Deltohelicoidal Surfaces

Erhan Güler

Bartın University, Faculty of Science, Department of Mathematics

eguler@bartin.edu.tr

Abstract

The surface theory has been working by many mathematicians, and also geometers for hundreds of years. We meet nice papers and books for the theory in the literature. In this paper, we consider deltohelicoidal surface in Euclidean 3-space $\mathbb{E}^3$*. We show some basic notions of three dimensional Euclidean geometry. Moreover, constructing a helicoidal surface, we define deltohelicoidal surface, and compute its Gauss map, the Gaussian curvature and the mean curvature. Finally, we reveal some results of the Gaussian curvature and the mean curvature of the deltohelicoidal surface in the three dimensional Euclidean space* $\mathbb{E}^3$.

Key Words: *Euclidean 3-Space, Deltohelicoidal Surface, Gauss Map, Gaussian Curvature, Mean Curvature*

1. Introduction

The surface theory has been worked by many geometers. We meet nice books for the theory in the literature, such as Eisenhart [1], Forsyth [2], Gray et al. [3], Hacısalihoğlu [4,5], Nitsche [6], Spivak [7].

In this paper, we consider the deltohelicoidal surface in Euclidean 3-space. In Section 2, we show some basic notions of three dimensional Euclidean geometry. We define helicoidal surface in Section 3. We give deltohelicoidal surface, and compute its Gaussian curvature and the mean curvature in the last section.

2. Preliminaries

We consider a vector $(\mathrm{a},\mathrm{b},\mathrm{c})$ with its transpose $(\mathrm{a},\mathrm{b},\mathrm{c})^t$, identially, in the rest of this work. We introduce the first and second fundamental forms, matrix of the shape operator **S**, Gaussian curvature K, and the mean curvature H of surface $\mathbf{M}=\mathbf{M}(u,v)$ in the three dimensional Euclidean space $\mathbb{E}^3$.

Let **M** be an isometric immersion of surface M^2 in $\mathbb{E}^3$. The vector product of $\vec{x}=(x_1,x_2,x_3)$ and $\vec{y}=(y_1,y_2,y_3)$ on $\mathbb{E}^3$ is defined by

$$\vec{x}\times\vec{y}=\det\begin{pmatrix} e_1 & e_2 & e_3 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{pmatrix}.$$

For a surface **M** in $\mathbb{E}^3$, we have following results

$$\det I=\det\begin{pmatrix} E & F \\ F & G \end{pmatrix}=EG-F^2,$$

and

$$\det II=\det\begin{pmatrix} L & M \\ M & N \end{pmatrix}=LN-M^2,$$

where

$$E = \mathbf{M}_u \cdot \mathbf{M}_u,\ F = \mathbf{M}_u \cdot \mathbf{M}_v,\ G = \mathbf{M}_v \cdot \mathbf{M}_v,$$
$$L = \mathbf{M}_{uu} \cdot e,\quad M = \mathbf{M}_{uv} \cdot e,\quad N = \mathbf{M}_{vv} \cdot e,$$

" $\cdot$ " is Euclidean inner product, e is the Gauss map

$$e = \frac{\mathbf{M}_u \times \mathbf{M}_v}{\|\mathbf{M}_u \times \mathbf{M}_v\|}.$$

We compute

$$I^{-1}.II,$$

and then it gives shape operator matrix **S** as follows

$$\mathbf{S} = \frac{1}{\det I}\begin{pmatrix} GL-FM & GM-FN \\ EM-FL & EN-FM \end{pmatrix}.$$

Therefore, we get the following the Gaussian and the mean curvature formulas, respectively,

$$K = \det(\mathbf{S}) = \frac{LN-M^2}{EG-F^2},$$

and

$$H = \frac{1}{2}tr(\mathbf{S}) = \frac{EN+GL-2FM}{2(EG-F^2)}.$$

A surface **M** is flat if $K = 0$, and it is minimal if $H = 0$, identically.

3. Helicoidal Surface

Next, we give the rotational surface and helicoidal surface in $\mathbb{E}^3$. For an open interval $I \subset \mathbb{R}$, let $\gamma : I \longrightarrow \Pi$ be a curve, and let ℓ be a line in Π.

We define a rotational surface in $\mathbb{E}^3$ as a surface rotating a profile curve γ around an axis ℓ. While a profile curve rotates around the ℓ, it simultaneously displaces parallel lines orthogonal to the ℓ, so that the speed of displacement is proportional to the speed of rotation. Final surface is called the *helicoidal surface* with axis ℓ, and pitch $p \in \mathbb{R}^+$.

We assume that ℓ is the line spanned by the vector $(0,0,1)^t$. The orthogonal matrix which fixes the above vector is

$$Z(v) = \begin{pmatrix} \cos v & -\sin v & 0 \\ \sin v & \cos v & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

where $v \in \mathbb{R}$. The matrix Z can be found by solving the following equations, simultaneously,

$$Z\ell = \ell,\quad Z^t Z = ZZ^t = I_3,\ detZ = 1.$$

When the axis of rotation is ℓ, there is an Euclidean transformation by which the axis is ℓ transformed to the x_3-axis of $\mathbb{E}^3$. Profile curve is

$$\gamma(u) = (f(u), 0, h(u)),$$

where $f(u), h(u) : I \subset \mathbb{R} \longrightarrow \mathbb{R}$ are differentiable functions for all $u \in I$.

So, a helicoidal surface spanned by the vector $(0,0,1)$ with pitch p, is as follows

$$\mathbf{H}(u,v) = Z(u)\gamma(u) + pv\ell^t,$$

where $u \in I$, $v \in [0, 2\pi)$.

Clearly, we write helicoidal surface as follows

$$\mathbf{H}(u,v) = \begin{pmatrix} f(u)\cos v \\ f(u)\sin v \\ h(u) + pv \end{pmatrix}.$$

When $p = 0$, helicoidal surface is transform to a *rotational surface*.

4. Deltohelicoidal Surface

In $\mathbb{E}^3$, a *deltohelicoidal surface* (see Figure 1) which is spanned by the vector $(0,0,1)$ with pitch $b \in \mathbb{R}^+$, (see Figure 2 for $b = 0$) is defined by as follows:

$$\mathfrak{D}(u,v) = \begin{pmatrix} a(\cos(2u - v) + 2\cos(u + v)) \\ -a(\sin(2u - v) - 2\sin(u + v)) \\ \varphi(u) + bv \end{pmatrix},$$

where parametrization of the profile space curve is given by

$$\gamma(u) = (2a\cos u + a\cos 2u, 2a\sin u - a\sin 2u, \varphi(u)),$$

$\varphi : I \subset \mathbb{R} \longrightarrow \mathbb{R}$ is differentiable function for all $u \in I$, $a, b \in \mathbb{R}$, and $v \in [0, 2\pi)$.

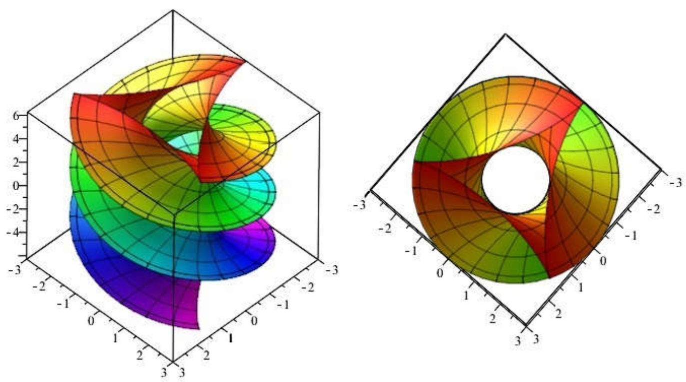

Figure 1. Left: Deltohelicoidal surface, **Right**: Its top view

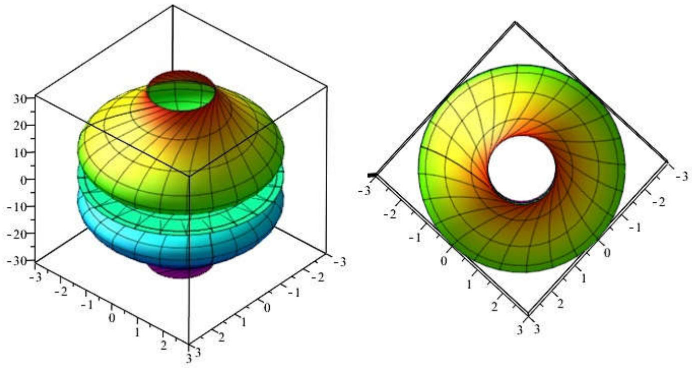

Figure 2. Left: Deltorotational surface, **Right**: Its top view

Using the first differentials of $\mathfrak{D}(u, v)$ with respect to u and v, we obtain the first quantities as follows

$$E = -8a^2\lambda + \varphi'^2,$$
$$F = -2a^2\lambda + b\varphi',$$
$$G = a^2(4\beta + 1) + b^2,$$

and then, we get

$$\det I = [(4\beta + 1)\varphi'^2 + 4b\lambda\varphi' + (9\beta + 2b^2)]a^2,$$

where

$$\lambda = (\cos u - 1)(2\cos u + 1)^2,$$

$$\beta = (\cos u + 1)(2\cos u - 1)^2.$$

The Gauss map of surface is as follows

$$e_{\mathfrak{D}} = \frac{1}{\sqrt{\det I}}\begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix},$$

where

$$\begin{aligned} e_1 &= 2b(1 + 2\cos u)(\cos u\cos v + \sin u\sin v - \cos v) \\ &\quad + \left\{ \begin{array}{c} -2\cos^2 u\cos v - 2(\sin u\sin v - \cos v)\cos u \\ +2\sin u\sin v + \cos v \end{array} \right\}\varphi', \end{aligned}$$

$$\begin{aligned} e_2 &= -2b(1 + 2\cos u)(\cos u\sin v - \sin u\cos v - \sin v) \\ &\quad + \left\{ \begin{array}{c} -2\cos^2 u\sin v + 2(\sin u\cos v - \sin v)\cos u \\ -2\sin u\cos v + \sin v \end{array} \right\}\varphi', \end{aligned}$$

$$e_3 = -6a(2\cos u - 1)(1 + 2\cos u).$$

Finally, the mean curvature of the deltohelicoidal surface is as follows

$$H = \frac{\eta_1 \varphi'' + \eta_2 \varphi'^3 + \eta_3 \varphi'^2 + \eta_4 \varphi' + \eta_5}{2(\det I)^{3/2}},$$

where

$$\begin{aligned}
\eta_1 &= 6(2\cos u - 1)(2\cos u + 1) \\
&\quad \left(a^2 \left(4(\cos u + 1)(2\cos u - 1)^2 + 1\right) + b^2\right) \sin u, \\
\eta_2 &= -a\left(4(\cos u + 1)(2\cos u - 1)^2 + 1\right), \\
\eta_3 &= -6ab(\cos u - 1)(2\cos u + 1)^2, \\
\eta_4 &= 2a^2(4\cos^2 u - 3) \\
&\quad .[20(4\cos^2 u - 3)(a-1)\cos u - 4a - 41]\cos u \\
&\quad - 2b^2[-4(8a-5)\cos^2 u + 24a^3 - 15]\cos u \\
&\quad - 8[(2a+1)b^2 + (4a+5)a^2],
\end{aligned}$$

$$\begin{aligned}
\eta_5 &= -4b(\cos u - 1)(2\cos u + 1)^2 \\
&\quad .[4a^2(4\cos^2 u - 3)(a-1)\cos u - 5a^2 - 4a^3 - b^2],
\end{aligned}$$

and the Gaussian curvature of the deltohelicoidal surface is as follows

$$K = \frac{\theta_1 \varphi' \varphi'' + \theta_2 \varphi'' + \theta_3 \varphi'^2 + \theta_4 \varphi' + \theta_5}{(\det I)^2},$$

where

$$\begin{aligned}
\theta_1 &= -6(2\cos u + 1)(2\cos u - 1)(4\beta + 1)\sin u, \\
\theta_2 &= -12b\lambda(2\cos u - 1)(2\cos u + 1)\sin u, \\
\theta_3 &= 18(2\beta - 1)(2\cos u + 1)\cos u, \\
\theta_4 &= 36b\lambda^2, \\
\theta_5 &= -72b^2\lambda^2.
\end{aligned}$$

Corollary 1. *Let* $\mathfrak{D} : \mathrm{M}^2 \longrightarrow \mathbb{E}^3$ *be an immersion given by* $\mathfrak{D}(\mathrm{u}, \mathrm{v})$. M^2 *is minimal iff*

$$\eta_1 \varphi'' + \eta_2 \varphi'^3 + \eta_3 \varphi'^2 + \eta_4 \varphi' + \eta_5 = 0.$$

Corollary 2. *Let* $\mathfrak{D} : \mathrm{M}^2 \longrightarrow \mathbb{E}^3$ *be an immersion given by* $\mathfrak{D}(\mathrm{u}, \mathrm{v})$. M^2 *is flat iff*

$$\theta_1 \varphi' \varphi'' + \theta_2 \varphi'' + \theta_3 \varphi'^2 + \theta_4 \varphi' + \theta_5 = 0.$$

Corollary 3. *Let* $\mathfrak{D}: \mathrm{M}^2 \longrightarrow \mathbb{E}^3$ *be an immersion given by* $\mathfrak{D}(\mathrm{u}, \mathrm{v})$. M^2 *has Weingarten relation as follows*

$$\begin{aligned} 0 = {} & a^3\left[(4\beta+1)\varphi'^2 + 4b\lambda\varphi' + (9\beta + 2b^2)\right]^{3/2} \\ & \cdot (\eta_1\varphi'' + \eta_2\varphi'^3 + \eta_3\varphi'^2 + \eta_4\varphi' + \eta_5)K \\ & - 2(\theta_1\varphi'\varphi'' + \theta_2\varphi'' + \theta_3\varphi'^2 + \theta_4\varphi' + \theta_5)H. \end{aligned}$$

References

[1] Eisenhart, L.P., *A Treatise on the Differential Geometry of Curves and Surfaces*. Dover Publications, N.Y. 1909.

[2] Forsyth, A.R., *Lectures on the Differential Geometry of Curves and Surfaces*, Cambridge Un. press, 2nd ed. 1920.

[3] Gray, A., Salamon, S. and Abbena, E., *Modern Differential Geometry of Curves and Surfaces with Mathematica*, Third ed. Chapman & Hall/CRC Press, Boca Raton, 2006.

[4] Hacısalihoğlu, H.H., *Diferensiyel Geometri I*. Ankara Ün., Ankara, 1982.

[5] Hacısalihoğlu, H.H., *2 ve 3 Boyutlu Uzaylarda Analitik Geometri*. Ertem Basım, Ankara, 2013.

[6] Nitsche, J.C.C., *Lectures on Minimal Surfaces. Vol. 1. Introduction, Fundamentals, Geometry and Basic Boundary Value Problems*. Cambridge Un. Press, Cambridge, 1989.

[7] Spivak, M., *A Comprehensive Introduction to Differential Geometry*, Vol. IV. Third edition. Publish or Perish, Inc., Houston, Texas, 1999.