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# On the Gauss Map of a Class of Hypersurfaces in $\mathbb{H}^4$

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**Abstract.** We consider hypersurfaces of Riemannian space forms in terms of the type of their Gauss map, spherical Gauss map or hyperbolic Gauss map. We give a brief summary of results on submanifolds with  $L_k$  finite type Gauss map for  $k > 0$ . We also obtain some classification results.

## INTRODUCTION

In the middle of 1980's, B.-Y. Chen started a program to understand the geometry of finite type submanifolds of Euclidean space by considering the spectral decomposition Laplace operator, [1, 2]. Then, the definition of finite type mappings is given in [3]. Namely, a mapping  $\phi$  from a submanifold of  $\mathbb{E}^n$  to another Euclidean space is said to be finite type if it can be expressed as a some of finitely many eigenvectors of Laplace operator.

In particular, the Gauss maps of submanifolds of semi-Euclidean spaces catch interests of a lot of geometers after B.-Y. Chen and Piccini presented the question 'To what extent does the type of the Gauss map of a submanifold of  $ET$  determine the submanifold?' in [4]. They made a general study on compact submanifolds of Euclidean spaces with finite type Gauss map, and for hypersurfaces they proved that a compact hypersurface  $M$  of  $\mathbb{E}^{n+1}$  has 1-type Gauss map if and only if  $M$  is a hypersphere in  $\mathbb{E}^{n+1}$ . Also many geometers studied submanifolds with finite type Gauss map ([5, 6, 7, 4, 8] etc.).

In 2006, Alias and Gürbüz consider similar notion by replacing the Laplace operator by a sequence of operators  $L_0, L_1, \dots, L_{n-1}$  with  $L_0 = \Delta$ . We note that  $L_1, \dots, L_{n-1}$  operators are a natural generalization of the Laplace operator because they extend some classical results corresponding to  $\Delta$  (See below). Then some

In this paper, we consider hypersurfaces of Riemannian space forms with negative constant curvature in terms of the type of their (hyperbolic) Gauss map. The organization of this paper is as follows. First, we describe our basic notation before we summarize some of the basic facts on hypersurfaces of Riemannian space forms and finite type mappings. Then, we give a brief summary of recent results on  $L_k$  finite type maps obtain our classification results. Finally, we obtain some results on hypersurfaces in  $\mathbb{H}^n$ .

## PRELIMINARIES

Let  $R^{n+1}(c)$  denote the  $n + 1$ -dimensional Riemannian space form with the curvature  $c \in \{-1, 0, 1\}$  and  $M$  an hypersurface of  $R^{n+1}(c)$  with the unit normal vector field  $N$ , where we put

$$R^m(c) = \begin{cases} \mathbb{S}^m & \text{if } c = 1, \\ \mathbb{E}^m & \text{if } c = 0, \\ \mathbb{H}^m & \text{if } c = -1 \end{cases}$$

are called  $m$ -sphere, Euclidean  $m$ -space and hyperbolic  $m$ -space, respectively and  $\mathbb{L}^{m+1}$  is the Lorentzian space of dimension  $m + 1$ .

Then, the Gauss and Weingarten formulas given by

$$\begin{aligned}\tilde{\nabla}_X Y &= \nabla_X Y + h(X, Y), \\ \tilde{\nabla}_X N &= -S(X),\end{aligned}$$

define the second fundamental form  $h$  and shape operator  $S$  of  $M$ , where  $\tilde{\nabla}$  and  $\nabla$  are Levi-Civita connections of  $R^{n+1}(c)$  and  $M$ , respectively.

The Gauss and Codazzi equations are given, respectively, by

$$\begin{aligned}R(X, Y)Z &= c(X \wedge Y)Z + A_{h(Y,Z)}X - A_{h(X,Z)}Y, \\ (\tilde{\nabla}_X h)(Y, Z) &= (\tilde{\nabla}_Y h)(X, Z),\end{aligned}\tag{1}$$

$$\tag{2}$$

where  $R$  is the curvature tensor associated with connection  $\nabla$  and  $\tilde{\nabla}h$  is the covariant derivative of  $h$ .

On the other hand, since the shape operator  $S$  is a self-adjoint operator, there exists an orthonormal frame field  $\{e_1, e_2, \dots, e_n\}$  of the tangent bundle of  $M$  such that  $S e_i = \lambda_i e_i$  for some smooth functions  $\lambda_i$  called principle curvatures of  $M$  corresponding to the principle direction  $e_i$ . From the Codazzi equation (1) we have

$$e_i(\lambda_j) = \omega_{ij}(e_j)(\lambda_i - \lambda_j),\tag{3}$$

$$\omega_{ij}(e_l)(\lambda_i - \lambda_j) = \omega_{il}(e_j)(\lambda_i - \lambda_l)\tag{4}$$

for distinct  $i, j, l = 1, 2, \dots, n$ , where  $\omega_{ij}$  are the connection forms of  $M$ .

### $L_k$ Operators

Let  $M$  be a hypersurface of  $R^{n+1}(c)$  with principle curvatures  $\lambda_1, \lambda_2, \dots, \lambda_n$ . Then, the algebraic invariants  $s_1, s_2, \dots, s_n$  of shape operator  $S$  of  $M$  take the form

$$s_k = \sigma_k^n(\lambda_1, \lambda_2, \dots, \lambda_n), \quad 0 \leq k \leq n,$$

where  $\sigma_k^n : \mathbb{R}^n \rightarrow \mathbb{R}$  is the  $k$ -th symmetric function in  $\mathbb{R}^n$  given by

$$\sigma_k^n(t_1, t_2, \dots, t_n) = \sum_{1 \leq i_1 < i_2 < \dots < i_k \leq n} t_{i_1} t_{i_2} \dots t_{i_k}.$$

By the definition, we put  $s_0 = 1$  and  $s_l = 1$  if  $l > n$ . Note that,  $H = \frac{s_1}{n}$  is called the (first) mean curvature of  $M$  while we are going to call  $s_k$  as  $k$ -th mean curvature of  $M$  if  $k > 1$  by an abuse of terminology.

On the other hand, one can define  $k$ -th Newton transformation  $P_k : \Gamma(M) \rightarrow \Gamma(M)$  by  $P_k = s_k P_0 - S \circ P_{k-1}$  and  $P_0 = I$ , where  $\Gamma(M, TN)$  denote all  $N$ -vector fields defined on  $M$  and  $\Gamma(M) = \Gamma(M, TM)$ . Note that because of Cayley-Hamilton theorem, we have  $P_n = 0$ . Then, by using these transformations, one can define the operator  $L_k : C^\infty(M) \rightarrow C^\infty(M)$  by

$$L_k(f) = \text{tr}(P_k \circ \nabla^2 f)$$

for  $k = 0, 1, 2, \dots, n-1$ , where  $\nabla^2 f$  is the Hessian of  $f$  ([9]). Then,  $L_0, L_1, \dots, L_{n-1}$  become a sequence of second order differential operators with  $L_0 = \Delta$  is the usual Laplace operator  $\Delta$  of  $M$  with respect to the induced metric from  $R^{n+1}(c)$  and  $L_1 = \square$  is called the Cheng-Yau operator, [10]. When  $c = 0$ , the operators  $L_1, L_2, \dots, L_{n-1}$  satisfy the following properties which can be seen naturally to generalize some of fundamental results about  $L_0 = \Delta$  (See Sect. b)).

The operator  $L_k$  can be naturally extend to  $C^\infty(M, \mathbb{E}_s^N)$  as following:

$$\begin{aligned}\hat{L}_k : C^\infty(M, \mathbb{E}_s^N) &\longrightarrow C^\infty(M, \mathbb{E}_s^N) \\ X &\longmapsto \hat{L}_k(X) : \langle L_k(X), C \rangle = L_k(\langle X, C \rangle) \text{ whenever } C \in \mathbb{R}^N,\end{aligned}\tag{5}$$

where  $\mathbb{E}_s^N$  denotes the semi-Euclidean  $N$  space with index  $s$  that is  $\mathbb{R}^N$  equipped with the standard non-degenerated inner product of index  $s$ . By abusing the notation, we will put  $L_k = \hat{L}_k$  (See, for example, [9, 11] for the same usage).

## $L_k$ Finite Type Mappings

First let us recall the definition of  $L_k$   $k$ -type mappings

**Definition 1.** [11] Let  $\phi : M \rightarrow E^N$  be a smooth map from a submanifold  $M$  of a Euclidean space. Then, is said to be of  $L_k$   $k$ -type if it can be expressed as

$$\phi = \phi_0 + \phi_1 + \cdots + \phi_k$$

for a constant map  $x_0$  and non-constant smooth maps  $x_1, \dots, x_k$  such that  $L_k x_i = \lambda_i x_i$  for some distinct eigenvalues  $\lambda_1, \lambda_2, \dots, \lambda_k$  of  $L_k$ .

Further, we have

**Definition 2.** [11] Let  $\phi : M \rightarrow E^N$  be a smooth map from a submanifold  $M$  of a Euclidean space. If  $\phi$  is  $k$ -type for a  $k \in \mathbb{N}$ , then it is also said to be  $L_k$  finite type.

From Definition 1 one can observe that a smooth map  $\phi : M \rightarrow E^N$  is  $L_k$  1-type if and only if the equation

$$L_k G = \lambda(G + C) \tag{6}$$

for a constant  $\lambda \in \mathbb{R}$  and a constant vector  $C$ , where  $M$  is a submanifold of a Euclidean space. When the codimension of  $M$  is 1, by replacing the condition of satisfying (6) with a weaker one, one can define pointwise 1-type Gauss map (See, for example [12, 13, 14]).

**Definition 3.** Let  $\phi : M \rightarrow R^{n+1}(c)$  be a map from a hypersurface  $M$  of Euclidean space  $R^{n+1}(c)$ .  $\phi$  is said to be  $L_k$  pointwise 1-type if it satisfies

$$L_k G = f(G + C) \tag{7}$$

for a smooth function  $f \in C^\infty(M)$  and a constant vector  $C \in E(n+1, c)$ . We also have the following definitions for the particular cases:

- An  $L_k$  pointwise 1-type map is said to be of the first kind if (7) is satisfied for  $C = 0$ ; otherwise, it is said to be of the second kind.
- If (7) is satisfied for  $f = 0$ , then  $\phi$  is called  $L_k$  harmonic
- An  $L_k$  pointwise 1-type map is said to be proper if (7) is satisfied for a non-constant function  $f$ .

## RECENT RESULTS ON $L_k$ OPERATORS

In this section, we would like to give a brief summary of the results on  $L_k$  finite type maps which have recently appeared. First we want to present some of results that shows why  $L_1, L_2, \dots, L_{n-1}$  operators can be seen as a natural generalization of  $L_0 = \Delta$ :

- The well-known that Laplace-Beltrami formula  $\Delta x = nHN = s_1 N$  is generalized in [9], where Alias and Gürbüz proved

$$L_k x = s_{k+1} N,$$

where  $x : M \hookrightarrow \mathbb{E}^{n+1} = R^{n+1}(0)$  is an isometric immersion.

- In [15] and [16], a generalization of classical Takahashi Theorem ([17]) had been obtained by considering isometric immersions of codimension 1 into  $\mathbb{E}^{n+1}$  satisfying  $\Delta x = Ax + B$  for some matrices  $A \in \mathbb{R}^{(n+1) \times (n+1)}$  and  $B \in \mathbb{R}^{(n+1) \times 1}$  (See citeDillenetal1990 for the case  $n = 2$ ). In [9], it is proved that an extension of this result holds if the operator  $L_0 = \Delta$  is replaced by  $L_k$ .
- The Gauss map  $G$  of a hypersurface  $M$  of  $\mathbb{E}^{n+1}$  satisfies

$$\Delta G = \nabla s_1 + (s_1^2 - 2s_2)G.$$

In [13, 14], it is proved that  $L_1 G$  also satisfies a similar formula.

## Hypersurfaces with $L_k$ Finite Type Gauss Map

- In [18], Dursun studied hypersurfaces in Minkowski space  $\mathbb{E}_1^{n+1}$  of arbitrary dimension and obtained the following results.

**Theorem 4.** [18] *If an oriented hypersurface  $M$  in the Minkowski space  $\mathbb{E}^{n+1}$  has proper pointwise 1-type Gauss map of the second kind, then the mean curvature  $\alpha$  of  $M$  is non-constant.*

**Theorem 5.** [18] *Let  $M$  be an oriented hypersurface in the Minkowski space  $\mathbb{E}^{n+1}$ . Then  $M$  has proper pointwise 1-type Gauss map of the first kind if and only if  $M$  has constant mean curvature and  $\|S\|^2$  is non-constant.*

- In [13], Kim and Turgay presented the definition of  $L_k$  pointwise 1-type Gauss map for the case  $k = 1$  and  $n = 2$ . In the same paper, authors state

**Open Problem.** Classify surfaces in  $\mathbb{E}^3$  with  $\square$ -1-type Gauss map. In particular they derive the formula for the Gauss map of a surface of  $\mathbb{E}^3$

$$\square G = -\nabla K - 2HKG, \quad (8)$$

where  $K$  and  $H$  are Gaussian and mean curvature of  $M$ . In [13], the following theorems obtained.

**Theorem 6.** [13] *An oriented surface  $M$  in  $\mathbb{E}^3$  has  $\square$ -harmonic Gauss map if and only if it is flat, i.e. its Gaussian curvature vanishes identically.*

**Theorem 7.** [13] *An oriented surface  $M$  in  $\mathbb{E}^3$  has  $\square$ -pointwise 1-type Gauss map of the first kind if and only if it has constant Gaussian curvature.*

**Theorem 8.** [13] *An oriented minimal surface  $M$  in  $\mathbb{E}^3$  has  $\square$ -pointwise 1-type Gauss map if and only if it is an open part of a plane.*

**Theorem 9.** [13] *Let  $M$  be a surface in  $\mathbb{E}^3$  with a constant principal curvature. Then,  $M$  has  $\square$ -pointwise 1-type Gauss map of the first kind if and only if it is either a flat surface or an open part of a sphere.*

- In [13] the following classification result obtained for surfaces with  $L_k$  finite type Gauss map

**Theorem 10.** [13] *Let  $M$  be a surface with constant mean curvature in  $\mathbb{E}^3$ . Then  $M$  has  $\square$ -1 type Gauss map if and only if it is an open part of a sphere, a right circular cylinder or a plane.*

- In [14], helicoidal surfaces of  $\mathbb{E}^3$  is studied in terms of having  $\square$ -pointwise 1-type Gauss map. It is proved that a helicoidal surface with  $\square$ -pointwise 1-type Gauss map of the second must necessarily be a rotational surface with a specifically chosen profile curve. Further, in [19] this study was moved into pseudo-Galilean space  $G_1^3$ .
- On the other hand, if the ambient space is Minkowskian, then the similar results have been very recently obtained by Kim and Turgay when the shape operator of the surface is diagonalizable. On the other hand, if the shape operator is non-diagonalizable then the following results obtained.

**Theorem 11.** [20] *Let  $M$  be a surface in  $\mathbb{E}_1^3$  with non-diagonalizable shape operator whose characteristic polynomial is of the form of  $Q(\lambda) = (\lambda - k)^2$  for a function  $k$ . Then, the followings are equivalent:*

- (i)  $M$  has  $\square$ -pointwise 1-type Gauss map,
- (ii)  $M$  has constant Gaussian curvature, i.e.,  $k$  is constant.
- (iii)  $M$  is a B-scroll.

**Theorem 12.** [20] *Let  $M$  be a surface in  $\mathbb{E}_1^3$  with constant mean curvature and non-diagonalizable shape operator whose characteristic polynomial has complex roots. Then,  $M$  has  $\square$ -pointwise 1-type Gauss map if and only if it has proper  $\square$ -pointwise 1-type Gauss map of the second kind.*

- in [21], Qian and Kim study canal surfaces in  $\mathbb{E}^3$  given by

$$x(s, \theta) = c(s) + r(s) \left( \sin \psi(s) \cos \theta N + \sin \psi(s) \sin \theta B + \cos \psi(s) T \right), \quad (9)$$

for an arc-length parametrized curve  $\alpha(s)$  in  $\mathbb{E}^3$  with the Frenet frame  $\{T, S, B\}$  and a smooth function  $r$  such that  $-r'(s) = \cos \psi(s)$ . They obtain the complete classification of such surfaces with  $L_1$ -pointwise 1-type Gauss map.

- Recently, some classification and characterization theorems for hypersurfaces of Euclidean spaces with  $L_k$  pointwise 1-type Gauss map has been obtained, [22]. In particular Theorem 10 were generalized for hypersurfaces of arbitrary dimensional Euclidean spaces under the restriction of having at most two distinct principle curvatures.
- Recently, in [23], the first named author obtained a classification of hypersurfaces given by

$$x(s, t, u) = \left( \frac{aA(s)^2}{s} + as(t^2 + u^2) + \frac{s}{4a} + \frac{a}{s}, st, su, A(s), \frac{aA(s)^2}{s} + as(t^2 + u^2) - \frac{s}{4a} + \frac{a}{s} \right). \quad (10)$$

with  $(\Delta)$  pointwise 1-type Gauss map.

## HYPERSURFACES IN SPACE FORMS

In this section, we first consider hypersurfaces of the Riemannian space form  $R^{n+1}(\varepsilon)$  for  $\varepsilon = \pm 1$ . Let  $E_\varepsilon^{n+2}$  denote the semi-Riemannian manifold  $(\mathbb{R}^{n+2}, g_\varepsilon = \langle \cdot, \cdot \rangle)$ , where

$$g_\varepsilon = \varepsilon dx_1^2 + dx_2^2 + dx_3^2 + \dots + dx_{n+2}^2$$

for a Cartesian coordinate system  $(x_1, x_2, \dots, x_{n+2})$ . Note that  $E_\varepsilon^{n+2}$  is either a Euclidean space or a Minkowski space subject to  $\varepsilon = 1$  or  $\varepsilon = -1$ . Let  $\hat{\nabla}$  denote the Levi-Civita connection of  $E_\varepsilon^{n+2}$ .

Let  $M$  be a hypersurface of  $R^{n+1}(\varepsilon)$ ,  $x : M \hookrightarrow R^{n+1}(\varepsilon)$  an isometric immersion and  $i : R^{n+1}(\varepsilon) \subset E_\varepsilon^{n+2}$  the canonical inclusion and  $\hat{x} = i \circ x$ .  $h$  and  $\hat{h}$  will stand for the second fundamental forms of  $x$  and  $\hat{x}$ , respectively. Then, we have

$$\hat{h}(X, Y) = i_*(h(X, Y)) - \varepsilon \langle X, Y \rangle \hat{x},$$

whenever  $X, Y$  tangent to  $M$ . Consequently, we have

$$\hat{\nabla}_{e_i} e_j = \nabla_{e_i} e_j + \delta_{ij} (\lambda_i N - \varepsilon \hat{x}), \quad (11)$$

$$\hat{\nabla}_{e_i} i_* N = -\lambda_i e_i, \quad (12)$$

where  $N$  is the unit normal vector field of  $x$ ,  $\lambda_1, \lambda_2, \dots, \lambda_n$  are principle curvatures of  $M$  with the corresponding principle directions  $e_1, e_2, \dots, e_n$ .

The (spherical or hyperbolic) Gauss map of  $M$  is defined by

$$\begin{aligned} G : M &\longrightarrow E_\varepsilon^{n+2} \\ p &\longmapsto N(p), \end{aligned} \quad (13)$$

or, equivalently,  $G = i_* N$ .

**Lemma 13.** *The Gauss map  $G$  of a hypersurface  $M$  of the Riemannian space form  $R^{n+1}(\varepsilon)$  satisfies*

$$L_k G = \nabla_{S_{k+1}} + (s_1 s_{k+1} - (k+2) s_{k+2}) G - \varepsilon (k+1) s_{k+1} \hat{x}. \quad (14)$$

*Proof.* We are going to use the notation

$$\mu_{k,i} = \sigma^{n-1}(\lambda_1, \lambda_2, \dots, \lambda_{i-1}, \lambda_{i+1}, \dots, \lambda_n).$$

Then, we have

$$\nabla_{S_{k+1}} = \sum_{i=1}^n \mu_{k,i} \nabla \lambda_i \quad (15)$$

and

$$L_k = \sum_{i=1}^n \mu_{k,i} (\hat{\nabla}_{\nabla_{e_i} e_i} - \hat{\nabla}_{e_i} \hat{\nabla}_{e_i}).$$

By a direct computation using (13), (11) and (12), we obtain

$$\begin{aligned}
L_k G &= \sum_{i=1}^n \mu_{k,i} \left( \hat{\nabla}_{\nabla_{e_i} e_i} i_* N - \hat{\nabla}_{e_i} \hat{\nabla}_{e_i} i_* N \right) \\
&= \sum_{i=1}^n \mu_{k,i} \left( - \sum_{j \neq i} \lambda_j \omega_{ij}(e_i) e_j + \hat{\nabla}_{e_i} (\lambda_i e_i) \right) \\
&= \sum_{i=1}^n \mu_{k,i} \left( \sum_{j \neq i} (\lambda_i - \lambda_j) \omega_{ij}(e_i) e_j + e_i (\lambda_i) e_i + \lambda_i^2 G - \varepsilon \lambda_i \hat{x} \right) \\
&= \sum_{i=1}^n \mu_{k,i} \nabla \lambda_i + \left( \sum_{i=1}^n \mu_{k,i} \lambda_i^2 \right) G - \left( \sum_{i=1}^n \mu_{k,i} \lambda_i \right) \hat{x},
\end{aligned}$$

where the last equality follows from the Codazzi equation (3). By combining this equation with  $\sum_{i=1}^n \mu_{k,i} \lambda_i^2 = s_1 s_{k+1} - (k+2)s_{k+2}$  and  $\sum_{i=1}^n \mu_{k,i} \lambda_i = (k+1)s_{k+1}$ , we get (14).  $\square$

Next we obtain the following classification of hypersurfaces of  $R^{n+1}(\varepsilon)$  with 1-type Gauss map.

**Theorem 14.** *Let  $M$  be a hypersurface of the Riemannian space form  $R^{n+1}(\varepsilon)$  and  $g$  the induced metric of  $M$  from  $R^{n+1}(\varepsilon)$ . Then,  $M$  has  $L_k$  1-type Gauss map if and only if it belongs to one of the following classes of hypersurfaces.*

- (i) *A totally geodesic hypersurface of  $R^{n+1}(\varepsilon)$ .*
- (ii) *Hypersurfaces with constant  $k+2$ th mean curvature and zero  $k+1$ th mean curvature;*
- (iii) *Totally umbilical hypersurface of  $R^{n+1}(\varepsilon)$  with principle curvatures  $r \neq 0$ , i.e.,  $h(X, Y) = rg(X, Y)N$ ;*

*Proof.* In order to prove the necessary condition, assume that  $M$  has  $L_k$  1-type Gauss map. Then, the equation

$$L_k G = \lambda G + C$$

is satisfied for a constant vector  $C \in E_\varepsilon^{n+2}$  and constant  $\lambda$ . From the above equation and (13), we obtain

$$\lambda G + C = \nabla s_{k+1} + (s_1 s_{k+1} - (k+2)s_{k+2})G - \varepsilon(k+1)s_{k+1} \hat{x}. \quad (16)$$

If  $C = 0$ , then we have case (ii) of the theorem.

Assume that (16) is satisfied for  $C \neq 0$ . Then, (16) gives

$$\langle C, e_i \rangle = \langle \nabla s_{k+1}, e_i \rangle = e_i(s_{k+1}), \quad (17)$$

$$\langle C, \hat{x} \rangle = -(k+1)s_{k+1}. \quad (18)$$

By differentiating (18) along  $e_i$  and taking into account  $\hat{\nabla}_{e_i} \hat{x} = e_i$ , we obtain

$$\langle C, e_i \rangle = -(k+1)e_i(s_{k+1}).$$

By combining the above equation with (17), we get  $e_i(s_{k+1}) = 0$  which yields  $s_{k+1} = c_0$  for a constant  $c_0 \in \mathbb{R}$ . Therefore, (16) becomes

$$C = (s_1 c_0 - (k+2)s_{k+2} - \lambda)G - \varepsilon(k+1)c_0 \hat{x}. \quad (19)$$

By differentiating (19) along  $e_i$ , we obtain

$$0 = \left( c_0 e_i(s_1) - (k+2)e_i(s_{k+2}) \right) G - \left( (s_1 c_0 - (k+2)s_{k+2} - \lambda) k_i - \varepsilon(k+1)c_0 \right) e_i. \quad (20)$$

From (20), we obtain

$$k_i = \frac{\varepsilon(k+1)c_0}{s_1 c_0 - (k+2)s_{k+2} - \lambda} \quad \text{for all } i$$

which yields that  $M$  is totally umbilical. Hence, we have the case (i) or the case (iii) of the theorem subject to  $r = 0$  or  $r \neq 0$ . Hence the proof of necessary condition is completed.

Proof of the sufficient condition follows from a direct computation.  $\square$

## Hypersurfaces in $\mathbb{H}^4$

In order to present an explicit example of hypersurfaces in  $\mathbb{H}^4$  with  $L_1$  1-type hyperbolic Gauss map, we would like to consider the hypersurface  $M$  given by (10) for a constant  $a \neq 0$  and smooth, non-constant function  $A$ . This family of hypersurfaces is obtained in [24].

Note that the principle curvatures of  $M$  are

$$\begin{aligned} k_1 &= \frac{A(3s^2A'^2+1)-3sA^2A'-s(sA''^2A'^3+A')A^3}{((A-sA')^2+1)^{3/2}}, \\ k_2 = k_3 &= \frac{A-sA'}{\sqrt{(A-sA')^2+1}} \end{aligned} \quad (21)$$

with corresponding principle directions  $e_1, e_2, e_3$ , proportional to  $\partial_s, \partial_t, \partial_u$ , respectively. Moreover, the Levi-Civita connection of  $M$

$$\begin{aligned} \nabla_{e_1}e_i = \nabla_{e_2}e_3 = \nabla_{e_3}e_2 = 0, & \quad i = 1, 2, 3, \\ \nabla_{e_\alpha}e_1 = \omega e_\alpha, & \quad \alpha = 2, 3, \quad \omega(s) = \frac{1}{\sqrt{(A(s)-sA'(s))^2+1}}. \end{aligned} \quad (22)$$

We have the following classification theorem.

**Theorem 15.** *Let  $M$  be the hypersurface given by (10) for a constant  $a \neq 0$  and smooth, non-constant function  $A$ . Then, we have the followings.*

- (i)  $M$  has  $L_1$ -pointwise 1-type Gauss map of the first kind if and only if it is 2-minimal.
- (ii)  $M$  has  $L_1$ -pointwise 1-type Gauss map of the second kind if and only if the equation

$$2s_2(k_1\omega e_1(s_2) + k_2e_1e_1(s_2)) + 4(k_1 - k_2)s_2^2 - 3k_2e_1(s_2)^2 = 0 \quad (23)$$

*Proof.* The proof of (i) is a direct consequent of Lemma 13. We will prove (ii). Assume that  $M$  has  $L_1$  pointwise 1-type Gauss map of the second kind. Then from Lemma 13 we have

$$e_1(s_2)e_1 + (s_1s_2 - 3s_3)G + 2s_2\hat{x} = f(G + C) \quad (24)$$

for a non-zero function  $f$  and non-zero constant vector  $C$  because of (21). Therefore, we have

$$C = C_1e_1 + C_2G + C_3\hat{x}$$

for some smooth functions  $C_1, C_2, C_3$ . By taking into account that  $C$  is a constant vector and considering (21), (22), we see that  $C_1, C_2, C_3$  satisfy

$$C_1\omega - C_2k_2 + C_3 = 0, \quad (25)$$

$$e_1(C_1) = C_2k_2 - C_3 \quad (26)$$

$$e_1(C_3) = -C_1 \quad (27)$$

On the other hand, (24) gives

$$fC_1 = e_1(s_2), \quad fC_3 = -2s_2. \quad (28)$$

By combining (27) with (28), we obtain  $f = c_1s_2^{3/2}$  for a non-zero constant  $c_1$ . Therefore, (28) implies

$$C_1 = \frac{e_1(s_2)}{c_1s_2^{3/2}}, \quad C_3 = -2\frac{s_2}{c_1s_2^{3/2}}. \quad (29)$$

By a direct computation using (25), (26) and (29) we obtain (23). The converse follows from a direct computation.  $\square$

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