Asian Research Journal of Mathematics

16(11): 62-68, 2020; Article no.ARJOM.63690
ISSN: 2456-477X

The Principal Curvatures and the Third Fundamental Form of Dini-Type Helicoidal Hypersurface in 4-Space

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Author's contribution

The sole author designed, analyzed, interpreted and prepared the manuscript.

Article Information

DOI: 10.9734/ARJOM/2020/v16i1130243
Editor(s):
(1) Dr. Ruben Dario Ortiz Ortiz, Universidad Michoacana de San Nicolas de Hidalgo, Mexico.
Reviewers:
(1) Andrews S, Mahendra Engineering College, India.
(2) Tuhid Pashaee Golmarz, Urmia University of Technology, Iran.
(3) Zsuzsanna Balajti, University of Miskolc, Hungary.
Complete Peer review History: http://www.sdiarticle4.com/review-history/63690

Received: 15 October 2020
Accepted: 22 December 2020
Published: 30 December 2020

Original Research Article

Abstract

We consider the principal curvatures and the third fundamental form of Dini-type helicoidal hypersurface $\mathbf{D}(u,v,w)$ in the four dimensional Euclidean space $\mathbb{E}^4$. We find the Gauss map e of helicoidal hypersurface in $\mathbb{E}^4$. We obtain characteristic polynomial of shape operator matrix S. Then, we compute principal curvatures $k_{i=1,2,3}$, and the third fundamental form matrix III of $\mathbf{D}$.

Keywords: Four dimensional; Dini-type helicoidal hypersurface; Gauss map; principal curvatures; the third fundamental form.

2020 Mathematics Subject Classification: Primary 53A35; Secondary 53C42.

1 Introduction

Theory of surfaces and hypersurfaces have been studied by many geometers for years such as [1 − 26].

In the rest of this paper, we identify a vector (a,b,c,d) with its transpose (a,b,c,d)^t. Let $\gamma : I \longrightarrow \Pi$ be a curve in a plane Π in $\mathbb{E}^4$, and let ℓ be a straight line in Π for an open interval $I \subset \mathbb{R}$. A

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rotational hypersurface in $\mathbb{E}^4$ is defined as a hypersurface rotating a curve γ (i.e. profile curve) around a line (i.e. axis) ℓ. Suppose that when a profile curve γ rotates around the axis ℓ, it simultaneously displaces parallel lines orthogonal to the axis ℓ, so that the speed of displacement is proportional to the speed of rotation. Resulting hypersurface is called the *helicoidal hypersurface* with axis ℓ and pitches $a, b \in \mathbb{R}\backslash\{0\}$.

Let ℓ be a line spanned by the vector $(0,0,0,1)^t$. The orthogonal matrix

$$\mathfrak{M}(v,w) = \begin{pmatrix} \cos v \cos w & -\sin v & -\cos v \sin w & 0 \\ \sin v \cos w & \cos v & -\sin v \sin w & 0 \\ \sin w & 0 & \cos w & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad v, w \in \mathbb{R},$$

fixes the vector ℓ. The matrix $\mathfrak{M}$ can be found by solving the following equations simultaneously; $\mathfrak{M}.\ell = \ell$, $\mathfrak{M}^t.\mathfrak{M} = \mathfrak{M}.\mathfrak{M}^t = I_4$, $\det \mathfrak{M} = 1$. When the axis of rotation is ℓ, there is an Euclidean transformation by which the axis is ℓ transformed to the x_4-axis of $\mathbb{E}^4$. Parametrization of the profile curve is given by $\gamma(u) = (u, 0, 0, \varphi(u))$, where $\varphi(u) : I \subset \mathbb{R} \longrightarrow \mathbb{R}$ is a differentiable function for all $u \in I$. So, the helicoidal hypersurface is given by $\mathbf{H}(u,v,w) = \mathfrak{M}.\gamma^t + (av + bw).\ell^t$, where $u \in I$, $v, w \in [0, 2\pi]$, $a, b \in \mathbb{R}\backslash\{0\}$. Clearly, we write helicoidal hypersurface as follows

$$\mathbf{H}(u,v,w) = (u \cos v \cos w, u \sin v \cos w, u \sin w, \varphi(u) + av + bw).$$

In this paper, we study the principal curvatures and the third fundamental form of the Ulisse Dini-type helicoidal hypersurface in Euclidean 4-space $\mathbb{E}^4$. We give some basic notions of four dimensional Euclidean geometry in section 2. In section 3, we give Ulisse Dini-type helicoidal hypersurface, and calculate its principal curvatures, and the third fundamental form in section 4. In addition, we give a conclusion in the last section.

2 Preliminaries

In this section, we introduce the fundamental form matrices I, II, III, the shape operator matrix $\mathbf{S}$, the Gaussian curvature K, and the mean curvature H of a hypersurface $\mathbf{M} = \mathbf{M}(u,v,w)$ in the Euclidean 4-space $\mathbb{E}^4$.

Let $\mathbf{M}$ be an isometric immersion of a hypersurface M^3 in the $\mathbb{E}^4$. The inner product of $\overrightarrow{x} = (x_1, x_2, x_3, x_4)$, $\overrightarrow{y} = (y_1, y_2, y_3, y_4)$, and the vector product of $\overrightarrow{x}$, $\overrightarrow{y}$, $\overrightarrow{z} = (z_1, z_2, z_3, z_4)$ on $\mathbb{E}^4$ are defined by

$$\begin{aligned} \overrightarrow{x} \cdot \overrightarrow{y} &= x_1 y_1 + x_2 y_2 + x_3 y_3 + x_4 y_4, \\ \overrightarrow{x} \times \overrightarrow{y} \times \overrightarrow{z} &= \det \begin{pmatrix} e_1 & e_2 & e_3 & e_4 \\ x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \\ z_1 & z_2 & z_3 & z_4 \end{pmatrix}, \end{aligned}$$

respectively. A hypersurface $\mathbf{M}$ in 4-space has the first and the second fundamental form matrices

$$I = \begin{pmatrix} E & F & A \\ F & G & B \\ A & B & C \end{pmatrix}, \quad II = \begin{pmatrix} L & M & P \\ M & N & T \\ P & T & V \end{pmatrix},$$

respectively. Here,

$$\begin{aligned} &E = \mathbf{M}_u \cdot \mathbf{M}_u, \ F = \mathbf{M}_u \cdot \mathbf{M}_v, \ G = \mathbf{M}_v \cdot \mathbf{M}_v, \ A = \mathbf{M}_u \cdot \mathbf{M}_w, \ B = \mathbf{M}_v \cdot \mathbf{M}_w, \ C = \mathbf{M}_w \cdot \mathbf{M}_w, \\ &L = \mathbf{M}_{uu} \cdot e, \quad M = \mathbf{M}_{uv} \cdot e, \quad N = \mathbf{M}_{vv} \cdot e, \quad P = \mathbf{M}_{uw} \cdot e, \quad T = \mathbf{M}_{vw} \cdot e, \quad V = \mathbf{M}_{ww} \cdot e, \end{aligned}$$

and e is the Gauss map

$$e = \frac{\mathbf{M}_u \times \mathbf{M}_v \times \mathbf{M}_w}{\|\mathbf{M}_u \times \mathbf{M}_v \times \mathbf{M}_w\|}.$$

Hence, $I^{-1}.II$ gives the shape operator matrix of $\mathbf{M}$

$$S = \frac{1}{\det I}\begin{pmatrix} s_{11} & s_{12} & s_{13} \\ s_{21} & s_{22} & s_{23} \\ s_{31} & s_{32} & s_{33} \end{pmatrix},$$

where

$$\det I = (EG - F^2)C - A^2G + 2ABF - B^2E,$$

$$\begin{aligned}
s_{11} &= ABM - CFM - AGP + BFP + CGL - B^2L, \\
s_{12} &= ABN - CFN - AGT + BFT + CGM - B^2M, \\
s_{13} &= ABT - CFT - AGV + BFV + CGP - B^2P, \\
s_{21} &= ABL - CFL + AFP - BPE + CME - A^2M, \\
s_{22} &= ABM - CFM + AFT - BTE + CNE - A^2N, \\
s_{23} &= ABP - CFP + AFV - BVE + CTE - A^2T, \\
s_{31} &= -AGL + BFL + AFM - BME + GPE - F^2P, \\
s_{32} &= -AGM + BFM + AFN - BNE + GTE - F^2T, \\
s_{33} &= -AGP + BFP + AFT - BTE + GVE - F^2V.
\end{aligned}$$

Therefore, using $II.S$, we get the third fundamental form matrix

$$III = \frac{1}{\det I}\begin{pmatrix} \Gamma & \Phi & \Omega \\ \Phi & \Psi & \Theta \\ \Omega & \Theta & \Delta \end{pmatrix},$$

where

$$\begin{aligned}
\Gamma &= -A^2M^2 + 2ABLM + 2AFMP - 2GALP - B^2L^2 + 2BFLP \\
&\quad -2EBMP - F^2P^2 - 2CFLM + CGL^2 + CEM^2 + GEP^2, \\
\Phi &= ABM^2 - CFM^2 - B^2LM - A^2MN - F^2PT + CMNE \\
&\quad -BNPE - BMTE + GPTE + ABLN - CFLN + CGLM \\
&\quad +AFNP - AGMP + BFMP + AFMT - AGLT + BFLT, \\
\Omega &= BFP^2 - AGP^2 - B^2LP - A^2MT - F^2PV + CMTE \\
&\quad -BMVE - BPTE + GPVE + ABMP + ABLT - CFMP \\
&\quad +CGLP - CFLT + AFMV - AGLV + BFLV + AFPT, \\
\Psi &= -A^2N^2 + 2ABMN + 2AFNT - 2GAMT - B^2M^2 + 2BFMT \\
&\quad -2EBNT - F^2T^2 - 2CFMN + CGM^2 + CEN^2 + GET^2, \\
\Theta &= AFT^2 - B^2MP - A^2NT - F^2TV - BT^2E + CNTE \\
&\quad -BNVE + GTVE + ABNP + ABMT - CFNP + CGMP \\
&\quad -CFMT + AFNV - AGMV + BFMV - AGPT + BFPT, \\
\Delta &= -A^2T^2 + 2ABPT + 2AFTV - 2GAPV - B^2P^2 + 2BFPV \\
&\quad -2EBTV - F^2V^2 - 2CFPT + CGP^2 + CET^2 + GEV^2.
\end{aligned}$$

3 The Principal Curvatures and the Third Fundamental Form of the Dini-Type Helicoidal Hypersurface

We consider Dini-type helicoidal hypersurface

$$\mathbf{D}(u,v,w)=\begin{pmatrix}\sin u\cos v\cos w\\ \sin u\sin v\cos w\\ \sin u\sin w\\ \cos u+\log\left(\tan\frac{u}{2}\right)+av+bw\end{pmatrix}, \tag{3.1}$$

where $u\in\mathbb{R}\backslash\{0\}$ and $0\le v,w\le 2\pi$. Using the first differentials of (3.1) with respect to u,v,w, we get the first quantities

$$I=\begin{pmatrix}\cot^2 u & a\cot u\cos u & b\cot u\cos u\\ a\cot u\cos u & \sin^2 u\cos^2 w+a^2 & ab\\ b\cot u\cos u & ab & \sin^2 u+b^2\end{pmatrix},$$

and then, we have $\det I=\left(\left(b^2+1\right)\cos^2 w+a^2\right)\sin^2 u\cos^2 u$. The Gauss map of (3.1) is given by

$$e_{\mathbf{D}}=\frac{1}{\sqrt{W}}\begin{pmatrix}\cos u\cos v\cos^2 w+a\sin v-b\cos v\sin w\cos w\\ \cos u\sin v\cos^2 w+a\cos v-b\sin v\sin w\cos w\\ (\cos u\sin w+b\cos w)\cos w\\ -\sin u\cos w\end{pmatrix}, \tag{3.2}$$

where $W=\left(b^2+1\right)\cos^2 w+a^2$. Using the second differentials of the (3.1) with respect to u,v,w, with (3.2) , we have the second quantities of the (3.1)

$$II=\frac{1}{W^{1/2}}\begin{pmatrix}\cot u\cos w & a\cos u\cos w & b\cos u\cos w\\ a\cos u\cos w & (b\sin w-\cos u\cos w)\sin u\cos^2 w & -a\sin u\sin w\\ b\cos u\cos w & -a\sin u\sin w & -\sin u\cos u\cos w\end{pmatrix}.$$

Computing $I^{-1}.S$, we obtain the shape operator matrix of (3.1)

$$S=\begin{pmatrix}\frac{\sin u\cos w}{W^{1/2}\cos u} & \frac{a\cos w}{W^{1/2}\cos u} & \frac{a^2(b\cos w+\cos u\sin w)+b\left(b^2+1\right)\cos^3 w}{W^{3/2}\cos u}\\ 0 & \frac{b\sin w-\cos u\cos w}{W^{1/2}\sin u} & -\frac{a\left(b^2+1\right)\sin w}{W^{3/2}\sin u}\\ 0 & -\frac{a\sin w}{W^{1/2}\sin u} & \frac{a^2(b\sin w-\cos u\cos w)-\left(b^2+1\right)\cos u\cos^3 w}{W^{3/2}\sin u}\end{pmatrix}. \tag{3.3}$$

Theorem 1. *Let* $\mathbf{D}:M^3\longrightarrow\mathbb{E}^4$ *be an immersion given by* (3.1). *Then, characteristic polynomial of the* (3.3) *of the* (3.1) *is given by*

$$X^3+pX^2+qX+r=0.$$

where

$$p=\frac{\left\{\begin{array}{c}\cos^2 u\cos^3 w+b^2\cos^2 u\cos^3 w+W\cos^2 u\cos w\\ -W\cos w\sin^2 u+a^2\cos^2 u\cos w\\ -bW\cos u\sin w-a^2 b\cos u\sin w\end{array}\right\}}{W^{3/2}\cos u\sin u},$$

$$q=\frac{\left\{\begin{array}{c}\cos^3 u\cos^4 w+a^2\cos^3 u\cos^2 w+b^2\cos^3 u\cos^4 w\\ -a^2\cos u\sin^2 w-\cos u\cos^4 w\sin^2 u\\ -W\cos u\cos^2 w\sin^2 u-a^2\cos u\cos^2 w\sin^2 u\\ -b^2\cos u\cos^4 w\sin^2 u-b^3\cos^2 u\cos^3 w\sin w\\ -b\cos^2 u\cos^3 w\sin w+bW\cos w\sin^2 u\sin w\\ -2a^2 b\cos^2 u\cos w\sin w+a^2 b\cos w\sin^2 u\sin w\end{array}\right\}}{W^2\cos u\sin^2 u},$$

$$r = \frac{\left\{ \begin{array}{c} a^2 \sin^2 w - \cos^2 u \cos^4 w - a^2 \cos^2 u \cos^2 w \\ -b^2 \cos^2 u \cos^4 w + b \cos u \cos^3 w \sin w \\ +b^3 \cos u \cos^3 w \sin w + 2a^2 b \cos u \cos w \sin w \end{array} \right\} \cos w}{W^{5/2} \sin u \cos u}.$$

Proof. Computing $\det(S - X.I_3) = 0$, we get the p, q, and r.

Corollary 1. *Let* $\mathbf{D} : M^3 \longrightarrow \mathbb{E}^4$ *be an immersion given by* (3.1). *Then,* (3.1) *has the principal curvatures*

$$k_1 = \frac{\sin u \cos w}{W^{1/2} \cos u}, \; k_2 = \frac{\beta_1}{2W^{3/2} \sin u}, \; k_3 = \frac{\beta_2}{2W^{3/2} \sin u},$$

where

$$\begin{aligned} \beta_1 &= \mathfrak{T}^{1/2} - 2W \cos u \cos w + \left(W + a^2\right) b \sin w, \\ \beta_2 &= -\mathfrak{T}^{1/2} - 2W \cos u \cos w + \left(W + a^2\right) b \sin w, \end{aligned}$$

and

$$\begin{aligned} \mathfrak{T} = \; & \left(-W + a^2\right)^2 (\cos u \cos w - 2b \sin w) \cos u \cos w \\ & + \left(4a^2 W + b^2 \left(W + a^2\right)^2\right) \sin^2 w \\ & -2\left(b^2 + 1\right)\left(-W + a^2\right)(\cos u \cos w + b \sin w) \cos u \cos^3 w \\ & + \left(b^2 + 1\right)^2 \cos^2 u \cos^6 w. \end{aligned}$$

Proof. Solving characteristic polynomial of S, we have eigenvalues of S.

Corollary 2. *Let* $\mathbf{D} : M^3 \longrightarrow \mathbb{E}^4$ *be an immersion given by* (3.1). *Then,* (3.1) *has the third fundamental form matrix*

$$III = \frac{\cos^2 w}{W} \begin{pmatrix} 1 & a \sin u & b \sin u \\ a \sin u & \frac{(b \sin w - \cos u \cos w)^2 \cos^2 w + a^2}{\cos^2 w} & \frac{a(b \cos 2w + \cos u \sin 2w)}{\cos^2 w} \\ b \sin u & \frac{a(b \cos 2w + \cos u \sin 2w)}{\cos^2 w} & \frac{a^2\left(b^2 + 1 - \cos^2 w \sin^2 w\right) + \left(b^2 + 1\right)\left(b^2 + \cos^2 u\right) \cos^4 w}{\cos^2 w} \end{pmatrix}.$$

Proof. Using $II.S$, we get the third fundamental form matrix of (3.1).

4 Conclusion

In this paper, we introduce the principal curvatures, and the third fundamental form of the Dini-type helicoidal hypersurface $\mathbf{D}(u, v, w)$ in the four dimensional Euclidean space $\mathbb{E}^4$. We calculate the Gauss map e of the $\mathbf{D}(u, v, w)$ in $\mathbb{E}^4$. We obtain the characteristic polynomial of the shape operator matrix S. After long calculations, we reveal the principal curvatures k_1, k_2, k_3, and the third fundamental form matrix III of the Dini-type helicoidal hypersurface.

5 Competing Interests

Author has declared that no competing interests exist.

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