Cardiohelicoidal Surfaces

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Abstract

Geometers have been working the curve theory and the surface theory for hundreds of years. We see good works for the theories in the literature. In this work, we introduce cardiohelicoidal surface in the three dimensional Euclidean space $\mathbb{E}^3$. We indicate basic notions of Euclidean geometry. Then, stating a helicoidal surface, we obtain cardiohelicoidal surface, and calculate its Gauss map, the Gaussian curvature and the mean curvature. In the end, we find some corollaries of the Gaussian curvature and the mean curvature of the cardiohelicoidal surface in $\mathbb{E}^3$.

Key Words: *Euclidean 3-Space, Cardiohelicoidal Surface, Gauss Map, Gaussian Curvature, Mean Curvature*

1. Introduction

In this work, we introduce the cardiohelicoidal surface in Euclidean 3-space $\mathbb{E}^3$. See some books Forsyth [1], Gray et al. [2], Hacısalihoğlu [3,4], Nitsche [5], Spivak [6] for cardioid curve and helicoidal surface.

We show some basic notions of three dimensional Euclidean geometry in this Section. We define helicoidal surface in Section 2. Finally, we give cardiohelicoidal surface, and compute its Gaussian curvature and the mean curvature in the last section.

Throughout the paper, we identify a vector $(\mathrm{a}, \mathrm{b}, \mathrm{c})$ with its transpose. We consider the first and second fundamental forms, matrix of the shape operator **S**, Gaussian curvature K, and the mean curvature H of surface $\mathbf{M}=\mathbf{M}(u, v)$ in Euclidean 3-space.

Let **M** be an isometric immersion of surface M^2 in $\mathbb{E}^3$. The inner product and the vector product of $\vec{x} = (x_1, x_2, x_3)$, $\vec{y} = (y_1, y_2, y_3)$ on $\mathbb{E}^3$ are defined by as follows, respectively,

$$\vec{x} \cdot \vec{y} = x_1 y_1 + x_2 y_2 + x_3 y_3$$

and

$$\vec{x} \times \vec{y} = (x_2 y_3 - x_3 y_2, -x_1 y_3 + x_3 y_1, x_1 y_2 - x_2 y_1).$$

For a surface **M** in the three dimensional Euclidean space, with the first and the second fundamental coefficients

$$E = \mathbf{M}_u \cdot \mathbf{M}_u,\ F = \mathbf{M}_u \cdot \mathbf{M}_v,\ G = \mathbf{M}_v \cdot \mathbf{M}_v,$$
$$L = \mathbf{M}_{uu} \cdot e,\quad M = \mathbf{M}_{uv} \cdot e,\quad N = \mathbf{M}_{vv} \cdot e,$$

we know

$$\det I = \det \begin{pmatrix} E & F \\ F & G \end{pmatrix} = EG - F^2,$$

and

$$\det II = \det\begin{pmatrix} L & M \\ M & N \end{pmatrix} = LN - M^2,$$

where

$$e = \frac{\mathbf{M}_u \times \mathbf{M}_v}{\|\mathbf{M}_u \times \mathbf{M}_v\|}.$$

is the Gauss map. Computing

$$I^{-1}.II,$$

we get following shape operator matrix

$$\mathbf{S} = \frac{1}{\det I}\begin{pmatrix} GL - FM & GM - FN \\ EM - FL & EN - FM \end{pmatrix}.$$

Hence, we have the following formulas of the Gaussian curvature and the mean curvature, respectively,

$$K = \det(\mathbf{S}) = \frac{LN - M^2}{EG - F^2},$$

and

$$H = \frac{1}{2}tr(\mathbf{S}) = \frac{EN + GL - 2FM}{2(EG - F^2)}.$$

A surface $\mathbf{M}$ is flat if $K = 0$, and it is minimal if $H = 0$, identically.

2. Helicoidal Surface

We define the rotational surface and helicoidal surface in $\mathbb{E}^3$. For an open interval $I \subset \mathbb{R}$, let $\gamma : I \longrightarrow \Pi$ be a curve in a plane Π, and let ℓ be a straight line in Π.

A rotational surface in $\mathbb{E}^3$ is defined as a surface rotating a curve γ around a line ℓ (these are called the *profile curve* and the *axis*, respectively). Suppose that when a profile curve γ rotates around the axis ℓ, it simultaneously displaces parallel lines orthogonal to the axis ℓ, so that the speed of displacement is proportional to the speed of rotation. Then, resulting surface is called the *helicoidal surface* with axis ℓ and pitch $b \in \mathbb{R}^+$.

We may suppose that ℓ is the line spanned by the vector $(0,0,1)^t$. The orthogonal matrix is as follows

$$O(v) = \begin{pmatrix} \cos v & -\sin v & 0 \\ \sin v & \cos v & 0 \\ 0 & 0 & 1 \end{pmatrix}, \ v \in \mathbb{R}.$$

The matrix O supplies following equations, simultaneously,

$$O\ell = \ell, \quad O^tO = OO^t = I_3, \ detO = 1.$$

When the axis of rotation is ℓ, there is an Euclidean transformation by which the axis is ℓ transformed to the x_3-axis of 3-space. The profile curve is given by as follows

$$\gamma(u) = (f(u), 0, h(u)).$$

Here $f(u), h(u) : I \subset \mathbb{R} \longrightarrow \mathbb{R}$ are differentiable functions for all $u \in I$.

Therefore, a helicoidal surface which is spanned by the vector $(0,0,1)$ with pitch b, is as follows

$$\mathbf{H}(u, v) = O(v)\gamma(u) + bv\ell^t,$$

where $u \in I$, $v \in [0, 2\pi)$.

More cleear form of the helicoidal surface is as follows

$$\mathbf{H}(u,v) = \big(f(u)\cos v,\ f(u)\sin v,\ h(u) + bv\big).$$

When $b = 0$, the surface is a *rotational surface*.

3. Cardiohelicoidal Surface

In $\mathbb{E}^3$, a *cardiohelicoidal surface* (see Figure 1) which is spanned by the vector $(0,0,1)$ with pitch $b \in \mathbb{R}^+$, (see Figure 2 for $b = 0$) is defined by as follows:

$$\mathfrak{C}(u,v) = \begin{pmatrix} a(1-\cos u)\cos u\cos v - a(1-\sin u)\sin u\sin v \\ a(1-\cos u)\cos u\sin v + a(1-\sin u)\sin u\cos v \\ \varphi(u) + bv \end{pmatrix},$$

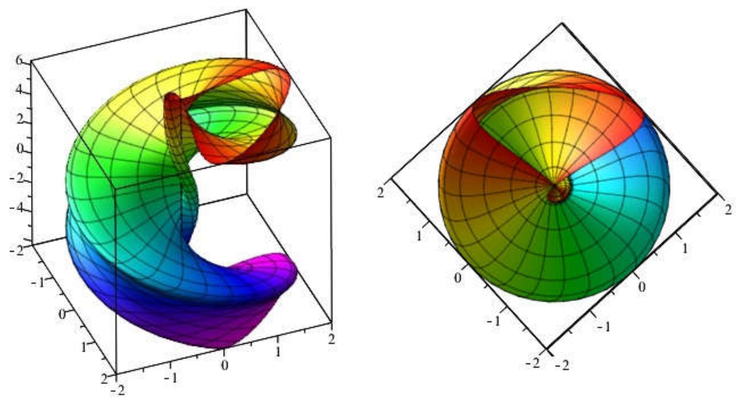

Figure 1. Left: Cardiohelicoidal surface, **Right**: Its top view

where profile space curve is given by

$$\gamma(u) = (a(1-\cos u)\cos u, a(1-\sin u)\sin u, \varphi(u)),$$

$\varphi : I \subset \mathbb{R} \longrightarrow \mathbb{R}$ is differentiable function for all $u \in I$, $a \in \mathbb{R}$, and $v \in [0, 2\pi)$.

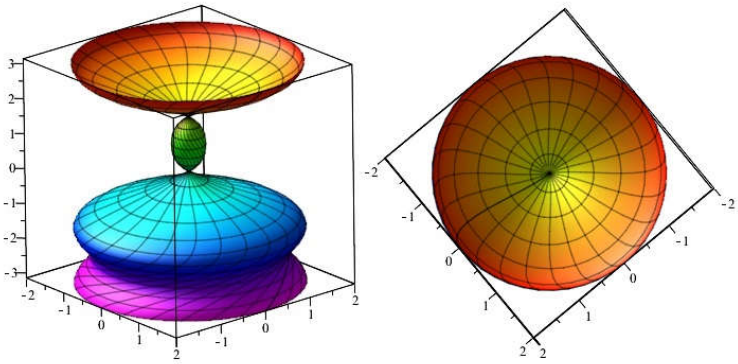

Figure 2. Left: Cardiorotational surface, **Right**: Its top view

Calculating the first differentials of $\mathfrak{C}(u,v)$ with respect to u and v, we have

$$\det I = A_1\varphi'^2 + A_2\varphi' + A_3,$$

where

$$A_1 = 2a^2(\cos u - 1)((\cos u + 1)((\cos u - 1)\cos u + \sin u) - 1),$$

$$A_2 = 2a^2 b(\cos u - 1)(-(\cos u + 1)\cos u + (\cos u - 1)\sin u + 1),$$

$$\begin{aligned} A_3 = {} & -a^4(\cos^2 u)(\cos u - 1)(\cos u + 1) \\ & \cdot\begin{bmatrix} 4(2\cos u - 1)(-2\cos u + 2\cos^2 u - 3)\cos u \\ +6((4\cos u - 3)\cos u - 2)\sin u + 13 \end{bmatrix} \\ & + 4b^2((\cos u - 1)(2\cos u - 1)(\cos u + 1) + \cos u \sin u)\cos u. \end{aligned}$$

The Gauss map of the cardiohelicoidal surface is as follows

$$e_{\mathfrak{C}} = \frac{1}{\sqrt{\det I}}\begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix},$$

where

$$\begin{aligned} e_1 = {} & [2(\sin v - \cos v)\sin u\cos u + (\cos u\cos v - \sin u\sin v)]b \\ & + [(1 - \sin u)\sin u\sin v + (\cos u - 1)\cos u\cos v]\varphi', \end{aligned}$$

$$\begin{aligned} e_2 = {} & [-2(\cos v + \sin v)\sin u\cos u + (\sin u\cos u + \sin v\cos u)]b \\ & + [(\sin u - 1)\sin u\cos v + (\cos u - 1)\cos u\sin v]\varphi', \end{aligned}$$

$$e_3 = a[(4\cos u + 1)(1 - \cos u) + 1 - 3\sin u]\sin u\cos u.$$

After long calculations, we obtain the Gaussian curvature of the cardiohelicoidal surface as follows

$$K = \frac{\delta_1 \varphi' \varphi'' + \delta_2 \varphi'' + \delta_3 \varphi'^2 + \delta_4 \varphi' + \delta_5}{(\det I)^2},$$

where $C := cosu, S := sin\ u$

$$\begin{aligned}
\delta_1 &= -2CS(C-1)(4C^5 - 3C^4 - 9C^3 - 4C^2 + 8C + 5) \\
&\quad + 2(7C^3 + C^2 - 8C - 5)(C-1)^2(C+1)C,
\end{aligned}$$

$$\begin{aligned}
\delta_2 &= b(2C^2 + 2C - 5)(2C^2 - 1)(C-1)CS \\
&\quad + b(2C^2 - 1)(C-1)^2(2C-5)(C+1)C,
\end{aligned}$$

$$\begin{aligned}
\delta_3 &= (16C^6 + 6C^5 - 36C^4 - 52C^3 + C^2 + 32C + 10)(C-1)^2 \\
&\quad + 2(13C^5 + 4C^4 - 22C^3 - 13C^2 + 11C + 5)(C-1)S,
\end{aligned}$$

$$\begin{aligned}
\delta_4 &= b[2(10C^5 - 33C^4 + 17C^3 + 33C^2 - 33C + 1)CS \\
&\quad - (20C^6 - 94C^4 + 71C^3 + 62C^2 - 71C + 2)C + 10(S-1)],
\end{aligned}$$

$$\begin{aligned}
\delta_5 &= b^2[-4C^5(16C^3 - 16C^2 - 32C + 33) - (3S-2) \\
&\quad - CS(64C^5 - 36C^4 - 60C^3 + 36C^2 - 9C - 6) \\
&\quad - C(30C^3 - 63C^2 + 34C - 2)].
\end{aligned}$$

And also we obtain the mean curvature as follows

$$H = \frac{\lambda_1 \varphi'' + \lambda_2 \varphi'^3 + \lambda_3 \varphi'^2 + \lambda_4 \varphi' + \lambda_5}{2(\det I)^{3/2}},$$

where $C := cosu, S := sin\ u$

$$\begin{aligned}
\lambda_1 &= 2a^2[(-8C + C^2 + 7C^3 - 5)(C^2 - 1) \\
&\quad - (4C^5 - 3C^4 - 9C^3 - 4C^2 + 8C + 5)]C(C-1) \\
&\quad + b^2[3(C-1)(C+1) - (4C^2 - 3C - 2)S]C,
\end{aligned}$$

$$\lambda_2 = 2(C-1)(-C + S + CS + C^3 - 1),$$

$$\lambda_3 = -3Cb(C^2 - 2) + 3b(S-1) + 3CSb(C-2),$$

$$\begin{aligned}
\lambda_4 &= a^2[-2(20C + 5C^2 - 20C^3 + 5C^4 + C^6 - 6)C - 10(S-1) \\
&\quad + 2(18C - 8C^3 + C^5 - 6)CS] + b^2[(-6C + 8C^2 + 3C^3 - 8C^4 + 5) \\
&\quad - 3(C^2 + 1)S],
\end{aligned}$$

$$\begin{aligned}
\lambda_5 &= a^2b[-(4C^6 + 4C^5 - 24C^4 + 10C^3 + 23C^2 - 11C - 5) \\
&\quad + (14C^4 - 8C^3 - 4C^2 + 6C + 5)(C-1)S](C-1) \\
&\quad + b^3(2S - 2C^2S + 2C^3 - 1).
\end{aligned}$$

Corollary 1. *We assume* $\mathfrak{C}: \mathrm{M}^2 \longrightarrow \mathbb{E}^3$ *be an immersion given by* $\mathfrak{C}(u,v)$. *So,* M^2 *is minimal iff*

$$\lambda_1\varphi'' + \lambda_2\varphi'^3 + \lambda_3\varphi'^2 + \lambda_4\varphi' + \lambda_5 = 0.$$

Corollary 2. *We assume* $\mathfrak{C}: \mathrm{M}^2 \longrightarrow \mathbb{E}^3$ *be an immersion given by* $\mathfrak{C}(u,v)$. *Hence,* M^2 *is flat iff*

$$\delta_1\varphi'\varphi'' + \delta_2\varphi'' + \delta_3\varphi'^2 + \delta_4\varphi' + \delta_5 = 0.$$

Corollary 3. *We assume* $\mathfrak{C}: \mathrm{M}^2 \longrightarrow \mathbb{E}^3$ *be an immersion given by* $\mathfrak{C}(u,v)$. *Therefore,* M^2 *has following Weingarten relation*

$$a^3(\det I)^{3/2}\,\eta\,K - 2\theta\,H = 0,$$

where η and θ are the numerator functions of H and K, respectively.

References

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