12-23-2014

# On Dual Toric Complete Intersection Codes

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## 1. Introduction

In this paper we consider a class of evaluation codes called toric complete intersection codes. They were introduced in [14] and are a natural generalization of evaluation codes on complete intersections in the projective space, previously studied by Duursma, Rentería, and Tapia-Recillas [7], Gold, Little, and Schenck [8], and Ballico and Fontanari [1].

Fix an integer $\ell \geq 1$. A toric complete intersection code $\mathcal{C}_{S,A}$ is constructed by evaluating $\ell$-variate Laurent polynomials supported in a given lattice polytope $A$ at the set $S$ of common zeroes of $\ell$ Laurent polynomials with given Newton polytopes $P_1, \dots, P_\ell$. In [14], the second author proved general bounds for the minimum distance of such codes in terms of $A$ and the $P_i$. The goal of this paper is to study duality for toric complete intersection codes. In particular, we give conditions on $A$ and $P_1, \dots, P_\ell$ when the code $\mathcal{C}_{S,A}$ is quasi-self-dual, see Theorem 3.3.

When $\ell = 2$ we give a combinatorial formula for the dimension of $\mathcal{C}_{S,A}$, thus reducing the above mentioned conditions to purely combinatorial ones (see Theorem 4.5). We show how restrictive this condition is when the polytopes $P_i$ are similar. In fact, in this case a quasi-self-dual code $\mathcal{C}_{S,A}$ exists if and only if the $P_i$ are $\mathrm{GL}(2,\mathbb{Z})$-equivalent to an integer multiple of one of 16 polygons as in Proposition 4.6. On the other hand, Theorem 4.9 provides a much less restrictive framework for constructing the polytopes $A$ and $P_1$, $P_2$ which produce quasi-self-dual codes $\mathcal{C}_{S,A}$.

The paper concludes with an algorithm for finding dual and quasi-self-dual toric complete intersection codes, and provides with a list of examples over the finite field of 16 elements.

## 2. Preliminaries

### 2.1. Dual codes

To set our notation we start with basic definitions about dual codes, following Ref. [15]. Throughout the paper, $\mathbb{F}_q$ denotes a finite field of $q$ elements and $\mathbb{F}_q^*$ its multiplicative group of non-zero elements. A linear code $\mathcal{C}$ over $\mathbb{F}_q$ of block length $n$, dimension $k$, and minimum distance $d$ is referred to as an $[n,k,d]_q$-code.

Fix a vector $y \in (\mathbb{F}_q^*)^n$. It defines a *$y$-dot product* on $\mathbb{F}_q^n$ given by $(u \cdot v)_y = \sum_{i=1}^n y_i u_i v_i$. If $y = (1, \dots, 1)$, it is the standard dot product. Define

$$\mathcal{C}^{\perp_y} = \left\{ v \in \mathbb{F}_q^n \;\middle|\; (u \cdot v)_y = 0 \ \forall u \in \mathcal{C} \right\}.$$

It is easy to see that $\mathcal{C}^{\perp_y}$ is equivalent to $\mathcal{C}^{\perp}$. In fact, $y\mathcal{C}^{\perp_y} = \mathcal{C}^{\perp}$ in the above notation.

**Definition 2.1.** A code $\mathcal{C}$ is called *quasi-self-dual* with respect to $y \in (\mathbb{F}_q^*)^n$ if $\mathcal{C} = \mathcal{C}^{\perp_y}$. If $y = (1, 1, \dots, 1)$, we say $\mathcal{C}$ is *self-dual.* A code equivalent to a self-dual code is called *isodual.*

Clearly, if $\mathcal{C}$ is a quasi-self-dual code with respect to $y = (x_1^2, \dots, x_n^2)$, for some $x_i \in \mathbb{F}_q^*$, then $x\mathcal{C}$ is a self-dual code. In particular, if $char(\mathbb{F}_q) = 2$, then any quasi-self-dual code is isodual.

*2.2. Toric complete intersection codes*

Recall the definition of a toric complete intersection code following Ref. [14]. Let $\mathbb{K}$ be a field, $\overline{\mathbb{K}}$ be its algebraic closure, and $\mathbb{K}^* = \mathbb{K} \setminus \{0\}$. We use standard terminology and notation from the theory of Newton polytopes. An element $f$ of the Laurent polynomial ring $\mathbb{K}[t_1^{\pm 1}, \dots, t_\ell^{\pm 1}]$ is a finite sum

$$f = \sum_{a \in \mathcal{A}} c_a t^a, \quad \text{where } t^a = t_1^{a_1} \cdots t_\ell^{a_\ell},\ \mathcal{A} \subseteq \mathbb{Z}^\ell,\ c_a \in \mathbb{K}.$$

The convex hull of the finite set $\mathcal{A} \subseteq \mathbb{Z}^\ell$ is called the *Newton polytope* of $f$ and will be denoted by $P(f)$.

For any set $A \subseteq \mathbb{R}^\ell$ let $A_\mathbb{Z} = A \cap \mathbb{Z}^\ell$ denote the set of lattice points in $A$. By a slight abuse of notation we use either $|A_\mathbb{Z}|$ or $|A|_\mathbb{Z}$ to denote the cardinality of the set $A_\mathbb{Z}$.

All polytopes considered in this paper are assumed to be *lattice polytopes*, i.e. convex hulls of finitely many points in $\mathbb{Z}^\ell$. A polytope of dimension $\ell$ will be called an $\ell$-polytope, for short.

The point-wise sum of two polytopes $P + Q = \{p + q \in \mathbb{R}^\ell \mid p \in P,\ q \in Q\}$ is called the Minkowski sum. Recall that any polytope (in fact, any convex body) $P$ is uniquely determined by its *support function* $l_P$ defined by

$$l_P(v) = \max\big\{(u \cdot v) \bigm| u \in P\big\} \quad \text{for all } v \in \mathbb{R}^\ell.$$

We will need the following basic properties of the support function: (1) $l_{P+Q} = l_P + l_Q$ and (2) $P \subseteq Q$ if and only if $l_P(v) \le l_Q(v)$ for every $v$ in $\mathbb{R}^\ell$.

We denote the Euclidean $\ell$-dimensional volume of $P$ by $V_\ell(P)$, or simply by $V(P)$ when the dimension is clear. We use $V(P_1, \dots, P_\ell)$ to denote the normalized *mixed volume* of $\ell$ lattice polytopes $P_1, \dots, P_\ell$. By definition,

$$V(P_1, \dots, P_\ell) = \sum_{I \subseteq \{1,\dots,\ell\}} (-1)^{\ell - |I|} V_\ell(P_I),$$

where $P_I = \sum_{i \in I} P_i$. The mixed volume $V(P_1, \dots, P_\ell)$ is non-negative, multilinear with respect to Minkowski addition, and coincides with $\ell! V_\ell(P)$ when $P_i = P$ for every $1 \le i \le \ell$. More about the mixed volume can be found in [4, Chapter 4].

Now fix a finite subset $S = \{p_1, \dots, p_n\}$ of the algebraic torus $(\mathbb{K}^*)^\ell$ and a finite-dimensional subspace $\mathcal{L}$ of $\mathbb{K}[t_1^{\pm 1}, \dots, t_\ell^{\pm 1}]$.

**Definition 2.2.** Define the *evaluation map*

$$ev_S : \mathcal{L} \to \mathbb{K}^n, \quad f \mapsto \big(f(p_1), \dots, f(p_n)\big).$$

The image of $ev_S$ is called the *evaluation code* corresponding to $S$ and $\mathcal{L}$. We will denote this code by $\mathcal{C}_{S,\mathcal{L}}$.

Clearly, $\mathcal{C}_{S,\mathcal{L}}$ is a linear code over $\mathbb{K}$ of block length $n$.

Toric complete intersection codes are special evaluation codes when $S$ is the solution set of a Laurent polynomial system satisfying some assumptions. Here is the precise definition.

**Definition 2.3.** Fix a collection of $\ell$-polytopes $P_1, \ldots, P_\ell$ in $\mathbb{R}^\ell$ and consider $\ell$ Laurent polynomials $f_1, \ldots, f_\ell$ over $\mathbb{K}$ with Newton polytopes $P_1, \ldots, P_\ell$ such that the solution set $S$ of the system $f_1 = \cdots = f_\ell = 0$ in $(\bar{\mathbb{K}}^*)^\ell$ satisfies the following:

*(1)* $|S| = V(P_1, \ldots, P_\ell)$,
*(2)* the set $S$ consists of $\mathbb{K}$-rational points i.e. $S \subseteq (\mathbb{K}^*)^\ell$.

Then $S$ is called a *toric complete intersection over* $\mathbb{K}$.

**Remark 2.4.** In general, the set $S$ is the intersection of $\ell$ hypersurfaces in a toric variety associated with the polytope $P$. According to the Bernstein–Kushnirenko–Khovanskii bound [3,11], if $S$ consists of isolated points, its cardinality $|S|$ cannot exceed the mixed volume $V(P_1, \ldots, P_\ell)$. Moreover, the bound is attained for systems with generic coefficients (having the $P_i$ fixed) in which case the hypersurfaces do not intersect outside of the torus $(\bar{\mathbb{K}}^*)^\ell$ and the intersections are transversal. This is guaranteed by the assumption *(1)*. In particular, this implies that the local intersection multiplicities equal one, and the ideal $\langle f_1, \ldots, f_\ell \rangle$ is radical.

The following toric analog of the Euler–Jacobi theorem by Khovanskii [9] is fundamental for our results about toric complete intersection codes and their duals. For a proof that works over arbitrary algebraically closed fields see [10, Section 14]. First we need a couple of definitions.

**Definition 2.5.** Let $P^\circ$ be the interior of $P = P_1 + \cdots + P_\ell$. Fix any subset $A$ of $P^\circ$. It defines a space of Laurent polynomials over $\mathbb{K}$:

$$\mathcal{L}(A) = \operatorname{span}_{\mathbb{K}}\{t^a \mid a \in A_{\mathbb{Z}}\} \subseteq \mathbb{K}[t_1^{\pm 1}, \ldots, t_\ell^{\pm 1}].$$

**Definition 2.6.** Let $f_1, \ldots, f_\ell \in \mathbb{K}[t_1^{\pm 1}, \ldots, t_\ell^{\pm 1}]$ be Laurent polynomials. The Laurent polynomial

$$J_f^{\mathbb{T}} = \det\left(t_j \frac{\partial f_i}{\partial t_j}\right)$$

is called the *toric Jacobian* of $f_1, \ldots, f_\ell$.

**Theorem 2.7.** *(See [9].) Let $S$ be a toric complete intersection over $\mathbb{K}$. Let $P = P_1 + \cdots + P_\ell$ be the Minkowski sum and $P^\circ$ be its interior. Then for any $h \in \mathcal{L}(P^\circ)$ we have*

$$\sum_{p \in S} \frac{h(p)}{J_f^{\mathbb{T}}(p)} = 0.$$

Note, since the local intersection multiplicities are equal to one, $J_f^{\mathbb{T}}(p) \neq 0$ for every $p \in S$, and the above sum makes sense. In fact, the above sum represents the *global residue*, which is the sum of local (Grothendieck) residues over the solution set $S$. For a connection between residue theory in toric varieties and higher-dimensional evaluation codes we refer to [14, Section 2.3] and the references therein.

**Definition 2.8.** Let $S$ be toric complete intersection over $\mathbb{K}$. Let $A \subseteq P^\circ$ and let $\mathcal{L}(A)$ be the corresponding polynomial space. The evaluation code $\mathcal{C}_{S,\mathcal{L}(A)}$ is called a *toric complete intersection code*, denoted simply by $\mathcal{C}_{S,A}$.

In [14] the second author gave lower bounds for the minimum distance of toric complete intersection codes. It turns out that the bound is significantly better if the solution set $S$ satisfies an extra assumption of "generic position". We formulate it below.

**Definition 2.9.** Let $Q$ be a polytope in $\mathbb{R}^\ell$. A subset $S \subset (\mathbb{K}^*)^\ell$ is said to be in *$Q$-generic position* if for any subset $T \subseteq S$ of size $|Q_\mathbb{Z}|$ the evaluation map $ev_T : \mathcal{L}(Q) \to \mathbb{K}^{|Q_\mathbb{Z}|}$ is an isomorphism.

In other words, $S$ is in $Q$-generic position if for any collection $T$ of size $|Q_\mathbb{Z}|$ there is a polynomial $h \in \mathcal{L}(Q)$ which takes the zero value at all but the last point of $T$. For example, when $Q = \Delta_\ell$ is the standard $\ell$-simplex, i.e. the convex hull of $\{0, e_1, \ldots, e_\ell\}$, where $\{e_1, \ldots, e_\ell\}$ is the standard basis for $\mathbb{R}^\ell$, this means that no $\ell + 1$ points of $S$ lie on a hyperplane.

Here is the lower bound on the minimum distance for toric complete intersection codes.

**Theorem 2.10.** *(See [14].) Let $S$ be a toric complete intersection in $Q$-generic position. Let $A$ be any set such that $A + mQ \subseteq P^\circ$ up to a lattice translation, for some $m \geq 0$. Then*

$$d(\mathcal{C}_{S,A}) \geq \big(|Q_\mathbb{Z}| - 1\big)m + 2.$$

## 3. Results in arbitrary dimension $\ell$

We begin this section with an immediate result about evaluation codes when $S$ is $Q$-generic and $A = Q$.

**Theorem 3.1.** *Let $S \subseteq (\mathbb{K}^*)^\ell$ be any subset in $Q$-generic position for some $\ell$-polytope $Q$. Then the evaluation code $\mathcal{C}_{S,Q}$ is an MDS code.*

**Proof.** Denote $\mathcal{C} := \mathcal{C}_{S,Q}$. We need to show that $\mathcal{C}$ is an $[n,k,n-k+1]_q$-code where $k = \dim(\mathcal{C})$ and $n = |S|$. (Here and everywhere in the paper dim denotes the dimension of a vector space over $\mathbb{K}$.) First, we show that $k = |Q_\mathbb{Z}|$. Consider the evaluation map

$$ev_S : \mathcal{L}(Q) \to \mathbb{K}^n, \quad f \mapsto \big(f(p_1), \dots, f(p_n)\big).$$

By definition $\mathcal{C} = \mathrm{Im}(ev_S)$. Since $\dim \mathcal{L}(Q) = |Q_\mathbb{Z}|$, it is enough to show that $ev_S$ is injective. If $f \in \mathrm{Ker}(ev_S)$ then $f \in \mathrm{Ker}(ev_T)$ for any subset $T \subseteq S$ of size $|Q_\mathbb{Z}|$. By Definition 2.9, $ev_T$ is an isomorphism, so $\mathrm{Ker}(ev_T)$ is trivial. Therefore $f = 0$.

Now we show that $d(\mathcal{C}) = n - k + 1$. By before, $\mathrm{Ker}(ev_T)$ is trivial for any $T \subseteq S$ of size $|Q_\mathbb{Z}|$. Therefore any non-zero $f \in \mathcal{L}(Q)$ can have at most $|Q_\mathbb{Z}| - 1$ zeroes in $S$. In other words, the image of $f$ under $ev_S$ has weight at least $n - |Q_\mathbb{Z}| + 1$. This shows that $d(\mathcal{C}) \geq n - k + 1$. On the other hand, by the Singleton bound $d(\mathcal{C}) \leq n - k + 1$. This proves that $\mathcal{C}$ is an $[n,k,n-k+1]_q$-code. □

**Corollary 3.2.** *The dual code $\mathcal{C}_{S,Q}^{\perp_y}$ is an MDS code.*

This follows from the fact that $\mathcal{C}^{\perp_y}$ is equivalent to $\mathcal{C}^\perp$ and that the dual of an MDS-code is also MDS.

The following theorem relates the toric complete intersection codes defined by $A \subseteq P^\circ$ and $B \subseteq P^\circ$ which satisfy $A + B \subseteq P^\circ$.

**Theorem 3.3.** *Let $S$ be a toric complete intersection. Let $A$, $B$ be subsets of $P^\circ$ such that $A + B \subseteq P^\circ$. If $\dim(\mathcal{C}_{S,B}) = |S| - \dim(\mathcal{C}_{S,A})$, then there exists $y \in (\mathbb{K}^*)^n$ such that*

$$\mathcal{C}_{S,B} = \mathcal{C}_{S,A}^{\perp_y}.$$

*In particular, if $|S|$ is even, $2A \subseteq P^\circ$, and $\dim(\mathcal{C}_{S,A}) = |S|/2$ then $\mathcal{C}_{S,A}$ is quasi-self-dual.*

**Proof.** Let $S = \{p_1, \dots, p_n\}$. First, for any $h = fg$, where $f \in \mathcal{L}(A)$ and $g \in \mathcal{L}(B)$, we have $h \in \mathcal{L}(A+B) \subseteq \mathcal{L}(P^\circ)$. By Theorem 2.7,

$$\sum_{i=1}^{n} \frac{h(p_i)}{J_f^\mathbb{T}(p_i)} = \sum_{i=1}^{n} \frac{f(p_i)g(p_i)}{J_f^\mathbb{T}(p_i)} = 0.$$

This implies that $ev_S(f)$ and $ev_S(g)$ are $y$-orthogonal where $y = (\frac{1}{J_f^\mathbb{T}(p_1)}, \dots, \frac{1}{J_f^\mathbb{T}(p_n)})$. Hence $\mathcal{C}_{S,B}$ is a subspace of $\mathcal{C}_{S,A}^{\perp_y}$. On the other hand, $\dim(\mathcal{C}_{S,B}) = n - \dim(\mathcal{C}_{S,A}) = \dim(\mathcal{C}_{S,A}^{\perp_y})$, and the first statement follows.

For the second part, let $B = A$. Then, $\dim(\mathcal{C}_{S,A}) = \dim(\mathcal{C}_{S,B}) = n/2$. By Definition 2.1, $\mathcal{C}_{S,A}$ is a quasi-self-dual code. □

As an immediate consequence of the above theorem we obtain the following.

**Corollary 3.4.** *Let* $char(\mathbb{K}) = 2$*. If* $|S|$ *is even,* $2A \subseteq P^\circ$*, and* $\dim(\mathcal{C}_{S,A}) = |S|/2$ *then* $\mathcal{C}_{S,A}$ *is isodual.*

Our ultimate goal is to give a description of the polytopes $P_1, \dots, P_\ell$ and the set $A$ such that generic systems produce quasi-self-dual codes. For this we need a way to compute the dimension $\dim(\mathcal{C}_{S,A})$. According to Definition 2.2, this amounts to computing the dimension of the kernel of the evaluation map,

$$\dim(\mathcal{C}_{S,A}) = \dim\big(\mathcal{L}(A)\big) - \dim \operatorname{Ker}(ev_S) = |A_{\mathbb{Z}}| - \dim \operatorname{Ker}(ev_S). \tag{3.1}$$

Let $J = \langle f_1, \dots, f_d \rangle$ be radical. Then polynomials in $\operatorname{Ker}(ev_S)$ are, in fact, elements of $\mathcal{L}(A) \cap J$. In other words, one has to compute an analog of the Hilbert function for the ideal $J$:

$$\operatorname{Hilb}_J(A) = \dim\big(\mathcal{L}(A) \cap J\big).$$

Although this can be done in some situation, there appears to be no simple formula for $\operatorname{Hilb}_J(A)$ in general. We explore $\ell = 2$ case in the next section. Also, in [12] the authors give a formula for $\dim(\mathcal{C}_{S,A})$ when the polynomials $(f_1, \dots, f_\ell)$ give rise to a regular sequence $(F_1, \dots, F_\ell)$ in the *homogeneous coordinate ring* of a toric variety. We plan to return to this problem in the future.

## 4. Results in dimension $\ell = 2$

In this section we concentrate on the case $\ell = 2$. We reserve the word "polygon" for any convex polytope of dimension at most two. Let $S \subseteq (\mathbb{K}^*)^2$ be a toric complete intersection defined by Laurent polynomial system $f_1 = f_2 = 0$ with lattice polygons $P_1$, $P_2$ as in Definition 2.3. As before, $P^\circ$ denotes the interior of $P = P_1 + P_2$, and $V(P, Q)$ the normalized mixed volume (mixed area) of $P$ and $Q$, i.e.

$$V(P, Q) = V(P + Q) - V(P) - V(Q),$$

where $V(P)$ is the Euclidean area of $P$. It is easy to check that $V(P, Q) = 0$ if and only if either one of the polygons is a point or $P$, $Q$ are parallel segments.

Our goal is to give a description of lattice polygons $P_1$, $P_2$ for which there exists $A$ satisfying

$$2A \subseteq (P_1 + P_2)^\circ, \quad \text{and} \quad \dim(\mathcal{C}_{S,A}) = V(P_1, P_2)/2. \tag{4.1}$$

Then, $\mathcal{C}_{S,A}$ is quasi-self-dual, by Theorem 3.3.

In Theorem 4.5 below we give a general geometric condition on $P_1$, $P_2$, and $A$ that guarantees that $\mathcal{C}_{S,A}$ is quasi-self-dual. Then we look at special cases (Proposition 4.6, Theorem 4.9) when we can construct $P_1$, $P_2$, and $A$ explicitly.

Intuitively, $A$ has to be just a bit "smaller" than the "average" of $P_1$ and $P_2$. Although we have the Minkowski addition on the space of lattice polygons, there is no subtraction, in general. To resolve this, we introduce the following analog of difference of (convex) sets.

Let $A$, $B$ be subsets of $\mathbb{R}^d$. Define

$$A - B = \{u \in \mathbb{R}^d \mid u + B \subseteq A\}.$$

It is easy to show that $A - B$ is convex if $A$ is convex. Also $(A - B) + B \subseteq A$, but not equal to $A$, in general. Rather, it is the largest subset in $A$ that has $B$ as a Minkowski summand.

Now we are ready to give a combinatorial formula for $\dim(\mathcal{C}_{S,A})$. We begin with a few lemmas.

**Lemma 4.1.** *Let $A \subset \mathbb{R}^2$. Then $V = \langle f_1 \rangle \cap \mathcal{L}(A)$ is a subspace of $\mathcal{L}(A)$ with a basis $\mathcal{B} = \{f_1 t^a \mid a \in (A - P_1)_{\mathbb{Z}}\}$.*

**Proof.** The fact that $V \subseteq \mathcal{L}(A)$ is a subspace is straightforward. Denote $R = A - P_1$. To show that $\mathcal{B}$ is linearly independent, suppose

$$\sum_{a \in R_{\mathbb{Z}}} \lambda_a f_1 t^a = 0, \quad \lambda_a \in \mathbb{K}.$$

Then $f_1(\sum_{a \in R_{\mathbb{Z}}} \lambda_a t^a) = 0$ in $\mathcal{L}(A)$. Since $\mathcal{L}(A)$ is a subset of $\mathbb{K}[t_1^{\pm 1}, t_2^{\pm 1}]$, it has no zero divisors. Thus, $\sum \lambda_a t^a = 0$, which implies that $\lambda_a = 0$, and so $\mathcal{B}$ is linearly independent.

To show $\mathcal{B}$ spans $V$, note that any $g \in V$ can be written as $g = hf_1 \in \mathcal{L}(A)$. We have $P(g) \subseteq A$ and $P(g) = P(h) + P_1$. Thus $P(h) \subseteq A - P_1 = R$. In particular, every monomial in $h$ has exponent lying in $R_{\mathbb{Z}}$, i.e. $h = \sum_{a \in R_{\mathbb{Z}}} \lambda_a t^a$. This shows that $g$ is a linear combination of elements in $\mathcal{B}$. $\square$

**Lemma 4.2.** *Let $f_1$ be absolutely irreducible and $A$ a lattice polygon. If $V(P_1, A) < V(P_1, P_2)$ then*

$$\langle f_1, f_2 \rangle \cap \mathcal{L}(A) = \langle f_1 \rangle \cap \mathcal{L}(A).$$

**Proof.** One inclusion $\langle f_1 \rangle \cap \mathcal{L}(A) \subseteq \langle f_1, f_2 \rangle \cap \mathcal{L}(A)$ is obvious. For the other one, consider $f \in \langle f_1, f_2 \rangle \cap \mathcal{L}(A)$. Clearly, $f$ vanishes at points in $S$. Now, the system $f_1 = f = 0$ has at least $|S| = V(P_1, P_2) > V(P_1, A)$ solutions. On the other hand, $f \in \mathcal{L}(A)$ implies $P(f) \subseteq A$, hence, by the Bernstein–Kushnirenko theorem, $f$ and $f_1$ must have a common component. Since $f_1$ is absolutely irreducible, $f_1$ divides $f$. Therefore, $f \in \langle f_1 \rangle \cap \mathcal{L}(A)$. This implies $\langle f_1, f_2 \rangle \cap \mathcal{L}(A) \subseteq \langle f_1 \rangle \cap \mathcal{L}(A)$, and the statement follows. □

**Proposition 4.3.** *Let $S$ be a toric complete intersection over $\mathbb{K}$ and suppose $f_1$ is absolutely irreducible. Let $A$ be a lattice polygon such that $V(P_1, A) < V(P_1, P_2)$. Then*

$$\dim(\mathcal{C}_{S,A}) = |A|_{\mathbb{Z}} - |A - P_1|_{\mathbb{Z}}.$$

**Proof.** By (3.1) we have $\dim(\mathcal{C}_{S,A}) = |A_{\mathbb{Z}}| - \dim \operatorname{Ker}(ev_S)$. Since $\langle f_1, f_2 \rangle$ is radical, it implies that $\operatorname{Ker}(ev_S) = \langle f_1, f_2 \rangle \cap \mathcal{L}(A)$. The latter equals $\langle f_1 \rangle \cap \mathcal{L}(A)$, by Lemma 4.2. The result now follows from Lemma 4.1. □

**Remark 4.4.** Note that $A = P_1$ corresponds to $A - P_1 = (0,0)$, the origin. In this case, $\dim(\mathcal{C}_{S,A}) = |A|_{\mathbb{Z}} - 1$. If $A$ does not contain any lattice translate of $P_1$ then $A - P_1$ is empty and $\dim(\mathcal{C}_{S,A}) = |A|_{\mathbb{Z}}$.

Now Proposition 4.3 and Theorem 3.3 provide the following geometric criterion.

**Theorem 4.5.** *Let $S$ be a toric complete intersection over $\mathbb{K}$ and suppose $f_1$ is absolutely irreducible. Let $A$ be a lattice polygon such that*

i. $V(P_1, A) < V(P_1, P_2)$,
ii. $2A \subseteq (P_1 + P_2)^\circ$,
iii. $|A|_{\mathbb{Z}} - |A - P_1|_{\mathbb{Z}} = V(P_1, P_2)/2$.

*Then $\mathcal{C}_{S,A}$ is a quasi-self-dual toric complete intersection code.*

To make our result more explicit we analyze the geometric conditions of Theorem 4.5 in some special cases. First we consider the so-called *unmixed* case, when $P_1$ and $P_2$ are integer dilates of the same lattice polygon $Q$. In other words, $P_1 = m_1 Q$ and $P_2 = m_2 Q$, for some positive integers $m_i$. Choose $A = aQ$ for some positive integer $a$. Then the conditions in Theorem 4.5 become

$$a < m_2, \qquad 2a < m_1 + m_2, \qquad \text{and} \qquad |aQ|_{\mathbb{Z}} - \big|(a - m_1)Q\big|_{\mathbb{Z}} = m_1 m_2 V(Q). \qquad (4.2)$$

We have the following result.

**Proposition 4.6.** *Let $P_1 = m_1 Q$, $P_2 = m_2 Q$, and $A = aQ$ for some lattice polygon $Q$ and positive integers $m_1$, $m_2$, and $a$. Suppose (4.2) holds. Then only the following three cases are possible.*

(1) *$Q$ is $\mathrm{GL}(2, \mathbb{Z})$-equivalent to the standard 2-simplex, $a = (m_1 + m_2 - 3)/2$, and $a \in \mathbb{N}$;*

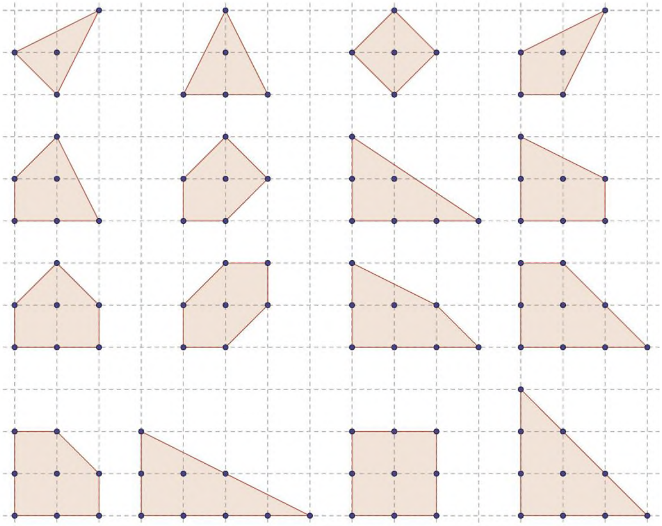

**Fig. 1.** The sixteen $\mathrm{GL}(2,\mathbb{Z})$-classes of Fano polygons.

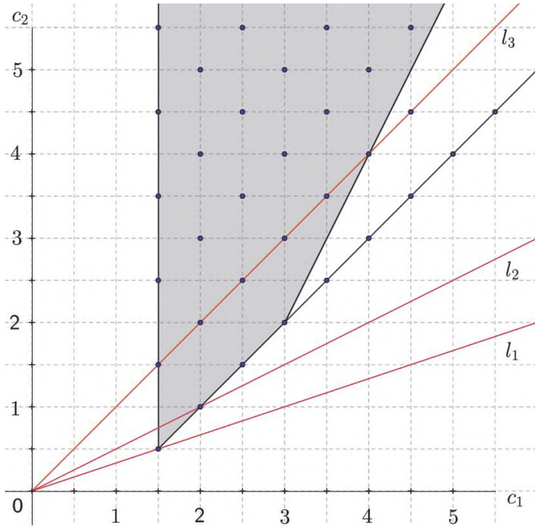


**Fig. 2.** The set of coefficients of Ehrhart polynomials.

(2) *$Q$ is $\mathrm{GL}(2,\mathbb{Z})$-equivalent to either the triangle with vertices $\{0, 2e_1, e_2\}$ or the standard square, $a = (m_1 + m_2 - 2)/2$, and $a \in \mathbb{N}$;*

(3) *$Q$ is $\mathrm{GL}(2,\mathbb{Z})$-equivalent to one of the sixteen Fano polygons in Fig. 1, $a = (m_1 + m_2 - 1)/2$, and $a \in \mathbb{N}$.*

**Proof.** According to Pick's formula $|aQ|_{\mathbb{Z}} = a^2V(Q) + \frac{a}{2}|\partial Q|_{\mathbb{Z}} + 1$, where $\partial Q$ denotes the boundary of $Q$. This is the Ehrhart polynomial of $Q$. Fig. 2 depicts the set of all $(c_1, c_2)$

(marked with dots) which are possible coefficients of Ehrhart polynomials, i.e. for which there exists a lattice polygon $Q$ with $c_1 = \frac{1}{2}|\partial Q|_{\mathbb{Z}}$ and $c_2 = V(Q)$, see [2].

These points $(c_1, c_2)$ have integer or half-integer coordinates and consist of points lying either in the shaded region or on the line $c_2 = c_1 - 1$ with the exception of a single point $(9/2, 9/2)$.

First, assume $a \geq m_1$. Applying Pick's formula to $|aQ|_{\mathbb{Z}}$ and $|(a-m_1)Q|_{\mathbb{Z}}$ and simplifying, we see that the equation in (4.2) is equivalent to

$$(m_1 + m_2 - 2a)V(Q) = \frac{1}{2}|\partial Q|_{\mathbb{Z}}.$$

It follows from Fig. 2 that the only lines $\lambda c_2 = c_1$ with $\lambda \in \mathbb{N}$ that intersect the set of possible coefficients are $3c_2 = c_1$, $2c_2 = c_1$, and $c_2 = c_1$, labeled by $l_1$, $l_2$, and $l_3$, respectively.

In the first case, $c_1 = 3/2$, $c_2 = 1/2$, which corresponds to $Q$ being $\mathrm{GL}(2,\mathbb{Z})$-equivalent to the standard 2-simplex. In this case $a = (m_1 + m_2 - 3)/2$ and it has to be a positive integer. In the second case, $c_1 = 2$, $c_2 = 1$, which corresponds to $Q$ being $\mathrm{GL}(2,\mathbb{Z})$-equivalent to either the triangle with vertices $\{0, 2e_1, e_2\}$ or the standard square. Here $a = (m_1 + m_2 - 2)/2$, and we must have $a \in \mathbb{N}$. Finally, $c_2 = c_1$ corresponds to lattice polygons with exactly one interior lattice point. These are Fano polygons and there are exactly sixteen classes of them up to $\mathrm{GL}(2,\mathbb{Z})$ equivalence. In this case $a = (m_1 + m_2 - 1)/2$, and $a$ must be in $\mathbb{N}$.

Now assume $a < m_1$. In this case $\dim(\mathcal{C}_{S,A}) = |A_{\mathbb{Z}}|$ by Remark 4.4, and the equation in (4.2) becomes $|aQ|_{\mathbb{Z}} = m_1 m_2 V(Q)$. Again, by using Pick's formula one can show that this is equivalent to the line $(m_1 m_2 - a^2)c_2 = ac_1 + 1$ having a non-trivial intersection with the set of lattice points in Fig. 2, which is impossible if $1 \leq a < m_1$. This completes the proof of the theorem. □

**Remark 4.7.** We point out that the last three polygons in the bottom row in Fig. 1 are, in fact, particular cases of (2) when both $m_1$ and $m_2$ are even, and (3) when both $m_1$ and $m_2$ are multiples of three.

Combining the results of Theorem 2.10, Theorem 4.5, and Proposition 4.6 we obtain the following.

**Corollary 4.8.** *Let $S$ be a toric complete intersection over $\mathbb{K}$ in $Q$-generic position and assume $f_1$ is absolutely irreducible. Let $P_1 = m_1 Q$, $P_2 = m_2 Q$ and $A = aQ$ be as (1), (2) or (3) in Proposition 4.6. Then $\mathcal{C}_{S,A}$ is quasi-self-dual with parameters*

(1) $n = m_1 m_2$, $k = n/2$, $d(\mathcal{C}_{S,A}) \geq (m_1 + m_2 + 1)/2$; *or*
(2) $n = 2m_1 m_2$, $k = n/2$, $d(\mathcal{C}_{S,A}) \geq m_1 + m_2$; *or*
(3) $n = 2V(Q)m_1 m_2$, $k = n/2$, $d(\mathcal{C}_{S,A}) \geq V(Q)(m_1 + m_2 - 1) + 2$,

*respectively.*

We note that the polygons $m_1Q$, $m_2Q$, for $1 < m_1 < m_2$, satisfy the geometric conditions in [14, Theorems 4.1 and 4.3] which implies that systems $f_1 = f_2 = 0$ with Newton polygons $m_1Q$, $m_2Q$ and *generic coefficients* in $\bar{\mathbb{K}}$ produce quasi-self-dual codes $\mathcal{C}_{S,A}$.

Our next situation is more general. Here we only assume that $P_1$ is a Minkowski summand of $A$, and $A$ is a Minkowski summand of $P_2$. In other words,

$$A = P_1 + R_1 \quad \text{and} \quad P_2 = A + R_2,$$

for some lattice polygons $R_1$, $R_2$ (we allow $R_1$ to be a point or a lattice segment).

Recall that $l_P(v)$ denotes the support function of $P$. Also, by $\mathrm{Fan}(P)$ we mean the set of primitive lattice vectors (i.e. whose entries are coprime) that are the outer normals to the edges of $P$.

**Theorem 4.9.** *Let $A = P_1 + R_1$ and $P_2 = A + R_2$ for some lattice polygons $P_1$, $R_1$, and $R_2$. Then i–iii in Theorem 4.5 hold if and only if $R_1 \subset R_2^\circ$ and $l_{R_2}(v) = l_{R_1}(v) + 1$ for all $v \in \mathrm{Fan}(P_1)$.*

**Proof.** The condition $2A \subseteq (P_1+P_2)^\circ$ written in terms of the support functions translates to $l_{2A}(v) < l_{P_1+P_2}(v)$ for all $v \in \mathbb{R}^2$. By properties of the support function this is equivalent to $l_{R_1}(v) < l_{R_2}(v)$ for all $v \in \mathbb{R}^2$, which means $R_1 \subset R_2^\circ$.

Next we look at condition iii:

$$2|A|_{\mathbb{Z}} - 2|A - P_1|_{\mathbb{Z}} = V(P_1, P_2). \tag{4.3}$$

Applying Pick's formula and linearity of the mixed volume, we can rewrite the left hand side as follows.

$$2|P_1 + R_1|_{\mathbb{Z}} - 2|R_1|_{\mathbb{Z}} = 2V(P_1 + R_1) - 2V(R_1) + |\partial P_1|_{\mathbb{Z}} = V(P_1, P_1 + 2R_1) + |\partial P_1|_{\mathbb{Z}},$$

where we used an obvious relation $|\partial(P_1+R_1)|_{\mathbb{Z}} = |\partial P_1|_{\mathbb{Z}}+|\partial R_1|_{\mathbb{Z}}$. Now (4.3) is equivalent to

$$V(P_1, R_1) + |\partial P_1|_{\mathbb{Z}} = V(P_1, R_2). \tag{4.4}$$

There is an "inductive" formula for computing the mixed volume [4, Chapter 4]. It can be adapted to the lattice situation. In dimension two it states the following. Let $P$ be a lattice polygon and $L_v$ be the lattice length of the edge of $P$ corresponding to $v \in \mathrm{Fan}(P)$. Then for any lattice polygon $R$

$$V(P, R) = \sum_{v \in \mathrm{Fan}(P)} l_R(v) L_v$$

Note that when all $l_R(v)$ equal one, the above sum is just $|\partial P_1|_{\mathbb{Z}}$. Therefore, (4.4) is equivalent to

$$\sum_{v\in \mathrm{Fan}(P_1)} (l_{R_1}(v)+1)L_v = \sum_{v\in \mathrm{Fan}(P_1)} l_{R_2}(v)L_v. \tag{4.5}$$

On the other hand, we have $l_{R_1}(v) < l_{R_2}(v)$ for all $v \in \mathbb{R}^2$, by condition ii. In particular, $l_{R_1}(v)+1 \le l_{R_2}(v)$ for $v \in \mathrm{Fan}(P_1)$, as $l_{R_i}(v)$ takes integer values for these $v$. Therefore (4.5) holds if and only if

$$l_{R_1}(v)+1 = l_{R_2}(v) \quad \text{for all } v \in \mathrm{Fan}(P_1).$$

Finally, the condition $V(P_1, A) < V(P_1, P_2)$ is the same as $V(P_1, R_2) > 0$, which is true since $P_1$ is 2-dimensional and $R_2$ is not just a point, otherwise $R_1 \subset R_2^\circ$ would be false. $\square$

**Remark 4.10.** In fact, we can restate the condition in Theorem 4.9 as follows:

$$2A \subset (P_1+P_2)^\circ \quad \text{and} \quad 2l_A(v) = l_{P_1}(v) + l_{P_2}(v) - 1 \quad \text{for all } v \in \mathrm{Fan}(P_1).$$

For this it is enough to only assume that $P_1$ is a Minkowski summand of $A$. This justifies what we said previously that $A$ has to be a bit smaller than the average of $P_1$ and $P_2$. However, this condition is not as convenient for constructing examples as the one in Theorem 4.9.

## 5. Algorithm and examples

In this section we present some examples of toric complete intersection codes illustrating constructions from the previous sections. All our examples were produced using MAGMA algebra system [5]. Our method is a rather straightforward random search for toric complete intersections. The algorithm which we put below works well for small polygons $P_1$. In all our examples we work over $\mathbb{F}_{16}$. More examples can be found in [6].

First we need a simple necessary condition for $S$ to be a toric complete intersection.

**Proposition 5.1.** *Let $S$ be a toric complete intersection with Newton polygons $P_1$, $P_2$. Then the rank of the evaluation map $ev_S$ satisfies*

$$rk(ev_S) \le |P_2|_{\mathbb{Z}} - |P_2 - P_1|_{\mathbb{Z}} - 1.$$

*In particular, when $P_1$ is a Minkowski summand of $P_2$ we have*

$$rk(ev_S) \le |S| - |P_1^\circ|_{\mathbb{Z}}.$$

**Proof.** Let $n = |S| = V(P_1, P_2)$ and consider the following sequence which is exact in the first two terms:

$$0 \to \mathrm{Ker}(ev_S) \to \mathcal{L}(P_2) \xrightarrow{ev_S} \mathbb{K}^n.$$

Clearly, $\langle f_1\rangle \cap \mathcal{L}(P_2)$ is a subspace of $\mathrm{Ker}(ev_S)$. On the other hand $f_2$ lies in $\mathrm{Ker}(ev_S)$ and has no common factors with $f_1$, so the inclusion is strict. Therefore, by Lemma 4.1, we have

$$|P_2 - P_1|_{\mathbb{Z}} = \dim\big(\langle f_1\rangle \cap \mathcal{L}(P_2)\big) < \dim \mathrm{Ker}(ev_S) = |P_2|_{\mathbb{Z}} - rk(ev_S),$$

and the first inequality follows.

Now if $P_2 = P_1 + R$ for some lattice polygon $R$ then applying Pick's formula,

$$|P_2|_{\mathbb{Z}} - |P_2 - P_1|_{\mathbb{Z}} - 1 = |P_1 + R|_{\mathbb{Z}} - |R|_{\mathbb{Z}} - 1 = V(P_1, R) + V(P_1) + \frac{1}{2}|\partial P_1|_{\mathbb{Z}} - 1.$$

By the linearity of the mixed volume $V(P_1, P_2) = V(P_1, P_1 + R) = 2V(P_1) + V(P_1, R)$, so we get

$$|P_2|_{\mathbb{Z}} - |P_2 - P_1|_{\mathbb{Z}} - 1 = V(P_1, P_2) - \left(V(P_1) - \frac{1}{2}|\partial P_1|_{\mathbb{Z}} + 1\right).$$

By Pick's formula again, the expression in the parentheses on the right is $|P_1^\circ|_{\mathbb{Z}}$. $\square$

Below is the algorithm we use to produce examples of toric complete intersections $S$. The input is lattice polygons $P_1$, $P_2$, and $Q$ if we wish $S$ to be in $Q$-generic position. The output is $S$ and the polynomials $f_1$, $f_2$.

**Algorithm.**

1. Choose a random absolutely irreducible Laurent polynomial $f_1$ whose Newton polytope is $P_1$.
2. Find the $\mathbb{K}$-rational points of $f_1 = 0$.
3. Choose a subset $S$ of $n = V(P_1, P_2)$ of the points in Step 2 in $Q$-generic position.
4. Check whether the rank of the evaluation map $ev_S : \mathcal{L}(P_2) \to \mathbb{K}^n$ satisfies the inequality in Proposition 5.1.
5. If yes, obtain $f_2$ with Newton polytope $P_2$ with coefficients from the matrix of the kernel of $ev_S$, stop. If no, go back to Step 3 (or Step 1).

A variation of this algorithm is to loop over all irreducible polynomials in Step 1 until a toric complete intersection is found. For $q = 16$ this is still feasible. This is why in most of our examples below $f_1$ does not look "random". In Step 3 we either run through all subsets of size $n$ or sample $10^5$ random subsets of size $n$, whichever is less, before we go back to Step 1.

Now we turn to examples. Our first two examples demonstrate the construction in Theorem 4.9, while the others come from polygons classified in Proposition 4.6. The

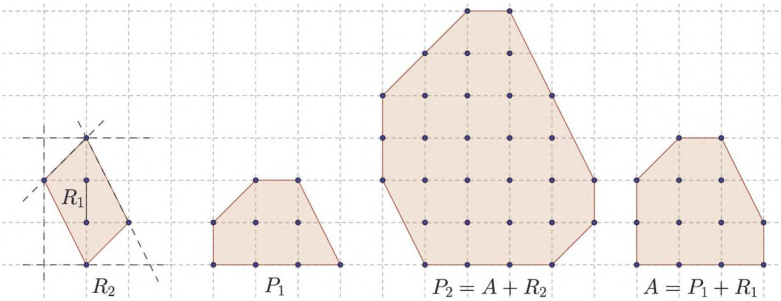


**Fig. 3.** Example of construction from Theorem 4.9.

first example is written in full detail, the reader may easily reconstruct details in the subsequent examples in a similar manner.

**Example 5.2.** Our first example illustrates the construction of $P_1$, $P_2$ and $A$ in Theorem 4.9. We choose $R_1$ to be the vertical unit segment and $R_2$ a parallelogram "around it", as in Fig. 3. We put $A = P_1 + R_1$ and $P_2 = A + R_2$.

Geometrically, $l_{R_2}(v) = l_{R_1}(v) + 1$, for all $v \in \mathrm{Fan}(P_1)$, means the following. Draw lines parallel to the sides of $P_1$ which are lattice distance one from $R_1$. We obtain, strictly speaking, a rational polygon (presented by dotted lines in Fig. 3). Then the above condition means that $R_2$ is inscribed in this rational polygon.

Let $\mathbb{K} = \mathbb{F}_{16}$ with a primitive generator $t$. Consider the following system with Newton polygons $P_1$, $P_2$.

$$\begin{aligned}
f_1 &= x^3 + x^2y^2 + t^7x^2y + tx^2 + xy^2 + x + y + 1 = 0,\\
f_2 &= t^7x^5y^2 + t^{11}x^5y + t^{11}x^4y^2 + t^9x^4y + t^7x^4 + tx^3y^6 + t^5x^3y + t^8x^3 + x^2y^6\\
&\quad + t^5x^2y^3 + t^7x^2y^2 + t^5x^2y + t^{12}xy^5 + xy^4 + t^5xy^3 + xy^2 + t^{11}xy + t^{13}x\\
&\quad + ty^4 + t^4y^3 + t^9y^2 = 0.
\end{aligned}$$

The solution set $S$ consists of $n = V(P_1, P_2) = 22$ points in $(\mathbb{F}_{16}^*)^2$ and is a toric complete intersection.

$$\begin{aligned}
S = \{&(1,t^{10}), (t,t^{10}), (t^3,1), (t^3,t), (t^4,t^{10}), (t^5,t^7), (t^5,t^{11}), (t^6,t), (t^6,t^{13}), (t^7,t^2),\\
&(t^7,t^7), (t^8,t^2), (t^8,t^{12}), (t^9,t^{11}), (t^9,t^{13}), (t^{10},t), (t^{10},t^6), (t^{12},t^7), (t^{12},t^8),\\
&(t^{13},t^{11}), (t^{13},t^{14}), (t^{14},t^4)\}.
\end{aligned}$$

By Proposition 4.3, $\dim(\mathcal{C}_{S,A}) = |A|_{\mathbb{Z}} - |R_1|_{\mathbb{Z}} = 13 - 2 = 11$ which, as predicted by Theorem 4.9, is exactly half the length of the code. By Theorem 4.5, $\mathcal{C}_{S,A}$ is an isodual code. According to MAGMA, its parameters are $[22, 11, 10]$. To find an equivalent self-dual code, first compute the vector $y$ of local residues:

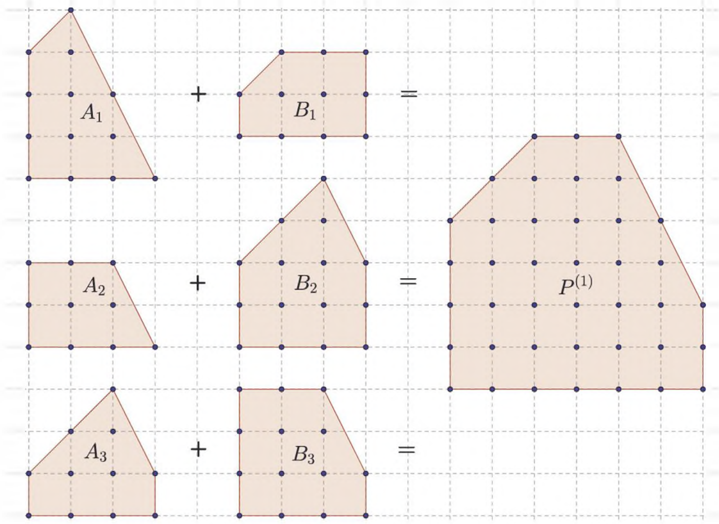


**Fig. 4.** Several Minkowski decompositions of $P^{(1)}$.

$$
\begin{aligned}
y &= \left(\frac{1}{J_f^{\mathbb{T}}(p_1)}, \dots, \frac{1}{J_f^{\mathbb{T}}(p_{22})}\right) \\
&= \left(t^9, t^2, t, t^8, t^{11}, 1, t^{11}, 1, t, t^9, t^{10}, t^{14}, t^6, t^{10}, t^{12}, t^3, t^9, t^9, t^7, t^{11}, t^9, t^6\right).
\end{aligned}
$$

This determines the vector $x$ such that $x_i^2 = y_i$ for $1 \le i \le 22$:

$$
x = \left(t^{12}, t, t^8, t^4, t^{13}, 1, t^{13}, 1, t^8, t^{12}, t^5, t^7, t^3, t^5, t^6, t^9, t^{12}, t^{12}, t^{11}, t^{13}, t^{12}, t^3\right).
$$

Finally, the code $x\mathcal{C}_{S,A}$ is a self-dual code with parameters $[22, 11, 10]$ over $\mathbb{F}_{16}$.

Next we look at some $y$-dual codes. Let $P^{(1)}$ denote the convex hull of the interior points of $P = P_1 + P_2$. Then we can decompose $P^{(1)}$ into Minkowski sum of two lattice polygons in several ways. They are depicted in Fig. 4. (Of course, there is also $2A = P^{(1)}$, which we do not include in the figure.)

An easy application of Proposition 4.3 shows that the codes $\mathcal{C}_{S,A_i}$ and $\mathcal{C}_{S,B_i}$ have complementary dimensions. Therefore, by Theorem 3.3, they are $y$-dual codes. We list their parameters in a table below.

| Polygons | Parameters | Properties of codes |
|---|---|---|
| $A_1$ | $[22, 12, 9]$ | |
| $B_1$ | $[22, 10, 11]$ | $y$-dual of $\mathcal{C}_{S,A_1}$ |
| $A_2$ | $[22, 9, 12]$ | |
| $B_2$ | $[22, 13, 8]$ | $y$-dual of $\mathcal{C}_{S,A_2}$ |
| $A_3$ | $[22, 10, 11]$ | |
| $B_3$ | $[22, 12, 9]$ | $y$-dual of $\mathcal{C}_{S,A_3}$ |
| $A$ | $[22, 11, 10]$ | isodual code |

**Example 5.3.** In our next example we consider rectangular boxes $P_1 = [0,3] \times [0,2]$ and $P_2 = [0,7] \times [0,4]$. This is a particular case of Theorem 4.9. First we choose a system with these Newton polygons which defines a toric complete intersection $S$ over $\mathbb{F}_{16}$ of size $V(P_1, P_2) = 26$:

$$\begin{aligned} f_1 &= x^3y^2 + t^4x^3y + x^3 + t^5x^2y^2 + t^2x^2y + x^2 + t^{11}xy^2 + txy + x + y^2 + y + 1 = 0, \\ f_2 &= x^7y^4 + t^{10}x^7y + t^2x^7 + t^{12}x^6y + t^8x^6 + t^{10}x^5y + t^3x^5 + t^{13}x^4y + t^{13}x^4 + t^9x^3y \\ &\quad + t^{13}x^3 + t^{11}x^2y^3 + t^8x^2y^2 + t^{12}x^2y + t^{14}x^2 + xy^4 + t^6xy^3 + t^3xy^2 + t^{12}x + t^6y^4 \\ &\quad + t^9y^2 + t^{13}y + t^5 = 0. \end{aligned}$$

Let $P^{(1)}$ be the convex hull of the interior lattice points of $P$, shifted to the origin, i.e. $P^{(1)} = [0,8] \times [0,4]$. Then, a shift of $A = [0,4] \times [0,2]$ defines an isodual code. We also try different subsets $A$ and $B$ such that $A + B = P^{(1)}$. Note that $A + B = P^{(1)}$ does not guarantee that $\mathcal{C}_{S,A}$ and $\mathcal{C}_{S,B}$ are $y$-dual since their dimensions might not be complementary. For example, if $A = [0,3] \times [0,4]$ and $B = [0,5] \times \{0\}$ then $A+B = P^{(1)}$. However, $\dim \mathcal{C}_{S,A} = 20 - 3 = 17$ and $\dim \mathcal{C}_{S,B} = 6$, by Proposition 4.3. In the table below all the codes have best known parameters as confirmed in [13].

| Polytopes | Parameters | Properties of codes |
|---|---|---|
| (a) $A = [0,3] \times [0,3]$ | $[26, 14, 11]$ | |
| $B = [0,5] \times [0,1]$ | $[26, 12, 13]$ | $y$-dual of $\mathcal{C}_{S,A}$ |
| (b) $A = [0,4] \times [0,2]$ | $[26, 13, 12]$ | isodual code |
| (c) $A = [0,2] \times [0,2]$ | $[26, 9, 16]$ | |
| $B = [0,6] \times [0,2]$ | $[26, 17, 8]$ | $y$-dual of $\mathcal{C}_{S,A}$ |
| (d) $A = [0,3] \times [0,1]$ | $[26, 8, 17]$ | |
| $B = [0,2] \times [0,4]$ | $[26, 18, 7]$ | $y$-dual of $\mathcal{C}_{S,A}$ |

Next we construct dual and isodual toric complete intersection codes using polygons classified in Proposition 4.6.

**Example 5.4.** Let $P_1 = 2Q_1$ and $P_2 = 3Q_1$ where $Q_1$ is the first Fano polygon as in Fig. 1 and consider the following system with these Newton polygons.

$$\begin{aligned} f_1 &= x^4y^4 + t^5x^3y^2 + t^{10}x^2y^3 + x^2y^2 + x^2y + x^2 + xy^2 + xy + y^2 = 0, \\ f_2 &= x^6y^6 + x^4y^4 + x^4y^3 + x^3y^4 + x^3y^3 + t^5x^3y^2 + t^5x^3y + t^5x^3 \\ &\quad + t^{10}x^2y^3 + t^{10}xy^3 + t^{10}y^3 = 0. \end{aligned}$$

Its solutions set $S$ is a toric complete intersection of size $n = 2V(Q_1) \cdot 6 = 18$. By Proposition 4.6, the code $\mathcal{C}_{S,A}$ with $A = 2Q_1$ is isodual. Now, notice that $Q_1$ and $3Q_1$ satisfy $Q + 3Q \subseteq P^{\circ}$. In fact, the corresponding codes have complementary dimensions. Indeed, $\dim(\mathcal{C}_{S,Q_1}) = |Q_1|_{\mathbb{Z}} = 4$, clearly. As for $\dim(\mathcal{C}_{S,3Q_1})$, Proposition 4.3 is not applicable since $V(A, P_2) < V(P_1, P_2)$ fails. But it's clear here that the kernel of the evaluation map $ev_S : \mathcal{L}(3Q_1) \to \mathbb{F}_{16}^{18}$ has one more basis element, namely, $P_2$ itself. Therefore, $\dim(\mathcal{C}_{S,3Q_1}) = |3Q_1|_{\mathbb{Z}} - |Q_1|_{\mathbb{Z}} - 1 = 19 - 4 - 1 = 14$. This justifies that $\mathcal{C}_{S,Q_1}$ and $\mathcal{C}_{S,3Q_1}$ are $y$-dual. We record the corresponding parameters below.

| Polygons | Parameters | Properties of codes |
|---|---|---|
| (a) $A = Q_1$ | $[18, 4, 13]$ | |
| $B = 3Q_1$ | $[18, 14, 4]$ | $y$-dual of $\mathcal{C}_{S,Q_1}$ |
| (b) $A = 2Q_1$ | $[18, 9, 8]$ | isodual code |

Similarly, we obtain toric complete intersection for $Q_2$, $Q_3$, and $Q_4$ (the polygons in the first row of Fig. 1). For $P_1 = 2Q_2$ and $P_2 = 3Q_2$ we take

$$\begin{aligned} f_1 &= x^4 + x^3y^2 + x^3y + x^3 + x^2y^4 + t^8x^2y^3 + t^3x^2y^2 + x^2y + x^2 + xy^2 \\ &\quad + xy + x + 1 = 0, \\ f_2 &= t^8x^6 + t^{13}x^5y^2 + t^{13}x^5y + x^5 + x^4y^2 + t^2x^4y + t^7x^4 + x^3y^6 + t^{13}x^3y^3 + t^{14}x^3y^2 \\ &\quad + t^{13}x^3y + t^2x^3 + x^2y^2 + t^2x^2y + t^7x^2 + t^{13}xy^2 + t^{13}xy + x + t^8 = 0. \end{aligned}$$

For $P_1 = 2Q_3$ and $P_2 = 3Q_3$ we take

$$\begin{aligned} f_1 &= x^4y^2 + t^{11}x^3y^3 + x^3y^2 + x^3y + x^2y^4 + t^9x^2y^3 + x^2y^2 + x^2y + x^2 + t^{11}xy^3 \\ &\quad + xy^2 + xy + y^2 = 0, \\ f_2 &= t^5x^6y^3 + t^4x^5y^3 + t^2x^5y^2 + t^6x^4y^4 + x^4y^3 + t^{12}x^4y^2 + t^7x^4y + x^3y^6 + t^{14}x^3y^3 \\ &\quad + t^7x^3y^2 + tx^3y + t^{13}x^3 + t^6x^2y^4 + x^2y^3 + t^{12}x^2y^2 + t^7x^2y + t^4xy^3 + t^2xy^2 \\ &\quad + t^5y^3 = 0. \end{aligned}$$

Finally, for $P_1 = 2Q_4$ and $P_2 = 3Q_4$ we take

$$\begin{aligned} f_1 &= x^4y^4 + t^8x^3y^3 + t^{13}x^3y^2 + t^{13}x^2y^3 + t^2x^2y^2 + x^2y + x^2 + xy^2 + xy + x \\ &\quad + y^2 + y + 1 = 0, \\ f_2 &= x^6y^6 + t^7x^4y^2 + t^{14}x^3y^3 + t^2x^3y^2 + t^7x^3y + tx^3 + t^7x^2y^4 + t^2x^2y^3 + t^8x^2y^2 \\ &\quad + t^2x^2y + t^3x^2 + t^7xy^3 + t^2xy^2 + t^{14}xy + t^4x + ty^3 + t^3y^2 + t^4y + t^{10} = 0. \end{aligned}$$

The corresponding codes happen to have the same parameters for each $i = 2, 3, 4$ and are listed below.

| Polygons | Parameters | Properties of codes |
|---|---|---|
| (a) $A = Q_i$ | $[24, 5, 16]$ | |
| $B = 3Q_i$ | $[24, 19, 4]$ | $y$-dual of $\mathcal{C}_{S,Q_i}$ |
| (b) $A = 2Q_i$ | $[24, 12, 8]$ | isodual code |

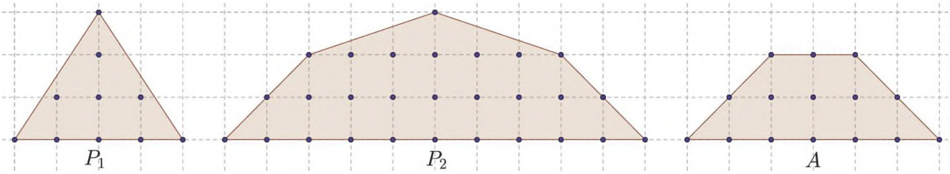


**Fig. 5.** Here $P_1$ is not a Minkowski summand of $A$.

**Example 5.5.** Our final example does not use the geometric construction of Theorem 4.9. The polygons are depicted in Fig. 5.

Clearly, $P_1$ is not a Minkowski summand of $A$, but one can check that the equality of the support functions in Remark 4.10 still holds. The toric complete intersection $S$ is defined over $\mathbb{F}_{16}$ by

$$\begin{aligned} f_1 &= x^4 + x^2y^3 + x^2 + x + 1 = 0, \\ f_2 &= t^8x^{10} + tx^9y + t^3x^9 + t^9x^8y^2 + t^{10}x^8y + t^{12}x^8 + t^4x^7y^2 + t^{14}x^7y + t^4x^7 + t^7x^6y^2 \\ &\quad + t^6x^6y + tx^6 + t^9x^5y^3 + tx^5y + t^4x^5 + t^7x^4y^2 + t^7x^4y + t^6x^4 + t^{10}x^3y^2 + x^3y \\ &\quad + t^6x^3 + x^2y^2 + x^2y + t^6x^2 + xy + t^{10}x + 1 = 0. \end{aligned}$$

The corresponding isodual code $\mathcal{C}_{S,A}$ has parameters $[30, 15, 12]$.

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