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 \mathcal{J} -Lacunary Statistical Convergence in Intuitionistic Fuzzy Normed Spaces**Ömer Kişi**

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sequence**Abstract**

In this study, first, we investigate the notions of \mathcal{J} -lacunary statistical convergence and strongly \mathcal{J} -lacunary convergence with regards to the intuitionistic fuzzy norm (IFN for short) (μ, ν) . Then, we investigate relationships among this new concepts and make important observations about them. Furthermore, we examine the relations among \mathcal{J} -lacunary statistical convergence and \mathcal{J} -statistical convergence in terms of IFN (μ, ν) in the corresponding intuitionistic fuzzy normed space.

Sezgisel Fuzzy Normlu Uzaylarda \mathcal{J} -Lacunary İstatistiksel Yakınsaklık**Anahtar kelimeler**İdeal yakınsaklık;
 \mathcal{J} -lacunary istatistiksel
yakınsaklık; Sezgisel
fuzzy normlu uzay;
Banach uzayı; Cauchy
dizisi**Öz**

Bu çalışmada, ilk olarak (μ, ν) sezgisel normuna göre \mathcal{J} -lacunary istatistiksel yakınsaklık ve kuvvetli \mathcal{J} -lacunary yakınsaklık kavramları tanımlandı. Daha sonra bu kavramlar arasındaki ilişkiler incelendi ve bu kavramlar üzerine önemli gözlemler yapıldı. Bununla birlikte, ilgili sezgisel fuzzy normlu uzayda (μ, ν) sezgisel normuna göre \mathcal{J} -lacunary istatistiksel yakınsaklık ile \mathcal{J} -istatistiksel yakınsaklık arasındaki ilişkiler incelendi.

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1. Introduction and Definitions

Statistical convergence of a real number sequence was firstly originated by Fast (1951). It became a notable topic in summability theory after the work of Fridy (1985) and Šalát (1980).

Theory of \mathcal{J} -convergence of sequences was given by Kostyrko et al. (2000). It is known that \mathcal{J} -convergence is a significant generality of statistical convergence. The other studies of ideals can be done by Das and Ghosal (2010), Das et al. (2011) and Savaş and Das (2011).

Using lacunary sequence, Fridy and Orhan (1993a) examined lacunary statistical convergence. Subsequently, it was studied by Fridy and Orhan (1993b), Li (2000), Mursaleen and Mohiuddine (2009), Bakery (2014). Lacunary ideal convergence for real sequences was investigated by Tripathy et al. (2012). The other studies of this concept can be examined by Yamancı and Gürdal (2014a, 2014b), Ulusu and Dündar (2014), Kişi and Dündar (2018), Savaş et al. (2019).

Several authors have studied invariant convergent sequences (see, Ulusu et al. (2018), Pancaroğlu Akın et al. (2018)).

Fuzzy set theory has become an important working area after the study of Zadeh (1965). Atanassov (1986) investigated intuitionistic fuzzy set; this concept was utilized by Atanassov et al. (2002) in the study of decision-making problems. The idea of an intuitionistic fuzzy metric space was put forward by Park (2004). Saadati and Park (2006) initially introduced the concept of an intuitionistic fuzzy normed space (IFNS). Several studies of the convergence of sequences in some normed linear spaces with a fuzzy setting might be revealed by the research of the Hosseini et al. (2008), Debnath (2012), Sen and Debnath (2011), Debnath and Sen (2014), Debnath (2015). Karakuş et al. (2008) defined statistical convergence in IFNS. Savaş and Gürdal (2015) studied the concept of $\mathcal{J} - [V, \lambda]$ -summability and $\mathcal{J} - \lambda$ -statistical convergence with regards to the IFN (μ, ν) .

2. Main Results

Definition 2.1 Let $(X, \Phi, \omega, *, \theta)$ be an IFNLS. A sequence $x = (x_k)$ is called to be *J*-lacunary statistically convergent to $\xi \in X$ in terms of IFN (μ, ν) , and is demonstrated by $S_\theta(\mathcal{J})^{(\mu, \nu)} - \lim x = \xi$ or $x_k \xrightarrow{(\mu, \nu)} \xi(S_\theta(\mathcal{J}))$, if for every $\varepsilon > 0$, every $\delta > 0$, and $t > 0$,

$$\left\{ r \in N: \frac{1}{h_r} |\{k \in I_r : \mu(x_k - \xi, t) \leq 1 - \varepsilon \text{ or } \nu(x_k - \xi, t) \geq \varepsilon\}| \geq \delta \right\} \in I.$$

We consider \mathcal{J}_f as the family of all finite subsets of \mathbb{N} . Then \mathcal{J}_f is an admissible ideal and *J*-lacunary statistically convergence coincides with lacunary statistically convergence introduced in the study of Mursaleen and Mohiuddine (2009).

Definition 2.2 Let $(X, \Phi, \omega, *, \theta)$ be an IFNLS. $x = (x_k)$ in X is called strongly *J*-lacunary convergent to $\xi \in X$ or $N_\theta(\mathcal{J})$ -convergent to $\xi \in X$ in terms of IFN (μ, ν) , and is demonstrated by $x_k \xrightarrow{(\mu, \nu)} \xi(N_\theta(\mathcal{J}))$, if for every $\delta > 0$, and $t > 0$,

$$\left\{ r \in N: \frac{1}{h_r} \sum_{k \in I_r} \mu(x_k - \xi, t) \leq 1 - \delta \text{ or } \frac{1}{h_r} \sum_{k \in I_r} \nu(x_k - \xi, t) \geq \delta \right\} \in I.$$

Theorem 2.1 Let $(X, \Phi, \omega, *, \theta)$ be an IFNLS, and $x = \{x_k\} \in X$, then

1.
 - a. If $x_k \xrightarrow{(\mu, \nu)} \xi(N_\theta(\mathcal{J}))$ then $x_k \xrightarrow{(\mu, \nu)} \xi(S_\theta(\mathcal{J}))$.
 - b. If $x \in m(X)$, the space of all bounded sequences of X and $x_k \xrightarrow{(\mu, \nu)} \xi(S_\theta(\mathcal{J}))$ then $x_k \xrightarrow{(\mu, \nu)} \xi(N_\theta(\mathcal{J}))$.
2. $S_\theta(\mathcal{J})^{(\mu, \nu)} \cap m(X) = N_\theta(\mathcal{J})^{(\mu, \nu)} \cap m(X)$.

Proof. 1 (a) By assumption, for each $\varepsilon > 0$, $\delta > 0$, and $t > 0$, let $x_k \xrightarrow{(\mu, \nu)} \xi(N_\theta(\mathcal{J}))$. We have

$$\begin{aligned} & \sum_{k \in I_r} (\mu(x_k - \xi, t) \text{ or } \nu(x_k - \xi, t)) \\ \geq & \sum_{\substack{k \in I_r \\ \mu(x_k - \xi, t) \leq 1 - \varepsilon \text{ or } \nu(x_k - \xi, t) \geq \varepsilon}} (\mu(x_k - \xi, t) \text{ or } \nu(x_k - \xi, t)) \\ \geq & \varepsilon \cdot |\{k \in I_r : \mu(x_k - \xi, t) \leq 1 - \varepsilon \text{ or } \nu(x_k - \xi, t) \geq \varepsilon\}|. \end{aligned}$$

Then, observe that

$$\begin{aligned} & \frac{1}{h_r} |\{k \in I_r : \mu(x_k - \xi, t) \leq 1 - \varepsilon \\ & \text{ or } \nu(x_k - \xi, t) \geq \varepsilon\}| \geq \delta \\ \Rightarrow & \frac{1}{h_r} \sum_{k \in I_r} \mu(x_k - \xi, t) \leq (1 - \varepsilon)\delta \\ & \text{ or } \frac{1}{h_r} \sum_{k \in I_r} \nu(x_k - \xi, t) \geq \varepsilon\delta, \end{aligned}$$

which implies

$$\begin{aligned} & \left\{ r \in N: \frac{1}{h_r} |\{k \in I_r : \mu(x_k - \xi, t) \leq 1 - \varepsilon \text{ or } \nu(x_k - \xi, t) \geq \varepsilon\}| \geq \delta \right\} \\ \subset & \left\{ r \in N: \frac{1}{h_r} \left\{ \sum_{k \in I_r} \mu(x_k - \xi, t) \leq 1 - \varepsilon \text{ or } \sum_{k \in I_r} \nu(x_k - \xi, t) \geq \varepsilon \right\} \geq \varepsilon\delta \right\}. \end{aligned}$$

Since $x_k \xrightarrow{(\mu, \nu)} \xi(N_\theta(\mathcal{J}))$, we see that $x_k \xrightarrow{(\mu, \nu)} \xi(S_\theta(\mathcal{J}))$, hence we get the result.

1 (b) Assume that $x_k \xrightarrow{(\mu, \nu)} \xi(S_\theta(\mathcal{J}))$ and $x \in l_\infty^{(\mu, \nu)}$. The inequalities $\mu(x_k - \xi, t) \geq 1 - M$ or $\nu(x_k - \xi, t) \leq M$ hold for all k . For $\varepsilon > 0$, we get

$$\begin{aligned} & \frac{1}{h_r} \sum_{k \in I_r} (\mu(x_k - \xi, t) \text{ or } \nu(x_k - \xi, t)) \\ = & \frac{1}{h_r} \sum_{\substack{k \in I_r \\ \mu(x_k - \xi, t) \leq 1 - \varepsilon \text{ or } \nu(x_k - \xi, t) \geq \varepsilon}} (\mu(x_k - \xi, t) \text{ or } \nu(x_k - \xi, t)) \\ + & \frac{1}{h_r} \sum_{\substack{k \in I_r \\ \mu(x_k - \xi, t) > 1 - \varepsilon \text{ or } \nu(x_k - \xi, t) < \varepsilon}} (\mu(x_k - \xi, t) \text{ or } \nu(x_k - \xi, t)) \\ \leq & \frac{M}{h_r} |\{k \in I_r : \mu(x_k - \xi, t) \leq 1 - \varepsilon \\ & \text{ or } \nu(x_k - \xi, t) \geq \varepsilon\}| + \varepsilon. \end{aligned}$$

Note that

$$\begin{aligned} A_{\mu, \nu}(\varepsilon, t) = & \left\{ r \in N: \frac{1}{h_r} |\{k \in I_r : \mu(x_k - \xi, t) \leq 1 - \varepsilon \text{ or } \nu(x_k - \xi, t) \geq \varepsilon\}| \geq \frac{\varepsilon}{M} \right\} \end{aligned}$$

belongs to \mathcal{J} . If $r \in (A_{\mu, \nu}(\varepsilon, t))^c$ then we have

$$\begin{aligned} & \frac{1}{h_r} \sum_{k \in I_r} \mu(x_k - \xi, t) > 1 - 2\varepsilon \\ & \text{ or } \frac{1}{h_r} \sum_{k \in I_r} \nu(x_k - \xi, t) < 2\varepsilon. \end{aligned}$$

Now

$$\begin{aligned} T_{\mu, \nu}(\varepsilon, t) = & \left\{ r \in N: \frac{1}{h_r} \sum_{k \in I_r} \mu(x_k - \xi, t) \leq 1 - 2\varepsilon \text{ or } \frac{1}{h_r} \sum_{k \in I_r} \nu(x_k - \xi, t) \geq 2\varepsilon \right\}. \end{aligned}$$

Hence, $T_{\mu,\nu}(\varepsilon, t) \subseteq B_{\mu,\nu}(\varepsilon, t)$ and so, by the definition of an ideal,

$$T_{\mu,\nu}(\varepsilon, t) \in \mathcal{J}.$$

Therefore, we conclude that $x_k \xrightarrow{(\mu,\nu)} \xi(N_\theta(\mathcal{J}))$.

2. This readily follows from 1 (a) and 1 (b).

Theorem 2.2 $(X, \Phi, \omega, *, \theta)$ be an IFNLS. If $\theta = \{k_r\}$ be a lacunary sequence with $\liminf q_r > 1$, then

$$x_k \xrightarrow{(\mu,\nu)} \xi(S(\mathcal{J})) \Rightarrow x_k \xrightarrow{(\mu,\nu)} \xi(S_\theta(\mathcal{J})).$$

Proof. Suppose first that $\liminf q_r > 1$, then there exists a $\alpha > 0$ such that $q_r \geq 1 + \alpha$ for quite large r , which emphasize that

$$\frac{h_r}{k_r} \geq \frac{\alpha}{1 + \alpha}.$$

If $x_k \xrightarrow{(\mu,\nu)} \xi(S(\mathcal{J}))$, then for every $\varepsilon > 0$, for each $x \in X$ and for quite large r , we have

$$\begin{aligned} & \frac{1}{k_r} |\{k \leq k_r : \mu(x_k - \xi, t) \leq 1 - \varepsilon \text{ or} \\ & \quad \nu(x_k - \xi, t) \geq \varepsilon\}| \\ & \geq \frac{1}{k_r} |\{k \in I_r : \mu(x_k - \xi, t) \leq 1 - \varepsilon \text{ or} \\ & \quad \nu(x_k - \xi, t) \geq \varepsilon\}| \\ & \geq \frac{\alpha}{1 + \alpha} \frac{1}{h_r} |\{k \in I_r : \mu(x_k - \xi, t) \leq 1 - \varepsilon \text{ or} \\ & \quad \nu(x_k - \xi, t) \geq \varepsilon\}| \end{aligned}$$

For any $\delta > 0$, we get

$$\begin{aligned} & \left\{ r \in \mathbb{N} : \frac{1}{h_r} |\{k \in I_r : \mu(x_k - \xi, t) \leq 1 - \varepsilon \text{ or} \right. \\ & \quad \left. \text{or } \nu(x_k - \xi, t) \geq \varepsilon\}| \geq \delta \right\} \\ & \subseteq \left\{ r \in \mathbb{N} : \frac{1}{k_r} |\{k \leq k_r : \mu(x_k - \xi, t) \leq 1 - \varepsilon \text{ or} \right. \\ & \quad \left. \text{or } \nu(x_k - \xi, t) \geq \varepsilon\}| \geq \frac{\delta \alpha}{(\alpha + 1)} \right\}. \end{aligned}$$

This shows that $x_k \xrightarrow{(\mu,\nu)} \xi(S_\theta(\mathcal{J}))$.

To prove of the following theorem, we accept that the lacunary sequence θ satisfies the situation that for any set $C \in F(\mathcal{J})$, $\cup\{n: k_{r-1} < n \leq k_r, r \in C\} \in F(\mathcal{J})$.

Theorem 2.3 $(X, \Phi, \omega, *, \theta)$ be an IFNLS. If $\theta = \{k_r\}$ be a lacunary sequence with $\limsup q_r < \infty$, then

$$x_k \xrightarrow{(\mu,\nu)} \xi(S_\theta(\mathcal{J})) \Rightarrow x_k \xrightarrow{(\mu,\nu)} \xi(S(\mathcal{J})).$$

Proof. If $\limsup q_r < \infty$ then without any loss of generalization we can accept that there exists a $0 < M < \infty$ such that $q_r < M$ for all $r \geq 1$. Suppose that

$$C = \left\{ r \in \mathbb{N} : \frac{1}{h_r} |\{k \in I_r : \mu(x_k - \xi, t) \leq 1 - \varepsilon \text{ or} \right. \\ \left. \text{or } \nu(x_k - \xi, t) \geq \varepsilon\}| < \delta \right\}$$

and

$$T = \left\{ n \in \mathbb{N} : \frac{1}{n} |\{k \leq n : \mu(x_k - \xi, t) \leq 1 - \varepsilon \text{ or} \right. \\ \left. \text{or } \nu(x_k - \xi, t) \geq \varepsilon\}| < \delta_1 \right\}.$$

It is clear from our assumption that $C \in F(\mathcal{J})$, the filter associated with the ideal \mathcal{J} . Observe that

$$K_j = \frac{1}{h_j} |\{k \in I_j : \mu(x_k - \xi, t) \leq 1 - \varepsilon \text{ or} \\ \text{or } \nu(x_k - \xi, t) \geq \varepsilon\}| < \delta$$

for all $j \in C$. Let $n \in \mathbb{N}$ be such that $k_{r-1} < n \leq k_r$ for some $r \in C$. Now

$$\begin{aligned} & \frac{1}{n} |\{k \leq n : \mu(x_k - \xi, t) \leq 1 - \varepsilon \text{ or} \\ & \quad \nu(x_k - \xi, t) \geq \varepsilon\}| \\ & \leq \frac{1}{k_{r-1}} |\{k \leq k_r : \mu(x_k - \xi, t) \leq 1 - \varepsilon \text{ or} \\ & \quad \nu(x_k - \xi, t) \geq \varepsilon\}| \\ & = \frac{1}{k_{r-1}} |\{k \in I_1 : \mu(x_k - \xi, t) \leq 1 - \varepsilon \text{ or} \\ & \quad \nu(x_k - \xi, t) \geq \varepsilon\}| \\ & \quad + \frac{1}{k_{r-1}} |\{k \in I_2 : \mu(x_k - \xi, t) \leq 1 - \varepsilon \text{ or} \\ & \quad \nu(x_k - \xi, t) \geq \varepsilon\}| \\ & \quad + \dots + \frac{1}{k_{r-1}} |\{k \in I_r : \mu(x_k - \xi, t) \leq 1 - \varepsilon \text{ or} \\ & \quad \nu(x_k - \xi, t) \geq \varepsilon\}| \\ & = \frac{k_1}{k_{r-1}} \frac{1}{h_1} |\{k \in I_1 : \mu(x_k - \xi, t) \leq 1 - \varepsilon \text{ or} \\ & \quad \nu(x_k - \xi, t) \geq \varepsilon\}| \\ & \quad + \frac{k_2 - k_1}{k_{r-1}} \frac{1}{h_2} |\{k \in I_2 : \mu(x_k - \xi, t) \leq 1 - \varepsilon \text{ or} \\ & \quad \nu(x_k - \xi, t) \geq \varepsilon\}| \\ & \quad + \dots + \frac{k_r - k_{r-1}}{k_{r-1}} \frac{1}{h_r} |\{k \in I_r : \mu(x_k - \xi, t) \leq 1 - \varepsilon \text{ or} \\ & \quad \nu(x_k - \xi, t) \geq \varepsilon\}| \\ & = \frac{k_1}{k_{r-1}} K_1 + \frac{k_2 - k_1}{k_{r-1}} K_2 + \dots + \frac{k_r - k_{r-1}}{k_{r-1}} K_r \\ & \leq \{ \sup_{j \in C} K_j \} \frac{k_r}{k_{r-1}} \\ & < M \delta. \end{aligned}$$

Choosing $\delta_1 = \frac{\delta}{M}$ and given the fact that

$$\bigcup \{n: k_{r-1} < n \leq k_r, r \in C\} \subset T$$

where $C \in F(\mathcal{J})$ it follows from our assumption on θ that the set $T \in F(\mathcal{J})$.

Combination of Theorem 2.2 and Theorem 2.3 we have

Theorem 2.4 $(X, \Phi, \omega, *, \theta)$ be an IFNLS. If $\theta = \{k_r\}$ be a lacunary sequence with $1 < \liminf q_r \leq \limsup q_r < \infty$, then

$$x_k \xrightarrow{(\mu,\nu)} \xi(S_\theta(\mathcal{J})) \Leftrightarrow x_k \xrightarrow{(\mu,\nu)} \xi(S(\mathcal{J}))$$

Proof. This is an immediate consequence of Theorem 2.2 and Theorem 2.3.

Theorem 2.5 $(X, \Phi, \omega, *, \Theta)$ be an IFNLS such that

$$\frac{1}{4} \varepsilon_n \Theta \frac{1}{4} \varepsilon_n < \frac{1}{2} \varepsilon_n$$

and

$$\left(1 - \frac{1}{4} \varepsilon_n\right) * \left(1 - \frac{1}{4} \varepsilon_n\right) > 1 - \frac{1}{2} \varepsilon_n.$$

If X is a Banach space, then $S_\theta(\mathcal{J})^{(\mu, \nu)} \cap m(x)$ is a closed subset of $m(x)$.

Proof. We first assume that $(x^n) \subset S_\theta(\mathcal{J})^{(\mu, \nu)} \cap m(x)$ is a convergent sequence that it converges to $x \in m(x)$. We have to show that $x \in S_\theta(\mathcal{J})^{(\mu, \nu)} \cap m(x)$. Suppose that $x^n \xrightarrow{(\mu, \nu)} L_n(S_\theta(\mathcal{J}))$ for all $n \in \mathbb{N}$. Consider a sequence $\{\varepsilon_n\}$ of reducing positive numbers converging to zero. We can determinate an $n \in \mathbb{N}$ such that $\sup_j v(x - x^j, t) < \frac{1}{4} \varepsilon_n$ for all $j \geq n$. Choose $0 < \beta < \frac{1}{5}$. Let

$$A_{\mu, \nu}(\varepsilon_n, t) = \left\{ \begin{array}{l} r \in N: \frac{1}{h_r} |\{k \in I_r : \\ \mu(x_k^n - L_n, t) \leq 1 - \frac{\varepsilon_n}{4} \text{ or } \\ v(x_k^n - L_n, t) \geq \frac{\varepsilon_n}{4}\}| < \delta \end{array} \right\}$$

belongs to $F(\mathcal{J})$ and

$$B_{\mu, \nu}(\varepsilon_n, t) = \left\{ \begin{array}{l} r \in N: \frac{1}{h_r} |\{k \in I_r : \\ \mu(x_k^{n+1} - L_{n+1}, t) \leq 1 - \frac{\varepsilon_n}{4} \text{ or } \\ v(x_k^{n+1} - L_{n+1}, t) \geq \frac{\varepsilon_n}{4}\}| < \delta \end{array} \right\}$$

belongs to $F(\mathcal{J})$. Since $A_{\mu, \nu}(\varepsilon_n, t) \cap B_{\mu, \nu}(\varepsilon_n, t) \in F(\mathcal{J})$ and $\emptyset \notin F(\mathcal{J})$, we can select $r \in A_{\mu, \nu}(\varepsilon_n, t) \cap B_{\mu, \nu}(\varepsilon_n, t)$. Then

$$\frac{1}{h_r} |\{k \in I_r : \mu(x_k^n - L_n, t) \leq 1 - \frac{\varepsilon_n}{4}$$

$$\text{or } v(x_k^n - L_n, t) \geq \frac{\varepsilon_n}{4} \vee$$

$$\mu(x_k^{n+1} - L_{n+1}, t) \leq 1 - \frac{\varepsilon_n}{4} \text{ or}$$

$$v(x_k^{n+1} - L_{n+1}, t) \geq \frac{\varepsilon_n}{4}\}| \leq 2\delta < 1.$$

Since $h_r \rightarrow \infty$ and $A_{\mu, \nu}(\varepsilon_n, t) \cap B_{\mu, \nu}(\varepsilon_n, t) \in F(\mathcal{J})$ is finite, we can select the above r so that $h_r > 5$. So, there must exist a $k \in I_r$ for which we get at the same time, $\mu(x_k^n - L_n, t) > 1 - \frac{\varepsilon_n}{4}$ or $v(x_k^n - L_n, t) < \frac{\varepsilon_n}{4}$ and $\mu(x_k^{n+1} - L_{n+1}, t) > 1 - \frac{\varepsilon_n}{4}$ or $v(x_k^{n+1} - L_{n+1}, t) < \frac{\varepsilon_n}{4}$. For a given $\varepsilon_n > 0$ select $\frac{\varepsilon_n}{2}$ such that $\left(1 - \frac{\varepsilon_n}{2}\right) * \left(1 - \frac{\varepsilon_n}{2}\right) > 1 - \varepsilon_n$ and $\frac{1}{4} \varepsilon_n \Theta \frac{1}{4} \varepsilon_n < \varepsilon_n$. Then, it follows that

$$\begin{aligned} v\left(L_n - x_k^n, \frac{t}{2}\right) \Theta v\left(L_{n+1} - x_k^{n+1}, \frac{t}{2}\right) \\ \leq \frac{\varepsilon_n}{4} \Theta \frac{\varepsilon_n}{4} < \frac{\varepsilon_n}{2} \end{aligned}$$

and

$$\begin{aligned} v(x_k^n - x_k^{n+1}, t) \leq \sup_n v\left(x - x^n, \frac{t}{2}\right) \\ \Theta \sup_n v\left(x - x^{n+1}, \frac{t}{2}\right) \\ \leq \frac{1}{4} \varepsilon_n \Theta \frac{1}{4} \varepsilon_n < \frac{1}{2} \varepsilon_n. \end{aligned}$$

Hence, we have

$$\begin{aligned} v(L_n - L_{n+1}, t) \leq \left[v\left(L_n - x_k^n, \frac{t}{3}\right) \right. \\ \Theta v\left(x_k^{n+1} - L_{n+1}, \frac{t}{3}\right) \\ \left. \Theta v\left(x_k^n - x_k^{n+1}, \frac{t}{3}\right) \right] \\ \leq \frac{1}{2} \varepsilon_n \Theta \frac{1}{2} \varepsilon_n < \varepsilon_n \end{aligned}$$

and likewise $\mu(L_n - L_{n+1}, t) > 1 - \varepsilon_n$. This implies that $\{L_n\}_{n \in \mathbb{N}}$ is a Cauchy sequence in X and let $L_n \rightarrow L \in X$ as $n \rightarrow \infty$. We have to show that $x \xrightarrow{(\mu, \nu)} L(S_\theta(\mathcal{J}))$. For any $\varepsilon > 0$ and $t > 0$, choose $n \in \mathbb{N}$ such that $\varepsilon_n < \frac{\varepsilon}{4}$, $\sup_n v(x - x^n, t) < \frac{1}{4} \varepsilon$, $\mu(L_n - L, t) > 1 - \frac{1}{4} \varepsilon$ or $v(L_n - L, t) < \frac{1}{4} \varepsilon$. Now, since

$$\begin{aligned} \frac{1}{h_r} |\{k \in I_r : v(x_k - L, t) \geq \varepsilon\}| \\ \leq \frac{1}{h_r} |\{k \in I_r : v\left(x_k - x_k^n, \frac{t}{3}\right) \Theta \\ \left[v\left(x_k^n - L_n, \frac{t}{3}\right) \Theta v\left(L_n - L, \frac{t}{3}\right) \right] \geq \varepsilon\}| \\ \leq \frac{1}{h_r} |\{k \in I_r : v\left(x_k^n - L_n, \frac{t}{3}\right) \geq \frac{\varepsilon}{2}\}| \end{aligned}$$

and similarly

$$\begin{aligned} \frac{1}{h_r} |\{k \in I_r : \mu(x_k - L, t) \leq 1 - \varepsilon\}| \\ > \frac{1}{h_r} |\{k \in I_r : \mu\left(x_k^n - L, \frac{t}{3}\right) \leq 1 - \frac{\varepsilon}{2}\}|. \end{aligned}$$

It follows that

$$\begin{aligned} \left\{ r \in N: \frac{1}{h_r} |\{k \in I_r : \mu(x_k - L, t) \leq 1 - \varepsilon \right. \\ \left. \text{or } v(x_k - L, t) \geq \varepsilon\}| \geq \delta \right\} \\ \subset \left\{ r \in N: \frac{1}{h_r} |\{k \in I_r : \mu\left(x_k^n - L, \frac{t}{3}\right) \leq \right. \\ \left. 1 - \frac{\varepsilon}{2} \text{ or } v\left(x_k^n - L, \frac{t}{3}\right) \geq \frac{\varepsilon}{2}\}| \geq \delta \right\} \end{aligned}$$

for given $\delta > 0$. Hence, we have $x \xrightarrow{(\mu, \nu)} L(S_\theta(\mathcal{J}))$.

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