Torus Type Helicoidal Hypersurface in 4-Space

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Abstract

We study torus-type helicoidal hypersurface in the four dimensional Euclidean space $\mathbb{E}^4$. We define torus-type helicoidal hypersurface. Then, we calculate its curvatures with some results.

Keywords: 4-space, torus-type helicoidal hypersurface, curvatures.

1 Introduction

Focusing on the rotational characters in the literature, we meet $[1-6, 8-18, 20, 21, 24-26, 28, 31, 32, 34, 35]$, and many others.

About helicoidal surfaces in Euclidean 3-space, Do Carmo and Dajczer [14] proved that there exists a two-parameter family of helicoidal surfaces isometric to a given helicoidal surface using a result of Bour [7].

Magid, Scharlach and Vrancken [28] introduced the affine umbilical surfaces in 4-space. Vlachos [35] considered hypersurfaces in $\mathbb{E}^4$ with harmonic mean curvature vector field. Scharlach [32] studied on affine geometry of surfaces and hypersurfaces in $\mathbb{E}^4$. Cheng and Wan [11] considered complete hypersurfaces of $\mathbb{E}^4$ with constant mean curvature. Arvanitoyeorgos, Kaimakamais and Magid [6] showed that if the mean curvature vector field of M_1^3 satisfies the equation $\Delta H = \alpha H$ (α a constant), then M_1^3 has constant mean curvature in Minkowski 4-space $\mathbb{E}_1^4$.

General rotational surfaces in $\mathbb{E}^4$ were introduced by Moore [29, 30]. Ganchev and Milousheva [17] considered the analogue of these surfaces in the Minkowski 4-space. Moruz and Munteanu [31] considered hypersurfaces in $\mathbb{E}^4$ defined as the sum of a curve and a surface whose mean curvature vanishes. Verstraelen, Walrave and Yaprak [34] studied on the minimal translation surfaces in $\mathbb{E}^n$ for arbitrary dimension n. Kim and Turgay [26] studied surfaces with L_1-pointwise 1-type Gauss map in the 4-dimensional Euclidean space $\mathbb{E}^4$.

Güler, Magid and Yaylı [21] studied Laplace Beltrami operator of a helicoidal hypersurface in $\mathbb{E}^4$. Güler, Hacisalihoglu and Kim [18] worked on the Gauss map and the third Laplace-Beltrami operator of rotational hypersurface in $\mathbb{E}^4$. Güler, Kaimakamis and Magid [19] introduced the helicoidal hypersurfaces in Minkowski 4-space $\mathbb{E}_1^4$. Güler and Turgay [22] studied Cheng-Yau operator and Gauss map of rotational hypersurfaces in $\mathbb{E}^4$. Moreover; Güler, Turgay and Kim [23] considered L_2 operator and Gauss map of rotational hypersurfaces in $\mathbb{E}^5$. Some relations among the Laplace-Beltrami operator and curvatures of the helicoidal surfaces were shown by Güler, Yaylı and Hacısalihoğlu [24]. Güler and Kişi [20] defined torus type rotational hypersurface in 4-space.

We study the torus-type helicoidal hypersurface in Euclidean 4-space $\mathbb{E}^4$. We give some basic notions of four dimensional Euclidean geometry in section 2. In section 3, we define

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helicoidal hypersurface of four-space. Moreover, we obtain torus-type helicoidal hypersurface, and calculate its curvatures in the last section.

2 Preliminaries

We shall identify a vector (a,b,c,d) with its transpose $(a,b,c,d)^t$ in the rest of this paper. Next, we introduce the first and second fundamental forms, matrix of the shape operator $\mathbf{S}$, Gaussian curvature K, and the mean curvature H of hypersurface $\mathbf{M} = \mathbf{M}(u, v, w)$ in Euclidean 4-space $\mathbb{E}^4$.

Let $\mathbf{M}$ be an isometric immersion of a hypersurface M^3 in $\mathbb{E}^4$. The triple vector product $\overrightarrow{x} \times \overrightarrow{y} \times \overrightarrow{z}$ of $\overrightarrow{x} = (x_1, x_2, x_3, x_4)$, $\overrightarrow{y} = (y_1, y_2, y_3, y_4)$, $\overrightarrow{z} = (z_1, z_2, z_3, z_4)$ on $\mathbb{E}^4$ is defined as follows

$$\begin{pmatrix} x_2y_3z_4 - x_2y_4z_3 - x_3y_2z_4 + x_3y_4z_2 + x_4y_2z_3 - x_4y_3z_2 \\ -x_1y_3z_4 + x_1y_4z_3 + x_3y_1z_4 - x_3z_1y_4 - y_1x_4z_3 + x_4y_3z_1 \\ x_1y_2z_4 - x_1y_4z_2 - x_2y_1z_4 + x_2z_1y_4 + y_1x_4z_2 - x_4y_2z_1 \\ -x_1y_2z_3 + x_1y_3z_2 + x_2y_1z_3 - x_2y_3z_1 - x_3y_1z_2 + x_3y_2z_1 \end{pmatrix}.$$

For a hypersurface $\mathbf{M}$ in $\mathbb{E}^4$ we have

$$\det I = \det \begin{pmatrix} E & F & A \\ F & G & B \\ A & B & C \end{pmatrix} = (EG - F^2)C - A^2G + 2ABF - B^2E,$$

and

$$\det II = \det \begin{pmatrix} L & M & P \\ M & N & T \\ P & T & V \end{pmatrix} = \left(LN - M^2\right)V - P^2N + 2PTM - T^2L,$$

where

$$\begin{array}{l} A = \mathbf{M}_u \cdot \mathbf{M}_w, \;\; B = \mathbf{M}_v \cdot \mathbf{M}_w, \;\; C = \mathbf{M}_w \cdot \mathbf{M}_w, \\ P = \mathbf{M}_{uw} \cdot e, \quad T = \mathbf{M}_{vw} \cdot e, \quad V = \mathbf{M}_{ww} \cdot e, \end{array}$$

e is the Gauss map (i.e., the unit normal vector field). We compute the matrix of the shape operator $\mathbf{S}$, as follows

$$\mathbf{S} = \frac{1}{\det I} \begin{pmatrix} s_{11} & s_{12} & s_{13} \\ s_{21} & s_{22} & s_{23} \\ s_{31} & s_{32} & s_{33} \end{pmatrix}, \tag{1}$$

where

$$\begin{array}{l} s_{11} = ABM - CFM - AGP + BFP + CGL - B^2L, \\ s_{12} = ABN - CFN - AGT + BFT + CGM - B^2M, \\ s_{13} = ABT - CFT - AGV + BFV + CGP - B^2P, \\ s_{21} = ABL - CFL + AFP - BPE + CME - A^2M, \\ s_{22} = ABM - CFM + AFT - BTE + CNE - A^2N, \\ s_{23} = ABP - CFP + AFV - BVE + CTE - A^2T, \\ s_{31} = -AGL + BFL + AFM - BME + GPE - F^2P, \\ s_{32} = -AGM + BFM + AFN - BNE + GTE - F^2T, \\ s_{33} = -AGP + BFP + AFT - BTE + GVE - F^2V. \end{array}$$

So, we get the following formulas of the Gaussian and the mean curvatures

$$\begin{array}{rcl} K & = & \det(\mathbf{S}) = \dfrac{\det II}{\det I} \\ & = & \dfrac{\left(LN - M^2\right)V + 2MPT - P^2N - T^2L}{(EG - F^2)C + 2ABF - A^2G - B^2E}, \end{array}$$

and

$$\begin{aligned} H &= \frac{1}{3} tr\left(\mathbf{S}\right) \\ &= \frac{1}{3\det I}[(EN+GL-2FM)C+(EG-F^2)V \\ &\quad -A^2N-B^2L-2(APG+BTE-ABM-ATF-BPF)]. \end{aligned}$$

A hypersurface $\mathbf{M}$ is minimal, if $H=0$ identically on $\mathbf{M}$.

3 Helicoidal Hypersurface

Next, we define the rotational hypersurface in $\mathbb{E}^4$. For an open interval $I\subset\mathbb{R}$, let $\gamma : I \longrightarrow \Pi$ be a curve in a plane Π in $\mathbb{E}^4$, and let ℓ be a straight line in Π.

A *rotational hypersurface* in $\mathbb{E}^4$ is defined as a hypersurface rotating a curve γ around a line ℓ (these are called the *profile curve* and the *axis*, respectively). Suppose that when a profile curve γ rotates around the axis ℓ, it simultaneously displaces parallel lines orthogonal to the axis ℓ, so that the speed of displacement is proportional to the speed of rotation. Then the resulting hypersurface is called the *helicoidal hypersurface* with axis ℓ and pitchs $b, d \in \mathbb{R}\backslash\{0\}$.

We may suppose that ℓ is the line spanned by the vector $(0,0,0,1)^t$. The orthogonal matrix which fixes the above vector is

$$Z(v,w)=\begin{pmatrix} \cos v\cos w & -\sin v & -\cos v\sin w & 0 \\ \sin v\cos w & \cos v & -\sin v\sin w & 0 \\ \sin w & 0 & \cos w & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{2}$$

where $v,w\in\mathbb{R}$. The matrix Z can be found by solving the following equations simultaneously;

$$Z\ell=\ell,\quad Z^tZ=ZZ^t=I_4,\ \ \det Z=1.$$

When the axis of rotation is ℓ, there is an Euclidean transformation by which the axis is ℓ transformed to the x_4-axis of $\mathbb{E}^4$. Parametrization of the profile curve is given by

$$\gamma(u)=\left(f\left(u\right),0,0,\varphi\left(u\right)\right),$$

where $f\left(u\right),\varphi\left(u\right) : I\subset\mathbb{R}\longrightarrow\mathbb{R}$ are differentiable functions for all $u\in I$. So, the helicoidal hypersurface which is spanned by the vector $(0,0,0,1)$ is as follows

$$\mathbf{H}(u,v,w)=Z(v,w)\gamma(u)^t+\left(bv+dw\right)\ell^t,$$

where $u\in I,\ v,w\in[0,2\pi]$. Clearly, we write helicoidal hypersurface as follows

$$\mathbf{H}(u,v,w)=\begin{pmatrix} f\left(u\right)\cos v\cos w \\ f\left(u\right)\sin v\cos w \\ f\left(u\right)\sin w \\ \varphi\left(u\right)+bv+dw \end{pmatrix}. \tag{3}$$

4 Torus-Type Helicoidal Hypersurface

Taking profile curve as

$$\gamma(u)=\left(a+c\cos u,0,0,c\sin u\right),$$

with the orthogonal matrix Z, then we get torus-type helicoidal hypersurface in $\mathbb{E}^4$ as follows

$$\mathfrak{T}(u,v,w) = \begin{pmatrix} (c+a\cos u)\cos v\cos w \\ (c+a\cos u)\sin v\cos w \\ (c+a\cos u)\sin w \\ a\sin u + bv + dw \end{pmatrix}, \tag{4}$$

where $a,b,c,d \in \mathbb{R}\backslash\{0\}$ and $0 \leq u,v,w \leq 2\pi$.

Using the first differentials of (4) with respect to u,v,w, we get the first quantities as follows

$$I = \begin{pmatrix} a^2 & ab\cos u & ad\cos u \\ ab\cos u & \beta_1 & bd \\ ad\cos u & bd & \beta_2 \end{pmatrix},$$

where

$$\begin{aligned} \beta_1 &= a\,(2c+a\cos u)\cos u\cos^2 w + b^2, \\ \beta_2 &= a\,(2c+a\cos u)\cos u + c^2 + d^2, \end{aligned}$$

and have

$$\det I = a^2\left(\left(2b^2d^2 - b^2\beta_2 - d^2\beta_1\right)\cos^2 u + \left(\beta_1\beta_2 - b^2d^2\right)\right).$$

Using the second differentials with respect to u,v,w, we have the second quantities as follows

$$II = \frac{1}{W}\begin{pmatrix} -a\phi & ab\sin^2 u & ad\sin^2 u \\ ab\sin^2 u & -\phi^2\cos u - d\phi\sin u & b\phi\sin u \\ ad\sin^2 u & b\phi\sin u & -\phi^2\cos u \end{pmatrix},$$

where $W = \sqrt{(a^2-2b^2-d^2)\cos^2 u + 2ac\cos u + a^2 + 2b^2 + d^2}$, $\phi = c + a\cos u$, and get

$$\det II = \frac{a\phi}{W^{3/2}}\begin{pmatrix} -\left(a\cos^2 u + c\cos u + d\sin u\right)\phi^3\cos u \\ +b^2\phi^2\sin^2 u + a\phi\left(b^2+d^2\right)\sin^4 u\cos u \\ +ad\left(2b^2+d^2\right)\sin^5 u \end{pmatrix}.$$

The Gauss map of the helicoidal hypersurface with spacelike axis is

$$e_{\mathfrak{T}} = \frac{1}{D}\begin{pmatrix} (\phi\cos u + d\sin u\sin w)\cos v\cos w + b\sin u\sin v \\ (\phi\cos u\cos w + d\sin u\sin w)\sin v\cos w - b\sin u\cos v \\ (\phi\cos u\sin w - d\sin u\cos w)\cos w \\ \phi\sin u\cos w \end{pmatrix}, \tag{5}$$

where $D = \sqrt{((a^2-d^2)\cos^2 u + 2ac\cos u)\cos^2 w + b^2\sin^2 u}$.

Finally, the Gaussian curvature of the torus-type helicoidal hypersurface is as follows

$$K = \frac{a\phi\Psi(u)}{W^{3/2}\det I},$$

where

$$\begin{aligned} \Psi = &-\left(a\cos^2 u + c\cos u + d\sin u\right)\phi^3\cos u + b^2\phi^2\sin^2 u \\ &+a\left(b^2+d^2\right)\phi\sin^4 u\cos u + ad\left(2b^2+d^2\right)\sin^5 u. \end{aligned}$$

and the mean curvature is as follows

$$H = -\frac{a\Omega(u,w)}{3W\det I},$$

where

$$\begin{aligned}\Omega &= a\phi^2\left(b^2\sin^2 u + a\left(2c + a\cos u\right)\cos u\cos^2 w\right)\cos u \\ &+ [b^2c^2 + a^2\left(a^2 - d^2\right)\cos^4 u - acd^2\cos^3 u + a^2\left(b^2 + 3c^2 + d^2\right)\cos^2 u \\ &+ ac\left(2b^2 + c^2 + d^2\right)\cos u + a^4\cos^4 u\cos^2 w - ad\left(2b^2 + d^2\right)\sin u\cos^2 u \\ &+ a^3c\left(4\cos^2 w + 3\right)\cos^3 u + ad\left(2b^2 + c^2 + d^2\right)\sin u \\ &+ ad\left(2c + a\cos u\right)\left(d\cos^2 w + a\sin u\right)\cos u]\phi \\ &+ 2a\left(b^2c^2 + a\left(b^2 + d^2\cos^2 w\right)\left(2c + a\cos u\right)\cos u\right)\cos u\sin^2 u.\end{aligned}$$

Corollary 1. *Let $\mathfrak{T} : M^3 \longrightarrow \mathbb{E}^4$ be an immersion given by* (4)*. Then M^3 has following Weingarten relation*

$$3\phi\Psi H + W^{1/2}\Omega K = 0.$$

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