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Generalized Helical Hypersurfaces Having Time-like Axis in Minkowski Spacetime

Erhan Güler

Department of Mathematics, Faculty of Sciences, Kutlubey Campus, Bartın University, 74100 Bartın, Turkey; eguler@bartin.edu.tr; Tel.: +90-378-5011000 (ext. 2275)

Abstract: In this paper, the generalized helical hypersurfaces $\mathbf{x} = \mathbf{x}(u, v, w)$ with a time-like axis in Minkowski spacetime $\mathbb{E}_1^4$ are considered. The first and the second fundamental form matrices, the Gauss map, and the shape operator matrix of $\mathbf{x}$ are calculated. Moreover, the curvatures of the generalized helical hypersurface $\mathbf{x}$ are obtained by using the Cayley–Hamilton theorem. The umbilical conditions for the curvatures of $\mathbf{x}$ are given. Finally, the Laplace–Beltrami operator of the generalized helical hypersurface with a time-like axis is presented in $\mathbb{E}_1^4$.

Keywords: Minkowski four-space; Lorentzian inner product; triple Lorentzian vector product; generalized helical hypersurface; Gauss map; curvature

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1. Introduction

Differential geometry has a very important place in mathematics, engineering, physics, and astrophysics. In differential geometry, the theory of hyper-surfaces has been worked on by many mathematicians (especially geometers), engineers, physicists, astrophysicists, etc., for hundreds of years.

For instance, Obata [1] revealed certain conditions for a Riemannian manifold to be isometric with a sphere; Takahashi [2] proved that a connected Euclidean submanifold is of the one-type, iff it is either minimal in $\mathbb{E}^m$ or minimal in some hypersphere of $\mathbb{E}^m$; Chern, do Carmo, and Kobayashi [3] gave the minimal submanifolds of a sphere with a second fundamental form of constant length; Cheng and Yau [4] considered the hypersurfaces with constant scalar curvature; Lawson [5] gave the minimal submanifolds and indicated the general definition of the Laplace–Beltrami operator.

Chen [6–9] studied the submanifolds of the finite-type whose immersion is into $\mathbb{E}^m$ (or $\mathbb{E}_\nu^m$) by using a finite number of eigenfunctions of their Laplacian. Some results of the two-type spherical closed submanifolds were given by [7,10,11]; Garay [12] researched the extension of Takahashi's theorem in $\mathbb{E}^m$. Chen and Piccinni [13] focused on the submanifolds with the finite-type Gauss map in $\mathbb{E}^m$.

In Euclidean three-space $\mathbb{E}^3$, when looking at ruled (helicoidal or helical) and rotational characters, it is seen that those are related by Bour's theorem [14]. Regarding helical surfaces in Euclidean three-space, do Carmo and Dajczer [15] proved that, by using a result of Bour [14], there exists a two-parameter family of helical surfaces isometric to a given helical surface. Takahashi [2] proved the minimal surfaces and spheres are the only surfaces satisfying the condition $\Delta r = \lambda r$, $\lambda \in \mathbb{R}$; Ferrandez, Garay, and Lucas [16] gave that the surfaces satisfying $\Delta H = AH$, $A \in Mat(3,3)$ are either minimal or an open piece of a sphere or of a right circular cylinder; Choi and Kim [17] studied the minimal helicoid in terms of the pointwise one-type Gauss map of the first kind; Garay [18] classified a certain class of finite-type surfaces of revolution; Dillen, Pas, and Verstraelen [19] focused on the only surfaces satisfying $\Delta r = Ar + B$, $A \in Mat(3,3)$, and $B \in Mat(3,1)$ being the minimal surfaces, the spheres, and the circular cylinders; Stamatakis and Zoubi [20] obtained the surfaces of revolution satisfying $\Delta^{III} x = Ax$; Senoussi and Bekkar [21] introduced helical surfaces M^2, which are of the finite-type with respect to the fundamental forms I, II, and III, i.e., their position vector field $r(u,v)$ satisfies the condition $\Delta^J r = Ar$, $J = I, II, III$, where $A \in Mat(3,3)$; Kim, Kim, and Kim [22] gave the Cheng–Yau operator and the Gauss map of the surfaces of revolution.

In Minkowski three-space $\mathbb{E}_1^3$, Beneki, Kaimakamis, and Papantoniou [23] studied helical surfaces with a space-like, time-like, and light-like axis in three-dimensional Minkowski space. Güler and Turgut Vanlı [24] worked the Bour's theorem; Güler [25] studied helical surfaces with a light-like profile curve using Bour's theorem in Minkowski geometry. Mira and Pastor [26] investigated helical maximal surfaces in Lorentz–Minkowski three-space. Kim and Yoon [27–29] worked with ruled and rotation surfaces in pseudo-Euclidean space. See also [2,24,30,31].

In Euclidean four-space $\mathbb{E}^4$, Moore [32,33] worked on general rotational surfaces; Hasanis and Vlachos [34] considered hypersurfaces with a harmonic mean curvature vector field; Cheng and Wan [35] gave the complete hypersurfaces with CMC; Arslan et al. [36] worked with the Vranceanu surface with the pointwise one-type Gauss map; Arslan et al. [37] studied generalized rotational surfaces; Magid, Scharlach, and Vrancken [38] introduced the affine umbilical surfaces in four-space. Scharlach [39] studied the affine geometry of surfaces and hypersurfaces in four-space. Arslan, Deszcz, and Yaprak [40] considered the Weyl pseudosymmetric hypersurfaces. Arslan, Bulca, and Milousheva [41] worked on meridian surfaces in four-space with the pointwise one-type Gauss map. Yoon [42] studied rotation surfaces with the finite-type Gauss map in four-space. Güler, Magid, and Yaylı [43] introduced helical hypersurfaces; Güler, Hacısalihoğlu, and Kim [44] studied the Gauss map and the third Laplace–Beltrami operator of rotational hypersurface; Güler [45] found the rotational hypersurfaces satisfying $\Delta^I R = AR$, where $A \in Mat(4,4)$. He [46] also introduced the fundamental form IV and the curvature formulas of the hypersphere.

In Minkowski four-space $\mathbb{E}_1^4$, Ganchev and Milousheva [47] indicated the analogue of the surfaces of [32,33]; Arvanitoyeorgos, Kaimakamais, and Magid [48] studied when the mean curvature vector field of M_1^3 satisfies the equation $\Delta H = \alpha H$ (α being a constant); then, M_1^3 has CMC; Arslan and Milousheva [49] considered meridian surfaces of the elliptic- or hyperbolic-type with the pointwise one-type Gauss map; Güler [50] gave helical hypersurfaces; recently, Iliadis [51] considered the fuzzy algebraic modelling of spatiotemporal time series paradoxes in cosmic-scale kinematics; Leuenberger [52] introduced the emergence of Minkowski spacetime by simple deterministic graph rewriting.

In this paper, the generalized helical hypersurfaces with a time-like axis in Minkowski four-space $\mathbb{E}_1^4$ are introduced. Some basic concepts of four-dimensional Minkowski geometry are given in Section 2. The i-th curvature formulas depend on the coefficients of the fundamental forms $(\mathfrak{g}_{ij})$ and $(\mathfrak{h}_{ij})$ in Minkowski four-space, which are obtained in Section 3. In Section 4, the definition of the generalized helical hypersurface having a time-like axis in $\mathbb{E}_1^4$ is indicated. The umbilic points of these kinds of hypersurfaces are described in Section 5. The generalized helical hypersurfaces with a time-like axis satisfying $\Delta \mathbf{x} = \mathcal{A}\mathbf{x}$ in $\mathbb{E}_1^4$ are given in Section 6. Finally, a summary of the paper is presented in the last section.

2. Preliminaries

In this section, some of the basic facts and definitions are given, then the notations used in this paper are described.

Let $\mathbb{E}_1^m$ denote the semi-Euclidean m-space with the semi-Euclidean metric tensor given by $\widetilde{g} = \langle \, , \, \rangle = \sum_{i=1}^{m-1} dx_i^2 - dx_m^2$, where $(x_1, x_2, \ldots, x_m)$ is an element of constantlength (or Lorentz metric) and x_i are the pseudo-Euclidean coordinates of type $(m-1, 1)$. Consider an m-dimensional semi-Riemannian submanifold M of the space $\mathbb{E}_1^m$. The Levi-Civita connections [53] of the manifold $\widetilde{M}$ and its submanifold M of $\mathbb{E}_1^m$ are denoted by $\widetilde{\nabla}$, ∇, respectively. Denoting the vector field tangent (respectively, normal) to M, the letters X, Y, Z, W (respectively, ξ, η) are used.

The Gauss and Weingarten formulas are given, respectively, by

$$\widetilde{\nabla}_X Y = \nabla_X Y + h(X,Y), \tag{1}$$
$$\widetilde{\nabla}_X \xi = -A_\xi(X) + D_X \xi, \tag{2}$$

where h, D, and A are the second fundamental form, the normal connection, and the shape operator of M, respectively.

For each $\xi \in T_p^\perp M$, the shape operator A_ξ is a symmetric endomorphism of the tangent space T_pM at $p \in M$. The shape operator and the second fundamental form are related by

$$\langle h(X,Y), \xi \rangle = \langle A_\xi X, Y \rangle.$$

The Gauss and Codazzi equations are given, respectively, by

$$\langle R(X,Y,)Z,W \rangle = \langle h(Y,Z), h(X,W) \rangle - \langle h(X,Z), h(Y,W) \rangle, \tag{3}$$
$$(\bar{\nabla}_X h)(Y,Z) = (\bar{\nabla}_Y h)(X,Z), \tag{4}$$

where R, R^D are the curvature tensors associated with connections ∇ and D, respectively, and $\bar{\nabla} h$ is defined by

$$(\bar{\nabla}_X h)(Y,Z) = D_X h(Y,Z) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z).$$

2.1. Hypersurfaces of Minkowski Space

Now, let M be an oriented hypersurface in Minkowski space $\mathbb{E}_1^{n+1}$, $\mathbf{S}$ its shape operator (i.e., the Weingarten map), and x its position vector. Consider a local orthonormal frame field $\{e_1, e_2, \ldots, e_n\}$ consisting of the principal directions of M corresponding to the principal curvature k_i for $i = 1, 2, \ldots n$. Let the dual basis of this frame field be $\{\theta_1, \theta_2, \ldots, \theta_n\}$. Then, the first Cartan structural equation is

$$d\theta_i = \sum_{i=1}^{n} \theta_j \wedge \omega_{ij}, \quad i,j = 1,2,\ldots,n, \tag{5}$$

where ω_{ij} denotes the connection forms corresponding to the chosen frame field. Denote the Levi-Civita connection of M of $\mathbb{E}_1^{n+1}$ by ∇. Then, from the Codazzi equation, the following holds:

$$e_i(k_j) = \omega_{ij}(e_j)(k_i - k_j),$$
$$\omega_{ij}(e_l)(k_i - k_j) = \omega_{il}(e_j)(k_i - k_l)$$

for distinct $i, j, l = 1, 2, \ldots, n$.

Put $s_j = \sigma_j(k_1, k_2, \ldots, k_n)$, where σ_j is the j-th elementary symmetric function given by

$$\sigma_j(a_1, a_2, \ldots, a_n) = \sum_{1 \le i_1 < i_2 < \ldots < i_j \le n} a_{i_1} a_{i_2} \ldots a_{i_j}.$$

The following notation is used

$$r_i^j = \sigma_j(k_1, k_2, \ldots, k_{i-1}, k_{i+1}, k_{i+2}, \ldots, k_n).$$

By the definition, $r_i^0 = 1$ and $s_{n+1} = s_{n+2} = \cdots = 0$. The function s_k is called the k-th mean curvature of M. Note that functions $H = \frac{1}{n}s_1$ and $K = s_n$ are called the mean curvature and Gauss–Kronecker curvature of M, respectively. In particular, M is said to be j-minimal if $s_j \equiv 0$ on M. See also [54,55].

In $\mathbb{E}_1^{n+1}$, finding the i-th curvature formulas $\mathfrak{C}_i$, where $i = 0, \ldots, n$, the following characteristic polynomial eq. of $\mathbf{S}$ is used:

$$P_{\mathbf{S}}(\lambda) = 0 = \det(\mathbf{S} - \lambda \mathcal{I}_n) = \sum_{k=0}^{n} (-1)^k s_k \lambda^{n-k}, \tag{6}$$

where $i = 0, \ldots, n$, $\mathcal{I}_n$ denotes the identity matrix of order n. Then, the curvature formulas are given by $\binom{n}{i}\mathfrak{C}_i = s_i$. That is, $\binom{n}{0}\mathfrak{C}_0 = s_0 = 1$ (by definition), $\binom{n}{1}\mathfrak{C}_1 = s_1, \ldots, \binom{n}{n}\mathfrak{C}_n = s_n$.

The k-th fundamental form of M is defined by $I\left(\mathbf{S}^{k-1}(X), Y\right) = \left\langle \mathbf{S}^{k-1}(X), Y \right\rangle$. Therefore,

$$\sum_{i=0}^{n} (-1)^i \binom{n}{i} \mathfrak{C}_i I\left(\mathbf{S}^{n-i}(X), Y\right) = 0. \tag{7}$$

In particular, one can obtain the classical result $\mathfrak{C}_0 III - 2\mathfrak{C}_1 II + \mathfrak{C}_2 I = 0$ of surface theory for $n = 2$. See [55] for the details.

For a Minkowskian submanifold x: $M \longrightarrow \mathbb{E}_1^m$, the immersion (M, x) is said to be the finite-type, if x can be expressed as a finite sum of the eigenfunctions of the Laplacian Δ of (M, x), i.e., $x = x_0 + \sum_{i=1}^{k} x_i$, where x_0 is a constant map, $x_1, \ldots, x_k$ non-constant maps, and $\Delta x_i = \lambda_i x_i$, $\lambda_i \in \mathbb{R}$, $i = 1, \ldots, k$. If λ_i are different, M is called *k-type*. See [7] for the Euclidean details.

2.2. Rotational Hypersurfaces

The definition of the rotational hypersurfaces in Riemannian space forms can be found in [56].

A rotational hypersurface $M \subset \mathbb{E}_1^{n+1}$ generated by a curve $\mathcal{C}$ around an axis $\mathfrak{r}$ that does not meet $\mathcal{C}$ is obtained by taking the orbit of $\mathcal{C}$ under those semi-orthogonal transformations of $\mathbb{E}_1^{n+1}$ that leaves $\mathfrak{r}$ pointwise fixed.

Throughout the paper, a vector and its transpose will be considered identical. Consider the case $n = 3$, and let $\mathcal{C}$ be the curve parametrized by $\gamma(u) = (f(u), 0, 0, g(u))$, where f, g are the differentiable functions. If $\mathfrak{r}$ is the x_4-axis, then a semi-orthogonal transformation of $\mathbb{E}_1^{n+1}$ that leaves $\mathfrak{r}$ pointwise fixed has the form

$$\mathcal{R}(v, w) = \begin{pmatrix} \mathcal{C}_1\mathcal{C}_2 & -\mathcal{S}_1 & -\mathcal{C}_1\mathcal{S}_2 & 0 \\ \mathcal{S}_1\mathcal{C}_2 & \mathcal{C}_1 & -\mathcal{S}_1\mathcal{S}_2 & 0 \\ \mathcal{S}_2 & 0 & \mathcal{C}_2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

where $\mathcal{C}_1 = \cos v$, $\mathcal{C}_2 = \cos w$, $\mathcal{S}_1 = \sin v$, $\mathcal{S}_2 = \sin w$, and $v, w \in [0, 2\pi)$. Therefore, the parametrization of the rotational hypersurface generated by a curve $\mathcal{C}$ around an axis $\mathfrak{r}$ is given by $\mathbf{x}(u, v, w) = \mathcal{R}(v, w)\gamma(u)$.

Definition 1. *Let $\mathbf{x} = \mathbf{x}(u, v, w)$ be an immersion from $M^3 \subset \mathbb{E}^3$ to $\mathbb{E}_1^4 = (\mathbb{R}^4, \langle ., . \rangle)$, where the Lorentzian inner product is given by*

$$\left\langle \overrightarrow{\mathfrak{x}}, \overrightarrow{\mathfrak{y}} \right\rangle = \mathfrak{x}_1\mathfrak{y}_1 + \mathfrak{x}_2\mathfrak{y}_2 + \mathfrak{x}_3\mathfrak{y}_3 - \mathfrak{x}_4\mathfrak{y}_4,$$

and the triple Lorentzian vector product is defined by

$$\overrightarrow{\mathfrak{x}} \times \overrightarrow{\mathfrak{y}} \times \overrightarrow{\mathfrak{z}} = \det \begin{pmatrix} e_1 & e_2 & e_3 & -e_4 \\ \mathfrak{x}_1 & \mathfrak{x}_2 & \mathfrak{x}_3 & \mathfrak{x}_4 \\ \mathfrak{y}_1 & \mathfrak{y}_2 & \mathfrak{y}_3 & \mathfrak{y}_4 \\ \mathfrak{z}_1 & \mathfrak{z}_2 & \mathfrak{z}_3 & \mathfrak{z}_4 \end{pmatrix}$$

with the vectors $\overrightarrow{\mathfrak{x}} = (\mathfrak{x}_1, \mathfrak{x}_2, \mathfrak{x}_3, \mathfrak{x}_4)$, $\overrightarrow{\mathfrak{y}} = (\mathfrak{y}_1, \mathfrak{y}_2, \mathfrak{y}_3, \mathfrak{y}_4)$, $\overrightarrow{\mathfrak{z}} = (\mathfrak{z}_1, \mathfrak{z}_2, \mathfrak{z}_3, \mathfrak{z}_4)$.

Definition 2. *Hypersurface*$\mathbf{x}$ *depends on three parameters in Minkowski four-space:*

$$\left(\mathfrak{g}_{ij}\right)_{3\times 3}, \ \left(\mathfrak{h}_{ij}\right)_{3\times 3}, \tag{8}$$

the first and the second fundamental form being symmetric matrices, respectively (i.e., **I** *and* **II**, *respectively): where* $\mathfrak{g}_{11} = \langle \mathbf{x}_u, \mathbf{x}_u \rangle$, $\mathfrak{g}_{12} = \langle \mathbf{x}_u, \mathbf{x}_v \rangle$, $\mathfrak{g}_{13} = \langle \mathbf{x}_u, \mathbf{x}_w \rangle$, $\mathfrak{g}_{22} = \langle \mathbf{x}_v, \mathbf{x}_v \rangle$, $\mathfrak{g}_{23} = \langle \mathbf{x}_v, \mathbf{x}_w \rangle$, $\mathfrak{g}_{33} = \langle \mathbf{x}_w, \mathbf{x}_w \rangle$, $\mathfrak{h}_{11} = \langle \mathbf{x}_{uu}, \mathcal{G} \rangle$, $\mathfrak{h}_{12} = \langle \mathbf{x}_{uv}, \mathcal{G} \rangle$, $\mathfrak{h}_{13} = \langle \mathbf{x}_{uw}, \mathcal{G} \rangle$, $\mathfrak{h}_{22} = \langle \mathbf{x}_{vv}, \mathcal{G} \rangle$, $\mathfrak{h}_{23} = \langle \mathbf{x}_{vw}, \mathcal{G} \rangle$, $\mathfrak{h}_{33} = \langle \mathbf{x}_{ww}, \mathcal{G} \rangle$. *Here,*

$$\mathcal{G} = \frac{\mathbf{x}_u \times \mathbf{x}_v \times \mathbf{x}_w}{\|\mathbf{x}_u \times \mathbf{x}_v \times \mathbf{x}_w\|} \tag{9}$$

is the unit normal (i.e., the Gauss map) of $\mathbf{x}$.

3. *i*-th Curvatures

Product matrices $(\mathfrak{g}_{ij})^{-1} \cdot (\mathfrak{h}_{ij})$ give the matrix of the shape operator $\mathbf{S}$ of hypersurface $\mathbf{x}$ in Minkowski four-space. See [43,44] for the details.

Therefore, the shape operator matrix of the hypersurface $\mathbf{x}$ is given by

$$\mathbf{S} = \frac{1}{\mathfrak{g}} \begin{pmatrix} \mathfrak{s}_{11} & \mathfrak{s}_{12} & \mathfrak{s}_{13} \\ \mathfrak{s}_{21} & \mathfrak{s}_{22} & \mathfrak{s}_{23} \\ \mathfrak{s}_{31} & \mathfrak{s}_{32} & \mathfrak{s}_{33} \end{pmatrix}, \tag{10}$$

where

$$\begin{aligned}
\mathfrak{s}_{11} &= \left(\mathfrak{g}_{22}\mathfrak{g}_{33} - \mathfrak{g}_{23}^2\right)\mathfrak{h}_{11} + (\mathfrak{g}_{13}\mathfrak{g}_{23} - \mathfrak{g}_{12}\mathfrak{g}_{33})\mathfrak{h}_{12} + (\mathfrak{g}_{12}\mathfrak{g}_{23} - \mathfrak{g}_{13}\mathfrak{g}_{22})\mathfrak{h}_{13}, \\
\mathfrak{s}_{12} &= \left(\mathfrak{g}_{22}\mathfrak{g}_{33} - \mathfrak{g}_{23}^2\right)\mathfrak{h}_{12} + (\mathfrak{g}_{13}\mathfrak{g}_{23} - \mathfrak{g}_{12}\mathfrak{g}_{33})\mathfrak{h}_{22} + (\mathfrak{g}_{12}\mathfrak{g}_{23} - \mathfrak{g}_{13}\mathfrak{g}_{22})\mathfrak{h}_{23}, \\
\mathfrak{s}_{13} &= \left(\mathfrak{g}_{22}\mathfrak{g}_{33} - \mathfrak{g}_{23}^2\right)\mathfrak{h}_{13} + (\mathfrak{g}_{13}\mathfrak{g}_{23} - \mathfrak{g}_{12}\mathfrak{g}_{33})\mathfrak{h}_{23} + (\mathfrak{g}_{12}\mathfrak{g}_{23} - \mathfrak{g}_{13}\mathfrak{g}_{22})\mathfrak{h}_{33}, \\
\mathfrak{s}_{21} &= (\mathfrak{g}_{13}\mathfrak{g}_{23} - \mathfrak{g}_{12}\mathfrak{g}_{33})\mathfrak{h}_{11} + \left(\mathfrak{g}_{11}\mathfrak{g}_{33} - \mathfrak{g}_{13}^2\right)\mathfrak{h}_{12} + (\mathfrak{g}_{12}\mathfrak{g}_{13} - \mathfrak{g}_{11}\mathfrak{g}_{23})\mathfrak{h}_{13}, \\
\mathfrak{s}_{22} &= (\mathfrak{g}_{13}\mathfrak{g}_{23} - \mathfrak{g}_{12}\mathfrak{g}_{33})\mathfrak{h}_{12} + \left(\mathfrak{g}_{11}\mathfrak{g}_{33} - \mathfrak{g}_{13}^2\right)\mathfrak{h}_{22} + (\mathfrak{g}_{12}\mathfrak{g}_{13} - \mathfrak{g}_{11}\mathfrak{g}_{23})\mathfrak{h}_{23}, \\
\mathfrak{s}_{23} &= (\mathfrak{g}_{13}\mathfrak{g}_{23} - \mathfrak{g}_{12}\mathfrak{g}_{33})\mathfrak{h}_{13} + \left(\mathfrak{g}_{11}\mathfrak{g}_{33} - \mathfrak{g}_{13}^2\right)\mathfrak{h}_{23} + (\mathfrak{g}_{12}\mathfrak{g}_{13} - \mathfrak{g}_{11}\mathfrak{g}_{23})\mathfrak{h}_{33}, \\
\mathfrak{s}_{31} &= (\mathfrak{g}_{12}\mathfrak{g}_{23} - \mathfrak{g}_{13}\mathfrak{g}_{22})\mathfrak{h}_{11} + (\mathfrak{g}_{12}\mathfrak{g}_{13} - \mathfrak{g}_{11}\mathfrak{g}_{23})\mathfrak{h}_{12} + \left(\mathfrak{g}_{11}\mathfrak{g}_{22} - \mathfrak{g}_{12}^2\right)\mathfrak{h}_{13}, \\
\mathfrak{s}_{32} &= (\mathfrak{g}_{12}\mathfrak{g}_{23} - \mathfrak{g}_{13}\mathfrak{g}_{22})\mathfrak{h}_{12} + (\mathfrak{g}_{12}\mathfrak{g}_{13} - \mathfrak{g}_{11}\mathfrak{g}_{23})\mathfrak{h}_{22} + \left(\mathfrak{g}_{11}\mathfrak{g}_{22} - \mathfrak{g}_{12}^2\right)\mathfrak{h}_{23}, \\
\mathfrak{s}_{33} &= (\mathfrak{g}_{12}\mathfrak{g}_{23} - \mathfrak{g}_{13}\mathfrak{g}_{22})\mathfrak{h}_{13} + (\mathfrak{g}_{12}\mathfrak{g}_{13} - \mathfrak{g}_{11}\mathfrak{g}_{23})\mathfrak{h}_{23} + \left(\mathfrak{g}_{11}\mathfrak{g}_{22} - \mathfrak{g}_{12}^2\right)\mathfrak{h}_{33},
\end{aligned}$$

and

$$\mathfrak{g} = \det(\mathfrak{g}_{ij}) = -\mathfrak{g}_{33}\mathfrak{g}_{12}^2 + 2\mathfrak{g}_{12}\mathfrak{g}_{13}\mathfrak{g}_{23} - \mathfrak{g}_{22}\mathfrak{g}_{13}^2 - \mathfrak{g}_{11}\mathfrak{g}_{23}^2 + \mathfrak{g}_{11}\mathfrak{g}_{22}\mathfrak{g}_{33}.$$

For computing the *i*-th curvature formulas $\mathfrak{C}_i$, where $i = 0, 1, 2, 3$, the characteristic polynomial eq. $P_{\mathbf{S}}(\lambda) = a\lambda^3 + b\lambda^2 + c\lambda + d = 0$, i.e., $P_{\mathbf{S}}(\lambda) = \det(\mathbf{S} - \lambda\mathcal{I}_3) = 0$, is used. Then, $\mathfrak{C}_0 = 1$ (by definition), $\binom{3}{1}\mathfrak{C}_1 = -\frac{b}{a}$, $\binom{3}{2}\mathfrak{C}_2 = \frac{c}{a}$, $\binom{3}{3}\mathfrak{C}_3 = -\frac{d}{a}$. See [46] for the details.

Therefore, the following *i*-th curvature formulas depend on the coefficients of the fundamental forms $(\mathfrak{g}_{ij})$ and $(\mathfrak{h}_{ij})$ in Minkowski four-space.

Theorem 1. *Any hypersurface* $\mathbf{x}$ *in* $\mathbb{E}_1^4$ *has the following curvature formulas,* $\mathfrak{C}_0 = 1$ *(by definition),*

$$\mathfrak{C}_1 = \frac{\left\{\begin{array}{c}(\mathfrak{g}_{11}\mathfrak{h}_{22}+\mathfrak{g}_{22}\mathfrak{h}_{11}-2\mathfrak{g}_{12}\mathfrak{h}_{12})\mathfrak{g}_{33}+(\mathfrak{g}_{11}\mathfrak{g}_{22}-\mathfrak{g}_{12}^2)\mathfrak{h}_{33}\\ -2(\mathfrak{g}_{13}\mathfrak{h}_{13}\mathfrak{g}_{22}-\mathfrak{g}_{23}\mathfrak{h}_{13}\mathfrak{g}_{12}-\mathfrak{g}_{13}\mathfrak{h}_{23}\mathfrak{g}_{12}\\ +\mathfrak{g}_{11}\mathfrak{g}_{23}\mathfrak{h}_{23}-\mathfrak{g}_{13}\mathfrak{g}_{23}\mathfrak{h}_{12})-\mathfrak{g}_{23}^2\mathfrak{h}_{11}-\mathfrak{g}_{13}^2\mathfrak{h}_{22}\end{array}\right\}}{3\left[(\mathfrak{g}_{11}\mathfrak{g}_{22}-\mathfrak{g}_{12}^2)\mathfrak{g}_{33}-\mathfrak{g}_{11}\mathfrak{g}_{23}^2+2\mathfrak{g}_{12}\mathfrak{g}_{13}\mathfrak{g}_{23}-\mathfrak{g}_{22}\mathfrak{g}_{13}^2\right]}, \tag{11}$$

$$\mathfrak{C}_2 = \frac{\left\{\begin{array}{c}(\mathfrak{g}_{11}\mathfrak{h}_{22}+\mathfrak{g}_{22}\mathfrak{h}_{11}-2\mathfrak{g}_{12}\mathfrak{h}_{12})\mathfrak{h}_{33}+(\mathfrak{h}_{11}\mathfrak{h}_{22}-\mathfrak{g}_{12}^2)\mathfrak{g}_{33}\\ -2(\mathfrak{g}_{13}\mathfrak{h}_{13}\mathfrak{h}_{22}-\mathfrak{g}_{23}\mathfrak{h}_{13}\mathfrak{h}_{12}-\mathfrak{g}_{13}\mathfrak{h}_{23}\mathfrak{h}_{12}\\ +\mathfrak{g}_{23}\mathfrak{h}_{23}\mathfrak{h}_{11}-\mathfrak{h}_{13}\mathfrak{h}_{23}\mathfrak{g}_{12})-\mathfrak{g}_{11}\mathfrak{h}_{23}^2-\mathfrak{g}_{22}\mathfrak{h}_{13}^2\end{array}\right\}}{3\left[(\mathfrak{g}_{11}\mathfrak{g}_{22}-\mathfrak{g}_{12}^2)\mathfrak{g}_{33}-\mathfrak{g}_{11}\mathfrak{g}_{23}^2+2\mathfrak{g}_{12}\mathfrak{g}_{13}\mathfrak{g}_{23}-\mathfrak{g}_{22}\mathfrak{g}_{13}^2\right]}, \tag{12}$$

$$\mathfrak{C}_3 = \frac{(\mathfrak{h}_{11}\mathfrak{h}_{22}-\mathfrak{h}_{12}^2)\mathfrak{h}_{33}-\mathfrak{h}_{11}\mathfrak{h}_{23}^2+2\mathfrak{h}_{12}\mathfrak{h}_{13}\mathfrak{h}_{23}-\mathfrak{h}_{22}\mathfrak{h}_{13}^2}{(\mathfrak{g}_{11}\mathfrak{g}_{22}-\mathfrak{g}_{12}^2)\mathfrak{g}_{33}-\mathfrak{g}_{11}\mathfrak{g}_{23}^2+2\mathfrak{g}_{12}\mathfrak{g}_{13}\mathfrak{g}_{23}-\mathfrak{g}_{22}\mathfrak{g}_{13}^2}. \tag{13}$$

Proof. The proof is the same as the Euclidean case. □

See [46] for the details of the Euclidean four-space. A hypersurface $\mathbf{x}$ in $\mathbb{E}_1^4$ is i-minimal, when $\mathfrak{C}_i = 0,\ i = 1, 2, 3$, identically on $\mathbf{x}$.

4. Generalized Helical Hypersurfaces Having Time-like Axis

In this section, the definition of the generalized helical hypersurface with a time-like axis in $\mathbb{E}_1^4$ is given. Let $\gamma : \mathbf{I} \subset \mathbb{R} \longrightarrow \Pi$ be a curve in a plane Π and ℓ be a straight line in Π in $\mathbb{E}_1^4$.

A rotational hypersurface in $\mathbb{E}_1^4$ is defined as a hypersurface rotating a curve γ around a line ℓ (called the *profile curve* and the *axis*, respectively). Suppose that when a profile curve γ rotates around the axis ℓ, it simultaneously displaces parallel lines orthogonal to the axis ℓ, so that the speed of displacement is proportional to the speed of rotation. Then, the resulting hypersurface is called the *generalized helical hypersurface* having axis ℓ and pitches $a, b \in \mathbb{R}\backslash\{0\}$.

Supposing that ℓ is the line spanned by the time-like vector $(0,0,0,1)^t$, the semi-orthogonal matrix fixing the above vector is defined by $\mathcal{R}(v, w)$, where $v, w \in \mathbb{R}$. The matrix $\mathcal{R}$ can be found by solving the following equations, simultaneously, $\det \mathcal{R} = 1$, $\mathcal{R}\ell = \ell$, $\mathcal{R}^t \varepsilon \mathcal{R} = \varepsilon$, where $\varepsilon = $diag$(1,1,1,-1)$. When the axis of rotation is ℓ, there is a Minkowskian transformation by which the axis ℓ is transformed to the x_4-axis of $\mathbb{E}_1^4$. The parametrization of the profile curve is given by $\gamma(u) = (f(u), 0, 0, g(u))$, where $g(u)$ is a differentiable function for all $u \in I$. Therefore, the generalized helical hypersurface spanned by the time-like vector $\ell = (0,0,0,1)$ with pitches $a, b \in \mathbb{R}\backslash\{0\}$ is as follows $\mathbf{x}(u, v, w) = \mathcal{R}(v, w)\gamma(u)^t + (av + bw)\ell^t$ in $\mathbb{E}_1^4$, where $u \in I$, $v, w \in [0, 2\pi)$. If $w = 0$, the generalized helical surface with a time-like axis as in three-dimensional Minkowski space $\mathbb{E}_1^3$ is obtained. When $a = b = 0$, the surface is just a generalized rotational hypersurface with a time-like axis.

Hence, the generalized helical hypersurface with a time-like axis is given by

$$\mathbf{x}(u, v, w) = (\ f(u)\cos v \cos w,\ f(u)\sin v \cos w,\ f(u)\sin w,\ g(u) + av + bw\ ), \tag{14}$$

where $u, a, b \in \mathbb{R} \setminus \{0\}$ and $0 \le v, w < 2\pi$.

See Figure 1 for the projection of the hypersurface $\mathbf{x}$ into the three-space.

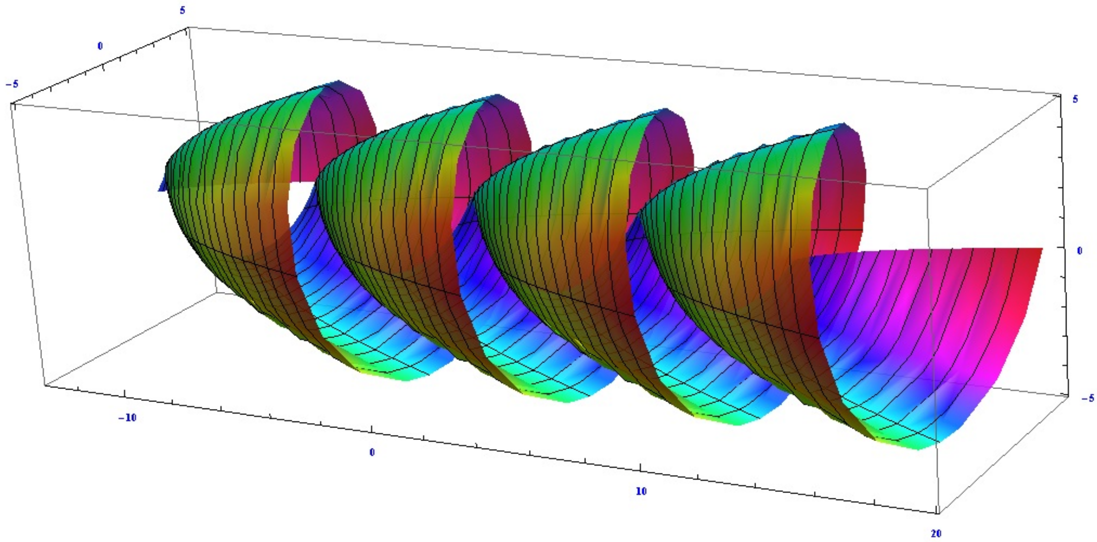

Figure 1. Helical surface having a time-like axis.

Using the first derivatives of (14) with respect to u, v, w, the first fundamental form matrix is presented as

$$(\mathfrak{g}_{ij}) = \begin{pmatrix} f'^2 - g'^2 & -ag' & -bg' \\ -ag' & f^2(\mathcal{C}_2)^2 - a^2 & -ab \\ -bg' & -ab & f^2 - b^2 \end{pmatrix}. \tag{15}$$

Here, $f = f(u)$, $f' = \frac{df}{du}$, $g = g(u)$, $g' = \frac{dg}{du}$. Then,

$$\mathfrak{g} = \det(\mathfrak{g}_{ij}) = f^2 W,$$

where $W = \left((f^2 - b^2)(\mathcal{C}_2)^2 - a^2\right)f'^2 - f^2 g'^2 (\mathcal{C}_2)^2$.

Definition 3. *For any curve $\gamma(u)$ or hypersurface $\mathbf{x} = \mathbf{x}(u, v, w)$ in Minkowski four-space, the following holds:*

i. *When $\langle \gamma', \gamma' \rangle > 0$ (respectively, $\mathfrak{g} > 0$), the curve γ (respectively, the hypersurface $\mathbf{x}$) is called space-like;*

ii. *When $\langle \gamma', \gamma' \rangle < 0$ (respectively, $\mathfrak{g} < 0$), the curve γ (respectively, the hypersurface $\mathbf{x}$) is called time-like;*

iii. *When $\langle \gamma', \gamma' \rangle = 0$ (respectively, $\mathfrak{g} = 0$), the curve γ (respectively, the hypersurface $\mathbf{x}$) is called light-like (or null).*

Here, $\gamma' = \frac{d\gamma}{du}$.

Corollary 1. *When the profile curve $\gamma(u) = (f(u), 0, 0, g(u))$ of the generalized helical hypersurface having time-like axis (14) is the unit speed curve, then $\langle \gamma', \gamma' \rangle = f'^2 - g'^2 = 1 > 0$, i.e., it is a space-like curve. Hence, the following holds:*

$$\mathfrak{g} = -f^2\left[\left(\left(b^2 - f^2\right)(\mathcal{C}_2)^2 + a^2\right) + \left(b^2(\mathcal{C}_2)^2 + a^2\right)g'^2\right].$$

Corollary 2. *While $\left(b^2(\mathcal{C}_2)^2 + a^2\right)g'^2 < (f^2 - b^2)(\mathcal{C}_2)^2 - a^2$, then $\mathfrak{g} > 0$, and then, the hypersurface is space-like.*

Corollary 3. *When $b^2 - f^2 > 0$, then $\mathfrak{g} < 0$, and then, the hypersurface is time-like.*

Corollary 4. *When the hypersurface having time-like axis (14) is the light-like hypersurface (i.e., $\mathfrak{g} = 0$), then it has the following:*

$$g(u) = \pm \int \sqrt{\frac{(f^2 - b^2)(\mathcal{C}_2)^2 - a^2}{b^2(\mathcal{C}_2)^2 + a^2}} du.$$

The Gauss map of the hypersurface (14) is described by

$$\mathcal{G} = \frac{1}{W^{1/2}} \begin{pmatrix} (fg'\mathcal{C}_2 - bf'\mathcal{S}_2)\mathcal{C}_1\mathcal{C}_2 - af'\mathcal{S}_1 \\ (fg'\mathcal{C}_2 - bf'\mathcal{S}_2)\mathcal{S}_1\mathcal{C}_2 + af'\mathcal{C}_1 \\ (fg'\mathcal{S}_2 + bf'\mathcal{C}_2)\mathcal{C}_2 \\ ff'\mathcal{C}_2 \end{pmatrix}. \tag{16}$$

Using the second derivatives of (14) with respect to u, v, w, the second fundamental form matrix is derived as

$$(\mathfrak{h}_{ij}) = \frac{1}{W^{1/2}} \begin{pmatrix} f(f''g' - f'g'')\mathcal{C}_2 & af'^2\mathcal{C}_2 & bf'^2\mathcal{C}_2 \\ af'^2\mathcal{C}_2 & f(bf'\mathcal{S}_2 - fg'\mathcal{C}_2)(\mathcal{C}_2)^2 & -aff'\mathcal{S}_2 \\ bf'^2\mathcal{C}_2 & -aff'\mathcal{S}_2 & -f^2g'\mathcal{C}_2 \end{pmatrix},$$

where $f'' = \frac{d^2 f}{du^2}$ and $g'' = \frac{d^2 g}{du^2}$. Then,

$$\mathfrak{h} = \det(\mathfrak{h}_{ij}) = \frac{f\mathcal{C}_2}{W^{3/2}} \left\{ \begin{array}{c} f^2\left(-f^2f'g'^2(\mathcal{C}_2)^4 + bff'^2g'(\mathcal{C}_2)^3\mathcal{S}_2 + a^2f'^3(\mathcal{S}_2)^2\right)g'' \\ +\left(f^4f''(\mathcal{C}_2)^4\right)g'^3 - \left(bf^3f'f''(\mathcal{C}_2)^3\mathcal{S}_2\right)g'^2 \\ +f\left(f'^4\left(a^2 + b^2(\mathcal{C}_2)^2\right)(\mathcal{C}_2)^2 - a^2ff'^2f''(\mathcal{S}_2)^2\right)g' \\ -bf'^5\left(2a^2 + b^2(\mathcal{C}_2)^2\right)\mathcal{C}_2\mathcal{S}_2 \end{array} \right\}.$$

Hence, by using (10), the shape operator matrix $\mathbf{S}$ of (14) has the following components:

$$\begin{aligned}
\mathfrak{s}_{11} &= -\frac{\left\{ff'\left[(b^2 - f^2)(\mathcal{C}_2)^2 + a^2\right]g'' + \left[-ff''\left((b^2 - f^2)(\mathcal{C}_2)^2 + a^2\right) + f'^2\left(b^2(\mathcal{C}_2)^2 + a^2\right)\right]g'\right\}\mathcal{C}_2}{W^{3/2}}, \\
\mathfrak{s}_{12} &= \frac{a\mathcal{C}_2}{W^{1/2}}, \\
\mathfrak{s}_{13} &= \frac{b[b^2f'^2 - f^2(f'^2 - g'^2)](\mathcal{C}_2)^3 + a^2f'(bf'\mathcal{C}_2 + fg'\mathcal{S}_2)}{W^{3/2}}, \\
\mathfrak{s}_{21} &= \frac{a[fg'(f'g'' - f''g') - f^{2\prime}(f'^2 - g'^2)]\mathcal{C}_2}{W^{3/2}}, \\
\mathfrak{s}_{22} &= \frac{bf'\mathcal{S}_2 - fg'\mathcal{C}_2}{fW^{1/2}}, \\
\mathfrak{s}_{23} &= \frac{af'[-b^2f'^2 + f^2(f'^2 - g'^2)]\mathcal{S}_2}{fW^{3/2}}, \\
\mathfrak{s}_{31} &= \frac{b[fg'(f'g'' - f''g') - f'^2(f'^2 - g'^2)](\mathcal{C}_2)^3}{W^{3/2}}, \\
\mathfrak{s}_{32} &= -\frac{af'\mathcal{S}_2}{fW^{1/2}}, \\
\mathfrak{s}_{33} &= \frac{f[-b^2f'^2 + f^2(f'^2 - g'^2)]g'(\mathcal{C}_2)^3 + a^2f'^2(bf'\mathcal{S}_2 - fg'\mathcal{C}_2)}{fW^{3/2}}.
\end{aligned}$$

Finally, the curvatures of the generalized helical hypersurface with time-like axis $\mathbf{x}$ are given by the following.

Theorem 2. *In $\mathbb{E}_1^4$, the generalized helical hypersurface having time-like axis (14) has the following curvatures, respectively, $\mathfrak{C}_0 = 1$ by definition,*

$$
\begin{aligned}
3\mathfrak{C}_1 &= \frac{p_1 g'' + p_2 g'^3 + p_3 g'^2 + p_4 g' + p_5}{fW^{3/2}}, \\
3\mathfrak{C}_2 &= \frac{(q_1 g'^3 + q_2 g'^2 + q_3 g' + q_4)g'' + (q_5 g'^3 + q_6 g'^2 + q_7 g' + q_8)g'^3 + q_9 g'^2 + q_{10} g' + q_{11}}{fW^3}, \\
\mathfrak{C}_3 &= \frac{(r_1 g'^2 + r_2 g' + r_3)g'' + (r_4 g'^2 + r_5 g' + r_6)g' + r_7}{fW^{5/2}},
\end{aligned}
$$

where

$$
\begin{aligned}
p_1 &= -f^2 f' \left[\left(b^2 - f^2 \right) (\mathcal{C}_2)^2 + a^2 \right] \mathcal{C}_2, \\
p_2 &= -2f^3 (\mathcal{C}_2)^3, \\
p_3 &= b f^2 f' (\mathcal{C}_2)^2 \mathcal{S}_2, \\
p_4 &= f \left[\left(a^2 + b^2 (\mathcal{C}_2)^2 \right) \left(f f'' - 3f'^2 \right) - f^3 f' (\mathcal{C}_2)^2 \right] \mathcal{C}_2, \\
p_5 &= b f'^3 \left[2a^2 + \left(b^2 - f^2 \right) (\mathcal{C}_2)^2 \right] \mathcal{S}_2,
\end{aligned}
$$

$$
\begin{aligned}
q_1 &= f^4 f' \left(\left(b^2 - 2f^2 \right) (\mathcal{C}_2)^2 + a^2 \right) (\mathcal{C}_2)^4, \\
q_2 &= -b f^3 f'^2 \left(\left(b^2 - f^2 \right) (\mathcal{C}_2)^2 + 2a^2 \right) (\mathcal{C}_2)^3 \mathcal{S}_2, \\
q_3 &= f^{3\prime} f^2 \left(\left(b^2 - f^2 \right) (\mathcal{C}_2)^2 + a^2 \right) \left(\left(b^2 - 2f^2 \right) (\mathcal{C}_2)^2 + a^2 \right) (\mathcal{C}_2)^2, \\
q_4 &= -b f f'^4 \left(\left(b^2 - f^2 \right) (\mathcal{C}_2)^2 + 2a^2 \right) \left(\left(b^2 - f^2 \right) (\mathcal{C}_2)^2 + a^2 \right) \mathcal{C}_2 \mathcal{S}_2, \\
q_5 &= f^5 (\mathcal{C}_2)^6, \\
q_6 &= -b f^4 f' (\mathcal{C}_2)^5 \mathcal{S}_2, \\
q_7 &= -f^3 \left\{ \begin{array}{c} f f'' \left((b^2 - 2f^2)(\mathcal{C}_2)^2 + a^2 \right) (\mathcal{C}_2)^2 \\ -f'^2 \left(a^2 \left(4(\mathcal{C}_2)^2 - 1 \right) + (3b^2 - 2f^2)(\mathcal{C}_2)^4 \right) \end{array} \right\} (\mathcal{C}_2)^2, \\
q_8 &= -b f^2 \left\{ \begin{array}{c} f'^3 \left((3b^2 - 2f^2)(\mathcal{C}_2)^2 + 5a^2 \right) \\ -f f' f'' \left((b^2 - f^2)(\mathcal{C}_2)^2 + 2a^2 \right) \end{array} \right\} (\mathcal{C}_2)^3 \mathcal{S}_2, \\
q_9 &= -f f'^2 \left\{ \begin{array}{c} f f'' \left((b^2 - 2f^2)(\mathcal{C}_2)^2 + a^2 \right) \left((b^2 - f^2)(\mathcal{C}_2)^2 + a^2 \right) (\mathcal{C}_2)^2 \\ +a^2 \left\{ \begin{array}{c} 2 \left((2b^2 - f^2)(\mathcal{C}_2)^2 + a^2 \right) (\mathcal{C}_2)^2 \\ - \left((b^2 - 2f^2)(\mathcal{C}_2)^2 + a^2 \right) (\mathcal{S}_2)^2 \end{array} \right\} \end{array} \right\}, \\
q_{10} &= -b f'^3 \left\{ \begin{array}{c} -f f'' (b^2 - f^2)^2 (\mathcal{C}_2)^4 \\ +f'^2 \left((2b^2 - f^2)(\mathcal{C}_2)^2 + 4a^2 \right) \left((b^2 - f^2)(\mathcal{C}_2)^2 + a^2 \right) \end{array} \right\} \\
&\quad + a^2 b f f'^3 f'' \left(3 \left(b^2 - f^2 \right) (\mathcal{C}_2)^2 + 2a^2 \right) \mathcal{C}_2 \mathcal{S}_2, \\
q_{11} &= f f'^6 \left(b^2 (\mathcal{C}_2)^4 + a^2 \right) \left(\left(b^2 - f^2 \right) (\mathcal{C}_2)^2 + a^2 \right),
\end{aligned}
$$

$$
\begin{aligned}
r_1 &= f^4 f' (\mathcal{C}_2)^5, \\
r_2 &= -b f^3 f'^2 (\mathcal{C}_2)^4 \mathcal{S}_2, \\
r_3 &= -a^2 f^2 f'^3 \mathcal{C}_2 (\mathcal{S}_2)^2, \\
r_4 &= -f^4 f'' (\mathcal{C}_2)^5, \\
r_5 &= b f^3 f' f'' (\mathcal{C}_2)^4 \mathcal{S}_2, \\
r_6 &= \left[a^2 f^2 f'^2 f'' (\mathcal{S}_2)^2 - f f'^4 \left(a^2 + b^2 (\mathcal{C}_2)^2 \right) (\mathcal{C}_2)^2 \right] \mathcal{C}_2, \\
r_7 &= b f'^5 \left(2a^2 + b^2 (\mathcal{C}_2)^2 \right) (\mathcal{C}_2)^2 \mathcal{S}_2, \\
W &= f'^2 \left[(f^2 - b^2)(\mathcal{C}_2)^2 - a^2 \right] - f^2 g'^2 (\mathcal{C}_2)^2.
\end{aligned}
$$

Proof. By using the Cayley–Hamilton theorem, the characteristic polynomial eq. $P_{\mathbf{S}}(\lambda) = 0$ of $\mathbf{S}$ for the generalized helical hypersurface with time-like axis $\mathbf{x}$, the following coefficients (in the theorem) of the eq. are obtained:

$$
\mathfrak{C}_0 \lambda^3 - 3\mathfrak{C}_1 \lambda^2 + 3\mathfrak{C}_2 \lambda - \mathfrak{C}_3 = 0.
$$

□

Corollary 5. *When $g = c = const.$, the curvatures are given by*

$$
\begin{aligned}
\mathfrak{C}_1 &= \frac{b\left[2a^2 + \left(b^2 - f^2\right)\cos^2 w\right]\sin w}{3f\left[\left(a^2 + \left(b^2 - f^2\right)\cos^2 w\right)\right]^{3/2}}, \\
\mathfrak{C}_2 &= -\frac{\left(a^2 + b^2 \cos^4 w\right)}{3\left[\left(a^2 + \left(b^2 - f^2\right)\cos^2 w\right)\right]^2}, \\
\mathfrak{C}_3 &= \frac{b\left(2a^2 + b^2 \cos^2 w\right)\cos^2 w \sin w}{f\left[\left(a^2 + \left(b^2 - f^2\right)\cos^2 w\right)\right]^{5/2}}.
\end{aligned}
$$

Corollary 6. *If $g = c = const.$, $w = k\pi$, $k = 0, 1, \ldots,$ the curvatures are described by*

$$
\mathfrak{C}_1 = 0,\ \mathfrak{C}_2 = -\frac{a^2 + b^2}{3\left(a^2 + b^2 - f^2\right)^2},\ \mathfrak{C}_3 = 0.
$$

Corollary 7. *While $g = c = const.$, $w = \pi/2 + k\pi$, $k = 0, 1, \ldots,$ the curvatures are found by*

$$
\mathfrak{C}_1 = \frac{2b}{3af},\ \mathfrak{C}_2 = -\frac{1}{3a^2},\ \mathfrak{C}_3 = 0.
$$

Corollary 8. *When $g = c = const$, $a = b = 0$, then $\mathfrak{C}_i = 0$, $i = 1, 2, 3$, i.e., the generalized helical hypersurface having time-like axis (14) is the i-minimal rotational hypersurface.*

Finally, the following holds:

Theorem 3. *Let $\gamma(u) = (f(u), 0, 0, g(u))$, $u \in I \subset \mathbb{R}$ be a generating curve of the generalized helical hypersurface having a time-like axis given by (14) immersed in $\mathbb{E}^4_1$. Then, the curvatures at the point $(f(u), 0, 0, g(u))$ are functions of the same variable u, i.e., $\mathfrak{C}_i = \mathfrak{C}_i(u)$, $i = 1, 2, 3$. Moreover, given constants $a, b, c \in I \subset \mathbb{R}$ and a function $\mathfrak{C}_i$, the family of curves $\gamma(u) \equiv \gamma(\mathfrak{C}_i, w = c)$ is defined.*

5. The Umbilical Hypersurfaces in Minkowski Four-Space

Before defining the umbilical hypersurface in Minkowski four-space, it can be seen that the curvatures and the principal curvatures of any hypersurface are related as follows:

$$\begin{aligned}\mathfrak{C}_0 &= 1,\\ 3\mathfrak{C}_1 &= k_1 + k_2 + k_3,\\ 3\mathfrak{C}_2 &= k_1k_2 + k_1k_3 + k_2k_3,\\ \mathfrak{C}_3 &= k_1k_2k_3.\end{aligned}$$

Then, the following holds:

Corollary 9. *The following are equivalent:*

$$\begin{aligned} k_1 = k_2 = k_3 &\Leftrightarrow k_1 - k_2 = 0 \wedge k_1 - k_3 = 0 \wedge k_2 - k_3 = 0\\ &\Leftrightarrow (k_1 - k_2)^2 = 0 \wedge (k_1 - k_3)^2 = 0 \wedge (k_2 - k_3)^2 = 0\\ &\Leftrightarrow \left\{\begin{array}{c} k_1^2 - 2k_1k_2 + k_2^2 = 0\\ k_1^2 - 2k_1k_3 + k_3^2 = 0\\ k_2^2 - 2k_2k_3 + k_3^2 = 0 \end{array}\right\}\\ &\Leftrightarrow k_1^2 + k_2^2 + k_3^2 = k_1k_2 + k_1k_3 + k_2k_3 \text{ (adding above eqs.)}\\ &\Leftrightarrow (\mathfrak{C}_1)^2 = \mathfrak{C}_2.\end{aligned}$$

Corollary 10. *The following are equivalent:*

$$\begin{aligned} k_1 = k_2 = k_3 &\Leftrightarrow k_1 - k_2 = 0 \wedge k_1 - k_3 = 0 \wedge k_2 - k_3 = 0\\ &\Leftrightarrow (k_1 - k_2)^2 = 0 \wedge (k_1 - k_3)^2 = 0 \wedge (k_2 - k_3)^2 = 0\\ &\Leftrightarrow \left\{\begin{array}{c} k_1^2 + 2k_1k_2 + k_2^2 = 4k_1k_2\\ k_1^2 + 2k_1k_3 + k_3^2 = 4k_1k_3\\ k_2^2 + 2k_2k_3 + k_3^2 = 4k_2k_3 \end{array}\right\}\\ &\Leftrightarrow (k_1 + k_2)^2 = 4k_1k_2 \wedge (k_1 + k_3)^2 = 4k_1k_3 \wedge (k_2 + k_3)^2 = 4k_2k_3\\ &\Leftrightarrow (k_1 + k_2)(k_1 + k_3)(k_2 + k_3) = 8k_1k_2k_3\\ &\Leftrightarrow \mathfrak{C}_1\mathfrak{C}_2 = \mathfrak{C}_3.\end{aligned}$$

Corollary 11. *Combining Corollary 9 with Corollary 10, the following holds:*

$$k_1 = k_2 = k_3 \Leftrightarrow (\mathfrak{C}_1)^3 = \mathfrak{C}_3.$$

Therefore, the following is given:

Definition 4. *The hypersurface M^3 immersed in a $\mathbb{E}_1^4$ is called the umbilical if all its points are umbilical, i.e., $k_1 = k_2 = k_3$ or, equivalently, $(\mathfrak{C}_1)^3 = \mathfrak{C}_3$ with $\mathfrak{C}_1\mathfrak{C}_2 = \mathfrak{C}_3$, $(\mathfrak{C}_1)^2 = \mathfrak{C}_2$.*

Remark 1. *The only umbilical hypersurfaces are (open) hyperplanes and hyperspheres in $\mathbb{E}_1^4$.*

An umbilical point is an important geometric attribute, closely related to the lines of curvature. It is a singularity of a line of curvature: a line of curvature will end at such points. This may partly be because there is an effective criterion for a smooth hypersurface defined by a formula, for both parametric or implicit hypersurfaces:

Lemma 1. *A point is an umbilical point on the hypersurface in Minkowski four-space if and only if $(\mathfrak{C}_1)^3 = \mathfrak{C}_3$, $\mathfrak{C}_1\mathfrak{C}_2 = \mathfrak{C}_3$, $(\mathfrak{C}_1)^2 = \mathfrak{C}_2$ at this point.*

Hence, the following comes out:

Problem 1. *Solve the following system of differential equations for the generalized helical hypersurface with time-like axis (14):*

$$\left\{\begin{array}{c} p_1 g'' \\ +p_2 g'^3 \\ +p_3 g'^2 + p_4 g' + p_5 \end{array}\right\}^3 = 27 f^2 W^2 \left\{\begin{array}{c} (r_1 g'^2 + r_2 g' + r_3) g'' \\ +(r_4 g'^2 + r_5 g' + r_6) g' \\ +r_7 \end{array}\right\},$$

$$\left\{\begin{array}{c} p_1 g'' \\ +p_2 g'^3 \\ +p_3 g'^2 + p_4 g' + p_5 \end{array}\right\}\left\{\begin{array}{c} (q_1 g'^3 + q_2 g'^2 + q_3 g' + q_4) g'' \\ +(q_5 g'^3 + q_6 g'^2 + q_7 g' + q_8) g'^3 \\ +q_9 g'^2 + q_{10} g' + q_{11} \end{array}\right\} = 9 f W^2 \left\{\begin{array}{c} (r_1 g'^2 + r_2 g' + r_3) g'' \\ +(r_4 g'^2 + r_5 g' + r_6) g' \\ +r_7 \end{array}\right\},$$

$$\left\{\begin{array}{c} p_1 g'' \\ +p_2 g'^3 \\ +p_3 g'^2 + p_4 g' + p_5 \end{array}\right\}^2 = 3 f \left\{\begin{array}{c} (q_1 g'^3 + q_2 g'^2 + q_3 g' + q_4) g'' \\ +(q_5 g'^3 + q_6 g'^2 + q_7 g' + q_8) g'^3 \\ +q_9 g'^2 + q_{10} g' + q_{11} \end{array}\right\}.$$

The $g = g(u)$ solutions of the problem will give the umbilic points of the hypersurface $\mathbf{x}$.

6. Generalized Helical Hypersurface Having Time-like Axis Satisfying $\Delta \mathbf{x} = \mathcal{A}\mathbf{x}$ in $\mathbb{E}^4_1$

In this section, the first Laplace–Beltrami operator (i.e., that depends on the first fundamental form) of a smooth function in $\mathbb{E}^4_1$ is given. Then, the Laplace–Beltrami operator is calculated by using the generalized helical hypersurface (14).

Definition 5. *The Laplace–Beltrami operator of a smooth function $\phi = \phi(x^1, x^2, x^3, x^4) \mid_{\mathbf{D}}$ ($\mathbf{D} \subset \mathbb{R}^4$) of class C^4 depends on the first fundamental form is defined by*

$$\Delta \phi = \frac{1}{\mathfrak{g}^{1/2}} \sum_{i,j=1}^{4} \frac{\partial}{\partial x^i}\left(\mathfrak{g}^{1/2} \mathfrak{g}^{ij} \frac{\partial \phi}{\partial x^j}\right), \quad (17)$$

where $\left(\mathfrak{g}^{ij}\right) = \left(\mathfrak{g}_{kl}\right)^{-1}$ *and* $\mathfrak{g} = \det\left(\mathfrak{g}_{ij}\right)$.

Hence, the inverse matrix $\left(\mathfrak{g}^{ij}\right)$ of the first fundamental form matrix $\left(\mathfrak{g}_{ij}\right)$ of any hypersurface having three parameters in Minkowski four-space is given by

$$\left(\mathfrak{g}^{ij}\right) = \frac{1}{\mathfrak{g}}\left(\begin{array}{ccc} \mathfrak{g}_{22}\mathfrak{g}_{33} - \mathfrak{g}_{23}^2 & \mathfrak{g}_{13}\mathfrak{g}_{23} - \mathfrak{g}_{12}\mathfrak{g}_{33} & \mathfrak{g}_{12}\mathfrak{g}_{23} - \mathfrak{g}_{13}\mathfrak{g}_{22} \\ \mathfrak{g}_{13}\mathfrak{g}_{23} - \mathfrak{g}_{12}\mathfrak{g}_{33} & \mathfrak{g}_{11}\mathfrak{g}_{33} - \mathfrak{g}_{13}^2 & \mathfrak{g}_{12}\mathfrak{g}_{13} - \mathfrak{g}_{11}\mathfrak{g}_{23} \\ \mathfrak{g}_{12}\mathfrak{g}_{23} - \mathfrak{g}_{13}\mathfrak{g}_{22} & \mathfrak{g}_{12}\mathfrak{g}_{13} - \mathfrak{g}_{11}\mathfrak{g}_{23} & \mathfrak{g}_{11}\mathfrak{g}_{22} - \mathfrak{g}_{12}^2 \end{array}\right),$$

where $\mathfrak{g} = -\mathfrak{g}_{33}\mathfrak{g}_{12}^2 + 2\mathfrak{g}_{12}\mathfrak{g}_{13}\mathfrak{g}_{23} - \mathfrak{g}_{22}\mathfrak{g}_{13}^2 - \mathfrak{g}_{11}\mathfrak{g}_{23}^2 + \mathfrak{g}_{11}\mathfrak{g}_{22}\mathfrak{g}_{33}$.

With the help of the above matrix, the inverse matrix of (15) is obtained by

$$\left(\mathfrak{g}^{ij}\right) = \frac{1}{W}\left(\begin{array}{ccc} (f^2 - b^2)(\mathcal{C}_2)^2 - a^2 & a g' & b g'(\mathcal{C}_2)^2 \\ a g' & \left[(f^2 - b^2) f'^2 - f^2 g'^2\right]/f^2 & a b f'^2/f^2 \\ b g'(\mathcal{C}_2)^2 & a b f'^2/f^2 & \left[(f^2(\mathcal{C}_2)^2 - a^2) f'^2 - f^2 g'^2 (\mathcal{C}_2)^2\right]/f^2 \end{array}\right),$$

where $W = \left((f^2 - b^2)(\mathcal{C}_2)^2 - a^2\right) f'^2 - f^2 g'^2 (\mathcal{C}_2)^2$.

By using (17) with the above matrix $\left(\mathfrak{g}^{ij}\right)$ of (14), the following holds.

Theorem 4. *The first Laplace–Beltrami operator of the generalized helical hypersurface (14) is given by*

$$\Delta \mathbf{x} = 4 \mathfrak{C}_1 \mathcal{G},$$

where $\mathfrak{C}_1$ is the mean curvature and $\mathcal{G}$ is the Gauss map of $\mathbf{x}$.

Proof. By directly computing (17) on $\mathbf{x}$, $\Delta \mathbf{x}$ is found. □

Theorem 5. *Let $\mathbf{x} : M^3 \subset \mathbb{E}^3 \longrightarrow \mathbb{E}^4_1$ be an immersion given by (14). $\Delta \mathbf{x} = \mathcal{A}\mathbf{x}$, where $\mathcal{A}$ is the matrix of order four if and only if $\mathbf{x}$ has $\mathfrak{C}_1 = 0$, i.e., it has zero mean curvature.*

Proof. Considering $4\mathfrak{C}_1\mathcal{G} = \mathcal{A}\mathbf{x}$, the following comes out:

$$\begin{aligned} f\mathcal{C}_1\mathcal{C}_2a_{11} + f\mathcal{S}_1\mathcal{C}_2a_{12} + f\mathcal{S}_2a_{13} + (g + av + bw)a_{14} &= \Psi\big((fg'\mathcal{C}_2 - bf'\mathcal{S}_2)\mathcal{C}_1\mathcal{C}_2 - af'\mathcal{S}_1\big), \\ f\mathcal{C}_1\mathcal{C}_2a_{21} + f\mathcal{S}_1\mathcal{C}_2a_{22} + f\mathcal{S}_2a_{23} + (g + av + bw)a_{24} &= \Psi\big((fg'\mathcal{C}_2 - bf'\mathcal{S}_2)\mathcal{S}_1\mathcal{C}_2 + af'\mathcal{C}_1\big), \\ f\mathcal{C}_1\mathcal{C}_2a_{31} + f\mathcal{S}_1\mathcal{C}_2a_{32} + f\mathcal{S}_2a_{33} + (g + av + bw)a_{34} &= \Psi(fg'\mathcal{S}_2 + bf'\mathcal{C}_2)\mathcal{C}_2, \\ f\mathcal{C}_1\mathcal{C}_2a_{41} + f\mathcal{S}_1\mathcal{C}_2a_{42} + f\mathcal{S}_2a_{43} + (g + av + bw)a_{44} &= -\Psi ff'\mathcal{C}_2, \end{aligned}$$

where $\mathcal{A}$ is the 4×4 matrix, $\Psi(u, w) = 4\mathfrak{C}_1/W^{1/2}$. Differentiating the above ODEs twice with respect to v, the following holds:

$$a_{14} = a_{24} = a_{34} = a_{44} = 0,\ \Psi(u, w) = 0. \tag{18}$$

Considering (18), the following appears:

$$\begin{aligned} -f\mathcal{C}_1\mathcal{C}_2a_{11} - f\mathcal{S}_1\mathcal{C}_2a_{12} &= 0, \\ -f\mathcal{C}_1\mathcal{C}_2a_{21} - f\mathcal{S}_1\mathcal{C}_2a_{22} &= 0, \\ -f\mathcal{C}_1\mathcal{C}_2a_{31} - f\mathcal{S}_1\mathcal{C}_2a_{32} &= 0, \\ -f\mathcal{C}_1\mathcal{C}_2a_{41} - f\mathcal{S}_1\mathcal{C}_2a_{42} &= 0. \end{aligned}$$

Taking into account that the functions sin and cos are linear independent of v, all the components of the matrix $\mathcal{A}$ are 0. Since $\Psi = 4\mathfrak{C}_1/\mathcal{W}^{1/2}$, then $\mathfrak{C}_1 = 0$. This means $\mathbf{x}$ is a one-minimal generalized helical hypersurface having a time-like axis. □

7. Summary

In this work, the definition of the generalized helical hypersurface having a time-like axis in four-dimensional Minkowski spacetime is given. The related differential geometric objects such as the fundamental form matrices, the Gauss map, the shape operator matrix, the curvatures, etc., of these kinds of hypersurfaces were calculated. The curvature formulas depend on the coefficients of the first and the second fundamental forms. The umbilical conditions of that kind of hypersurface were also indicated. The Laplace–Beltrami operator of the generalized helical hypersurfaces having a time-like axis was given. Thanks to three-dimensional projections, visual graphics were added to this work.

Time-like worldlines in general relativity correspond to physical, causal trajectories of mass particles. Therefore, we hope that this paper can be used for real-world applications in the near future.

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References

1. Obata, M. Certain conditions for a Riemannian manifold to be isometric with a sphere. *J. Math. Soc. Jpn.* **1962**, *14*, 333–340. [CrossRef]
2. Takahashi, T. Minimal immersions of Riemannian manifolds. *J. Math. Soc. Jpn.* **1966**, *18*, 380–385. [CrossRef]
3. Chern, S.S.; Do Carmo, M.P.; Kobayashi, S. *Minimal Submanifolds of a Sphere with Second Fundamental Form of Constant Length, Functional Analysis and Related Fields*; Springer: Berlin/Heidelberg, Germany, 1970.
4. Cheng, S.Y.; Yau, S.T. Hypersurfaces with constant scalar curvature. *Math. Ann.* **1977**, *225*, 195–204. [CrossRef]
5. Lawson, H.B. *Lectures on Minimal Submanifolds*, 2nd ed.; Mathematics Lecture Series 9; Publish or Perish, Inc.: Wilmington, DE, USA, 1980.
6. Chen, B.Y. On submanifolds of finite type. *Soochow J. Math.* **1983**, *9*, 65–81.
7. Chen, B.Y. *Total Mean Curvature and Submanifolds of Finite Type*; World Scientific: Singapore, 1984.
8. Chen, B.Y. *Finite Type Submanifolds and Generalizations*; University of Rome: Rome, Italy , 1985.
9. Chen, B.Y. Finite type submanifolds in pseudo-Euclidean spaces and applications. *Kodai Math. J.* **1985**, *8*, 358–374. [CrossRef]

10. Barros, M.; Chen, B.Y. Stationary 2-type surfaces in a hypersphere. *J. Math. Soc. Jpn.* **1987**, *39*, 627–648. [CrossRef]
11. Barros, M.; Garay, O.J. 2-type surfaces in S^3. *Geom. Dedicata* **1987**, *24*, 329–336. [CrossRef]
12. Garay, O.J. An extension of Takahashi's theorem. *Geom. Dedicata* **1990**, *34*, 105–112. [CrossRef]
13. Chen, B.Y.; Piccinni, P. Submanifolds with finite type Gauss map. *Bull. Aust. Math. Soc.* **1987**, *35*, 161–186. [CrossRef]
14. Bour, E. Theorie de la deformation des surfaces. *J. Ecole Imp. Polytech.* **1862**, *22*, 1–148.
15. Do Carmo, M.P.; Dajczer, M. Helicoidal surfaces with constant mean curvature. *Tohoku Math. J.* **1982**, *34*, 351–367. [CrossRef]
16. Ferrandez, A.; Garay, O.J.; Lucas, P. On a certain class of conformally at Euclidean hypersurfaces. In *Global Analysis and Global Differential Geometry*; Springer: Berlin/Heidelberg, Germany, 1990; pp. 48–54.
17. Choi, M.; Kim, Y.H. Characterization of the helicoid as ruled surfaces with pointwise 1-type Gauss map. *Bull. Korean Math. Soc.* **2001**, *38*, 753–761.
18. Garay, O.J. On a certain class of finite type surfaces of revolution. *Kodai Math. J.* **1988**, *11*, 25–31. [CrossRef]
19. Dillen, F.; Pas, J.; Verstraelen, L. On surfaces of finite type in Euclidean 3-space. *Kodai Math. J.* **1990**, *13*, 10–21. [CrossRef]
20. Stamatakis, S.; Zoubi, H. Surfaces of revolution satisfying $\Delta^{III}x = Ax$. *J. Geom. Graph.* **2010**, *14*, 181–186.
21. Senoussi, B.; Bekkar, M. Helicoidal surfaces with $\Delta^{J}r = Ar$ in 3-dimensional Euclidean space. *Stud. Univ. Babeş-Bolyai Math.* **2015**, *60*, 437–448.
22. Kim, D.S.; Kim, J.R.; Kim, Y.H. Cheng–Yau operator and Gauss map of surfaces of revolution. *Bull. Malays. Math. Sci. Soc.* **2016**, *39*, 1319–1327. [CrossRef]
23. Beneki, C.C.; Kaimakamis, G.; Papantoniou, B.J. Helicoidal surfaces in three-dimensional Minkowski space. *J. Math. Anal. Appl.* **2002**, *275*, 586–614. [CrossRef]
24. Güler, E.; Turgut Vanlı, A. Bour's theorem in Minkowski 3-space. *J. Math. Kyoto Univ.* **2006**, *46*, 47–63. [CrossRef]
25. Güler, E. Bour's theorem and lightlike profile curve. *Yokohama Math. J.* **2007**, *54*, 55–77.
26. Mira, P.; Pastor, J.A. Helicoidal maximal surfaces in Lorentz-Minkowski space. *Monatsh. Math.* **2003**, *140*, 315–334. [CrossRef]
27. Kim, Y.H.; Yoon, D.W. Classification of ruled surfaces in Minkowski 3-spaces. *J. Geom. Phys.* **2004**, *49*, 89–100. [CrossRef]
28. Kim, Y.H.; Yoon, D.W. Classifications of rotation surfaces in pseudo-Euclidean space. *J. Korean Math. Soc.* **2004**, *41*, 379–396. [CrossRef]
29. Kim, Y.H.; Yoon, D.W. On the Gauss map of ruled surfaces in Minkowski space. *Rocky Mountain J. Math.* **2005**, *35*, 1555–1581. [CrossRef]
30. Ji, F.; Kim, Y.H. Mean curvatures and Gauss maps of a pair of isometric helicoidal and rotation surfaces in Minkowski 3-space. *J. Math. Anal. Appl.* **2010**, *368*, 623–635. [CrossRef]
31. Ji, F.; Kim, Y.H. Isometries between minimal helicoidal surfaces and rotation surfaces in Minkowski space. *Appl. Math. Comput.* **2013**, *220*, 1–11. [CrossRef]
32. Moore, C. Surfaces of rotation in a space of four dimensions. *Ann. Math.* **1919**, *21*, 81–93. [CrossRef]
33. Moore, C. Rotation surfaces of constant curvature in space of four dimensions. *Bull. Am. Math. Soc.* **1920**, *26*, 454–460. [CrossRef]
34. Hasanis, T.; Vlachos, T. Hypersurfaces in $\mathbb{E}^4$ with harmonic mean curvature vector field. *Math. Nachr.* **1995**, *172*, 145–169. [CrossRef]
35. Cheng, Q.M.; Wan, Q.R. Complete hypersurfaces of $\mathbb{R}^4$ with constant mean curvature. *Monatsh. Math.* **1994**, *118*, 171–204. [CrossRef]
36. Arslan, K.; Bayram, B.K.; Bulca, B.; Kim, Y.H.; Murathan, C.; Öztürk, G. Vranceanu surface in $\mathbb{E}^4$ with pointwise 1-type Gauss map. *Indian J. Pure Appl. Math.* **2011**, *42*, 41–51. [CrossRef]
37. Arslan, K.; Bayram, B.K.; Bulca, B.; Öztü rk, G. Generalized rotation surfaces in $\mathbb{E}^4$. *Results Math.* **2012**, *61*, 315–327. [CrossRef]
38. Magid, M.; Scharlach, C.; Vrancken, L. Affine umbilical surfaces in $\mathbb{R}^4$. *Manuscripta Math.* **1995**, *88*, 275–289. [CrossRef]
39. Scharlach, C. Affine Geometry of Surfaces and Hypersurfaces in $\mathbb{R}^4$. In *Symposium on the Differential Geometry of Submanifolds*, Dillen, F., Simon, U., Vrancken, L.O., Eds.; Un. Valenciennes: Valenciennes, France, 2007; Volume 124, pp. 251–256.
40. Arslan, K.; Deszcz, R.; Yaprak, Ş. On Weyl pseudosymmetric hypersurfaces. *Colloq. Math.* **1997**, *72*, 353–361. [CrossRef]
41. Arslan, K.; Bulca, B.; Milousheva, V. Meridian surfaces in $\mathbb{E}^4$ with pointwise 1-type Gauss map. *Bull. Korean Math. Soc.* **2014**, *51*, 911–922. [CrossRef]
42. Yoon, D.W. Rotation surfaces with finite type Gauss map in $\mathbb{E}^4$. *Indian J. Pure Appl. Math.* **2001**, *32*, 1803–1808.
43. Güler; E.; Magid, M.; Yaylı, Y. Laplace–Beltrami operator of a helicoidal hypersurface in four-space. *J. Geom. Symmetry Phys.* **2016**, *41*, 77–95. [CrossRef]
44. Güler, E.; Hacısalihoğlu, H.H.; Kim, Y.H. The Gauss map and the third Laplace–Beltrami operator of the rotational hypersurface in 4-space. *Symmetry* **2018**, *10*, 398. [CrossRef]
45. Güler, E. Rotational hypersurfaces satisfying $\Delta^{I}R = AR$ in the four-dimensional Euclidean space. *J. Polytech.* **2021**, *24*, 517–520.
46. Güler, E. Fundamental form IV and curvature formulas of the hypersphere. *Malaya J. Mat.* **2020**, *8*, 2008–2011. [CrossRef]
47. Ganchev, G.; Milousheva, V. General rotational surfaces in the 4-dimensional Minkowski space. *Turk. J. Math.* **2014**, *38*, 883–895. [CrossRef]
48. Arvanitoyeorgos, A.; Kaimakamis, G.; Magid, M. Lorentz hypersurfaces in $\mathbb{E}^4_1$ satisfying $\Delta H = \alpha H$. *Illinois J. Math.* **2009**, *53*, 581–590. [CrossRef]
49. Arslan, K.; Milousheva, V. Meridian surfaces of elliptic or hyperbolic type with pointwise 1-type Gauss map in Minkowski 4-space. *Taiwan. J. Math.* **2016**, *20*, 311–332. [CrossRef]

50. Güler, E. Helical hypersurfaces in Minkowski geometry $\mathbb{E}_1^4$. *Symmetry* **2020**, *12*, 1206. [CrossRef]
51. Iliadis, L. Fuzzy algebraic modelling of spatiotemporal timeseries' paradoxes in cosmic scale kinematics. *Mathematics* **2022**, *10*, 622. [CrossRef]
52. Leuenberger, G. Emergence of Minkowski spacetime by simple deterministic graph rewriting. *Universe* **2022**, *8*, 149. [CrossRef]
53. Levi-Civita, T. Famiglie di superficie isoparametriche nellordinario spacio euclideo. *Rend. Acad. Lincei* **1937**, *26*, 355–362.
54. Alias, L.J.; Gürbüz, N. An extension of Takashi theorem for the linearized operators of the highest order mean curvatures. *Geom. Dedicata* **2006**, *121*, 113–127. [CrossRef]
55. Kühnel, W. *Differential Geometry, Curves-Surfaces-Manifolds*; 3rd ed.; Translated from the 2013 German ed.; AMS: Providence, RI, USA, 2015.
56. Do Carmo, M.P.; Dajczer, M. Rotation hypersurfaces in spaces of constant curvature. *Trans. Am. Math. Soc.* **1983**, *277*, 685–709. [CrossRef]