ENERGY-DEPENDENT FRACTIONAL STURM-LIOUVILLE IMPULSIVE PROBLEM

by

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In study, we show the existence and integral representation of solution for energy-dependent fractional Sturm-Liouville impulsive problem of order with $\alpha \in (1,2]$ *impulsive and boundary conditions. An existence theorem is proved for energy-dependent fractional Sturm-Liouville impulsive problem by using Schaefer fixed point theorem. Furthermore, in the last part of the article, an application is given for the problem and visual results are shown by figures.*

Key words: *Sturm-Liouville problem, energy dependent, fractional, impulsive condition, Caputo derivative*

Introduction

Sturm-Liouville problems is the main problem of applied science and have a lot of applications in engineering, physics, and mathematics. During history, many researchers have studied many subjects about this topic to develop the theory of classical Sturm-Liouville [1-4]. In recent years, however, the theory of fractional Sturm-Liouville which is become more advantageous than the classical models has drawn interest so much. Some initial study has done about fractional Sturm-Liouville problem [5-11]. In [5, 6], Klimek and Agrawal investigated fractional Sturm-Liouville operator involving left-sided Riemann-Liouville derivative (RLD) and right-sided Caputo derivative (CD). Furthermore, fractional Sturm-Liouville problem including different types of fractional operators is figured out numerical methods [12]. Differential equations involving impulses, firstly considered by Milman and Myshkis [13] in the 1960's, are used to characterize sudden changes in the real world phenomena and many evolutionary processes involving short term perturbations. This process is modeled naturally with impulsive differential equations. Therefore, impulsive differential equations always attract comprehensive interest and find many application areas in literature. The basic results are given by Lakshmikantham *et al.* [14]. There are many works on this problem in diverse fields such as medicine, biotechnology, pharmacokinetics, environmental science, pest control, hematopoiesis. For examples, by using radial acceleration control of the satellite orbit, periodic treatment of some diseases, death in the populations, non-autonomous equations, industrial robotics [15].

Fractional differential equations have developed as a popular and interesting research field and have great interest, because they have both theoretical and physical, biological applications in various sciences. Furthermore, modeling by fractional calculus become more advan-

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tageous than the integer order models [16-20]. The growing development on fractional differential equations has motivated the recent papers. However, fractional differential impulsive problem has not explored in any respect yet and these problems play a significant role in theory and applications recently. A lot of authors have handled fractional differential impulsive equations for diverse equations and conditions [21-25]. The works on this topic concern the existence solutions and many significant results have been reported. Agarwal *et al.* [26] handled the existence of solutions of fractional differential impulsive problem including the Caputo fractional derivatives. Furthermore, Zhou and Feng [27] studied fractional Sturm-Liouville problem under impulsive condition. However, this study is firstly about this topic. Diffusion equation, one of the natural laws of physics, arises in numerous situations of science. The simplest description of the diffusion equation in physics is the transport of fluid, chemical species or energy from the area of higher concentration to one of lower concentration. One of the main problems in physics is that describing interactions among colliding particles. A well-known theoretical model can express to this problem. Therefore, from collision experiments, s-wave scattering matrix to s-wave binding energies are assumed to be known precisely. The s-wave Schroedinger equation with a radial static potential, V, depending energy in some way is given:

$$u'' + \left[E - V(r)\right]u = 0,\ r \geq 0$$

Let $U(r), Q(r)$ are complex functions. The V has equation of energy dependence:

$$V\left[(r,E)\right] = U(r) + \left[2\sqrt{E}Q(r)\right]$$

Several scientists use contraction mapping principle, Banach fixed point theorem or Krasnoselskii's fixed point theorem to give the existence results of solutions for fractional differential impulsive problem. Here, we obtain results of our problem by using Schafer fixed point theorem.

In this study, we analyze energy-dependent fractional Sturm-Liouville impulsive problem:

$$-\mathrm{D}^{\alpha}_{\pi-}h(r)\,{}^{C}\mathrm{D}^{\alpha}_{0+}u(r) + \{[2\lambda p(r) + q(r)]\}u(r) = 0 \tag{1}$$

$$\Delta u\big|_{r=r_k} = I_k[u(r_k)],\ \Delta u'\big|_{r=r_k} = I_k^*[u(r_k)],\ r_k \in (0,\pi),\ k = 1,2,..,n \tag{2}$$

$$\alpha_1 u(0) + \beta_1 u'(0) = 0,\ \ \alpha_2 u(\pi) + \beta_2 u'(\pi) = 0 \tag{3}$$

where $\mathrm{D}^{\alpha}_{\pi}$ is the RLD [17], ${}^{C}\mathrm{D}^{\alpha}_{0+}$ – the (CD) [17], $I^{1-\alpha}_{a+}$, $I^{1-\alpha}_{b-}$ – the Riemann-Liouville integral [17], h – real-valued continuous positive function in interval $[0,\pi]$, $p \in W$, $q \in L^2[0,\pi]$, I_k, I_k^* : $R \to R$ and $\alpha_1,\alpha_2,\beta_1,\beta_2 \in R$, and $\alpha_1\alpha_2 + \alpha_1\beta_2 + \alpha_2\beta_1 \neq 0$, $0 = r_0 < r_1 < ... < r_n < r_{n+1} = \pi$, $\Delta u|_{r=r_k} = [u(r_k^+) - u(r_k^-)], u(r_k^-) = \lim_{h\to 0^-} u[r_k + h]$ stand for the right and left limits of $u(r)$ at $r = r_k$, $k = 1,2,..,n$, $\Delta u'\big|_{r=r_k}$ is a similar meaning for $u'(r)$.

Firstly, we obtain the integral representation for solution of energy-dependent farctional Sturm-Liouville impulsive problem and secondly we establish an operator, H, for system of eqs. (1)-(3), then demonstrate the compactness of H by applying Arzela Ascoli theorem and we prove existence result for problem (1)-(3).

Preliminaries

Let PC Banach space is $PC(X,R) = \{u : X \to R : u \in C((r_k, r_{k+1}], R)$ and there exist $[u(r_k^-)]$ and $[u(r_k^+)],\ u(r_k^-) = u(r_k), k = 1,2,...,n\}$ where $X = [0,\pi]$ and the norm is defined:

$$\|u\|_{PC} = \sup_{r\in X}|u(r)|$$

Definition 1. [28] If K is a compact metric space then a subset $F \subset C(K)$ of the space of continuous functions on K equipped with the uniform distance, is compact if and only if it is closed, bounded, equicontinuous.

Lemma 2. [23] Let, $c_i \in \mathrm{R}$. Then ${}^{C}\mathrm{D}^{\alpha}h(r) = 0,$ has solution:

$$h(r) = c_0 + c_1 r + c_2 r^2 + ... + c_n r^{n-1},\ (i=0,1,2,...,n)\ \ n = [\alpha]+1$$

Lemma 3. [23] Let $\alpha > 0$. Then:

$$I^{\alpha}\,{}^{C}\mathrm{D}^{\alpha}h(r) = h(r) + c_0 + c_1 r + c_2 r^2 + ... + c_n r^{n-1}$$

for some $n = [\alpha]+1$.

Lemma 4. [17] $\mathrm{Re}(\alpha) > 0,\ \ n = \mathrm{Re}(\alpha)+1$ and let $f_{n-\alpha}(x) = (I_{b-}^{n-\alpha} f)(x)$ be the fractional integral of order $n-\alpha$.

If $1 \le p \le \infty$ and $f(x) \in I_{b-}^{\alpha}(L_p)$, then:

$$[I_{b-}^{\alpha}\mathrm{D}_{b-}^{\alpha} f](x) = f(x)$$

where

$$I_{b-}^{\alpha}(L_p) = \{f / f = I_{b-}^{\alpha}\phi,\ \phi \in L_p(L_p)\}$$

If $f_{n,...,\alpha}(x) \in AC^n[a,b]$ and $f(x) \in L_1(a,b)$ then the equality:

$$[I_{b-}^{\alpha}\mathrm{D}_{b-}^{\alpha} f](x) = f(x) - \left[\sum_{j=1}^{n} \frac{f_{n-\alpha}^{(n-j)}(a)}{\Gamma(\alpha-j+1)}(x-a)^{\alpha-j}\right]$$

holds almost everywhere on $[a,b]$.

Lemma 5. [28] (Schaefer fixed point theorem)

Let $T: J \to J$ is a compact and continuous mapping where J is a Banach space. If the set:

$$\{x \in T : x = \lambda T(x) \text{ for some } \lambda \in [0,1]\}$$

is bounded, then H has a fixed point.

Main results

Theorem 6. Let $1 < \alpha \le 2$ and let $u \in PC(X,R)$. The solution of energy-dependent fractional Sturm-Liouville impulsive problem has integral representation given with:

$$u(r) = \int_{r_k}^{r}\left[\frac{(r-\zeta)^{\alpha-1}}{\Gamma(\alpha)}\frac{1}{h(\zeta)} I_{\pi,-}^{\alpha}\left(2\lambda p(\zeta)+q(\zeta)\right)u(\zeta)\right]\mathrm{d}\zeta +$$

$$+\sum_{i=1}^{n}\left[\int_{r_{i-1}}^{r_i}\left(\frac{\left((r-r_i)(r_i-\zeta)^{\alpha-2}\right)}{\Gamma(\alpha-1)} + \frac{(r_i-\zeta)^{\alpha-1}}{\Gamma(\alpha)}\right)\frac{1}{h(\zeta)} I_{\pi,-}^{\alpha}\left(2\lambda p(\zeta)+q(\zeta)\right)u(\zeta)\mathrm{d}\zeta\right] -$$

$$-\frac{1}{\mu}\alpha_2\beta_1\int_{r_n}^{\pi}\frac{(\pi-\zeta)^{\alpha-1}}{\Gamma(\alpha)}\frac{1}{h(\zeta)} I_{\pi,-}^{\alpha}\left(2\lambda p(\zeta)+q(\zeta)\right)u(\zeta)\mathrm{d}\zeta -$$

$$-\frac{\alpha_2\beta_1}{\mu}\sum_{i=1}^{n}\left(\int_{r_{i-1}}^{r_i}\left(\frac{\left((\pi-r_i)(r_i-\zeta)^{\alpha-2}\right)}{\Gamma(\alpha-1)}+\frac{(r_i-\zeta)^{\alpha-1}}{\Gamma(\alpha)}\right)\frac{1}{h(\zeta)}I_{\pi,-}^{\alpha}\begin{pmatrix}2\lambda p(\zeta)+\\+q(\zeta)\end{pmatrix}u(\zeta)\mathrm{d}\zeta\right)-$$

$$-\frac{1}{\mu}\alpha_2\beta_1\sum_{i=1}^{n}\left[I_i\left(u(r_i)\right)+I_i^{*}\left(u(r_i)\right)(\pi-r_i)\right]-$$

$$-\frac{\beta_1\beta_2}{\mu}\int_{r_n}^{\pi}\frac{(\pi-\zeta)^{\alpha-2}}{\Gamma(\alpha-1)}\frac{1}{h(\zeta)}I_{\pi,-}^{\alpha}\left(2\lambda p(\zeta)+q(\zeta)\right)u(\zeta)\mathrm{d}\zeta-$$

$$-\frac{1}{\mu}\beta_1\beta_2\left[\sum_{i=1}^{n}\frac{1}{\Gamma(\alpha-1)}\int_{r_{i-1}}^{r_i}\left(r_i-\zeta\right)^{\alpha-2}\frac{1}{h(\zeta)}\left(I_{\pi,-}^{\alpha}\left(2\lambda p(\zeta)+q(\zeta)\right)u(\zeta)\mathrm{d}\zeta\right)\right]-$$

$$-\frac{1}{\mu}\beta_1\beta_2\sum_{i=1}^{n}I_i^{*}\left(u(r_i)\right)+\frac{\alpha_1\alpha_2 r}{\mu}\int_{r_n}^{\pi}\frac{(\pi-\zeta)^{\alpha-1}}{\Gamma(\alpha)}\frac{1}{h(\zeta)}I_{\pi,-}^{\alpha}\left(2\lambda p(\zeta)+q(\zeta)\right)u(\zeta)\mathrm{d}\zeta+$$

$$+\frac{1}{\mu}\alpha_1\alpha_2 r\sum_{i=1}^{n}\left[\int_{r_{i-1}}^{r_i}\left(\frac{\left((\pi-r_i)(r_i-\zeta)^{\alpha-2}\right)}{\Gamma(\alpha-1)}+\frac{(r_i-\zeta)^{\alpha-1}}{\Gamma(\alpha)}\right)\frac{1}{h(\zeta)}I_{\pi,-}^{\alpha}\left(2\lambda p(\zeta)+q(\zeta)\right)u(\zeta)\mathrm{d}\zeta\right]+$$

$$+\frac{\alpha_1\alpha_2 r}{\mu}\sum_{i=1}^{n}\left[I_i\left(u(r_i)\right)+I_i^{*}\left(u(r_i)\right)(\pi-r_i)\right]+$$

$$+\frac{\alpha_1\beta_2 r}{\mu}\int_{r_k}^{\pi}\frac{(\pi-\zeta)^{\alpha-2}}{\Gamma(\alpha-1)}\frac{1}{h(\zeta)}\left(I_{\pi,-}^{\alpha}\left(2\lambda p(\zeta)+q(\zeta)\right)u(\zeta)\right)\mathrm{d}\zeta+$$

$$+\frac{\alpha_1\beta_2 r}{\mu}\left\{\sum_{i=1}^{n}\left[\frac{1}{\Gamma(\alpha-1)}\int_{r_{i-1}}^{r_i}\left(r_i-\zeta\right)^{\alpha-2}\frac{1}{h(\zeta)}I_{\pi,-}^{\alpha}\left(2\lambda p(\zeta)+q(\zeta)\right)u(\zeta)\mathrm{d}\zeta+I_i^{*}\left(u(r_i)\right)\right]\right\}+$$

$$+\sum_{i=1}^{n}\left[I_i\left(u(r_i)\right)+I_i^{*}\left(u(r_i)\right)(r-r_i)\right] \tag{4}$$

if and only if u is a solution of the eqs. (1)-(3) problem, where $\mu=\alpha_2\beta_1-\alpha_1\alpha_2\pi-\alpha_1\beta_2$.

Proof. Assuming u satisfies (1)-(3). If $r\in[0,r_1]$, then:

$$\mathrm{D}_{\pi,-}^{\alpha}h(r)\,{}^{C}\mathrm{D}_{0,+}^{\alpha}u(r)=\left[2\lambda p(r)+q(r)\right]u(r)$$

by means of *Lemma 4*

$$h(r)\,{}^{C}\mathrm{D}_{0,+}^{\alpha}u(r)=\left[I_{\pi,-}^{\alpha}\left(2\lambda p(r)+q(r)\right)u(r)\right]$$

$${}^{C}\mathrm{D}_{0,+}^{\alpha}u(r)=\frac{1}{h(r)}I_{\pi,-}^{\alpha}\left[2\lambda p(r)+q(r)\right]u(r)$$

if $b_0,b_1\in\mathrm{R}$ and *Lemma 3* implies:

$$u(r)=I_{0+}^{\alpha}\frac{1}{h(r)}I_{\pi,-}^{\alpha}\left(2\lambda p(r)+q(r)\right)u(r)+b_0+b_1r=$$

$$= \int_0^r \frac{(r-\zeta)^{\alpha-1}}{\Gamma(\alpha)} \frac{1}{h(\zeta)} I_{\pi,-}^{\alpha} \left[2\lambda p(\zeta) + q(\zeta) \right] u(\zeta) \mathrm{d}\zeta + b_0 + b_1 r \tag{5}$$

It follows from of eq. (5) that:

$$u'(r) = \int_0^r \frac{(r-\zeta)^{\alpha-2}}{\Gamma(\alpha-1)} \frac{1}{h(\zeta)} I_{\pi,-}^{\alpha} \left(2\lambda p(\zeta) + q(\zeta) \right) u(\zeta) \mathrm{d}\zeta + b_1$$

If $r \in (r_1, r_2]$, then, $c_0, c_1 \in \mathrm{R}$, we can write:

$$u(r) = \int_{r_1}^r \frac{(r-\zeta)^{\alpha-1}}{\Gamma(\alpha)} \frac{1}{h(\zeta)} I_{\pi,-}^{\alpha} \left(2\lambda p(\zeta) + q(\zeta) \right) u(\zeta) \mathrm{d}\zeta + c_0 + c_1 (r - r_1)$$

$$u'(r) = \frac{1}{\Gamma(\alpha-1)} \int_{r_1}^r (r-\zeta)^{\alpha-2} \frac{1}{h(\zeta)} I_{\pi,-}^{\alpha} \left(2\lambda p(\zeta) + q(\zeta) \right) u(\zeta) \mathrm{d}\zeta + c_1$$

Using the impulse conditions $\Delta u\big|_{r=r_1} = I_1(u(r_1))$ and $\Delta u'\big|_{r=r_1} = I_1^*(u(r_1))$, we find that:

$$c_0 = \int_0^{r_1} \frac{(r_1-\zeta)^{\alpha-1}}{\Gamma(\alpha)} \frac{1}{h(\zeta)} I_{\pi,-}^{\alpha} \left(2\lambda p(\zeta) + q(\zeta) \right) u(\zeta) \mathrm{d}\zeta + b_0 + b_1 r + I_1\left(u(r_1)\right)$$

$$c_1 = \int_0^{r_1} \frac{(r_1-\zeta)^{\alpha-2}}{\Gamma(\alpha-1)} \frac{1}{h(\zeta)} I_{\pi,-}^{\alpha} \left(2\lambda p(\zeta) + q(\zeta) \right) u(\zeta) \mathrm{d}\zeta + b_1 + I_1^*\left(u(r_1)\right)$$

Thus:

$$u(r) = \int_{r_1}^r \frac{(r-\zeta)^{\alpha-1}}{\Gamma(\alpha)} \frac{1}{h(\zeta)} I_{\pi,-}^{\alpha} \left(2\lambda p(\zeta) + q(\zeta) \right) u(\zeta) \mathrm{d}\zeta +$$

$$+ \int_0^{r_1} \left(\frac{(r-r_1)(r_1-\zeta)^{\alpha-2}}{\Gamma(\alpha-1)} + \frac{(r_1-\zeta)^{\alpha-1}}{\Gamma(\alpha)} \right) \frac{1}{h(\zeta)} I_{\pi,-}^{\alpha} \left(2\lambda p(\zeta) + q(\zeta) \right) u(\zeta) \mathrm{d}\zeta +$$

$$+ b_0 + b_1 r + I_1\left(u(r_1)\right) + I_1^*\left(u(r_1)\right)(r - r_1)$$

and

$$u'(r) = \frac{1}{\Gamma(\alpha-1)} \int_{r_1}^r (r-\zeta)^{\alpha-2} \frac{1}{h(\zeta)} I_{\pi,-}^{\alpha} \left(2\lambda p(\zeta) + q(\zeta) \right) u(\zeta) \mathrm{d}\zeta +$$

$$+ \frac{1}{\Gamma(\alpha-1)} \int_0^{r_1} \left(r_1 - \zeta \right)^{\alpha-2} \frac{1}{h(\zeta)} I_{\pi,-}^{\alpha} \left(2\lambda p(\zeta) + q(\zeta) \right) u(\zeta) \mathrm{d}\zeta + b_1 + I_1^*\left(u(r_1)\right)$$

repeating the previous procedure for $r \in (r_k, r_{k+1}]$, we obtain:

$$u(r) = \int_{r_k}^r \left[\frac{(r-\zeta)^{\alpha-1}}{\Gamma(\alpha)} \frac{1}{h(\zeta)} I_{\pi,-}^{\alpha} \left(2\lambda p(\zeta) + q(\zeta) \right) u(\zeta) \right] \mathrm{d}\zeta +$$

$$+\sum_{i=1}^{n}\left[\int_{r_{i-1}}^{r_i}\left(\frac{\left((r-r_i)(r_i-\zeta)^{\alpha-2}\right)}{\Gamma(\alpha-1)}+\frac{(r_i-\zeta)^{\alpha-1}}{\Gamma(\alpha)}\right)\frac{1}{h(\zeta)}\left[I_{\pi,-}^{\alpha}\left(2\lambda p(\zeta)+q(\zeta)\right)u(\zeta)\right]\mathrm{d}\zeta\right]+$$

$$+b_0+b_1r+\sum_{i=1}^{n}\left[I_i\left(u(r_i)\right)+I_i^*\left(u(r_i)\right)(r-r_i)\right] \tag{6}$$

Let $u(0)=b_0$, $u'(0)=b_1$ and:

$$u(\pi)=\int_{r_k}^{\pi}\frac{(\pi-\zeta)^{\alpha-1}}{\Gamma(\alpha)}\frac{1}{h(\zeta)}I_{\pi,-}^{\alpha}\left(2\lambda p(\zeta)+q(\zeta)\right)u(\zeta)\mathrm{d}\zeta+$$

$$+\sum_{i=1}^{n}\left[\int_{r_{i-1}}^{r_i}\left(\frac{(\pi-r_i)(r_i-\zeta)^{\alpha-2}}{\Gamma(\alpha-1)}+\frac{(r_i-\zeta)^{\alpha-1}}{\Gamma(\alpha)}\right)\frac{1}{h(\zeta)}\left(I_{\pi,-}^{\alpha}\left[2\lambda p(\zeta)+q(\zeta)\right]\right)u(\zeta)\mathrm{d}\zeta\right]+$$

$$+b_0+b_1\pi+\sum_{i=1}^{n}\left[I_i\left(u(r_i)\right)+I_i^*\left(u(r_i)\right)(\pi-r_i)\right]$$

$$u'(\pi)=\frac{1}{\Gamma(\alpha-1)}\int_{r_k}^{\pi}\left[(\pi-\zeta)^{\alpha-2}\frac{1}{h(\zeta)}\left(I_{\pi,-}^{\alpha}\left(2\lambda p(\zeta)+q(\zeta)\right)u(\zeta)\right)\right]\mathrm{d}\zeta+$$

$$+\sum_{i=1}^{n}\left(\frac{1}{\Gamma(\alpha-1)}\int_{r_{i-1}}^{r_i}(r_i-\zeta)^{\alpha-2}\frac{1}{h(\zeta)}I_{\pi,-}^{\alpha}\left(2\lambda p(\zeta)+q(\zeta)\right)u(\zeta)\mathrm{d}\zeta\right)+$$

$$+b_1+\sum_{i=1}^{n}I_i^*\left(u(r_i)\right)$$

Using boundary conditions, we have:

$$b_1=\frac{1}{\mu}\left\{\alpha_1\alpha_2\left[\int_{r_k}^{\pi}\frac{(\pi-\zeta)^{\alpha-1}}{\Gamma(\alpha)}\frac{1}{h(\zeta)}I_{\pi,-}^{\alpha}\left(2\lambda p(\zeta)+q(\zeta)\right)u(\zeta)\mathrm{d}\zeta+\right.\right.$$

$$+\sum_{i=1}^{n}\left[\int_{r_{i-1}}^{r_i}\left[\frac{\left[(\pi-r_i)(r_i-\zeta)^{\alpha-2}\right]}{\Gamma(\alpha-1)}+\frac{(r_i-\zeta)^{\alpha-1}}{\Gamma(\alpha)}\right]\frac{1}{h(\zeta)}I_{\pi,-}^{\alpha}\left(2\lambda p(\zeta)+q(\zeta)\right)u(\zeta)\mathrm{d}\zeta+\right.$$

$$\left.\left.+\left(I_i\left(u(r_i)\right)+I_i^*\left(u(r_i)\right)(\pi-r_i)\right)\right]\right]+$$

$$+\alpha_1\beta_2\left[\frac{1}{\Gamma(\alpha-1)}\int_{r_k}^{\pi}(\pi-\zeta)^{\alpha-2}\frac{1}{h(\zeta)}I_{\pi,-}^{\alpha}\left[2\lambda p(\zeta)+q(\zeta)\right]u(\zeta)\mathrm{d}\zeta+\right.$$

$$\left.\left.+\sum_{i=1}^{n}\left(\frac{1}{\Gamma(\alpha-1)}\int_{r_{i-1}}^{r_i}(r_i-\zeta)^{\alpha-2}\frac{1}{h(\zeta)}\left[I_{\pi,-}^{\alpha}\left(2\lambda p(\zeta)+q(\zeta)\right)u(\zeta)\right]\mathrm{d}\zeta+I_i^*\left(u(r_i)\right)\right)\right]\right\} \tag{7}$$

$$b_0=-\frac{1}{\mu}\left\{\alpha_2\beta_1\left[\int_{r_k}^{\pi}\frac{(\pi-\zeta)^{\alpha-1}}{\Gamma(\alpha)}\frac{1}{h(\zeta)}I_{\pi,-}^{\alpha}\left[2\lambda p(\zeta)+q(\zeta)\right]u(\zeta)\mathrm{d}\zeta+\right.\right.$$

$$+\sum_{i=1}^{n}\left[\int_{i-1}^{r_i}\left(\frac{\left((\pi-r_i)(r_i-\zeta)^{\alpha-2}\right)}{\Gamma(\alpha-1)}+\frac{(r_i-\zeta)^{\alpha-1}}{\Gamma(\alpha)}\right)\frac{1}{h(\zeta)}I_{\pi,-}^{\alpha}\left[2\lambda p(\zeta)+q(\zeta)\right]u(\zeta)\mathrm{d}\zeta+\right.$$

$$\left.+\left[I_i\left(u(r_i)\right)+I_i^*\left[u(r_i)\right](\pi-r_i)\right]\right]+$$

$$+\beta_1\beta_2\left[\int_{r_k}^{\pi}\frac{(\pi-r_i)^{\alpha-2}}{\Gamma(\alpha-1)}\frac{1}{h(\zeta)}I_{\pi,-}^{\alpha}\left[2\lambda p(\zeta)+q(\zeta)\right]u(\zeta)\mathrm{d}\zeta+\right.$$

$$\left.\left.+\sum_{i=1}^{n}\left(\int_{r_{i-1}}^{r_i}\frac{(r_i-\zeta)^{\alpha-2}}{\Gamma(\alpha-1)}\frac{1}{h(\zeta)}I_{\pi,-}^{\alpha}\left(2\lambda p(\zeta)+q(\zeta)\right)u(\zeta)\mathrm{d}\zeta+I_i^*\left(u(r_i)\right)\right)\right]\right\} \tag{8}$$

Substituting eqs. (7) and (8) into eq. (6) we obtain eq. (4).

Theorem 7. Presume that h is real-value continuous function, $q\in L[0,\pi]$, $p\in W$. There are constants $M,N,R,m>0$ satisfying:

$$|p(r)|\le R,|\lambda|\le N,\min|h(r)|=m,\ |q(r)|\le M,\quad \text{for each } r\in X$$

and there exist constant $c_1,c_2>0$ satisfying:

$$|I_k(u)|<c_1,\ |I_k^*(u)|<c_2,\quad k=1,...,n,$$

where the functions $I_k,I_k^*:\mathrm{R}\to\mathrm{R}$ are continuous.

Then the problem (1)-(3) has at least one solution on X.

Proof. Defining following operator we can change the (1)-(3) problem into a fixed point problem:

$$H\ :\ PC(X,R)\to PC(X,R)$$

given by:

$$Hu(r)=\int_{r_n}^{r}\frac{(r-\zeta)^{\alpha-1}}{\Gamma(\alpha)}\frac{1}{h(\zeta)}\left(I_{\pi,-}^{\alpha}\left(2\lambda p(\zeta)+q(\zeta)\right)u(\zeta)\right)\mathrm{d}\zeta+$$

$$+\sum_{i=1}^{n}\left[\int_{r_{i-1}}^{r_i}\left(\frac{((r-r_i)(r_i-\zeta)^{\alpha-2})}{\Gamma(\alpha-1)}+\frac{(r_i-\zeta)^{\alpha-1}}{\Gamma(\alpha)}\right)\frac{1}{h(\zeta)}I_{\pi,-}^{\alpha}\left[2\lambda p(\zeta)+q(\zeta)\right]u(\zeta)\mathrm{d}\zeta\right]-$$

$$-\frac{1}{\mu}\alpha_2\beta_1\int_{r_n}^{\pi}\frac{(\pi-\zeta)^{\alpha-1}}{\Gamma(\alpha)}\frac{1}{h(\zeta)}I_{\pi,-}^{\alpha}\left(2\lambda p(\zeta)+q(\zeta)\right)u(\zeta)\mathrm{d}\zeta-$$

$$-\frac{1}{\mu}\alpha_2\beta_1\sum_{i=1}^{n}\left[\int_{r_{i-1}}^{r_i}\left(\frac{(r_i-\zeta)^{\alpha-1}}{\Gamma(\alpha)}+\frac{\left((\pi-r_i)(r_i-\zeta)^{\alpha-2}\right)}{\Gamma(\alpha-1)}\right)\frac{1}{h(\zeta)}I_{\pi,-}^{\alpha}\left(2\lambda p(\zeta)+q(\zeta)\right)u(\zeta)\mathrm{d}\zeta+\right.$$

$$\left.+I_i\left(u(r_i)\right)+I_i^*\left(u(r_i)\right)(\pi-r_i)\right]-$$

$$-\frac{1}{\mu}\beta_1\beta_2\left[\int_{r_n}^{\pi}\frac{(\pi-\zeta)^{\alpha-2}}{\Gamma(\alpha-1)}\frac{1}{h(\zeta)}I_{\pi,-}^{\alpha}\left(2\lambda p(\zeta)+q(\zeta)\right)u(\zeta)\mathrm{d}\zeta\right]-$$

$$-\frac{\beta_1\beta_2}{\mu}\sum_{i=1}^{n}\left[\frac{1}{\Gamma(\alpha-1)}\int_{r_{i-1}}^{r_i}(r_i-\zeta)^{\alpha-2}\frac{1}{h(\zeta)}I_{\pi,-}^{\alpha}\left(2\lambda p(\zeta)+q(\zeta)\right)u(\zeta)\mathrm{d}\zeta+I_i^*\left(u(r_i)\right)\right]+$$

$$+\frac{\alpha_1\alpha_2 r}{\mu}\int_{r_n}^{\pi}\frac{(\pi-\zeta)^{\alpha-1}}{\Gamma(\alpha)}\frac{1}{h(\zeta)}\left(I_{\pi,-}^{\alpha}\left(2\lambda p(\zeta)+q(\zeta)\right)u(\zeta)\right)\mathrm{d}\zeta+$$

$$+\frac{\alpha_1\alpha_2 r}{\mu}\sum_{i=1}^{n}\left[\int_{r_{i-1}}^{r_i}\left(\frac{(r_i-\zeta)^{\alpha-1}}{\Gamma(\alpha)}+\frac{\left((\pi-r_i)(r_i-\zeta)^{\alpha-2}\right)}{\Gamma(\alpha-1)}\right)\frac{1}{h(\zeta)}\left(I_{\pi,-}^{\alpha}\left(2\lambda p(\zeta)+q(\zeta)\right)u(\zeta)\right)\mathrm{d}\zeta+\right.$$

$$\left.+I_i\left(u(r_i)\right)+I_i^*\left(u(r_i)\right)(\pi-r_i)\right]+$$

$$+\frac{\alpha_1\beta_2 r}{\mu}\int_{r_n}^{\pi}\frac{(\pi-\zeta)^{\alpha-2}}{\Gamma(\alpha-1)}\frac{1}{h(\zeta)}I_{\pi,-}^{\alpha}\left(2\lambda p(\zeta)+q(\zeta)\right)u(\zeta)\mathrm{d}\zeta+$$

$$+\sum_{i=1}^{n}\left[\frac{r\alpha_1\beta_2}{\mu}\int_{r_{i-1}}^{r_i}\left[\frac{(r_i-\zeta)^{\alpha-2}}{\Gamma(\alpha-1)}\frac{1}{h(\zeta)}I_{\pi,-}^{\alpha}\left[2\lambda p(\zeta)+q(\zeta)\right]u(\zeta)\right]\mathrm{d}\zeta+\frac{\alpha_1\beta_2 r}{\mu}I_i^*\left(u(r_i)\right)+\right.$$

$$\left.+I_i\left(u(r_i)\right)+I_i^*\left(u(r_i)\right)\left(r-r_i\right)\right]$$

Now, using Schaefer fixed point theorem, let as prove that H has a fixed point and it will be proven in four steps.

First Step. Let as show H is continuous.

$u\in PC(X,R)$, $\{u_n\}$ is a sequence satisfying $u_n\to u$. For each $r\in X$:

$$\left|H(u_n)(r)-H(u)(r)\right|\le\int_{r_n}^{r}\frac{(r-\zeta)^{\alpha-1}}{\Gamma(\alpha)}\frac{1}{h(\zeta)}\left|I_{\pi,-}^{\alpha}\left(2\lambda p(\zeta)+q(\zeta)\right)\left(u_n(\zeta)-u(\zeta)\right)\right|\mathrm{d}\zeta+$$

$$+\sum_{i=1}^{n}\left[\int_{r_{i-1}}^{r_i}\left[\frac{(r_i-\zeta)^{\alpha-2}(r-r_i)}{\Gamma(\alpha-1)}+\frac{(r_i-\zeta)^{\alpha-1}}{\Gamma(\alpha)}\right]\frac{1}{h(\zeta)}\left|I_{\pi,-}^{\alpha}\left(2\lambda p(\zeta)+q(\zeta)\right)\left(u_n(\zeta)-u(\zeta)\right)\right|\mathrm{d}\zeta\right]+$$

$$+\frac{\alpha_2\beta_1}{\mu}\int_{r_k}^{\pi}\frac{(\pi-\zeta)^{\alpha-1}}{\Gamma(\alpha)}\frac{1}{h(\zeta)}\left|I_{\pi,-}^{\alpha}\left(2\lambda p(\zeta)+q(\zeta)\right)\left(u_n(\zeta)-u(\zeta)\right)\right|\mathrm{d}\zeta+$$

$$+\frac{1}{\mu}\alpha_2\beta_1\sum_{i=1}^{n}\left[\int_{r_{i-1}}^{r_i}\left[\frac{((r_i-\zeta)^{\alpha-2}(\pi-r_i))}{\Gamma(\alpha-1)}+\frac{(r_i-\zeta)^{\alpha-1}}{\Gamma(\alpha)}\right]\frac{1}{h(\zeta)}\left|\begin{matrix}I_{\pi,-}^{\alpha}\left(2\lambda p(\zeta)+q(\zeta)\right)\cdot\\ \cdot\left(u_n(\zeta)-u(\zeta)\right)\end{matrix}\right|\mathrm{d}\zeta\right]+$$

$$+\frac{1}{\mu}\alpha_2\beta_1\sum_{i=1}^{n}\left[\left|I_i\left[u_n(r_i)\right]-I_i\left[u(r_i)\right]\right|+\left|I_i^*\left[u_n(r_i)\right]-I_i^*\left[u(r_i)\right]\right|(\pi-r_i)\right]+$$

$$+\frac{\beta_1\beta_2}{\mu}\int_{r_k}^{\pi}\frac{(\pi-\zeta)^{\alpha-2}}{\Gamma(\alpha-1)}\frac{1}{h(\zeta)}\left|I_{\pi,-}^{\alpha}\left(2\lambda p(\zeta)+q(\zeta)\right)\left(u_n(\zeta)-u(\zeta)\right)\right|\mathrm{d}\zeta+$$

$$+\frac{1}{\mu}\beta_1\beta_2\sum_{i=1}^{n}\left[\int_{r_{i-1}}^{r_i}\frac{(r_i-\zeta)^{\alpha-2}}{\Gamma(\alpha-1)}\frac{1}{h(\zeta)}\left|I_{\pi,-}^{\alpha}\left[2\lambda p(\zeta)+q(\zeta)\right]\left(u_n(\zeta)-u(\zeta)\right)\right|\mathrm{d}\zeta\right]+$$

$$+\frac{1}{\mu}\beta_1\beta_2\sum_{i=1}^{n}\left|I_i^*\left(u_n\left(r_i\right)\right)-I_i^*\left(u\left(r_i\right)\right)\right|+$$

$$+\frac{\alpha_1\alpha_2 r}{\mu}\int_{r_k}^{\pi}\frac{(\pi-\zeta)^{\alpha-1}}{\Gamma(\alpha)}\frac{1}{h(\zeta)}\left|I_{\pi,-}^{\alpha}\left(2\lambda p(\zeta)+q(\zeta)\right)\left(u_n(\zeta)-u(\zeta)\right)\right|\mathrm{d}\zeta+$$

$$+\frac{\alpha_1\alpha_2 r}{\mu}\sum_{i=1}^{n}\int_{r_{i-1}}^{r_i}\left[\frac{(r_i-\zeta)^{\alpha-2}(\pi-r_i)}{\Gamma(\alpha-1)}+\frac{(r_i-\zeta)^{\alpha-1}}{\Gamma(\alpha)}\right]\frac{1}{h(\zeta)}\left|\begin{matrix}I_{\pi,-}^{\alpha}\left(2\lambda p(\zeta)+q(\zeta)\right)\cdot\\ \cdot\left(u_n(\zeta)-u(\zeta)\right)\end{matrix}\right|\mathrm{d}\zeta+$$

$$+\frac{\alpha_1\alpha_2 r}{\mu}\sum_{i=1}^{n}\left[\left|I_i\left[u_n\left(r_i\right)\right]-I_i\left[u\left(r_i\right)\right]\right|+\left|I_i^*\left[u_n\left(r_i\right)\right]-I_i^*\left[u\left(r_i\right)\right]\right|\left(\pi-r_i\right)\right]+$$

$$+\frac{\alpha_1\beta_2 r}{\mu}\int_{r_k}^{\pi}\frac{(\pi-\zeta)^{\alpha-2}}{\Gamma(\alpha-1)}\frac{1}{h(\zeta)}\left|I_{\pi,-}^{\alpha}\left(2\lambda p(\zeta)+q(\zeta)\right)\left(u_n(\zeta)-u(\zeta)\right)\right|\mathrm{d}\zeta+$$

$$+\frac{\alpha_1\beta_2 r}{\mu}\sum_{i=1}^{n}\left[\int_{r_{i-1}}^{r_i}\frac{(r_i-\zeta)^{\alpha-2}}{\Gamma(\alpha-1)}\frac{1}{h(\zeta)}\left|I_{\pi,-}^{\alpha}\left(2\lambda p(\zeta)+q(\zeta)\right)\left(u_n(\zeta)-u(\zeta)\right)\right|\mathrm{d}\zeta\right]+$$

$$+\frac{\alpha_1\beta_2 r}{\mu}\sum_{i=1}^{n}\left|I_i^*\left(u_n\left(r_i\right)\right)-I_i^*\left(u\left(r_i\right)\right)\right|+$$

$$+\sum_{i=1}^{n}\left[\left|I_i\left[u_n\left(r_i\right)\right]-I_i\left[u\left(r_i\right)\right]\right|+\left|I_i^*\left[u_n\left(r_i\right)\right]-I_i^*\left[u\left(r_i\right)\right]\right|\left(r-r_i\right)\right]$$

Since I_k^*, I_k are continuous $(k=1,...,n)$, we get $\left\|H(u_n)-H(u)\right\|_{\infty}\to 0$, as $n\to\infty$.

Second Step. Let as show H operator is bounded into bounded sets in $PC(X,R)$.

Absolutely, we will demonstrate that there exists a real constant $\gamma>0$ satisfying for each $u\in B=\{u\in PC(X,R)\mid \|u\|_{\infty}<\eta\}$ for any $\eta>0$ and we get $\|H(u)\|_{\infty}\le\gamma$. There exists constant $K>0$ satisfying $\|I_{\pi,-}^{\alpha}(2\lambda p(\zeta)+q(\zeta))u(\zeta)\|\le K$. Due to expression of the theorem, we have for each $r\in X$:

$$\left|Hu(r)\right|\le\frac{K\pi^{\alpha}}{m\mu}\left[\frac{\left[\alpha_2\beta_1(1+n)+\mu(n+1)\right]}{\Gamma(\alpha+1)}+\frac{\left[(n+1)\alpha_1\beta_2+n(\mu+\alpha_2\beta_1)\right]}{\Gamma(\alpha)}\right]+$$

$$+\frac{K\pi^{\alpha-1}}{m\mu\Gamma(\alpha)}\left[\beta_1\beta_2(n+1)\right]+\frac{K\pi^{\alpha+1}}{m\mu}\left[\frac{\alpha_1\alpha_2(n+1)}{\Gamma(\alpha+1)}+\frac{\alpha_1\alpha_2 n}{\Gamma(\alpha)}\right]+$$

$$+\frac{nc_2}{\mu}\left(\alpha_2\beta_1\pi+\beta_1\beta_2+\alpha_1\alpha_2\pi^2+\alpha_1\beta_2\pi+\pi\mu\right)+\frac{nc_1}{\mu}\left(\alpha_2\beta_1+\alpha_1\alpha_2\pi+\mu\right)$$

The right of the last inequality is constant, so we can write the following inequality:

$$\left\|H(u)\right\|_{\infty}\le\gamma$$

Proof is completed.

Third Step. H bounded into equicontinuous sets of $PC(X,R)$.

Let $\tau_1,\tau_2\in X$, $\tau_1<\tau_2$, B be a bounded set of $PC(X,R)$ as in *Step 2*, and let $u\in B$. Then:

$$\left|H(u)(\tau_2)-H(u)(\tau_1)\right| \le \frac{1}{\Gamma(\alpha)}\int_{r_k}^{\tau_1}\left|(\tau_2-\zeta)^{\alpha-1}-(\tau_1-\zeta)^{\alpha-1}\right|\left|\frac{1}{h(\zeta)}I^{\alpha}_{\pi,-}\left(2\lambda p(\zeta)+q(\zeta)\right)u(\zeta)\right|\mathrm{d}\zeta +$$

$$+\frac{1}{\Gamma(\alpha)}\int_{\tau_1}^{\tau_2}\left|(\tau_2-\zeta)^{\alpha-1}\right|\left|\frac{1}{h(\zeta)}I^{\alpha}_{\pi,-}\left(2\lambda p(\zeta)+q(\zeta)\right)u(\zeta)\right|\mathrm{d}\zeta +$$

$$+(\tau_2-\tau_1)\sum_{i=1}^{n}\left[\frac{1}{\Gamma(\alpha-1)}\int_{r_{i-1}}^{r_i}(r_i-\zeta)^{\alpha-2}\left|\frac{1}{h(\zeta)}I^{\alpha}_{\pi,-}\left(2\lambda p(\zeta)+q(\zeta)\right)u(\zeta)\right|\mathrm{d}\zeta\right] +$$

$$+\frac{\alpha_1\alpha_2}{\mu}(\tau_2-\tau_1)\int_{r_k}^{\pi}\frac{(\pi-\zeta)^{\alpha-1}}{\Gamma(\alpha)}\left|\frac{1}{h(\zeta)}I^{\alpha}_{\pi,-}\left(2\lambda p(\zeta)+q(\zeta)\right)u(\zeta)\right|\mathrm{d}\zeta +$$

$$+\frac{\alpha_1\alpha_2}{\mu}(\tau_2-\tau_1)\sum_{i=1}^{n}\left[\int_{r_{i-1}}^{r_i}\left[\frac{(r_i-\zeta)^{\alpha-2}(\pi-r_i)}{\Gamma(\alpha-1)}+\frac{(r_i-\zeta)^{\alpha-1}}{\Gamma(\alpha)}\right]\left|\frac{1}{h(\zeta)}I^{\alpha}_{\pi,-}\left(2\lambda p(\zeta)+q(\zeta)\right)u(\zeta)\right|\mathrm{d}\zeta +\right.$$

$$\left.+\left|I_i\left(u(r_i)\right)\right|+\left|I_i^*\left(u(r_i)\right)(\pi-r_i)\right|\right] +$$

$$+\frac{\alpha_1\beta_2}{\mu}(\tau_2-\tau_1)\int_{r_k}^{\pi}\frac{(\pi-\zeta)^{\alpha-2}}{\Gamma(\alpha-1)}\left|\frac{1}{h(\zeta)}I^{\alpha}_{\pi,-}\left[2\lambda p(\zeta)+q(\zeta)\right]u(\zeta)\right|\mathrm{d}\zeta +$$

$$+\frac{\alpha_1\beta_2}{\mu}(\tau_2-\tau_1)\sum_{i=1}^{n}\left[\int_{r_{i-1}}^{r_i}\frac{(r_i-\zeta)^{\alpha-2}}{\Gamma(\alpha-1)}\left|\frac{1}{h(\zeta)}I^{\alpha}_{\pi,-}\left(2\lambda p(\zeta)+q(\zeta)\right)u(\zeta)\right|\mathrm{d}\zeta +\right.$$

$$\left.+\left|I_i^*\left(u(r_i)\right)\right|\right]+\sum_{i=1}^{n}\left|I_i^*\left(u(r_i)\right)(\tau_2-\tau_1)\right|$$

As $\tau_2 \to \tau_1$, the right-hand of the inequality above goes to zero. As a result of *Step 1* to *Step 3* by Arzela-Ascoli theorem, *H* operator is completely continuous and continuous.

Fourth Step. Now, let as demonstrate that the set:

$$L=\left\{u\in PC[X,R]\;\middle|\;\; u=\vartheta H(u),\; 0<\vartheta<1\right\}$$

is bounded.

Let $u\in L$. Then $u=\vartheta H(u)$, for some $0<\vartheta<1$. Thus for each $r\in X,$ we have:

$$u(r)=\vartheta\int_{r_n}^{r}\frac{(r-\zeta)^{\alpha-1}}{\Gamma(\alpha)}\frac{1}{h(\zeta)}I^{\alpha}_{\pi,-}\left(2\lambda p(\zeta)+q(\zeta)\right)u(\zeta)\mathrm{d}\zeta +$$

$$+\vartheta\sum_{i=1}^{n}\left[\int_{r_{i-1}}^{r_i}\left(\frac{(r_i-\zeta)^{\alpha-1}}{\Gamma(\alpha)}+\frac{(r-r_i)(r_i-\zeta)^{\alpha-2}}{\Gamma(\alpha-1)}\right)\frac{1}{h(\zeta)}I^{\alpha}_{\pi,-}\left[2\lambda p(\zeta)+q(\zeta)\right]u(\zeta)\mathrm{d}\zeta\right] -$$

$$-\frac{1}{\mu}\vartheta\alpha_2\beta_1\int_{r_n}^{\pi}\frac{(\pi-\zeta)^{\alpha-1}}{\Gamma(\alpha)}\frac{1}{h(\zeta)}I^{\alpha}_{\pi,-}\left[2\lambda p(\zeta)+q(\zeta)\right]u(\zeta)\mathrm{d}\zeta -$$

$$-\frac{1}{\mu}\vartheta\alpha_2\beta_1\sum_{i=1}^{n}\left[I_i\left(u(r_i)\right)+I_i^*\left(u(r_i)\right)(\pi-r_i)+\right.$$

$$+\int_{r_{i-1}}^{r_i}\left(\frac{\left[(r_i-\zeta)^{\alpha-2}(\pi-r_i)\right]}{\Gamma(\alpha-1)}+\frac{(r_i-\zeta)^{\alpha-1}}{\Gamma(\alpha)}\right)\frac{1}{h(\zeta)}I^{\alpha}_{\pi,-}\left(2\lambda p(\zeta)+q(\zeta)\right)u(\zeta)\mathrm{d}\zeta\Bigg]-$$

$$-\frac{1}{\mu}\vartheta\beta_1\beta_2\int_{r_k}^{\pi}\frac{(\pi-\zeta)^{\alpha-2}}{\Gamma(\alpha-1)}\frac{1}{h(\zeta)}I^{\alpha}_{\pi,-}\left(2\lambda p(\zeta)+q(\zeta)\right)u(\zeta)\mathrm{d}\zeta-$$

$$-\frac{1}{\mu}\vartheta\beta_1\beta_2\sum_{i=1}^{n}\left[\int_{r_{i-1}}^{r_i}\frac{(r_i-\zeta)^{\alpha-2}}{\Gamma(\alpha-1)}\frac{1}{h(\zeta)}I^{\alpha}_{\pi,-}\left(2\lambda p(\zeta)+q(\zeta)\right)u(\zeta)\mathrm{d}\zeta+I_i^{*}\left(u(r_i)\right)\right]+$$

$$+\frac{\vartheta\alpha_1\alpha_2 r}{\mu}\int_{r_n}^{\pi}\frac{(\pi-\zeta)^{\alpha-1}}{\Gamma(\alpha)}\frac{1}{h(\zeta)}I^{\alpha}_{\pi,-}\left(2\lambda p(\zeta)+q(\zeta)\right)u(\zeta)\mathrm{d}\zeta$$

$$+\frac{\vartheta\alpha_1\alpha_2 r}{\mu}\sum_{i=1}^{n}\int_{r_{i-1}}^{r_i}\left[\frac{\left((r_i-\zeta)^{\alpha-2}(\pi-r_i)\right)}{\Gamma(\alpha-1)}+\frac{(r_i-\zeta)^{\alpha-1}}{\Gamma(\alpha)}\right]\frac{1}{h(\zeta)}I^{\alpha}_{\pi,-}\left(2\lambda p(\zeta)+q(\zeta)\right)u(\zeta)\mathrm{d}\zeta+$$

$$+\frac{\vartheta\alpha_1\alpha_2 r}{\mu}\sum_{i=1}^{n}\left[I_i\left(u(r_i)\right)+I_i^{*}\left(u(r_i)\right)(\pi-r_i)\right]+$$

$$+\frac{\vartheta\alpha_1\beta_2 r}{\mu}\int_{r_k}^{\pi}\frac{(\pi-\zeta)^{\alpha-2}}{\Gamma(\alpha-1)}\frac{1}{h(\zeta)}I^{\alpha}_{\pi,-}\left(2\lambda p(\zeta)+q(\zeta)\right)u(\zeta)\mathrm{d}\zeta+$$

$$+\frac{\vartheta\alpha_1\beta_2 r}{\mu}\sum_{i=1}^{n}\left[\int_{r_{i-1}}^{r_i}\frac{(r_i-\zeta)^{\alpha-2}}{\Gamma(\alpha-1)}\frac{1}{h(\zeta)}I^{\alpha}_{\pi,-}\left(2\lambda p(\zeta)+q(\zeta)\right)u(\zeta)\mathrm{d}\zeta+I_i^{*}\left(u(r_i)\right)\right]+$$

$$+\vartheta\sum_{i=1}^{n}\left[I_i\left(u(r_i)\right)+I_i^{*}\left(u(r_i)\right)(r-r_i)\right]$$

This implies by expression of theorem that for each $r\in X$ we have:

$$|u(r)|\le\frac{K\pi^{\alpha}}{m\mu}\Bigg[\frac{\left[\alpha_2\beta_1(1+n)+\mu(n+1)\right]}{\Gamma(\alpha+1)}+$$

$$+\frac{\left[n(\mu+\alpha_2\beta_1)+(n+1)\alpha_1\beta_2\right]}{\Gamma(\alpha)}\Bigg]+\frac{K\pi^{\alpha-1}}{m\mu\Gamma(\alpha)}\left[\beta_1\beta_2(n+1)\right]+\frac{K\pi^{\alpha+1}}{m\mu}\left[\frac{\alpha_1\alpha_2 n}{\Gamma(\alpha)}+\frac{(n+1)\alpha_1\alpha_2}{\Gamma(\alpha+1)}\right]+$$

$$+\frac{nc_2}{\mu}\left(\alpha_2\beta_1\pi+\beta_1\beta_2+\alpha_1\alpha_2\pi^2+\alpha_1\beta_2\pi+\pi\mu\right)+\frac{nc_1}{\mu}\left(\alpha_2\beta_1+\alpha_1\alpha_2\pi+\mu\right)=\gamma$$

Thus for every $r\in X$ we have $\|u\|_{\infty}\le\gamma$.

So, the set L is bounded. H has a fixed point in the solution of energy-dependent fractional Sturm-Lioville impulsive problem (1)-(3) by Schaefer's fixed point theorem.

Application

Let as take into consideration the following energy-dependent fractional Sturm-Liouville problem:

$$\begin{gathered} -\mathrm{D}_{0+}^{\alpha}\,{}^{C}\mathrm{D}_{0+}^{\alpha}u(r)+[2\lambda p(r)+q(r)]u(r)=0 \\ I_{0+}^{1-\alpha}\,{}^{C}\mathrm{D}_{0+}^{\alpha}u(r)\Big|_{r=0}=c_1,\quad u(0)=c_2 \end{gathered} \tag{9}$$

Using Laplace transform for eq. (9), we obtain the following equalities:

$$-\mathcal{L}\left\{\mathrm{D}_{0+}^{\alpha}\left[{}^{C}\mathrm{D}_{0+}^{\alpha}u(r)\right]\right\}+\left(2\lambda p(r)+q(r)\right)u(r)=0$$

$$-s^{\alpha}\mathcal{L}\left[{}^{C}\mathrm{D}_{0+}^{\alpha}u(r)\right]-I_{0+}^{1-\alpha}\,{}^{C}\mathrm{D}_{0+}^{\alpha}u(r)\Big|_{r=0}+\left[2\lambda p(r)+q(r)\right]\mathcal{L}\left[u(r)\right]=0$$

$$-s^{\alpha}\left\{s^{\alpha}\mathcal{L}\left[u(r)\right]-s^{\alpha-1}u(0)\right\}-c_1+\left[2\lambda p(r)+q(r)\right]\mathcal{L}\left[u(r)\right]=0$$

$$-s^{2\alpha}\mathcal{L}\left[u(r)\right]+s^{2\alpha-1}c_2-c_1+\left[2\lambda p(r)+q(r)\right]\mathcal{L}\left[u(r)\right]=0$$

$$\mathcal{L}\left\{u(r)\right\}=\frac{c_1}{\left(2\lambda p+q\right)-s^{2\alpha}}-c_2\frac{s^{2\alpha-1}}{\left(2\lambda p+q\right)-s^{2\alpha}}=$$

$$=\frac{-c_1}{s^{2\alpha}-\left(2\lambda p+q\right)}-\frac{c_2}{s}\left[\frac{s^{2\alpha}}{s^{2\alpha}-\left(2\lambda p+q\right)}\right]$$

By taking the inverse Laplace transform and using initial conditions, we obtain the analytical solution:

$$u(r)=\frac{-c_1}{2\sqrt{2\lambda p+q}}\left[r^{\alpha-1}E_{\alpha,\alpha}\left(\sqrt{2\lambda p+q}r^{\alpha}\right)-r^{\alpha-1}E_{\alpha,\alpha}\left(-\sqrt{2\lambda p+q}r^{\alpha}\right)\right]+$$

$$+c_2\left\{1+\frac{1}{2}\left[E_{\alpha}\left(\sqrt{2\lambda p+q}r^{\alpha}\right)-E_{\alpha}\left(-\sqrt{2\lambda p+q}r^{\alpha}\right)\right]\right\}$$

Conclusion

In this study, we use different fractional composition operators including CD and RLD. The classical Sturm-Liouville problem under the impulsive condition is analyzed for the situation of our energy-dependent fractional Sturm-Liouville problem. We show an explicit representation of the solution of energy-dependent fractional Stum-Liouville impulsive problem order $\alpha\in(1,2]$ and by using Schaefer fixed point theorem and also we proved it. At last, we give symbolic an application for the aforementioned problem.

By means of figures, we also show the behaviors of solutions visually. According to these solutions, in the figs. 1 and 2, we see curvilinear behavior of solution for $\alpha=0.6,\alpha=0.7,$ $\alpha=0.8,\alpha=0.9,$ where $(\lambda=2,p=1,q=0)$ and $(\lambda=2,p=1,q=1),$ recpectively. For fixed $\alpha=0,6,$ according to other data, we show behaviors of solutions by figs. 3 and 4.

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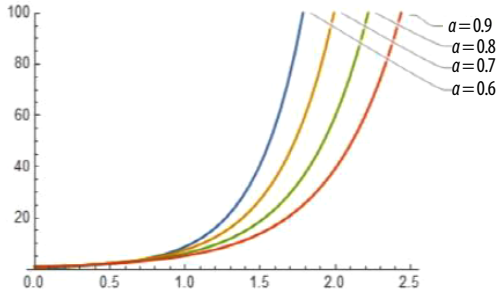

Figure 1. Comparasion of eigenfunction of problem (9), $c_1 = c_2 = 1$, $\lambda = 2$, $p = 1$, $q = 0$

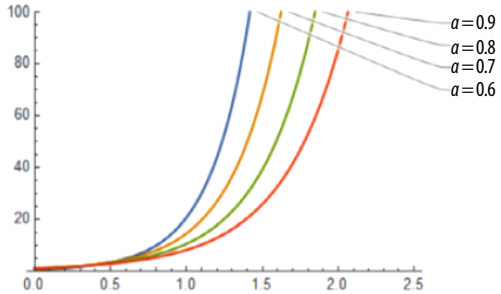

Figure 2. Comparasion of eigenfunction of problem (9), $c_1 = c_2 = 1$, $\lambda = 2$, $p = 1$, $q = 1$

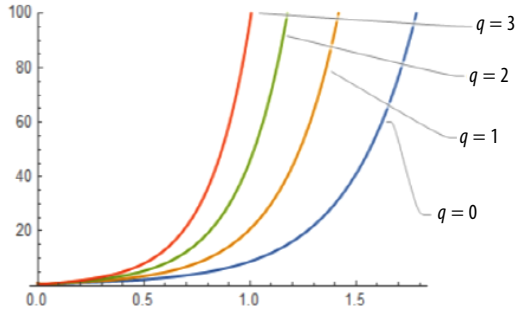

Figure 3. Comparasion of eigenfunction of problem (9), $c_1 = c_2 = 1$, $\lambda = 2$, $p = 1$, $\alpha = 0{,}6$

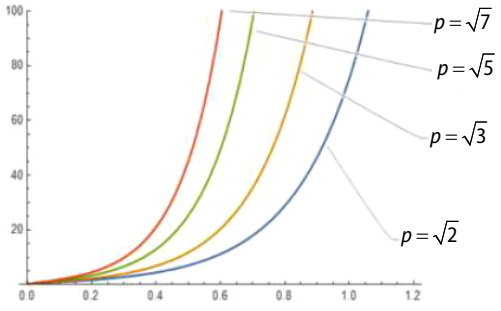

Figure 4 . Comparasion of eigenfunction of problem (9), $c_1 = c_2 = 1$, $\lambda = 2$, $q = 1$, $\alpha = 0{,}6$

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