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I_2-Convergence of Double Sequences in Topological Groups

Topolojik Gruplarda Çift Dizilerin I_2-Yakınsaklığı

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Abstract

Let $2^{N\times N}$ be a family of all subsets of N×N. Following the definition of ideal convergence in a metric space by Kostyrko et al. in 2000, ideal convergence for double sequences in a metric space was introduced by Das et al. (2008). In this paper, I investigate I_2-convergence and I_2-convergence of double sequences in a topological space and establish some basic teorems. Furthermore we introduce of I_2-Cauchy and I_2-Cauchy notions for double sequences in topological groups.

Keywords: I_2-convergence, I_2-convergence, topological group, double sequence, ideal

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Öz

$2^{N\times N,}$ N×N kümesinin tüm alt kümelerinin ailesi olsun. Kostyrko ve arkadaşlarının 2000'de bir metrik uzayda ideal yakınsaklığı tanımlamalarının ardından, çift diziler için ideal yakınsaklık Das ve arkadaşları tarafından tanımlandı (2008). Bu makalede bir topolojik uzayda çift dizilerin I_2-yakınsaklığı ve I_2-yakınsaklığı incelenmiş ve bazı önemli teoremler inşa edilmiştir. Ayrıca topolojik uzaylarda çift diziler için I_2-Cauchy ve I_2-Cauchy kavramları tanımlanmıştır.

Anahtar Kelimeler: I_2-yakınsaklık, I_2-yakınsaklık, topolojik grup, çift diziler, ideal

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1. Introduction

By an ideal on a set X we mean a nonempty family of subsets of X closed under taking finite unions and subsets of its elements. In other words, a non-empty set $I \subseteq 2^N$ is called an ideal on N if;

i $B \in I$ whenever $B \subseteq A$ for some $A \in I$. (closed unders subsets)

ii $A \cup B \in I$ whenever $A, B \in I$. (closed under unions)

If $N \notin I$ then we say that this ideal is a proper ideal. Similarly an ideal is proper and also contains all finite subsets then we say that this ideal is admissible. Filter is a dual notion of ideal and generally we will use ideals in our proofs but if the notion is more familiar for filters, we will use the notion of filter.

Similarly, a non-empty set $F \subseteq 2^N$ is called a filter on N if;

i $B \in F$ whenever $B \supseteq A$ for some $A \in F$. (closed unders supersets)

ii $A \cap B \in F$ whenever $A, B \in F$. (closed under intersections)

Proposition 1.1. If I is a non-trivial ideal in N, then the family of sets $F = F(I) = \{M = N \setminus A : A \in I\}$

is a filter in N and it is called the filter associated with the ideal.

Definition 1.1. Let $x = (x_k)$ be a real sequence. This sequence is said to be I- convergent to $L \in R$ if for each $\varepsilon > 0$ the set

$$A_\varepsilon = \{k \in N : |x_k - L| \geq \varepsilon\}$$

belongs to I. In this definition the number L is I-limit of the x.

I-convergence generalizes ordinary convergence and statistical convergence. This means that if we choose two

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special ideals, we have ordinary convergence and statistical convergence such that;

Example 1.1. I_f is an admissible ideal and I_f-convergence coincides with the usual convergence.

In recent years, important concepts such as statistical *I*-convergence, convergence etc. began to be defined for double sequences like:

(i) Mursaleen and Edely (2003) studied statistical convergence for double sequences.

(ii) Tripathy (2005) has a paper which introduced I-convergent double sequences.

(iii) Kumar (2007) defined the notions *I*- and I^*- convergence of double sequence and studied some properties of these notions.

(iV) In a metric space, Das et all. (2008) introduced the concepts of *I*- and I^*-convergence of double sequence.

(V) Dündar and Altay (2011 and 2014) have two papers about I_2-convergence of double sequences and I_2-Cauchy sequences.

In 2005, Lahiri and his friends introduced *I*-convergence in topological groups and also Savas (2014) gave new definitions in topological groups by using ideal. Gezer and Karakuş (2005) investigated *I*-pointwise and uniform convergence and I^*-pointwise and uniform convergence of function sequences and then they examined the relation between them.

Now lets remind the convergence of a double sequence. Convergence of a double sequence means the convergence in Pringsheim's sense. Let $x=(x_{kl})$ be a double sequence and *L* be a number of real sequences. $x=(x_{kl})$ has a Pringsheim limit provided that given an $\varepsilon>0$, there exists an $n\in N$ such that $|x_{kl}-L|<\varepsilon$ whenever $k,l>n$ and we denote by *P*-lim *x*=*L*. Generally we say that $x=(x_{kl})$ is *P*-convergent fort his situation.

Before giving Mursaleen and Edely's definition about statistical convergence for double sequences, lets recall two-deimensional analogue of natural density can be defined as follows:

$K(m,n)$ is the number of (*i*, *j*) in $K\subset N\times N$ such that $i\leqslant m$ and $j\leq n$. In case the sequence $\frac{K(m,n)}{mn}$ has a limit in Pringsheim's sense then *K* has double natural density and is defined by

$$P-\lim_{m,n\to\infty}\frac{K(m,n)}{mn}=\delta_2(K).$$

Example 1.2. Let $K=\{(i^2,j^2):i,j\in N\}$. Then

$$\delta_2(K)=\lim_{m,n\to\infty}\frac{K(m,n)}{mn}\leq\lim_{m,n\to\infty}\frac{\sqrt{m}\sqrt{n}}{mn}=0$$

i.e. the set $\varkappa$ has double natural density zero.

Definition 1.2 : (Mursaleen and Edely, 2003) Let $x=(x_{kl})$ be a real double sequence. This sequence is statistically convergent to the number *L* if for each $\varepsilon>0$ the set

$$\{(k,l),k\leq m,l\leq n:|x_{k,l}-L|\geq\varepsilon\}$$

has double natural density zero.

Now we will talk about *I*-convergence for double sequences according to Kumar's paper.

Throughout the paper $X=N\times N$ and *I* will denote the ideal of subsets of $N\times N$. The following proposition gives us the relation between an ideal and a fitler for double sequences.

Proposition 1.2. Let $I\subset 2^{N\times N}$ be a non-trivial ideal. Then $F(I)=\{(N\times N)\backslash A:A\in I\}$ is a filter on $N\times N$.

Definition 1.3. (Kumar, 2007) Let $I\subset 2^{N\times N}$ be a non-trivial ideal and $x=(x_{kl})$ be a double sequence. $x=(x_{kl})$ is *I*-convergent to the number *L* if for each $\varepsilon>0$ the set

$$A(\varepsilon)=\{(k,l)\in N\times N:|x_{kl}-L|\geq\varepsilon\}\in I.$$

It is denoted by $I-\lim_{k,l\to\infty}x_{kl}=L.$

Example 1.3. Lets take the ideal,

$I=\{E\subset N\times N: E\ \textit{is the form}\ (N\times A)\cup A\times N\}$ where A is a finite subset of *N*. Then *I*-convergence is equivalent to the usual Pringsheim's convergence.

Throughout the paper we take I_2 as a nontrivial admissible ideal in $N\times N$ and *X* be a Hausdorff topological abelian group written additively. A nontrivial ideal I_2 on $N\times N$ is called strongly admissible if $\{i\}\times N$ and $N\times\{i\}$ belongs to I_2 for each $i\in N$. It is evident that a strongly admissible ideal is admissible also.

2. Main Results

We will start with the definitions of I_2-convergence and I^*_2-convergence of double sequences in a topological group.

Definition 2.1. Let $x=(x_{kl})$ be a real double sequence. $x=(x_{kl})$ is said to be I_2-convergent to the number *L* if for each neighborhood U of 0,

$$\{(k,l)\in N\times N:x_{kl}-L\notin U\}\in I_2.$$

That is,

$\{(k,l)\in N\times N: x_{kl}-L\in U\}\in F(I_2)$.

It is denoted by $I_2-\lim_{k,l\to\infty} x_{kl}=L.$

Definition 2.2. Let $x=(x_{kl})$ be a real double sequence. $x=(x_{kl})$ is said to be I_2^*-convergent to the number L provided that for each neighborhood U of 0, there is a set

$M_2=\{(k,l)\in N\times N: k,l=1,2,...\}\in F(I_2)$

(i.e., $(N\times N)\backslash M_2\in I_2$) such that $\lim_{k,l\to\infty} x_{kl}=L.$ It is denoted by $I_2^*-\lim_{k,l\to\infty} x_{kl}=L$.

Definition 2.3. A real double sequences $x=(x_{kl})$ is said to be I_2-Cauchy in X if for each neighborhood U of 0, there exists $q,r\in N$ such that for all $k,m\geq q$ and $l,n\geq r$,

$\{(k,l)\in N\times N: x_{kl}-x_{mn}\notin U\}\in I_2$.

Definition 2.4. A real double sequences $x=(x_{kl})$ is said to be I_2^*-Cauchy in X if for each neighborhood U of 0, there is set

$M_2=\{(k,l)\in N\times N: k,l=1,2,...\}\in F(I_2)$

(i.e., $(N\times N)\backslash M_2\in I_2$ such that for every $(k,l),(p,q)\in N\times N$, we have

$\{(k,l),(p,q)\in N\times N: x_{kl}-x_{pq}\in U\}\in F(I_2)$.

Theorem 2.1. Let I_2 be an arbitrary strongly admissible ideal and X be a Hausdorff topological abelian group. Then $I_2-\lim_{k,l\to\infty} x_{kl}=L$ implies that (x_{kl}) is I_2-Cauchy double sequence.

Proof: Let I_2 be an arbitrary strongly admissible ideal and U be an arbitrary neighborhood of 0. Choose V, W such that $W+W\subset V\subset U$. Since $I_2-\lim_{k,l\to\infty} x_{k,l}=L$, we have

$A(W)=\{(k,l)\in N\times N: x_{kl}-L\notin W\}\in I_2$

for each neighborhood W of 0. Then for any $(k,l),(m,n)\in X\backslash A(W)$,

$x_{kl}-x_{mn}=x_{kl}-L+L-x_{mn}\in W+W\subset V\subset U$. Hence it follows that $\{(k,l)\in N\times N: x_{kl}-x_{mn}\notin U\}\subset A(W)$

where $(m,n)\in X\backslash A(W)$ is fixed. This shows the existence of $(m,n)\in X$ for which $\{(k,l)\in N\times N: x_{kl}-x_{mn}\notin U\}\in I_2$.

As this holds for each neighbourd U of 0, (x_{kl}) is I_2-Cauchy double sequence.

Theorem 2.2. Let I_2 be an arbitrary strongly admissible ideal, and X be a Hausdorff topological abelian group. If (x_{kl}) is I_2^*-Cauchy double sequence then it is I_2-Cauchy double sequence.

Proof: Let (x_{kl}) is I_2-Cauchy sequence then by the definition, there exits a set

$M_2=\{(k,l)\in N\times N: k,l=1,2,...\}\in F(I_2)$

(i.e., $(N\times N)\backslash M_2\in I_2$) such that

$\{(k,l),(p,q)\in N\times N: x_{kl}-x_{pq}\in U\}\in F(I_2)$,

for every $(k,l),(p,q)\in M_2, k,l,p,q>r=r(U)$ and $r\in N$. Then, we have

$$A(U)=\{(k,l)\in N\times N: x_{kl}-x_{pq}\notin U\}$$
$$\subset H\cup(M_2\cap[(\{1,2,...,(r-1)\}\times N)$$
$$\cup(N\times\{1,2,...,(r-1)\})]).$$

Since

$$H\cup(M_2\cap[(\{1,2,...,(r-1)\}\times N)$$
$$\cup(N\times\{1,2,...,(r-1)\})])\in I_2$$

then we have $A(U)\in I_2$. Thus (x_{kl}) is I_2-Cauchy double sequence.

Theorem 2.3. Take an arbitrary strongly admissible ideal I_2. If $x=(x_{kl})$ is I_2^*-convergent then it is I_2-Cauchy.

Proof: Let $I_2^*-\lim_{k,l\to\infty} x_{kl}=L$. Then by definition there exists a set

$M_2=\{(k,l)\in N\times N: k,l=1,2,...\}\in F(I_2)$

(i.e., $(N\times N)\backslash M_2\in I_2$ such that $\lim_{k,l\to\infty} x_{kl}=L$ for each $(k,l)\backslash M_2$. Then for each neighbourd U of 0, we have

$$\{(k,l)\in N\times N: x_{kl}-x_{mm}\in U\}$$
$$\subset\{(k,l)\in N\times N: x_{kl}-L\in U\}$$
$$\cup\{(m,n)\in N\times N: x_{mm}-L\in U\}$$

Therefore, for each $(k,l),(m,n)\in M_2$ we have

$\{(k,l),(m,n)\in N\times N: x_{kl}-x_{mn}\in \cup\}\in F(I_2)$

Hence (x_{kl}) is a I_2-Cauchy double sequence.

Definition 2.5. Let $I_2\subset 2^{N\times N}$ be an admissible ideal. I_2 satisfies (AP_2) if for every sequence $(A_n)_{n\in N}$ of pairwise disjoint sets from I_2 there exist sets $B_n\subset N, n\in N$ such that the symetric difference $A_n\Delta B_n$ is a finite set for every n and $\bigcup_{n\in N} B_n\in I_2$.

Theorem 2.4. If the ideal I_2 has the property (AP_2) then I_2-convergence for double sequence implies I_2^*-convergence .

Proof: Suppose that I_2 satisfies property (AP_2). Let $I_2-\lim_{k,l\to\infty} x_{kl}=L$. Then

$$T(U)=\{(k,l)\in N\times N: x_{kl}-L\notin U\}\in I_2 \quad (1)$$

for each neighborhood U of 0. Put

$$T_1(U)=\{(k,l)\in N\times N: x_{kl}-L\notin U\}$$

and

$$T_k(U)=(k,l)\in N\times N: U_{\frac{1}{k}}\notin x_{kl}-L\in U_{\frac{1}{k-1}}$$

for $k\geq 2$ and $k\in N$. Obviously $T_i\cap T_j=\phi$ for $i\neq j$ and $T_i(U)\in I_2$ for each $i\in N$. By property (AP_2) there exits a sequence of sets $\{V_k\}_{k\in N}$ such that $T_j\Delta V_j$ is included in finite union of rows and columbs in $N\times N$ for $j\in N$ and $V=\bigcup_{j=1}^{\infty}V_j\in I_2$. We shall prove that for $M_2=(N\times N)\backslash V, M_2\in F(I_2)$ we have $\lim_{k,l\to\infty}x_{kl}=L$.

Let $\gamma>0$. Choose $k\in N$ such that $\frac{1}{k}<\gamma$. Then

$$\{(k,l)\in N\times N: x_{kl}-L\notin U_\gamma\}\subset\bigcup_{j=1}^{k}T_j.$$

Since $T_j\Delta V_j, j=1,2,\ldots$, are included in finite union of rows and columbs, there exists $n_0\in N$ such that

$$\begin{aligned}&\bigcup_{j=1}^{k}T_j\cap\{(k,l)\in N\times N: k\geq n_0\ and\ l\geq n_0\}\\&=\bigcup_{j=1}^{k}V_j\cap\{(k,l)\in N\times N: k\geq n_0\ and\ l\geq n_0\}\end{aligned}\tag{2}$$

If $k,l>n_0$ and $(k,l)\notin V$, so $(k,l)\notin\bigcup_{j=1}^{k}V_j$ and from (2), $(k,l)\notin\bigcup_{j=1}^{k}T_j$. This implies that $\{(k,l)\in N\times N: x_{kl}-L\in U_\gamma\}\in F(I_2)$; so we have the proof.

Lemma 2.1. Let $\{P_i\}_{i=1}^{\infty}$ be a countable collection subsets of $N\times N$ such that $\{P_i\}_{i=1}^{\infty}\in F(I_2)$ is a filter associate with a strongly admissible ideal I_2 with property (AP_2). Then there exists a set $P\subset N\times N$ such that $P\in F(I_2)$ and the set $P\backslash P_i$ is finite for all i.

Now we will prove that, a I_2-Cauchy double sequence coincides with a I_2^*-Cauchy double sequence for strongly admissible ideals with property (AP_2).

Theorem 2.5. Let I_2 be strongly admissible ideal and it satisfies (AP_2). Then the concepts I_2-Cauchy double sequence and I_2^*-Cauchy double sequence coincide.

Proof: Even if a double sequence has not the property (AP_2), if is I_2^*-Cauchy then it is I_2-Cauchy by theorem 2.2. Now it is sufficient to prove that (x_{kl}), I_2^*-Cauchy double sequence under assumption that (x_{kl}) is a I_2-Cauchy double sequence. Let (x_{kl}) i s a I_2-Cauchy double sequence. Then by definition, there exits $q,r\in N$ such that, for all $k,m\geq q$ and $l,n\geq r$,

$$\{(k,l)\in N\times N: x_{kl}-x_{mn}\notin U\}\in I_2$$

for each neighborhood U of 0. Let

$$P_j=\{(k,l)\in N\times N: x_{kl}-x_{mn}\in U\}, j=1,2,\ldots$$

It is clear that $P_j\in F(I_2)$ for $j=1,2,\ldots$ Since I_2 has (AP_2) property, then from Lemma 2.1 there exists a set $P\subset N$ such that $P\in F(I_2)$ and $P\backslash P_j$ is finite for all j. Now we will show that

$$\{(k,l),(p,q)\in N\times N: x_{kl}-x_{pq}\in U\}\in F(I_2)$$

for every $(k,l),(p,q)\in P$. Let $j\in N$ such that $(m_j,n_j)\in N\times N$. If $(k,l),(p,q)\in P$ then $P\backslash P_j$ is finite set, there exists $k=k_j$ such that

$$\{(k,l)\in N\times N: x_{kl}-x_{m_jn_j}\in U\}\in F(I_2)$$

and

$$\{(p,q)\in N\times N: x_{pq}-x_{m_jn_j}\in U\}\in F(I_2)$$

for all $k,l,m,n>k_j$. Hence it follows that

$$\begin{aligned}&\{(k,l),(p,q)\in N\times N: x_{kl}-x_{pq}\in U\}\\&\subseteq\{(k,l)\in N\times N: x_{kl}-x_{m_jn_j}\in U\}\\&\cup\{(p,q)\in N\times N: x_{pq}-x_{m_jn_j}\in U\}\end{aligned}$$

For $k,l,m,n>k(U)$. Therefore there exists $(k,l),(p,q)\in P\in F(I_2)$ such that $\{(k,l),(p,q)\in N\times N: x_{kl}-x_{pq}\in U\}\in F(I_2)$ and this proves the theorem.

3. Functions Preserving I_2-Convergence in Topological Groups

Definition 3.1. Let X be a Hausdorff topological abelian group and I_2 be an arbitrary strongly admissible ideal. For a sequence (x_{kl}) in x, we say that the function $f:X\to X$ preserves I_2-convergence in x if $I_2-\lim_{k,l\to\infty}x_{kl}=L$ then $I_2-\lim_{k,l\to\infty}f(x_{kl})=f(L)$.

As is not difficult to the predict we have the following.

Theorem 3.1. If a function $f:X\to X$ is continous on X, then it preserves I_2-convergence in X. (for an arbitrary strongly ideal I_2)

Proof: Let $I_2-\lim_{k,l\to\infty}x_{kl}=L$. If f is continous, then for each neighborhood U_η, U_δ of 0 such that $x\in B(L,\delta)=U_\delta$. Then $f(x)\in B(f(L),\eta)=U_\eta$. But we have

$$\begin{aligned}&\{(k,l)\in N\times N: x_{kl}-L\in U_\delta\}\\&\subset\{(k,l)\in N\times N: f(x_{kl})-f(L)\in U_\eta\}\end{aligned}$$

and

$$\{(k,l)\in N\times N: f(x_{kl})-f(x_{kl})-f(L)\in U_\eta\}\in F(I_2)$$

since

$$\{(k,l)\in N\times N: x_{kl}-L\in U_\delta\}\in F(I_2)$$

Hence $I_2-\lim_{k,l\to\infty}f(x_{kl})=f(L)$ and f preserves I_2-convergence.

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