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NEW DEFINITONS ABOUT $A^{\mathcal{I}}$- STATISTICAL CONVERGENCE WITH RESPECT TO A SEQUENCE OF MODULUS FUNCTIONS AND LACUNARY SEQUENCES

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Abstract. In this paper, we generalize the concept of $\mathcal{I}-$statistical convergence, which is a recently introduced summability method, with an infinite matrix of complex numbers and a modulus function. Lacunary sequences will also be included in our definitions. The name of our new method will be $A^{\mathcal{I}}$-lacunary statistical convergence with respect to a sequence of modulus functions. We also define strongly $A^{\mathcal{I}}$-lacunary convergence and we give some inclusion relations between these methods.

1. Introduction

As is known, convergence is one of the most important notions in Mathematics. Statistical convergence extends the notion. We can easily show that any convergent sequence is statistically convergent, but not conversely. Let E be a subset of $\mathbb{N}$, the set of all natural numbers. $d(E) := \lim_n \frac{1}{n} \sum\limits_{j=1}^{n} \chi_E(j)$ is said to be natural density of E whenever the limit exists, where χ_E is the characteristic function of E.

Statistical convergence was given by Zygmund in the first edition of his monograph published in Warsaw in 1935 [23]. It was formally introduced by Fast [6] and Steinhaus [21] and later was reintroduced by Schoenberg [20]. It has become an active area of research in recent years. This concept has applications in different fields of mathematics such as number theory by Erdös and Tenenbaum [5], measure theory by Miller [14], trigonometric series by Zygmund [23], summability theory by Freedman and Sember [7], etc.

Following this very important definition, the concept of lacunary statistical convergence was defined by Fridy and Orhan [9]. Also, Fridy and Orhan gave the relationships between the lacunary statistical convergence and the Cesàro summability. Freedman and Sember. established the connection between the strongly Cesàro summable sequences space $|\sigma_1|$ and the strongly lacunary summable sequences space N_θ in their work [7] published in 1978.

$\mathcal{I}-$convergence has emerged as a kind of generalization form of many types of convergence. This means that, if we choose different ideals we will have different convergences. Koystro et. al. introduced this concept in a metric space [11]. We will explain this situation with two examples later. Before defining $\mathcal{I}-$convergence, the definitions of ideal and filter will be needed.

An ideal is a family of sets $\mathcal{I} \subseteq 2^{\mathbb{N}}$ such that (i) $\emptyset \in \mathcal{I}$, (ii) $A, B \in \mathcal{I}$ implies $A \cup B \in \mathcal{I}$, $iii)$ For each $A \in \mathcal{I}$ and each $B \subseteq A$ implies $B \in \mathcal{I}$. An ideal is called

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non-trivial if $\mathbb{N} \notin \mathcal{I}$ and a non-trivial ideal is called admissible if $\{n\} \in \mathcal{I}$ for each $n \in \mathbb{N}$.

A filter is a family of sets $\mathcal{F} \subseteq 2^{\mathbb{N}}$ such that (i) $\emptyset \notin \mathcal{F}$, (ii) $A, B \in \mathcal{F}$ implies $A \cap B \in \mathcal{F}$, $iii)$ For each $A \in \mathcal{F}$ and each $A \subseteq B$ implies $B \in \mathcal{F}$.

If $\mathcal{I}$ is an ideal in $\mathbb{N}$ then the collection,

$$F(\mathcal{I}) = \{A \subset \mathbb{N} : \mathbb{N} \backslash A \in \mathcal{I}\}$$

forms is a filter in $\mathbb{N}$ which is called the filter associated with $\mathcal{I}$.

Now let's remember the definition of modulus function. The notion of a modulus function was introduced by Nakano [15]. We recall that a modulus f is a function from $[0, \infty)$ to $[0, \infty)$ such that (i) $f(x) = 0$ if and only if $x = 0$, (ii) $f(x + y) = f(x) + f(y)$ for $x, y \geq 0$, (iii) f is increasing and (iv) f is continuous from the right at 0. It follows that f must be continuous on $[0, \infty)$. Connor [2], Bilgin [1], Maddox [13], Kolk [10], Pehlivan and Fisher [16] and Ruckle [17] used a modulus function to construct sequence spaces. Now let S be the space of sequence of modulus function $F = (f_k)$ such that $\lim_{x \to 0^+} \sup_k f_k(x) = 0$. Throughout this paper the set of all modulus functions determined by F and it will be denoted by $F = (f_k) \in S$ for every $k \in \mathbb{N}$.

In this paper, we intend to unify these approaches and use ideals to introduce the notion of $A^{\mathcal{I}}$-lacunary statistically convergence with respect to a sequence of modulus functions.

2. Definitions and Notations

First we recall some of the basic concepts which will be used in this paper.

Let $A = (a_{ki})$ be an infinite matrix of complex numbers. We write $Ax = (A_k(x))$, if $A_k(x) = \sum_{i=1}^{\infty} a_{ki} x_k$ converges for each k.

Definition 2.1. [6]*A number sequence $x = (x_k)$ is said to be statistically convergent to the number L if for every $\varepsilon > 0$,*

$$\lim_{n \to \infty} \frac{1}{n} |\{k \leq n : |x_k - L| \geq \varepsilon\}| = 0.$$

In this case we write $st - \lim x_k = L$. As we said before, statistical convergence is a natural generalization of ordinary convergence i.e. if $\lim x_k = L$, then $st - \lim x_k = L$.

By a lacunary sequence we mean an increasing integer sequence $\theta = \{k_r\}$ such that $k_0 = 0$ and $h_r = k_r - k_{r-1} \to \infty$ as $r \to \infty$. Throughout this paper the intervals determined by θ will be denoted by $I_r = (k_{r-1}, k_r]$.

Definition 2.2. [9] *A sequence $x = (x_k)$ is said to be lacunary statistically convergent to the number L if for every $\varepsilon > 0$,*

$$\lim_{r \to \infty} \frac{1}{h_r} |\{k \in I_r : |x_k - L| \geq \varepsilon\}| = 0.$$

In this case we write $S_\theta - \lim x_k = L$ or $x_k \to L(S_\theta)$.

Definition 2.3. [9] *The sequence space N_θ, which is defined by*

$$N_\theta = \left\{ (x_k) : \lim_{r \to \infty} \frac{1}{h_r} \sum_{k \in I_r} |x_k - L| = 0 \right\}.$$

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Definition 2.4. [11] *Let $\mathcal{I}\subset 2^{\mathbb{N}}$ be a proper admissible ideal in $\mathbb{N}$. The sequence (x_n) of elements of $\mathbb{R}$ is said to be $\mathcal{I}-$convergent to $L\in\mathbb{R}$ if for each $\varepsilon>0$ the set*

$$A(\varepsilon)=\{n\in\mathbb{N}:|x_n-L|\geq\varepsilon\}\in\mathcal{I}.$$

Example 2.1. *Define the set of all finite subsets of $\mathbb{N}$ by $\mathcal{I}_f$. Then, $\mathcal{I}_f$ is a non-trivial admissible ideal and $\mathcal{I}_f$ $-$convergence coincides with the usual convergence.*

Example 2.2. *Define the set $\mathcal{I}_d$ by $\mathcal{I}_d=\{A\subset\mathbb{N}:d(A)=0\}$. Then, $\mathcal{I}_d$ is an admissible ideal and $\mathcal{I}_d-$convergence gives the statistical convergence.*

Following the line of Savas et. al.[3], some authors obtained more general results about statistical convergence by using A matrix and they called this new method by $A^{\mathcal{I}}-$statistical convergence. (Bilgin [1]; Yamanci, Gürdal and Saltan, [22]).

Definition 2.5. [22] *Let $A=(a_{ki})$ be an infinite matrix of complex numbers and (f_k) be a sequence of modulus functions in S. A sequence $x=(x_k)$ is said to be $A^{\mathcal{I}}$-statistically convergent to $L\in X$ with respect to a sequence of modulus functions, for each $\varepsilon>0$, for every $x\in X$ and $\delta>0$,*

$$\left\{n\in\mathbb{N}:\frac{1}{n}|\{k\leq n:f_k\left(|A_k\left(x\right)-L|\right)\geq\varepsilon\}|\geq\delta\right\}\in\mathcal{I}.$$

In this case we write $x_k\to L\left(S^A\left(\mathcal{I},F\right)\right)$.

3. Inclusions Between $S^A(\mathcal{I},F)$, $S_\theta^A(\mathcal{I},F)$ and $N_\theta^A(\mathcal{I},F)$ Spaces

We now consider our main results. We begin with the following definitions.

Definition 3.1. *Let $A=(a_{ki})$ be an infinite matrix of complex numbers, $\theta=\{k_r\}$ be a lacunary sequence and $F=(f_k)$ be a sequence of modulus functions in S. A sequence $x=(x_k)$ is said to be $A^{\mathcal{I}}$-lacunary statistically convergent to $L\in X$ with respect to a sequence of modulus functions, for each $\varepsilon>0$, for each $x\in X$ and $\delta>0$,*

$$\left\{r\in\mathbb{N}:\frac{1}{h_r}|\left\{k\in I_r:f_k\left(|A_k\left(x\right)-L|\right)\geq\varepsilon\right\}|\geq\delta\right\}\in\mathcal{I}.$$

Definition 3.2. *Let $A=(a_{ki})$ be an infinite matrix of complex numbers, $\theta=\{k_r\}$ be a lacunary sequence and $F=(f_k)$ be a sequence of modulus functions in S. A sequence $x=(x_k)$ is said to be strongly $A^{\mathcal{I}}$-lacunary convergent to $L\in X$ with respect to a sequence of modulus functions, if, for each $\varepsilon>0$, for each $x\in X$,*

$$\left\{r\in\mathbb{N}:\frac{1}{h_r}\sum_{k\in I_r}f_k\left(|A_k\left(x\right)-L|\right)\geq\varepsilon\right\}\in\mathcal{I}.$$

We shall denote by $S_\theta^A(\mathcal{I},F)$, $N_\theta^A(\mathcal{I},F)$ the collections of all $A^{\mathcal{I}}$-lacunary statistically convergent and strongly $A^{\mathcal{I}}-$lacunary convergent sequences, respectively.

Theorem 3.1. *Let $A=(a_{ki})$ be an infinite matrix of complex numbers and (f_k) be a sequence of modulus functions in S. $\left(S_\theta^A(\mathcal{I},F)\right)\cap m(X)$ is a closed subset of $m(X)$ if X is a Banach space where $m(X)$ is the space of all bounded sequences of X.*

Proof. Suppose that $(x^n)\subset\left(S_\theta^A(\mathcal{I},F)\right)\cap m(X)$ is a convergent sequence and it converges to $x\in m(X)$. We need to show that $x\in\left(S_\theta^A(\mathcal{I},F)\right)\cap m(X)$. Assume that $x^n\to L_n\left(S_\theta^A(\mathcal{I},F)\right)$, $\forall n\in\mathbb{N}$. Take a sequence $\{\varepsilon_r\}_{n\in\mathbb{N}}$ of strictly decreasing

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positive numbers converging to zero. We can find an $r \in \mathbb{N}$ such that $\left\|x - x^j\right\|_\infty < \frac{\varepsilon_r}{4}$ for all $j \geq r$. Choose $0 < \delta < \frac{1}{5}$.

Now

$$A = \left\{ r \in \mathbb{N} : \frac{1}{h_r} \left| \left\{ k \in I_r : f_k \left(|A_k(x^n) - L_n| \right) \geq \frac{\varepsilon_r}{4} \right\} \right| < \delta \right\} \in \mathcal{F}(\mathcal{I})$$

and

$$B = \left\{ r \in \mathbb{N} : \frac{1}{h_r} \left| \left\{ k \in I_r : f_k \left(\left|A_k\left(x^{n+1}\right) - L_{n+1}\right| \right) \geq \frac{\varepsilon_r}{4} \right\} \right| < \delta \right\} \in \mathcal{F}(\mathcal{I}).$$

Since $A \cap B \in \mathcal{F}(\mathcal{I})$ and $\emptyset \notin \mathcal{F}(\mathcal{I})$, we can choose $r \in A \cap B$. Then

$$\frac{1}{h_r} \left| \left\{ k \in I_r : f_k \left(|A_k(x^n) - L_n| \right) \geq \frac{\varepsilon_r}{4} \vee f_k \left(\left|A_k\left(x^{n+1}\right) - L_{n+1}\right| \right) \geq \frac{\varepsilon_r}{4} \right\} \right| \leq 2\delta < 1.$$

Since $h_r \to \infty$ and $A \cap B \in \mathcal{F}(\mathcal{I})$ is infinite, we can actually choose the above r so that $h_r > 5$. Hence there must exist a $k \in I_r$ for which we have simultaneously, $|x_k^n - L_n| < \frac{\varepsilon_r}{4}$ and $\left|x_k^{n+1} - L_{n+1}\right| < \frac{\varepsilon_r}{4}$.

Then it follows that

$$\begin{aligned} |L_n - L_{n+1}| &\leq |L_n - x_k^n| + \left|x_k^n - x_k^{n+1}\right| + \left|x_k^{n+1} - L_{n+1}\right| \\ &\leq |x_k^n - L_n| + \left|x_k^{n+1} - L_{n+1}\right| + \|x - x^n\|_\infty + \left\|x - x^{n+1}\right\|_\infty \\ &\leq \tfrac{\varepsilon_r}{4} + \tfrac{\varepsilon_r}{4} + \tfrac{\varepsilon_r}{4} + \tfrac{\varepsilon_r}{4} = \varepsilon_r. \end{aligned}$$

This implies that $\{L_n\}_{n \in \mathbb{N}}$ is a Cauchy sequence in X. Since X is a Banach space we can write $L_n \to L \in X$ as $n \to \infty$. We shall prove that $x_k \to L\left(S_\theta^A(\mathcal{I}, F)\right)$. Choose $\varepsilon > 0$ and $r \in \mathbb{N}$ such that $\varepsilon_r < \frac{\varepsilon}{4}$, $\|x - x_n\|_\infty < \frac{\varepsilon}{4}$. Now since

$$\begin{aligned} &\tfrac{1}{h_r} \left| \{ k \in I_r : f_k \left(|A_k(x) - L| \right) \geq \varepsilon \} \right| \\ &\leq \tfrac{1}{h_r} \left| \{ k \in I_r : f_k \left(|A_k(x - x_n)| \right) + f_k \left(|A_k(x^n) - L_n| \right) + f_k \left(|L_n - L| \right) \geq \varepsilon \} \right| \\ &\leq \tfrac{1}{h_r} \left| \left\{ k \in I_r : f_k \left(|A_k(x^n) - L_n| \right) \geq \tfrac{\varepsilon}{2} \right\} \right|. \end{aligned}$$

It follows that

$$\begin{aligned} &\left\{ r \in \mathbb{N} : \tfrac{1}{h_r} \left| \{ k \in I_r : f_k \left(|A_k(x) - L| \right) \geq \varepsilon \} \right| \geq \delta \right\} \\ &\subset \left\{ r \in \mathbb{N} : \tfrac{1}{h_r} \left| \left\{ k \in I_r : f_k \left(|A_k(x) - L| \right) \geq \tfrac{\varepsilon}{2} \right\} \right| \geq \delta \right\} \end{aligned}$$

for given $\delta > 0$. This shows that $x_k \to L\left(S_\theta^A(\mathcal{I}, F)\right)$ and this completes the proof of the theorem. □

Theorem 3.2. *Let $A = (a_{ki})$ be an infinite matrix of complex numbers, $\theta = \{k_r\}$ be a lacunary sequence and (f_k) be a sequence of modulus functions in S.*

(i) *If $x_k \to L\left(N_\theta^A(\mathcal{I}, F)\right)$ then $x_k \to L\left(S_\theta^A(\mathcal{I}, F)\right)$ and $N_\theta^A(\mathcal{I}, F) \subset S_\theta^A(\mathcal{I}, F)$ is proper for every ideal $\mathcal{I}$.*

(ii) *If $x \in m(X)$, the space of all bounded sequences of X and $x_k \to L\left(S_\theta^A(\mathcal{I}, F)\right)$ then $x_k \to L\left(N_\theta^A(\mathcal{I}, F)\right)$.*

(iii) *$S_\theta^A(\mathcal{I}, F) \cap m(X) = N_\theta^A(\mathcal{I}, F) \cap m(X)$.*

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Proof. (*i*) Let $\varepsilon > 0$ and $x_k \to L\left(N_\theta^A(\mathcal{I}, F)\right)$. Then we can write

$$\begin{aligned}\sum_{k\in I_r} f_k\left(|A_k(x)-L|\right) &\geq \sum_{\substack{k\in I_r \\ f_k(|A_k(x)-L|)\geq\varepsilon}} f_k\left(|A_k(x)-L|\right) \\ &\geq \varepsilon.\left|\{k\in I_r : f_k\left(|A_k(x)-L|\right)\geq\varepsilon\}\right|.\end{aligned}$$

So for given $\delta > 0$,

$$\frac{1}{h_r}\left|\{k\in I_r : f_k\left(|A_k(x)-L|\right)\geq\varepsilon\}\right| \geq \delta \Longrightarrow \frac{1}{h_r}\sum_{k\in I_r} f_k\left(|A_k(x)-L|\right)\geq\varepsilon.\delta,$$

i.e.

$$\left\{r\in\mathbb{N} : \frac{1}{h_r}\left|\{k\in I_r : f_k\left(|A_k(x)-L|\right)\geq\varepsilon\}\right|\geq\delta\right\}\subseteq\left\{r\in\mathbb{N} : \frac{1}{h_r}\sum_{k\in I_r} f_k\left(|A_k(x)-L|\right)\geq\varepsilon.\delta\right\}.$$

Since $x_k \to L\left(N_\theta^A(\mathcal{I}, F)\right)$, the set on the right-hand side belongs to $\mathcal{I}$ and so it follows that $x_k \to L\left(S_\theta^A(\mathcal{I}, F)\right)$.

To show that $\left(S_\theta^A(\mathcal{I}, F)\right) \subsetneq \left(N_\theta^A(\mathcal{I}, F)\right)$, take a fixed $K \in \mathcal{I}$. Define $x = (x_k)$ by

$$(x_k) = \begin{cases} ku, & \text{for } k_{r-1} < k \leq k_{r-1} + \left[\sqrt{h_r}\right],\ r = 1,2,3...,\ r\notin K, \\ ku, & \text{for } k_{r-1} < k \leq k_{r-1} + \left[\sqrt{h_r}\right],\ r = 1,2,3...,\ r\in K, \\ \theta, & \text{otherwise,}\end{cases}$$

where $u \in X$ is a fixed element with $\|u\| = 1$ and θ is the null element of X. Then $x \notin m(X)$ and for every $0 < \varepsilon < 1$ since

$$\frac{1}{h_r}\left|\{k\in I_r : f_k\left(|A_k(x)-0|\right)\geq\varepsilon\}\right| = \frac{\left[\sqrt{h_r}\right]}{\sqrt{h_r}} \to 0$$

As $r \to \infty$ and $r \notin K$, for every $\delta > 0$,

$$\left\{r\in\mathbb{N} : \frac{1}{h_r}\left|\{k\in I_r : f_k\left(|A_k(x)-0|\right)\geq\varepsilon\}\right|\geq\delta\right\}\subset M\cup\{1,2,...,m\}$$

for some $m \in \mathbb{N}$. Since $\mathcal{I}$ is admissible, it follows that $x_k \to \theta\left(S_\theta^A(\mathcal{I}, F)\right)$. Obviously

$$\frac{1}{h_r}\sum_{k\in I_r} f_k\left(|A_k(x)-\theta|\right)\to\infty$$

i.e. $x_k \nrightarrow \theta\left(N_\theta^A(\mathcal{I}, F)\right)$. Note that if $K \in \mathcal{I}$ is finite then $x_k \nrightarrow \theta\left(S_\theta^A\right)$. This example shows that $A^{\mathcal{I}}$-lacunary statistical convergence is more general than lacunary statistical convergence.

(*ii*) Suppose that $x \in l_\infty$ and $x_k \to L\left(S_\theta^A(\mathcal{I}, F)\right)$. Then we can assume that

$$f_k\left(|A_k(x)-L|\right)\leq M$$

for each $x \in X$ and all k.

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Given $\varepsilon > 0$, we get

$$\begin{aligned}\frac{1}{h_r}\sum_{k\in I_r} f_k\left(|A_k(x)-L|\right) &= \frac{1}{h_r}\sum_{\substack{k\in I_r\\ f_k(|A_k x-L|)\geq\varepsilon}} f_k\left(|A_k(x)-L|\right)\\ &\quad+\frac{1}{h_r}\sum_{\substack{k\in I_r\\ f_k(|A_k x-L|)<\varepsilon}} f_k\left(|A_k(x)-L|\right)\\ &\leq \frac{M}{h_r}\left|\left\{k\in I_r : f_k\left(|A_k(x)-L|\right)\geq\varepsilon\right\}\right|+\varepsilon.\end{aligned}$$

Note that

$$A(\varepsilon)=\left\{r\in\mathbb{N} : \frac{1}{h_r}\left|\left\{k\in I_r : f_k\left(|A_k(x)-L|\right)\geq\varepsilon\right\}\right|\geq\frac{\varepsilon}{M}\right\}\in\mathcal{I}.$$

If $n\in(A(\varepsilon))^c$ then

$$\frac{1}{h_r}\sum_{k\in I_r} f_k|A_k(x)-L|<2\varepsilon.$$

Hence

$$\left\{r\in\mathbb{N} : \frac{1}{h_r}\sum_{k\in I_r} f_k|A_k(x)-L|\geq 2\varepsilon\right\}\subset A(\varepsilon)$$

and so belongs to $\mathcal{I}$. This shows that $x_k\to L\left(N_\theta^A(\mathcal{I},F)\right)$.This completes the proof.

(iii) This is an immediate consequence of (i) ve (ii). □

Theorem 3.3. *Let $A=(a_{ki})$ be an infinite matrix of complex numbers and (f_k) be a sequence of modulus functions in S. If $\theta=\{k_r\}$ be a lacunary sequence with $\liminf_r q_r>1$, then*

$$x_k\to L\left(S^A(\mathcal{I},F)\right)\Rightarrow x_k\to L\left(S_\theta^A(\mathcal{I},F)\right).$$

Proof. Suppose first that $\liminf_r q_r>1$, then there exists a $\delta>0$ such that $q_r\geq 1+\delta$ for sufficiently large r, which implies that

$$\frac{h_r}{k_r}\geq\frac{\delta}{1+\delta}.$$

If $x_k\to L\left(S_\theta^A(\mathcal{I},F)\right)$, then for every $\varepsilon>0$, for each $x\in X$ and for sufficiently large r, we have

$$\begin{aligned}\frac{1}{k_r}\left|\left\{k\leq k_r : f_k\left(|A_k(x)-L|\right)\geq\varepsilon\right\}\right| &\geq \frac{1}{k_r}\left|\left\{k\in I_r : f_k\left(|A_k(x)-L|\right)\geq\varepsilon\right\}\right|\\ &\geq \frac{\delta}{1+\delta}\frac{1}{h_r}\left|\left\{k\in I_r : f_k\left(|A_k(x)-L|\right)\geq\varepsilon\right\}\right|;\end{aligned}$$

Then for any $\delta>0$, we get

$$\begin{aligned}&\left\{r\in\mathbb{N} : \tfrac{1}{h_r}\left|\left\{k\in I_r : f_k\left(|A_k(x)-L|\right)\geq\varepsilon\right\}\right|\geq\delta\right\}\\ &\subseteq\left\{r\in\mathbb{N} : \tfrac{1}{k_r}\left|\left\{k\leq k_r : f_k\left(|A_k(x)-L|\right)\geq\varepsilon\right\}\right|\geq\tfrac{\delta\alpha}{(\alpha+1)}\right\}\in\mathcal{I}.\end{aligned}$$

This completes the proof. □

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For the next result we assume that the lacunary sequence θ satisfies the condition that for any set $C \in \mathcal{F}(\mathcal{I})$, $\left\{\bigcup n : k_{r-1} < n \leq k_r, r \in C\right\} \in \mathcal{F}(\mathcal{I})$.

Theorem 3.4. *Let $A = (a_{ki})$ be an infinite matrix of complex numbers and (f_k) be a sequence of modulus functions in S. If $\theta = \{k_r\}$ be a lacunary sequence with* $\limsup_r q_r < \infty$, *then*

$$x_k \to L\left(S_\theta^A(\mathcal{I}, F)\right) \text{ implies } x_k \to L\left(S^A(\mathcal{I}, F)\right).$$

Proof. If $\limsup_r q_r < \infty$ then without any loss of generality we can assume that there exists a $0 < M < \infty$ such that $q_r < M$ for all $r \geq 1$. Suppose that $x_k \to L\left(S_\theta^A(\mathcal{I}, F)\right)$, and for ε, δ, $\delta_1 > 0$ define the sets

$$C = \left\{r \in \mathbb{N} : \frac{1}{h_r}\left|\{k \in I_r : f_k(|A_k(x) - L|) \geq \varepsilon\}\right| < \delta\right\}$$

and

$$T = \left\{n \in \mathbb{N} : \frac{1}{n}\left|\{k \leq n : f_k(|A_k(x) - L|) \geq \varepsilon\}\right| < \delta_1\right\}.$$

It is obvious from our assumption that $C \in \mathcal{F}(\mathcal{I})$, the filter associated with the ideal $\mathcal{I}$. Further observe that

$$K_j = \frac{1}{h_j}\left|\{k \in I_j : f_k(|A_k(x) - L|) \geq \varepsilon\}\right| < \delta$$

for all $j \in C$. Let $n \in \mathbb{N}$ be such that $k_{r-1} < n \leq k_r$ for some $r \in C$. Now

$$\begin{aligned}
\frac{1}{n}\left|\{k \leq n : f_k|A_k(x) - L| \geq \varepsilon\}\right| &\leq \frac{1}{k_{r-1}}\left|\{k \leq k_r : f_k(|A_k(x) - L|) \geq \varepsilon\}\right| \\
&= \frac{1}{k_{r-1}}\left\{\left|\{k \in I_1 : f_k(|A_k(x) - L|) \geq \varepsilon\}\right|\right\} \\
&\quad + \frac{1}{k_{r-1}}\left\{\left|\{k \in I_2 : f_k(|A_k x - L|) \geq \varepsilon\}\right|\right\} \\
&\quad + \ldots + \frac{1}{k_{r-1}}\left\{\left|\{k \in I_r : f_k|A_k(x) - L| \geq \varepsilon\}\right|\right\} \\
&= \frac{k_1}{k_{r-1}}\frac{1}{h_1}\left|\{k \in I_1 : f_k|A_k(x) - L| \geq \varepsilon\}\right| \\
&\quad + \frac{k_2 - k_1}{k_{r-1}}\frac{1}{h_2}\left|\{k \in I_2 : f_k(|A_k(x) - L|) \geq \varepsilon\}\right| \\
&\quad + \ldots + \frac{k_r - k_{r-1}}{k_{r-1}}\frac{1}{h_r}\left|\{k \in I_r : f_k(|A_k(x) - L|) \geq \varepsilon\}\right| \\
&= \frac{k_1}{k_{r-1}}K_1 + \frac{k_2 - k_1}{k_{r-1}}K_2 + \ldots + \frac{k_r - k_{r-1}}{k_{r-1}}K_r \\
&\leq \left\{\sup\nolimits_{j \in C} K_j\right\}\frac{k_r}{k_{r-1}} < M\delta.
\end{aligned}$$

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Choosing $\delta_1 = \dfrac{\delta}{M}$ and in view of the fact that $\bigcup \{n : k_{r-1} < n \le k_r,\ r \in C\} \subset T$ where $C \in \mathcal{F}(\mathcal{I})$ it follows from our assumption on θ that the set T also belongs to $\mathcal{F}(\mathcal{I})$ and this completes the proof of the theorem. □

Combining Theorem 3.3 and Theorem 3.4 we get the following theorem.

Theorem 3.5. *Let $A = (a_{ki})$ be an infinite matrix of complex numbers and (f_k) be a sequence of modulus functions in S. If $\theta = \{k_r\}$ be a lacunary sequence with $1 < \liminf_r q_r \le \limsup_r q_r < \infty$, then*

$$x_k \to L\left(S_\theta^A(\mathcal{I}, F)\right) = x_k \to L\left(S_\theta^A(\mathcal{I}, F)\right).$$

4. Cesàro summability for $A^{\mathcal{I}}$

Definition 4.1. *Let $A = (a_{ki})$ be an infinite matrix of complex numbers and (f_k) be a sequence of modulus functions in S. A sequence $x = (x_k)$ is said to be $A^{\mathcal{I}}$-Cesàro summable to L if for each $\varepsilon > 0$ and for each $x \in X$,*

$$\left\{ n \in \mathbb{N} : \left| \frac{1}{n} \sum_{k=1}^{n} f_k\left(A_k(x) - L\right) \right| \ge \varepsilon \right\} \in \mathcal{I}.$$

In this case, we write $x_k \to L\left((\sigma_1)_\theta^A(\mathcal{I}, F)\right)$.

Definition 4.2. *Let $A = (a_{ki})$ be an infinite matrix of complex numbers and (f_k) be a sequence of modulus functions in S. A sequence $x = (x_k)$ is said to be strongly $A^{\mathcal{I}}$-Cesàro summable to L if for each $\varepsilon > 0$ and for each $x \in X$,*

$$\left\{ n \in \mathbb{N} : \frac{1}{n} \sum_{k=1}^{n} f_k\left(|A_k(x) - L|\right) \ge \varepsilon \right\} \in \mathcal{I}.$$

In this case, we write $x_k \to L\left(|\sigma_1|_\theta^A(\mathcal{I}, F)\right)$.

Theorem 4.1. *Let θ be a lacunary sequence. if $\liminf_r q_r > 1$ then,*

$$x_k \to L\left(|\sigma_1|_\theta^A(\mathcal{I}, F)\right) \Rightarrow x_k \to L\left(N_\theta^A(\mathcal{I}, F)\right).$$

Proof. If $\liminf_r q_r > 1$, then there exists $\delta > 0$ such that $q_r \ge 1 + \delta$ for all $r \ge 1$. Since $h_r = k_r - k_{r-1}$, we have $\dfrac{k_r}{h_r} \le \dfrac{1+\delta}{\delta}$ and $\dfrac{k_{r-1}}{h_r} \le \dfrac{1}{\delta}$. Let $\varepsilon > 0$ and define the set

$$S = \left\{ k_r \in \mathbb{N} : \frac{1}{k_r} \sum_{k=1}^{k_r} f_k\left(|A_k(x) - L|\right) < \varepsilon \right\}.$$

We can easily say that $S \in \mathcal{F}(\mathcal{I})$, which is a filter of the ideal $\mathcal{I}$,

$$\begin{aligned}
\tfrac{1}{h_r} \sum_{k \in I_r} f_k\left(|A_k(x) - L|\right) &= \tfrac{1}{h_r} \sum_{k=1}^{k_r} f_k\left(|A_k(x) - L|\right) - \tfrac{1}{h_r} \sum_{k=1}^{k_{r-1}} f_k\left(|A_k(x) - L|\right) \\
&= \tfrac{k_r}{h_r} \cdot \tfrac{1}{k_r} \sum_{k=1}^{k_r} f_k\left(|A_k(x) - L|\right) - \tfrac{k_{r-1}}{h_r} \cdot \tfrac{1}{k_{r-1}} \sum_{k=1}^{k_{r-1}} f_k\left(|A_k(x) - L|\right) \\
&\le \left(\frac{1+\delta}{\delta}\right)\varepsilon - \frac{1}{\delta}\varepsilon'
\end{aligned}$$

for each $k_r \in S$. Choose $\eta = \left(\frac{1+\delta}{\delta}\right)\varepsilon - \frac{1}{\delta}\varepsilon'$. Therefore,

$$\left\{r \in \mathbb{N} : \frac{1}{h_r}\sum_{k\in I_r} f_k\left(\left|A_k\left(x\right) - L\right|\right) < \eta\right\} \in \mathcal{F}(\mathcal{I})$$

and it completes the proof. □

Theorem 4.2. *Let $A = (a_{ki})$ be an infinite matrix of complex numbers and (f_k) be a sequence of modulus functions in S. If $(x_k) \in m(X)$ and $x_k \to L\left(S_\theta^A\left(\mathcal{I}, F\right)\right)$, then $x_k \to L\left(\left(\sigma_1\right)_\theta^A\left(\mathcal{I}, F\right)\right)$.*

Proof. Suppose that $(x_k) \in m(X)$ and $x_k \to L\left(S_\theta^A\left(\mathcal{I}, F\right)\right)$. Then we can assume that

$$f_k\left(\left|A_k x - L\right|\right) \leq M$$

for all $k \in \mathbb{N}$. Also for each $\varepsilon > 0$, we can write

$$\begin{aligned}
\left|\frac{1}{n}\sum_{k=1}^{n} f_k\left(A_k\left(x\right) - L\right)\right| &\leq \frac{1}{n}\sum_{k=1}^{n} f_k\left(\left|A_k\left(x\right) - L\right|\right) \\
&\leq \frac{1}{n}\sum_{\substack{k=1 \\ f_k\left(\left|A_k(x)-L\right|\right)\geq\frac{\varepsilon}{2}}}^{n} f_k\left(\left|A_k\left(x\right) - L\right|\right) \\
&\quad + \frac{1}{n}\sum_{\substack{k=1 \\ f_k\left(\left|A_k(x)-L\right|\right)<\frac{\varepsilon}{2}}}^{n} f_k\left(\left|A_k\left(x\right) - L\right|\right) \\
&\leq M.\frac{1}{n}\left|\left\{k \leq n : f_k\left(\left|A_k\left(x\right) - L\right|\right) \geq \varepsilon\right\}\right| + \varepsilon.
\end{aligned}$$

Consequently, if $\delta > \varepsilon > 0$, δ and ε are independent, put $\delta_1 = \delta - \varepsilon > 0$, we have

$$\left\{n \in \mathbb{N} : \left|\frac{1}{n}\sum_{k,l=1}^{n} f_k\left(A_k\left(x\right) - L\right)\right| \geq \delta\right\}$$

$$\subseteq \left\{n \in \mathbb{N} : \tfrac{1}{n}\left|\left\{k \leq n : f_k\left(\left|A_k\left(x\right) - L\right|\right) \geq \varepsilon\right\}\right| \geq \tfrac{\delta_1}{M}\right\} \in \mathcal{I}.$$

This shows that $x_k \to L\left(\left(\sigma_1\right)_\theta^A\left(\mathcal{I}, F\right)\right)$. □

Theorem 4.3. *Let θ be a lacunary sequence. if $\limsup_r q_r < \infty$ then,*

$$x_k \to L\left(N_\theta^A\left(\mathcal{I}, F\right)\right) \Rightarrow x_k \to L\left(\left|\sigma_1\right|_\theta^A\left(\mathcal{I}, F\right)\right).$$

Proof. If $\limsup_r q_r < \infty$, then there exists $M > 0$ such that $q_r < M$ for all $r \geq 1$. Let $x_k \to L\left(N_\theta^A\left(\mathcal{I}, F\right)\right)$ and define the sets T and R such that

$$T = \left\{r \in \mathbb{N} : \frac{1}{h_r}\sum_{k\in I_r} f_k\left(\left|A_k\left(x\right) - L\right|\right) < \varepsilon_1\right\}$$

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and

$$R=\left\{n\in\mathbb{N}:\frac{1}{n}\sum_{k=1}^{n}f_k\left(\left|A_k\left(x\right)-L\right|\right)<\varepsilon_2\right\}.$$

Let

$$A_j=\frac{1}{h_j}\sum_{k\in I_j}f_k\left(\left|A_k\left(x\right)-L\right|\right)<\varepsilon_1$$

for all $j\in T$. It is obvious that $T\in\mathcal{F}(\mathcal{I})$. Choose n is any integer with $k_{r-1}<n<k_r$, where $r\in T$,

$$\begin{aligned}
\tfrac{1}{n}\sum_{k=1}^{n}f_k\left(\left|A_k\left(x\right)-L\right|\right) &\leq \tfrac{1}{k_{r-1}}\sum_{k=1}^{k_r}f_k\left(\left|A_k\left(x\right)-L\right|\right)\\
&= \tfrac{1}{k_{r-1}}\Bigg(\sum_{k\in I_1}f_k\left(\left|A_k\left(x\right)-L\right|\right)+\sum_{k\in I_2}f_k\left(\left|A_k\left(x\right)-L\right|\right)\\
&\quad +...+\sum_{k\in I_r}f_k\left(\left|A_k\left(x\right)-L\right|\right)\Bigg)\\
&= \tfrac{k_1}{k_{r-1}}\left(\tfrac{1}{h_1}\sum_{k\in I_1}f_k\left(\left|A_k\left(x\right)-L\right|\right)\right)+\tfrac{k_2-k_1}{k_{r-1}}\left(\tfrac{1}{h_2}\sum_{k\in I_2}f_k\left(\left|A_k\left(x\right)-L\right|\right)\right)\\
&\quad +...+\tfrac{k_r-k_{r-1}}{k_{r-1}}\left(\tfrac{1}{h_r}\sum_{k\in I_r}f_k\left(\left|A_k\left(x\right)-L\right|\right)\right)\\
&= \tfrac{k_1}{k_{r-1}}A_1+\tfrac{k_2-k_1}{k_{r-1}}A_2+...+\tfrac{k_r-k_{r-1}}{k_{r-1}}A_r\\
&\leq \left(\sup\nolimits_{j\in T}A_j\right)\tfrac{k_1}{k_{r-1}}\\
&< \varepsilon_1.M.
\end{aligned}$$

Choose $\varepsilon_2=\frac{\varepsilon_1}{M}$ and in view of the fact that $\cup\{n:k_{r-1}<n<k_r,\ r\in T\}\subset R$, where $T\in\mathcal{F}(\mathcal{I})$, ıt follows from our assumption on θ thet the set R also belongs to $\mathcal{F}(\mathcal{I})$ and this completes the proof of the theorem. □

Theorem 4.4. *If* $x_k\to L\left(\left|\sigma_1\right|_\theta^A\left(\mathcal{I},F\right)\right)$, *then,* $x_k\to L\left(S^A\left(\mathcal{I},F\right)\right)$.

Proof. Let $x_k\to L\left(\left|\sigma_1\right|_\theta^A\left(\mathcal{I},F\right)\right)$, and $\varepsilon>0$ given. Then

$$\begin{aligned}
\sum_{k=1}^{n}f_k\left(\left|A_k\left(x\right)-L\right|\right) &\geq \sum_{\substack{k=1\\ f_k\left(\left|A_kx-L\right|\right)\geq\varepsilon}}^{n}f_k\left(\left|A_k\left(x\right)-L\right|\right)\\
&\geq \varepsilon.\left|\left\{k\leq n:f_k\left(\left|A_k\left(x\right)-L\right|\right)\geq\varepsilon\right\}\right|
\end{aligned}$$

and so

$$\frac{1}{\varepsilon.n}\sum_{k=1}^{n}f_k\left(\left|A_k\left(x\right)-L\right|\right)\geq\frac{1}{n}\left|\left\{k\leq n:f_k\left(\left|A_k\left(x\right)-L\right|\right)\geq\varepsilon\right\}\right|.$$

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So for a given $\delta > 0$,

$$\left\{n \in \mathbb{N} : \frac{1}{n}\left|\{k \leq n : f_k\left(|A_k(x) - L|\right) \geq \varepsilon\}\right| \geq \delta\right\}$$

$$\subseteq \left\{n \in \mathbb{N} : \frac{1}{n}\sum_{k=1}^{n} f_k\left(|A_k(x) - L|\right) \geq \varepsilon\delta\right\} \in \mathcal{I}.$$

Therefore $x_k \to L\left(S^A(\mathcal{I}, F)\right)$. □

Theorem 4.5. *Let* $(x_k) \in m(X)$. *If* $x_k \to L\left(S^A(\mathcal{I}, F)\right)$ *then,* $x_k \to L\left(|\sigma_1|_\theta^A(\mathcal{I}, F)\right)$.

Proof. Suppose that (x_k) is bounded and $x_k \to L\left(S^A(\mathcal{I}, F)\right)$. Then there is a M such that $f_k\left(|A_k(x) - L|\right) \leq M$ for all k. Given $\varepsilon > 0$, we have

$$\begin{aligned}
\frac{1}{n}\sum_{k=1}^{n} f_k\left(|A_k(x) - L|\right) &= \frac{1}{n}\sum_{\substack{k=1 \\ f_k(|A_k(x)-L|)\geq\varepsilon}}^{n} f_k\left(|A_k(x) - L|\right) \\
&+ \frac{1}{n}\sum_{\substack{k=1 \\ f_k(|A_k(x)-L|)<\varepsilon}}^{n} f_k\left(|A_k(x) - L|\right) \\
&\leq \frac{1}{n}M\left|\{k \leq n : f_k\left(|A_k(x) - L|\right) \geq \varepsilon\}\right| \\
&\quad + \frac{1}{n}\varepsilon\left|\{k \leq n : f_k\left(|A_k(x) - L|\right) < \varepsilon\}\right| \\
&\leq \frac{M}{n}\left|\{k \leq n : f_k\left(|A_k(x) - L|\right) \geq \varepsilon\}\right| + \varepsilon.
\end{aligned}$$

Then for any $\delta > 0$,

$$\left\{n \in \mathbb{N} : \frac{1}{n}\sum_{k=1}^{n} f_k\left(|A_k(x) - L|\right) \geq \delta\right\}$$

$$\subseteq \left\{n \in \mathbb{N} : \frac{1}{n}\left|\{k \leq n : f_k\left(|A_k(x) - L|\right) \geq \varepsilon\}\right| \geq \frac{\delta}{M}\right\} \in \mathcal{I}.$$

Therefore $x_k \to L\left(|\sigma_1|_\theta^A(\mathcal{I}, F)\right)$. □

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