The Principal Curvatures and the Third Fundamental Form of Dini-Type Helicoidal Hypersurface in 4-Space

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The sole author designed, analyzed, interpreted and prepared the manuscript.

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Abstract
We consider the principal curvatures and the third fundamental form of Dini-type helicoidal hypersurface \( D(u, v, w) \) in the four dimensional Euclidean space \( \mathbb{E}^4 \). We find the Gauss map \( e \) of helicoidal hypersurface in \( \mathbb{E}^4 \). We obtain characteristic polynomial of shape operator matrix \( S \). Then, we compute principal curvatures \( k_i=1, 2, 3 \), and the third fundamental form matrix \( III \) of \( D \).

Keywords: Four dimensional; Dini-type helicoidal hypersurface; Gauss map; principal curvatures; the third fundamental form.

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1 Introduction
Theory of surfaces and hypersurfaces have been studied by many geometers for years such as [1 – 26].

In the rest of this paper, we identify a vector \((a, b, c, d)\) with its transpose \((a, b, c, d)^t\). Let \( \gamma : I \rightarrow \Pi \) be a curve in a plane \( \Pi \) in \( \mathbb{E}^4 \), and let \( \ell \) be a straight line in \( \Pi \) for an open interval \( I \subset \mathbb{R} \). A

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rotational hypersurface in $\mathbb{E}^4$ is defined as a hypersurface rotating a curve $\gamma$ (i.e. profile curve) around a line (i.e. axis) $\ell$. Suppose that when a profile curve $\gamma$ rotates around the axis $\ell$, it simultaneously displaces parallel lines orthogonal to the axis $\ell$, so that the speed of displacement is proportional to the speed of rotation. Resulting hypersurface is called the helicoidal hypersurface with axis $\ell$ and pitches $a, b \in \mathbb{R}\setminus\{0\}$.

Let $\ell$ be a line spanned by the vector $(0, 0, 1)^T$. The orthogonal matrix

$$M(v, w) = \begin{pmatrix} \cos v \cos w & -\sin v & -\cos v \sin w & 0 \\ \sin v \cos w & \cos v & \sin v \sin w & 0 \\ \sin w & 0 & \cos w & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad v, w \in \mathbb{R},$$

fixes the vector $\ell$. The matrix $M$ can be found by solving the following equations simultaneously; $M\ell = \ell$, $M^T M = M M^T = I_4$, det $M = 1$. When the axis of rotation is $\ell$, there is an Euclidean transformation by which the axis is $\ell$ transformed to the $x_4$-axis of $\mathbb{E}^4$. Parametrization of the profile curve is given by $\gamma(u) = (u, 0, 0, \varphi(u))$, where $\varphi(u) : I \subset \mathbb{R} \rightarrow \mathbb{R}$ is a differentiable function for all $u \in I$. So, the helicoidal hypersurface is given by $H(u, v, w) = M(u, v, w) = M(u, v, w) \gamma(u) + (av + bw) \ell$. Here, $u, v, w \in [0, 2\pi]$, $a, b \in \mathbb{R}\setminus\{0\}$. Clearly, we write helicoidal hypersurface as follows

$$H(u, v, w) = (u \cos v \cos w, u \sin v \cos w, u \sin w, \varphi(u) + av + bw).$$

In this paper, we study the principal curvatures and the third fundamental form of the Ulisse Dini-type helicoidal hypersurface in Euclidean 4-space $\mathbb{E}^4$. We give some basic notions of four dimensional Euclidean geometry in section 2. In section 3, we give Ulisse Dini-type helicoidal hypersurface, and calculate its principal curvatures, and the third fundamental form in section 4. In addition, we give a conclusion in the last section.

## 2 Preliminaries

In this section, we introduce the fundamental form matrices $I$, $II$, $III$, the shape operator matrix $S$, the Gaussian curvature $K$, and the mean curvature $H$ of a hypersurface $M = M(u, v, w)$ in the Euclidean 4-space $\mathbb{E}^4$.

Let $M$ be an isometric immersion of a hypersurface $M^4$ in the $\mathbb{E}^4$. The inner product of $\vec{z} = (x_1, x_2, x_3, x_4)$, $\vec{y} = (y_1, y_2, y_3, y_4)$, and the vector product of $\vec{z}$, $\vec{y}$, $\vec{z} \times \vec{y}$ on $\mathbb{E}^4$ are defined by

$$\vec{z} \cdot \vec{y} = x_1 y_1 + x_2 y_2 + x_3 y_3 + x_4 y_4,$$

$$\vec{z} \times \vec{y} = \det \begin{pmatrix} x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \\ z_1 & z_2 & z_3 & z_4 \end{pmatrix},$$

respectively. A hypersurface $M$ in 4-space has the first and the second fundamental form matrices

$$I = \begin{pmatrix} E & F & A \\ F & G & B \\ A & B & C \end{pmatrix}, \quad II = \begin{pmatrix} L & M & P \\ M & N & T \\ P & T & V \end{pmatrix},$$

respectively. Here,

$$E = M_u \cdot M_u, \quad F = M_u \cdot M_v, \quad G = M_v \cdot M_v, \quad A = M_u \cdot M_w, \quad B = M_v \cdot M_w, \quad C = M_w \cdot M_w,$$

$$L = M_{uu} \cdot e, \quad M = M_{uv} \cdot e, \quad N = M_{vw} \cdot e, \quad P = M_{uw} \cdot e, \quad T = M_{vw} \cdot e, \quad V = M_{ww} \cdot e, \quad e,$$
and \( e \) is the Gauss map
\[
\varepsilon = \frac{M_v \times M_w}{\|M_v \times M_w\|}.
\]
Hence, \( I^{-1}II \) gives the shape operator matrix of \( M \)
\[
S = \frac{1}{\det I} \begin{pmatrix}
  s_{11} & s_{12} & s_{13} \\
  s_{21} & s_{22} & s_{23} \\
  s_{31} & s_{32} & s_{33}
\end{pmatrix},
\]
where
\[
\det I = (EG - F^2)C - A^2G + 2ABF - B^2E,
\]
\[
s_{11} = ABM - CFM - AGP + BFP + CGL - B^2L,
\]
\[
s_{12} = ABN - CFN - AGT + BFT + CGM - B^2M,
\]
\[
s_{13} = ABT - CFT - AGV + BFM + CGP - B^2P,
\]
\[
s_{21} = ABL - CFL + AFE - BPE + CME - A^2M,
\]
\[
s_{22} = ABM - CFM + AFT - BTE + CNE - A^2N,
\]
\[
s_{23} = ABP - CFP + AFV - BVE + CTE - A^2T,
\]
\[
s_{31} = -AGL + BFL + AFM - BME + GPE - F^2P,
\]
\[
s_{32} = -AGM + BFM + AFN - BNE + GTE - F^2T,
\]
\[
s_{33} = -AGP + BFP + AFT - BTE + GVE - F^2V.
\]
Therefore, using \( II.S \), we get the third fundamental form matrix
\[
III = \frac{1}{\det I} \begin{pmatrix}
  \Gamma & \Phi & \Omega \\
  \Phi & \Psi & \Theta \\
  \Omega & \Theta & \Delta
\end{pmatrix},
\]
where
\[
\Gamma = -A^2M^2 + 2ABLM + 2AFMP - 2GALP - B^2L^2 + 2BFLP - 2EBMP - F^2P^2 - 2CFLM + CGL^2 + CEM^2 + GEP^2,
\]
\[
\Phi = ABM^2 - CFM^2 - B^2LM - A^2MN - F^2PT + CMNE - BNPE - BMTE + GPTE + ABLN - CFLN + CGLM + AFNP - AGMP + BFMP + AFMT - AGLT + BFLT,
\]
\[
\Omega = BFP^2 - AGP^2 - B^2LP - A^2MT - F^2PV + CMTE - BMVE - BPTE + GVE + ABMP + ABLT - CFMP + CGLP - CFLT + AFMV - AGLV + BFLV + AFPT,
\]
\[
\Psi = -A^2N^2 + 2ABMN + 2AFNT - 2GAMT - B^2M^2 + 2BFMT - 2EBNT - F^2T^2 - 2CFMN + CGM^2 + CEN^2 + GET^2,
\]
\[
\Theta = AFT^2 - B^2MP - A^2NT - F^2TV - BT^2E + CNTE - BNVE + GTVE + ABNP + ABMT - CFNP + CGMP - CFMT + AFNV - AGMV + BFMV - AGPT + BFPT,
\]
\[
\Delta = -A^2T^2 + 2ABPT + 2AFTV - 2GAPV - B^2P^2 + 2BFPV - 2EBTV - F^2V^2 - 2CFPT + CGP^2 + CET^2 + GEV^2.
\]
3 The Principal Curvatures and the Third Fundamental Form of the Dini-Type Helicoidal Hypersurface

We consider Dini-type helicoidal hypersurface

\[ \mathbf{D}(u, v, w) = \begin{pmatrix} \sin u \cos v \cos w \\ \sin u \sin v \cos w \\ \sin u \sin w \\ \cos u + \log (\tan \frac{\pi}{2} u) + av + bw \end{pmatrix}, \quad (3.1) \]

where \( u \in \mathbb{R}\setminus\{0\} \) and \( 0 \leq v, w \leq 2\pi \). Using the first differentials of (3.1) with respect to \( u, v, w \), we get the first quantities

\[ I = \begin{pmatrix} \cot^2 u & a \cot u \cos u & b \cot u \cos u \\ a \cot u \cos u & \sin^2 u \cos^2 w + a^2 & ab \\ b \cot u \cos u & ab & \sin^2 u + b^2 \end{pmatrix}, \]

and then, we have \( \det I = ((b^2 + 1) \cos^2 w + a^2) \sin^2 u \cos^2 u \). The Gauss map of (3.1) is given by

\[ e_D = \frac{1}{\sqrt{W}} \begin{pmatrix} \cos u \cos v \cos^2 w + a \sin v - b \cos u \sin w \cos w \\ \cos u \sin v \cos^2 w + a \cos v - b \sin u \sin w \cos w \\ (\cos u \sin w + b \cos w) \cos w - \sin u \cos w \end{pmatrix}, \quad (3.2) \]

where \( W = (b^2 + 1) \cos^2 w + a^2 \). Using the second differentials of the (3.1) with respect to \( u, v, w \), with (3.2), we have the second quantities of the (3.1)

\[ II = \frac{1}{W^{1/2}} \begin{pmatrix} \cot u \cos w & a \cos u \cos w & b \cos u \cos w \\ a \cos u \cos w & (b \sin w - \cos u \cos w) \sin u \cos^2 w - a \sin u \sin w \\ b \cos u \cos w & -a \sin u \sin w & -\sin u \cos u \cos w \end{pmatrix}. \]

Computing \( I^{-1} S \), we obtain the shape operator matrix of (3.1)

\[ S = \begin{pmatrix} \sin u \cos w & a \cos u \cos w & a^2(b \cos w + \cos u \sin w + (b^2 + 1) \cos^2 w) \\ \cos u \cos w & \frac{a \cos u \cos w}{W^{1/2}} & \frac{a(2 \cos^2 w + \sin^2 u)}{W^{1/2}} \\ 0 & \frac{a \cos u \cos w}{W^{1/2}} & \frac{a^2(2 \cos w - \cos u \sin u + (b^2 + 1) \cos^2 w)}{W^{1/2}} \end{pmatrix}. \quad (3.3) \]

**Theorem 1.** Let \( \mathbf{D} : M^3 \rightarrow \mathbb{E}^4 \) be an immersion given by (3.1). Then, characteristic polynomial of the (3.3) of the (3.1) is given by

\[ X^3 + pX^2 + qX + r = 0, \]

where

\[ p = \frac{\cos^2 u \cos^3 w + b^2 \cos^2 u \cos^4 w + W \cos^2 u \cos w}{W^{3/2} \cos u \sin u}, \]

\[ q = \frac{\cos^3 u \cos^4 w + a^2 \cos^3 u \cos^2 w + b^2 \cos^2 u \cos^4 w}{W^{3/2} \cos u \sin u}, \]

and

\[ r = \frac{-W \cos u \sin^2 u + a^2 \cos u \sin^2 u - b^2 \cos^2 u \sin u - b^3 \cos^3 u \cos \sin w}{W^{3/2} \cos u \sin u}. \]
Let $S$ be an immersion given by (3.1). Then, (3.1) has the principal curvatures
\[ k_1 = \frac{\sin u \cos w}{W^{1/2} \cos u}, \quad k_2 = \frac{\beta_1}{2W^{3/2} \sin u}, \quad k_3 = \frac{\beta_2}{2W^{3/2} \sin u}, \]
where
\[ \beta_1 = -\frac{1}{2} - 2W \cos u \cos w + (W + a^2) b \sin w, \]
\[ \beta_2 = -\frac{1}{2} - 2W \cos u \cos w + (W + a^2) b \sin w, \]
and
\[ T = (W + a^2)^2 (\cos u \cos w - 2b \sin w) \cos u \cos w + \left(4a^2W + b^2 (W + a^2)^2\right) \sin^2 w - 2(b^2 + 1) (W + a^2) (\cos u \cos w + b \sin w) \cos u \cos^3 w + (b^2 + 1)^2 \cos^2 u \cos^6 w. \]

Proof. Solving characteristic polynomial of $S$, we have eigenvalues of $S$.

Corollary 2. Let $D : M^3 \to \mathbb{H}^4$ be an immersion given by (3.1). Then, (3.1) has the third fundamental form matrix
\[ III = \frac{\cos^2 w}{W} \begin{pmatrix} 1 & a \sin u & b \sin u \\ a \sin u & (\sin w - \cos u \cos w)^2 \cos^2 w + a^2 & a(b \cos 2w + \cos u \sin 2w) \\ b \sin u & a(b \cos 2w + \cos u \sin 2w) & a^2 (b^2 + 1) (\cos^2 w + \cos^4 u) \cos^4 w \end{pmatrix}. \]

Proof. Using II.S, we get the third fundamental form matrix of (3.1).

4 Conclusion

In this paper, we introduce the principal curvatures, and the third fundamental form of the Dini-type helicoidal hypersurface $D(u, v, w)$ in the four dimensional Euclidean space $\mathbb{E}^4$. We calculate the Gauss map $e$ of the $D(u, v, w)$ in $\mathbb{E}^4$. We obtain the characteristic polynomial of the shape operator matrix $S$. After long calculations, we reveal the principal curvatures $k_1, k_2, k_3$, and the third fundamental form matrix $III$ of the Dini-type helicoidal hypersurface.

5 Competing Interests

Author has declared that no competing interests exist.

References


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