659(-0.5041)	5.5584(4.5832)	1.4002(0.7548)
2	0.5	0.5	0.5	3.3082(2.7212)	1.2161(0.8963)	0.2638(-0.1328)	0.1554(-0.0918)
			2	3.9330(3.4712)	1.5652(0.9914)	0.2548(-0.1850)	0.6803(-0.3760)
		2	0.5	3.8817(2.7577)	1.3568(0.8767)	1.2491(-0.8684)	0.2400(-0.0014)
			2	4.8144(3.9835)	1.4284(0.9528)	1.4198(-1.0774)	0.9639(-0.0570)
	2	0.5	0.5	1.2974(0.9302)	1.3425(-0.5898)	0.2744(-0.1765)	0.1415(-0.0189)
			2	6.1680(5.0724)	1.4405(-0.3255)	0.3450(-0.2417)	0.5937(-0.2579)
		2	0.5	1.9517(0.4237)	1.2904(-0.6366)	1.2888(-0.1639)	0.2325(0.0222)
			2	3.8792(2.1480)	1.3012(-0.5746)	2.0012(-0.2870)	0.8848(0.0131)
				$n = 100$			
a	b	λ	β	$\widehat{a}$	$\widehat{b}$	$\widehat{\lambda}$	$\widehat{\beta}$
0.5	0.5	0.5	0.5	0.6984(0.5157)	0.8228(0.5976)	0.3988(0.1350)	0.1259(0.0088)
			2	2.2732(2.0378)	0.8827(0.6825)	0.2750(-0.1164)	0.4365(-0.1576)
		2	0.5	0.2478(-0.1042)	0.8163(0.5685)	2.6498(2.5582)	0.2004(0.0745)
			2	0.2510(-0.1708)	0.7751(0.5468)	3.5308(3.2911)	0.7697(0.3039)
	2	0.5	0.5	2.1308(1.3486)	1.1379(0.0088)	0.7162(0.3108)	0.1243(-0.0283)
			2	4.2373(3.1837)	1.2842(0.1886)	0.3986(-0.0354)	0.4844(-0.2807)
		2	0.5	0.4532(-0.2162)	1.0907(-0.7466)	6.0575(5.2429)	0.2566(0.1595)
			2	0.3899(-0.2272)	1.0826(-0.7194)	5.6719(4.8325)	0.9865(0.6047)
2	0.5	0.5	0.5	3.3765(2.8702)	1.0738(0.8939)	0.2199(-0.1364)	0.1459(-0.1202)
			2	3.8684(3.5125)	1.1455(0.8891)	0.2219(-0.1864)	0.5946(-0.4770)
		2	0.5	3.7872(2.7268)	1.0359(0.8137)	1.1995(-0.8884)	0.1339(-0.0544)
			2	5.3304(4.5278)	1.1271(0.8814)	1.3917(-1.1916)	0.5577(-0.2734)
	2	0.5	0.5	1.3448(0.9571)	1.0008(-0.7451)	0.2462(-0.1902)	0.1048(-0.0323)
			2	6.6949(5.6048)	1.0449(-0.5586)	0.3269(-0.2692)	0.4853(-0.2997)
		2	0.5	1.9048(0.4223)	1.0070(-0.7714)	1.2094(-0.1563)	0.1472(-0.0127)
			2	3.4734(1.8249)	1.0026(-0.7083)	2.0118(-0.1833)	0.5410(-0.1369)

Table 2: The estimated RMSEs (and estimated biases in the parentheses) of the DWLS estimators.

				$n = 50$			
a	b	λ	β	$\widetilde{a}$	$\widetilde{b}$	$\widetilde{\lambda}$	$\widetilde{\beta}$
0.5	0.5	0.5	0.5	1.5508(1.1871)	1.3368(0.7639)	0.4925(0.0647)	0.1546(-0.0098)
			2	1.2283(0.9701)	1.4142(0.8108)	0.6144(0.1245)	0.6385(-0.0179)
		2	0.5	0.5545(0.1668)	1.4583(0.8242)	1.9554(1.3829)	0.3310(0.0905)
			2	1.4617(0.7345)	1.2854(0.7723)	2.7509(0.7635)	1.0899(0.2333)
	2	0.5	0.5	1.3486(0.4047)	1.7081(0.0584)	1.2857(0.5722)	0.2112(0.0328)
			2	1.3462(0.4911)	1.9837(0.2174)	1.0377(0.4695)	0.7192(0.0244)
		2	0.5	1.0177(0.4995)	1.6451(-0.2880)	1.0789(0.1122)	0.3366(0.1272)
			2	1.5766(0.9861)	1.7766(-0.1384)	1.1645(-0.2817)	1.1441(0.3409)
2	0.5	0.5	0.5	2.7922(2.4763)	1.4728(0.9317)	0.2645(-0.1427)	0.1691(-0.0929)
			2	1.2922(0.7175)	1.6103(0.9438)	0.3530(-0.0073)	0.6715(-0.3045)
		2	0.5	1.8798(0.3513)	1.4759(0.8901)	1.2327(-0.2933)	0.2930(0.0207)
			2	1.8731(-0.1371)	1.5865(0.9555)	1.5480(0.1844)	1.2023(0.0953)
	2	0.5	0.5	1.2213(-1.0602)	1.4575(-0.7242)	0.7888(0.2064)	0.1891(0.0521)
			2	1.5793(1.0025)	1.4705(-0.4747)	0.2923(-0.1461)	0.6231(-0.1353)
		2	0.5	1.6422(0.6151)	1.4656(-0.4779)	1.0541(-0.5127)	0.2311(-0.0089)
			2	1.7862(0.7235)	1.5695(-0.4350)	1.0961(-0.4992)	0.9756(-0.0216)

				$n = 100$			
a	b	λ	β	$\widetilde{a}$	$\widetilde{b}$	$\widetilde{\lambda}$	$\widetilde{\beta}$
0.5	0.5	0.5	0.5	1.6825(1.3376)	0.8775(0.6326)	0.4220(0.0184)	0.1150(-0.0207)
			2	1.1659(0.9605)	0.9849(0.6638)	0.4265(0.0617)	0.4749(-0.0524)
		2	0.5	0.9156(0.6578)	1.0719(0.7232)	1.0480(0.0808)	0.1764(0.0212)
			2	1.1645(0.7254)	1.0108(0.6991)	1.9127(0.4071)	0.6616(0.0795)
	2	0.5	0.5	1.1599(0.3782)	1.4185(0.0085)	1.1146(0.5438)	0.1627(0.0109)
			2	1.1085(0.3756)	1.5878(0.0317)	1.2288(0.5405)	0.6696(0.0589)
		2	0.5	0.7909(0.4332)	1.1883(-0.4782)	0.9183(0.0933)	0.2272(0.0919)
			2	1.2723(0.9240)	1.2488(-0.3797)	0.8461(-0.4066)	0.7656(0.2392)
2	0.5	0.5	0.5	2.6732(2.4344)	1.1712(0.8706)	0.2218(-0.1417)	0.1475(-0.1170)
			2	1.0960(0.6855)	1.1870(0.8525)	0.2587(-0.0124)	0.5613(-0.3973)
		2	0.5	1.5690(0.3074)	1.1767(0.8303)	1.0729(-0.3305)	0.1608(-0.0342)
			2	1.3958(-0.2881)	1.2080(0.8402)	1.2041(0.1169)	0.6917(-0.0962)
	2	0.5	0.5	1.1563(-1.0778)	1.1559(-0.8427)	0.5980(0.1623)	0.1300(0.0307)
			2	1.6585(1.1611)	1.0928(-0.6483)	0.2569(-0.1743)	0.4895(-0.1694)
		2	0.5	1.4510(0.5394)	1.0958(-0.6620)	0.9666(-0.5524)	0.1552(-0.0314)
			2	1.6042(0.6319)	1.1295(-0.6293)	1.0122(-0.5179)	0.6263(-0.1413)

performance of the EG-PL distribution by means of two real data sets: (i) the Weibull-power Lindley (W-PL), Bourguignon et al. [6], with the following pdf

$$
\begin{aligned}
f(x) &= \frac{a\,b\,\beta\,\lambda^2\,(\lambda+1)^b}{(\lambda+1+\lambda x)^{b+1}}\,(1+x^\beta)\,x^{\beta-1}\mathrm{e}^{b\,\lambda x^\beta}\left[1-\big(1+\frac{\lambda}{1+\lambda}\,x^\beta\big)\mathrm{e}^{-\lambda x^\beta}\right]^{b-1}\\
&\quad\times\exp\left\{-a\left(\frac{1-\big(1+\frac{\lambda}{1+\lambda}\,x^\beta\big)\mathrm{e}^{-\lambda x^\beta}}{\big(1+\frac{\lambda x^\beta}{\lambda+1}\big)\mathrm{e}^{-\lambda x^\beta}}\right)^b\right\}, \qquad x>0, \quad a,\, b,\, \beta,\, \lambda>0,
\end{aligned}
$$

(ii) the exponentiated generalized Lindley (EG-L) distribution, (iii) the power Lindley (PL) distribution, (iv) the generalized Lindley (GL) distribution, Nadarajah et al. [19] and (v) the Lindley distribution. We used formal goodness-of-fit tests to compare the fits of the mentioned distributions. To this end, we considered the minimum value of the minus log-likelihood function ($-\ell$), the Kolmogorov-Smirnov (K-S) statistic and its corresponding p-value and the Cramér-von Mises (W^*) and Anderson-Darling (A^*) test statistics (see Chen and Balakrishnan [7]). Generally speaking, the smaller values of $-\ell$, K-S, A^* and W^*, the better fit to a data set. All the computations were carried out using the software R (see [23] and Pinheiro et al. [22]).

Note that initial values of model parameters are quite important to obtain the correct ML estimates of the parameters. To avoid local minima problem, first, we obtain the parameter estimate of the Lindley distribution. Then, the estimated parameter of the Lindley distribution is used as the initial value of the parameter λ of the PL and GL distributions. The estimated parameters of the PL distribution, λ and β, are then used as the initial values of the EG-PL distribution. This approach is useful to obtain correct parameter estimates of extended models.

8.1 First Application

The first data set represents the maintenance data with 46 observations reported on active repair times (hours) for an airborne communication transceiver given by Von Alven [25], (see also Chhikara and Folks, [8]). Table 3 gives the parameter ML estimates and their corresponding errors, the W^* and A^* statistics, the values of $-\ell$, the K-S statistics and their corresponding p-values. From Table 3, we see that the EG-PL distribution provides the overall best fit and therefore it could be chosen as the most suitable model among the considered models for modelling the first data set. In addition, the profile log-likelihood functions of the EG-PL distribution are plotted in Figure 5. These plots reveal that the likelihood equations of the EG-PL distribution have solutions that are maximizers.

The estimated asymptotic covariance matrix of the ML estimators of the EG-PL parameters for the first data set, which is $\widehat{\mathbf{I}}_F^{-1}(\theta)$, is given by

$$
\begin{pmatrix}
127.1104071 & 23.743125 & -28.21501521 & -0.531001858\\
23.7431247 & 475.513271 & 1.24656304 & -1.490675208\\
-28.2150152 & 1.246563 & 6.35507816 & 0.098233944\\
-0.5310019 & -1.490675 & 0.09823394 & 0.007096786
\end{pmatrix}.
$$

The %95 asymptotic confidence intervals for a, b, λ and β are given by $[3.838 \pm 6.581]$, $[21.496 \pm 9.152]$, $[1.175 \pm 3.111]$ and $[0.267 \pm 0.568]$, respectively.

Table 3: The parameter ML estimates (standard errors in the parentheses) and the goodness-of-fit test statistics for the first data set.

Models	a	b	λ	β	A^*	W^*	$-\ell$	K-S	p-value
EG-PL	3.838	21.496	1.175	0.267	0.316	0.050	99.910	0.0905	0.845
	(11.274)	(21.806)	(2.521)	(0.084)					
W-PL	0.032	10.358	1.255	0.051	0.959	0.138	105.049	0.125	0.433
	(0.026)	6.075	(4.42E-04)	(1.007E-05)					
EG-L	0.176	0.805	1.596	1	1.116	0.162	105.525	0.156	0.181
	(0.037)	(0.154)	(0.002)						
PL	1	1	0.675	0.758	0.963	0.1403	105.013	0.126	0.423
			(0.101)	(0.074)					
GL	1	0.664	0.367	1	1.388	0.205	107.848	0.166	0.140
		(0.135)	(0.064)						
Lindley	1	1	0.466	1	1.302	0.192	109.984	0.233	0.011
			(0.049)						

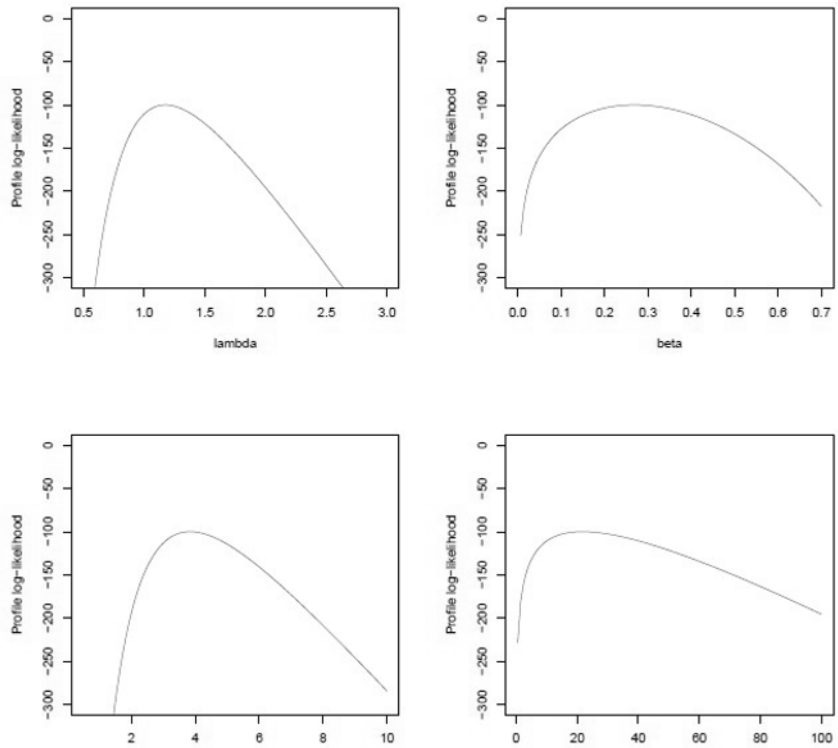

Figure 5: The profile log-likelihood functions of the EG-PL distribution for the first data set.

Table 4: The LR test results for the first data set.

	Hypotheses	LR	p-value
EG-PL versus EG-L	$H_0 : \beta = 1$	11.239	0.0008
EG-PL versus PL	$H_0 : a = b = 1$	10.206	0.0006
EG-PL versus GL	$H_0 : a = \beta = 1$	15.876	0.0003
EG-PL versus Lindley	$H_0 : a = b = \beta = 1$	20.148	0.0001

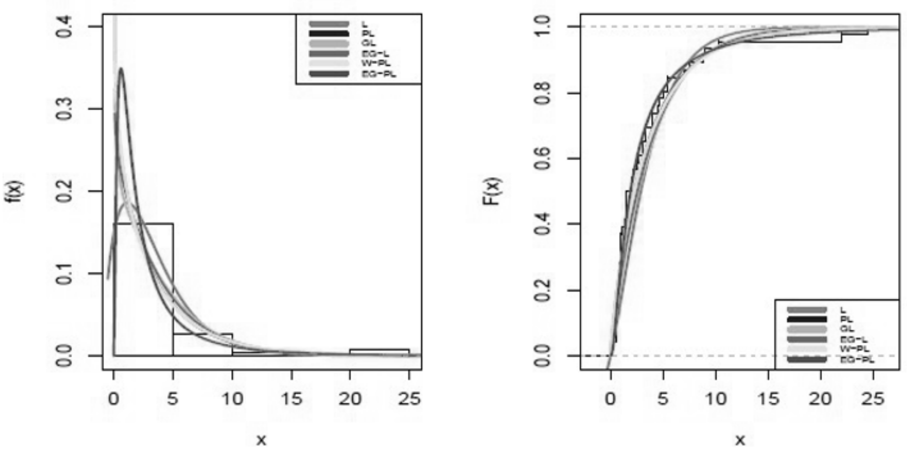

Figure 6: The fitted pdfs and cdfs of the considered distributions for the first data set.

Here, we also applied likelihood ratio (LR) tests. The LR tests can be used for comparing the EG-PL distribution with its sub-models. For example, the test of $H_0 : \beta = 1$ against $H_1 : \beta \neq 1$ is equivalent to comparing the EG-PL and EG-L distributions with each other. For this test, the LR statistic can be calculated by the following relation

$$LR = 2\left[\ell\left(\widehat{a}, \widehat{b}, \widehat{\lambda}, \widehat{\beta}\right) - \ell\left(\widehat{a}^*, \widehat{b}^*, \widehat{\lambda}^*, 1\right)\right],$$

where $\widehat{a}^*$, $\widehat{b}^*$ and $\widehat{\lambda}^*$ are the ML estimators of a, b and λ, respectively, that will be obtained under H_0. Under the regularity conditions and if H_0 is assumed to be true, the LR test statistic converges in distribution to a chi square with r degrees of freedom, where r equals the difference between the number of parameters estimated under H_0 and the number of parameters estimated in general, (for $H_0 : \beta = 1$, we have $r = 1$). Table 4 gives the LR statistics and the corresponding p-values for the first data set.

From Table 4, we observe that the computed p-values are too small, so we reject all the null hypotheses and conclude that the EG-PL fits the first data better than the considered sub-models in the sense of the LR criterion.

We also plotted the fitted pdfs and cdfs of the considered models for the sake of visual comparison, in Figure 6. Figure 6 suggests that the EG-PL fits the skewed data very well. In addition, we presented the probability-probability (P-P) plots for the fitted models in Figure 7. These plots reveal that the EG-PL distribution has the best fit.

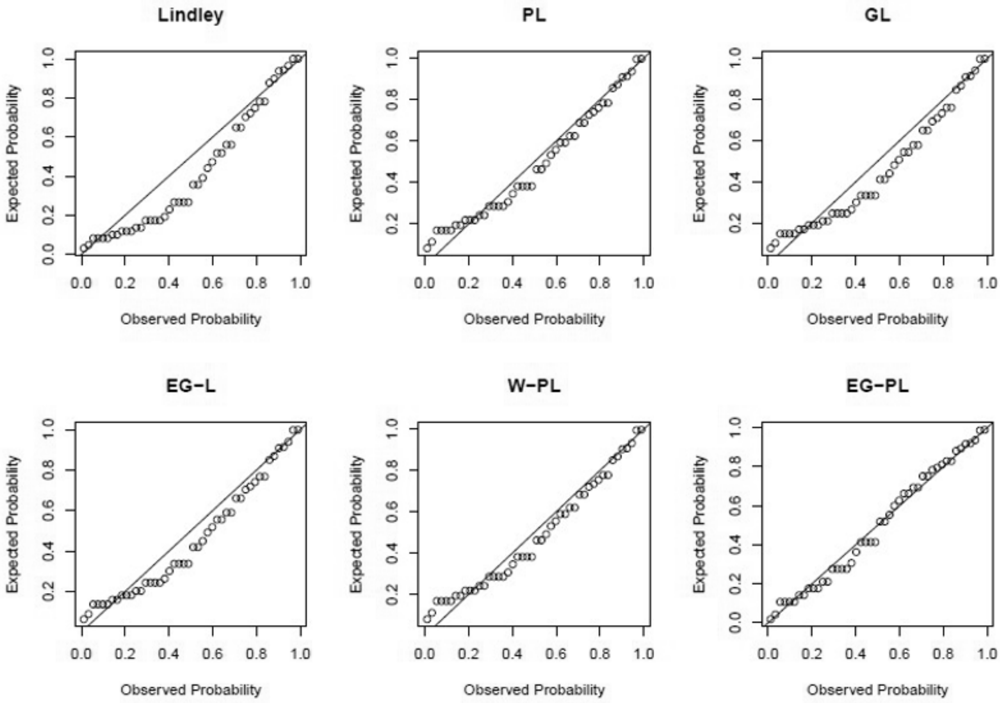

Figure 7: The P-P plots of the fitted models for the first data set.

8.2 Second Application

The second data set consists of prices ($\times 10^4$ dollars) of 428 new cars and trucks (Kiplinger's Personal Finance, Dec 2003), see for details Oluyede et al. [20], page 299. Table 5 presents the parameter ML estimates and their corresponding errors and the goodness-of-fit test statistics for the second data. We see that the EG-PL distribution outperforms the considered models according to the considered goodness-of-fit criteria. In addition, the plots of the profile log-likelihood functions of the EG-PL distribution reveal that the likelihood equations of the EG-PL distribution have solutions that are maximizers (the plots are not presented here).

The estimated asymptotic covariance matrix of the ML estimators of the EG-PL parameters for the second data set is given by

$$\begin{pmatrix} 0.20489890 & 1.1337013 & -0.3604887122 & -0.0109507550 \\ 1.13370131 & 135.2597012 & 1.8595302322 & -0.6476292267 \\ -0.36048871 & 1.8595302 & 0.7520200103 & 0.0009059048 \\ -0.01095076 & -0.6476292 & 0.0009059048 & 0.0036418419 \end{pmatrix}.$$

The %95 asymptotic confidence intervals for a, b, λ and β are given by $[1.024 \pm 1.319]$, $[28.687 \pm 6.684]$, $[2.412 \pm 1.825]$ and $[0.593 \pm 0.480]$, respectively. The LR test results for the second data set are given in Table 6. The null hypotheses are all rejected in favor of the EG-PL distribution since the p-values are less than 0.0001. We plotted the fitted pdfs and cdfs of the

Table 5: The parameter ML estimates (standard errors in the parentheses) and the goodness-of-fit test statistics for the second data set.

Models	a	b	λ	β	A^*	W^*	$-\ell$	K-S	p-value
EG-PL	1.024	28.687	2.412	0.593	0.481	0.055	747.995	0.086	0.897
	(0.453)	(11.630)	(0.867)	(0.060)					
W-PL	0.994	30.163	1.071	0.039	9.305	1.481	821.780	0.101	<0.001
	(0.013)	(2.808)	(0.002)	(0.002)					
EG-L	0.471	4.937	1.824	1	2.508	0.359	764.827	0.058	0.104
	(0.023)	(0.451)	(0.023)						
PL	1	1	0.277	1.480	8.563	1.347	812.019	0.091	0.001
			(0.018)	(0.043)					
GL	1	4.464	0.955	1	2.825	0.409	767.799	0.061	0.076
		(0.444)	(0.039)						
Lindley	1	1	0.507	1	5.344	0.812	878.145	0.222	<0.001
			(0.017)						

Table 6: The LR test results for the second data set.

	Hypotheses	LR	p-value
EG-PL versus EG-L	$H_0 : \beta = 1$	33.664	<0.0001
EG-PL versus PL	$H_0 : a = b = 1$	128.048	<0.0001
EG-PL versus GL	$H_0 : a = \beta = 1$	39.608	<0.0001
EG-PL versus Lindley	$H_0 : a = b = \beta = 1$	260.300	<0.0001

considered models in Figure 8. In addition, the P-P plots of the fitted models are displayed in Figure 9. Visual comparisons confirm the superiority of the EG-PL distribution.

§9 Concluding Remarks

In this article, we proposed a new generalization of the power Lindley distribution i.e. the EG-PL distribution. We discussed several important properties of the new distribution. The hazard rate function of the new model has the advantage of taking various forms depending on the values of the parameters. We also discussed the estimation problem of the unknown parameters of the new distribution using the ML and DWLS methods. A simulation study was performed and we observed that both estimation methods worked rather well for most considered cases. We analyzed two real data sets and observed that the new model fitted the data sets better than the considered distributions in the sense of some well-known goodness of fit criteria. Altogether, we may conclude that the new distribution can be a nice choice in real situations for analyzing lifetime, engineering, economics and many other types of data sets.

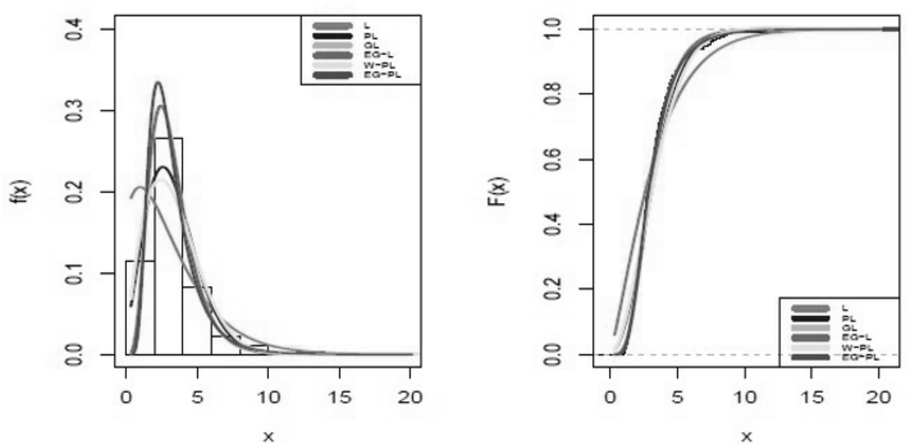

Figure 8: The fitted pdfs and cdfs of the considered distributions for the second data set.

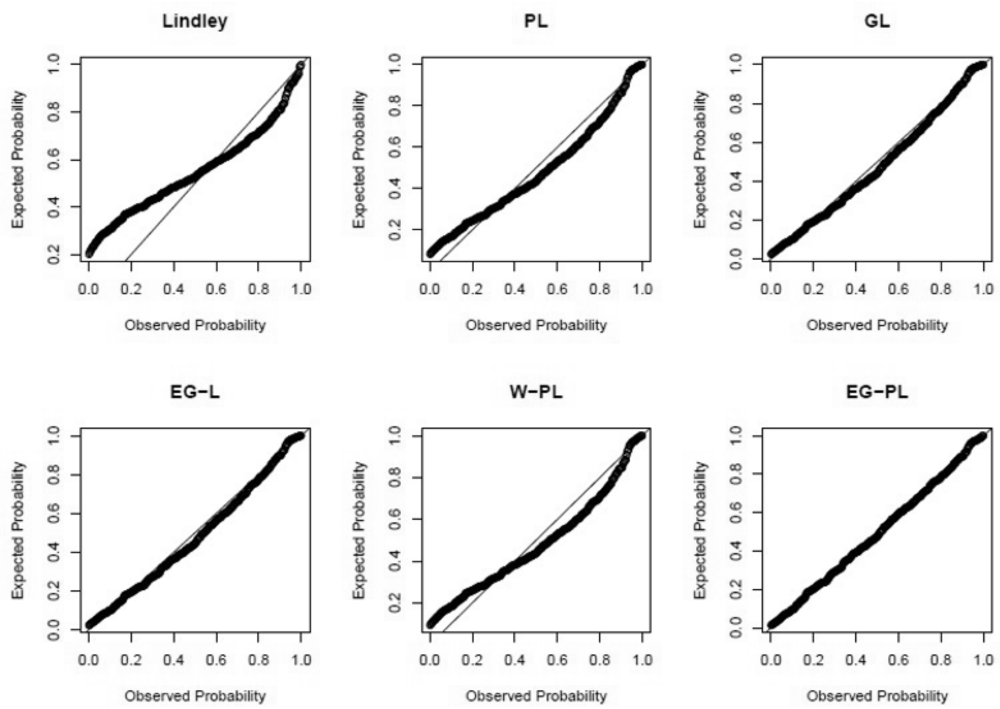

Figure 9: The P-P plots of the fitted models for the second data set.

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