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Preface

We are very pleased to introduce the abstracts of the 3rd International Intuitionistic Fuzzy Sets and Contemporary Mathematics Conference (IFSCOM2016).

As for previous conferences, the theme was the link between the Mathematics by many valued logics and its applications.

In this context, there is a need to discuss the relationships and interactions between many valued logics and contemporary mathematics.

Finally, in the previous conference, it made successful activities to communicate with scientists working in similar fields and relations between the different disciplines.

This conference has papers in different areas; multi-valued logic, geometry, algebra, applied mathematics, theory of fuzzy sets, intuitionistic fuzzy set theory, mathematical physics, mathematics applications, etc

Thank you to the scientists offering the most significant contribution to this conference.

Thank you to the Scientific Committee Members, Referee Committee Members, Local Committee Members, University Administrators and Mersin University Mathematics Department Scientists.

<div align="center">

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[1]Proceeding book designed by Maya ALTINOK

CONTENTS

SECTINON I
FULLTEXTS

NEW INTEGRAL INEQUALITIES INVOLVING BETA
FUNCTION VIA $P\varphi$-PREINVEX CONVEXITY

SEDA KILINÇ, SÜMEYYE ERMEYDAN, AND HÜSEYIN YILDIRIM

ABSTRACT. In this note, we establish some inequalities, involving the Euler Beta function, of the integral $\int_a^b (x-a)^p (b-x)^q f(x)\,dx$ for functions when a power of the absolute value is $P\varphi-$ preinvex.

1. INTRODUCTION

[27] , Wenjun Lio introduced new inequalities for P−convexity. We establish new Hermite-Hadamard inequalities for quasi-preinvex and P$_\varphi$−preinvex functions.

Let I be an interval in $\mathbb{R}$. Then $f : I \to \mathbb{R}$ is said to be preinvex convex if

$$(1.1) \qquad f\left(x + (1-t)\,e^{i\varphi}\eta(y,x)\right) \le tf(x) + (1-t)f(y)$$

holds for all $x, y \in I$ and $t \in [0,1]$.

The notion of quasi-preinvex functions generalizes the notion P$_\varphi$−preinvex functions. More precisely, a function $f : [a,b] \to \mathbb{R}$ is said to be quasi-preinvex on $[a,b]$ if,

$$(1.2) \qquad f\left(x + (1-t)\,e^{i\varphi}\eta(y,x)\right) \le \max\{f(x), f(y)\}$$

holds for any $x, y \in [a,b]$ and $t \in [0,1]$. Clearly, any preinvex function is a quasi-preinvex function. Furthermore, there exist quasi-preinvex functions which are not preinvex.

The generalized quadrature formula of Gauss-Jacobi type has the form

$$\int_a^b (x-a)^p (b-x)^q f(x)\,dx = \sum_{k=0}^m B_{m,k} f(\gamma_k) + \Re_m[f]$$

for certain B_{m,k,γ_k} and rest term $\Re_m[f]$ $(see\,[22])$.

Let $\mathbb{R}^n$ be Euclidian space and K is said to a nonempty closed in $\mathbb{R}^n$. Let $f : K \to \mathbb{R}$, $\varphi : K \to \mathbb{R}$ and $\eta : K \times K \to \mathbb{R}$ be a continuous functions.

Definition 1.1. ([13]) Let $u \in K$. The set K is said to be $\varphi-$invex at u with respect to η and φ if

$$(1.3) \qquad u + te^{i\varphi}\eta(v,u) \in K$$

for all $u, v \in K$ and $t \in [0,1]$.

Remark 1.1. Some special cases of Definition 2 are as follows.

(1) If $\varphi = 0$, then K is called an invex set.
(2) If $\eta(v, u) = v - u$, then K is called a $\varphi-$convex set.
(3) If $\varphi = 0$ and $\eta(v, u) = v - u$, then K is called a convex set.

Definition 1.2. (*see* [13]) The set $K_{\varphi\eta}$ in $\mathbb{R}^n$ is said to be $\varphi-$invex at u with respect to $\varphi(.)$, if there exists a bifunction $\eta(.,.) : K_{\varphi\eta} \times K_{\varphi\eta} \to \mathbb{R}$, such that

(1.4) $$u + te^{i\varphi}\eta(v, u) \in K_{\varphi\eta}, \quad \forall u, v \in K_{\varphi\eta}, \ t \in [0, 1].$$

Where $\mathbb{R}^n$ Euclidian space. The $\varphi-$invex set $K_{\varphi\eta}$ is also called $\varphi\eta-$connected set. Note that the convex set with $\varphi = 0$ and $\eta(v, u) = v - u$ is a $\varphi-$invex set, but the converse is not true.

Definition 1.3. Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a nonnegative function. A function f on the set $K_{\varphi\eta}$ is said to be $P_\varphi-$preinvex function according to φ and bifunction η. Let $\forall u, v \in I, \ \eta(v, u) > 0$ and $t \in (0, 1)$, then

(1.5) $$f\left(u + te^{i\varphi}\eta(v, u)\right) \leq f(u) + f(v).$$

2. Main Results

In this section, we will give lemma which we use later in this work.

Lemma 2.1. *Let* $f : [a, b] \subset [0, \infty) \to \mathbb{R}$ *be continuous on* $[a, b]$. *Where is* $f \in L\left(\left[a, a + e^{i\varphi}\eta(b, a)\right]\right)$, $p, q > 0$ *and* $a < b$. *Then the following equality holds,*

$$\int_a^{a+e^{i\varphi}\eta(b,a)} (x - a)^p \left(a + e^{i\varphi}\eta(b, a) - x\right)^q f(x)\, dx$$
$$= \left[e^{i\varphi}\eta(b, a)\right]^{p+q+1} \int_0^1 (1 - t)^p t^q f\left(a + (1 - t)e^{i\varphi}\eta(b, a)\right) dt.$$

Proof. By using Definition 3, if left-hand side of equality use $x = a + (1 - t)e^{i\varphi}\eta(b, a)$, we have

$$\int_a^{a+e^{i\varphi}\eta(b,a)} (x - a)^p \left(a + e^{i\varphi}\eta(b, a) - x\right)^q f(x)\, dx$$
$$= \int_0^1 \left((1 - t)e^{i\varphi}\eta(b, a)\right)^p \left(te^{i\varphi}\eta(b, a)\right)^q f\left(a + (1 - t)e^{i\varphi}\eta(b, a)\right) e^{i\varphi}\eta(b, a)\, dt$$
$$= \left[e^{i\varphi}\eta(b, a)\right]^{p+q+1} \int_0^1 (1 - t)^p t^q f\left(a + (1 - t)e^{i\varphi}\eta(b, a)\right) dt,$$

the proof is done. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\Box$

Remark 2.1. If we consider $\eta(b, a) = b - a$ and $\varphi = 0$ in Lemma 1, we obtain Lemma 1 in [27],

$$\int_a^b (x - a)^p (b - x)^q f(x)\, dx$$
$$= [b - a]^{p+q+1} \int_0^1 (1 - t)^p t^q f(at + (1 - t)b)\, dt.$$

Theorem 2.1. *Let* $f : [a, b] \to \mathbb{R}$ *be continuous on* $[a, b]$.
Where is $f \in L\left([a, a + e^{i\varphi}\eta(b, a)]\right)$ *and* $0 \le a < b < \infty$. *If* f *is quasi-preinvex on* $[a, b]$ *,then for some fixed* $p, q > 0$, *we have*

$$
\int\limits_a^{a+e^{i\varphi}\eta(b,a)} (x - a)^p \left(a + e^{i\varphi}\eta(b, a) - x\right)^q f(x)\, dx
$$
$$
\le \left(e^{i\varphi}\eta(b, a)\right)^{p+q+1} \beta(p + 1, q + 1) \max\{f(a), f(b)\},
$$

here $\beta(x, y)$ is the Euler Beta function.

Proof. By using inequality in (1.2), if left-hand side of equality use $x = a + (1 - t)e^{i\varphi}\eta(b, a)$, we have

$$
\int\limits_a^{a+e^{i\varphi}\eta(b,a)} (x - a)^p \left(a + e^{i\varphi}\eta(b, a) - x\right)^q f(x)\, dx
$$
$$
= \int\limits_0^1 \left((1 - t)e^{i\varphi}\eta(b, a)\right)^p \left(te^{i\varphi}\eta(b, a)\right)^q f\left(a + (1 - t)e^{i\varphi}\eta(b, a)\right) e^{i\varphi}\eta(b, a)\, dt
$$
$$
= \left[e^{i\varphi}\eta(b, a)\right]^{p+q+1} \int\limits_0^1 (1 - t)^p t^q f\left(a + (1 - t)e^{i\varphi}\eta(b, a)\right) dt
$$
$$
\le \left(e^{i\varphi}\eta(b, a)\right)^{p+q+1} \beta(p + 1, q + 1) \max\{f(a), f(b)\},
$$

the proof is done. $\qquad\square$

Remark 2.2. If we consider $\eta(b, a) = b - a$ and $\varphi = 0$ in Theorem 1, we obtain Theorem 1 in [27]

$$
\int\limits_a^b (x - a)^p (b - x)^q f(x)\, dx
$$
$$
\le (b - a)^{p+q+1} \beta(p + 1, q + 1) \max\{f(a), f(b)\}.
$$

Theorem 2.2. *Let* $f : [a, b] \to \mathbb{R}$ *be continuous on* $[a, b]$.
Where is $f \in L\left([a, a + e^{i\varphi}\eta(b, a)]\right)$, $p, q > 0$ *and* $0 \le a < b < \infty$. *If* $|f|$ *is* P_φ-*preinvex on* $[a, b]$, *then following inequality, we have*

$$
\int\limits_a^{a+e^{i\varphi}\eta(b,a)} (x - a)^p \left(a + e^{i\varphi}\eta(b, a) - x\right)^q f(x)\, dx
$$
$$
\le \left(e^{i\varphi}\eta(b, a)\right)^{p+q+1} \beta(p + 1, q + 1)\left(|f(a)| + |f(b)|\right),
$$

Proof. By using Definition 3, if left-hand side of equality use $x = a + (1 - t)e^{i\varphi}\eta(b, a)$, we have

$$
\int\limits_a^{a+e^{i\varphi}\eta(b,a)} (x - a)^p \left(a + e^{i\varphi}\eta(b, a) - x\right)^q f(x)\, dx
$$
$$
\le \left[e^{i\varphi}\eta(b, a)\right]^{p+q+1} \int\limits_0^1 |(1 - t)^p t^q|\left|f\left(a + (1 - t)e^{i\varphi}\eta(b, a)\right)\right| dt
$$
$$
\le \left[e^{i\varphi}\eta(b, a)\right]^{p+q+1} \int\limits_0^1 \left[(1 - t)^p t^q\right]\left(|f(a)| + |f(b)|\right) dt
$$
$$
\le \left(e^{i\varphi}\eta(b, a)\right)^{p+q+1} \beta(p + 1, q + 1)\left(|f(a)| + |f(b)|\right),
$$

the proof is done. $\qquad\square$

Remark 2.3. If we consider $\eta(b,a) = b - a$ and $\varphi = 0$ in Theorem 2, we obtain Theorem 4 in [27],

$$\int_a^b (x-a)^p (b-x)^q f(x)\, dx$$
$$\leq (b-a)^{p+q+1} \beta(p+1, q+1) \left(|f(a)| + |f(b)| \right).$$

Theorem 2.3. *Let $f : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$.*
Where is $f \in L\left([a, a+e^{i\varphi}\eta(b,a)]\right)$, $p, q > 0$ and $0 \leq a < b < \infty$. If $|f|^{\frac{k}{k-1}}$ is quasi-preinvex on $[a,b]$ and $k > 1$, then following inequality, we have

$$\int_a^{a+e^{i\varphi}\eta(b,a)} (x-a)^p \left(a + e^{i\varphi}\eta(b,a) - x\right)^q f(x)\, dx$$
$$\leq \left(e^{i\varphi}\eta(b,a)\right)^{p+q+1} [\beta(kp+1, kq+1)]^{\frac{1}{k}} \left(\max\left\{ |f(a)|^{\frac{k}{k-1}}, |f(b)|^{\frac{k}{k-1}} \right\} \right)^{\frac{k-1}{k}}.$$

Proof. By using lemma 1, quasi-preinvex of $|f|^{\frac{k}{k-1}}$ and Hölder's inequality, we obtain

$$\int_a^{a+e^{i\varphi}\eta(b,a)} (x-a)^p \left(a + e^{i\varphi}\eta(b,a) - x\right)^q f(x)\, dx$$
$$\leq \left[e^{i\varphi}\eta(b,a)\right]^{p+q+1} \int_0^1 \left|(1-t)^p t^q\right| \left| f\left(a + (1-t) e^{i\varphi}\eta(b,a)\right)\right| dt$$
$$\leq \left[e^{i\varphi}\eta(b,a)\right]^{p+q+1} \left(\int_0^1 \left|(1-t)^p t^q\right|^k dt\right)^{\frac{1}{k}} \left(\int_0^1 \left| f\left(a + (1-t) e^{i\varphi}\eta(b,a)\right)\right|^{\frac{k}{k-1}} dt\right)^{\frac{k-1}{k}}$$
$$\leq \left[e^{i\varphi}\eta(b,a)\right]^{p+q+1} \left(\int_0^1 (1-t)^{kp} t^{kq} dt\right)^{\frac{1}{k}} \left(\int_0^1 \max\left(|f(a)|^{\frac{k}{k-1}}, |f(b)|^{\frac{k}{k-1}}\right) dt\right)^{\frac{k-1}{k}}$$
$$\leq \left(e^{i\varphi}\eta(b,a)\right)^{p+q+1} \beta^{\frac{1}{k}}(kp+1, kq+1) \left[\max\left(|f(a)|^{\frac{k}{k-1}}, |f(b)|^{\frac{k}{k-1}}\right)\right]^{\frac{k-1}{k}},$$

the proof is done. $\square$

Remark 2.4. If we consider $\eta(b,a) = b - a$ and $\varphi = 0$ in Theorem 3, we obtain Theorem 2 in [27],

$$\int_a^b (x-a)^p (b-x)^q f(x)\, dx$$
$$\leq (b-a)^{p+q+1} [\beta(kp+1, kq+1)]^{\frac{1}{k}} \left(\max\left\{ |f(a)|^{\frac{k}{k-1}}, |f(b)|^{\frac{k}{k-1}} \right\} \right)^{\frac{k-1}{k}}.$$

Theorem 2.4. *Let $f : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$.*
Where is $f \in L\left([a, a+e^{i\varphi}\eta(b,a)]\right)$, $p, q > 0$ and $0 \leq a < b < \infty$. If $|f|^{\frac{k}{k-1}}$ is P_φ-preinvex on $[a,b]$ and $k > 1$, then following inequality, we have

$$\int_a^{a+e^{i\varphi}\eta(b,a)} (x-a)^p \left(a + e^{i\varphi}\eta(b,a) - x\right)^q f(x)\, dx$$
$$\leq \left(e^{i\varphi}\eta(b,a)\right)^{p+q+1} [\beta(kp+1, kq+1)]^{\frac{1}{k}} \left(|f(a)|^{\frac{k}{k-1}} + |f(b)|^{\frac{k}{k-1}} \right)^{\frac{k-1}{k}}.$$

Proof. By using lemma 1, P_φ-preinvex of $|f|^{\frac{k}{k-1}}$ and Hölder's inequality, we obtain

$$\int_a^{a+e^{i\varphi}\eta(b,a)} (x-a)^p \left(a + e^{i\varphi}\eta(b,a) - x\right)^q f(x)\,dx$$

$$\leq \left[e^{i\varphi}\eta(b,a)\right]^{p+q+1} \int_0^1 \left|(1-t)^p t^q\right| \left|f\left(a + (1-t)e^{i\varphi}\eta(b,a)\right)\right| dt$$

$$\leq \left[e^{i\varphi}\eta(b,a)\right]^{p+q+1} \left(\int_0^1 \left|(1-t)^p t^q\right|^k dt\right)^{\frac{1}{k}} \left(\int_0^1 \left|f\left(a + (1-t)e^{i\varphi}\eta(b,a)\right)\right|^{\frac{k}{k-1}} dt\right)^{\frac{k-1}{k}}$$

$$\leq \left[e^{i\varphi}\eta(b,a)\right]^{p+q+1} \left(\int_0^1 (1-t)^{kp} t^{kq} dt\right)^{\frac{1}{k}} \left(\int_0^1 \left(|f(a)|^{\frac{k}{k-1}} + |f(b)|^{\frac{k}{k-1}}\right) dt\right)^{\frac{k-1}{k}}$$

$$\leq \left(e^{i\varphi}\eta(b,a)\right)^{p+q+1} \beta^{\frac{1}{k}} (kp+1, kq+1) \left[|f(a)|^{\frac{k}{k-1}} + |f(b)|^{\frac{k}{k-1}}\right]^{\frac{k-1}{k}},$$

The proof is done. $\qquad\square$

Remark 2.5. If we consider $\eta(b,a) = b - a$ and $\varphi = 0$ in Theorem 4, we obtain Theorem 5 in [27],

$$\int_a^b (x-a)^p (b-x)^q f(x)\,dx$$

$$\leq (b-a)^{p+q+1} \left[\beta(kp+1, kq+1)\right]^{\frac{1}{k}} \left(|f(a)|^{\frac{k}{k-1}} + |f(b)|^{\frac{k}{k-1}}\right)^{\frac{k-1}{k}}.$$

Theorem 2.5. *Let $f : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$.*
Where is $f \in L\left([a, a + e^{i\varphi}\eta(b,a)]\right)$, $p, q > 0$ and $0 \leq a < b < \infty$. If $|f|^l$ is quasi-preinvex on $[a,b]$ and $l \geq 1$, then following inequality, we have

$$\int_a^{a+e^{i\varphi}\eta(b,a)} (x-a)^p \left(a + e^{i\varphi}\eta(b,a) - x\right)^q f(x)\,dx$$

$$\leq \left(e^{i\varphi}\eta(b,a)\right)^{p+q+1} \left[\beta(p+1, q+1)\right] \left(\max\left\{|f(a)|^l, |f(b)|^l\right\}\right)^{\frac{1}{l}}.$$

Proof. By using lemma 1, quasi-preinvex of $|f|^l$ and Power Mean inequality, we obtain

$$\int_a^{a+e^{i\varphi}\eta(b,a)} (x-a)^p \left(a + e^{i\varphi}\eta(b,a) - x\right)^q f(x)\,dx$$

$$\leq \left[e^{i\varphi}\eta(b,a)\right]^{p+q+1} \int_0^1 \left|(1-t)^p t^q\right| \left|f\left(a + (1-t)e^{i\varphi}\eta(b,a)\right)\right| dt$$

$$\leq \left[e^{i\varphi}\eta(b,a)\right]^{p+q+1} \left(\int_0^1 \left|(1-t)^p t^q\right| dt\right)^{1-\frac{1}{l}} \left(\int_0^1 \left|(1-t)^p t^q\right| \left|f\left(a + (1-t)e^{i\varphi}\eta(b,a)\right)\right|^l dt\right)^{\frac{1}{l}}$$

$$\leq \left[e^{i\varphi}\eta(b,a)\right]^{p+q+1} \left(\beta(p+1, q+1)\right)^{1-\frac{1}{l}} \left(\int_0^1 \left|(1-t)^p t^q\right| \max\left(|f(a)|^l, |f(b)|^l\right) dt\right)^{\frac{1}{l}}$$

$$\leq \left(e^{i\varphi}\eta(b,a)\right)^{p+q+1} \beta(p+1, q+1) \left[\max\left(|f(a)|^l, |f(b)|^l\right)\right]^{\frac{1}{l}},$$

the proof is done. $\qquad\square$

Remark 2.6. If we consider $\eta(b,a) = b - a$ and $\varphi = 0$ in Theorem 5, we obtain Theorem 3 in [27],

$$\int_a^b (x-a)^p (b-x)^q f(x)\, dx$$

$$\leq (b-a)^{p+q+1} [\beta(p+1, q+1)] \left(\max\left\{ |f(a)|^l, |f(b)|^l \right\} \right)^{\frac{1}{l}}.$$

Theorem 2.6. *Let* $f : [a,b] \to \mathbb{R}$ *be continuous on* $[a,b]$.
Where is $f \in L\left(\left[a, a + e^{i\varphi}\eta(b,a) \right] \right)$, $p, q > 0$ *and* $0 \leq a < b < \infty$. *If* $|f|^l$ *is quasi-preinvex on* $[a,b]$ *and* $l \geq 1$, *then following inequality, we have*

$$\int_a^{a+e^{i\varphi}\eta(b,a)} (x-a)^p \left(a + e^{i\varphi}\eta(b,a) - x \right)^q f(x)\, dx$$

$$\leq \left(e^{i\varphi}\eta(b,a) \right)^{p+q+1} [\beta(p+1, q+1)] \left(|f(a)|^l + |f(b)|^l \right)^{\frac{1}{l}}.$$

Proof. By using lemma 1, P_φ-preinvex of $|f|^l$ and Power Mean inequality, we obtain

$$\int_a^{a+e^{i\varphi}\eta(b,a)} (x-a)^p \left(a + e^{i\varphi}\eta(b,a) - x \right)^q f(x)\, dx$$

$$\leq \left[e^{i\varphi}\eta(b,a) \right]^{p+q+1} \int_0^1 |(1-t)^p t^q| \left| f\left(a + (1-t) e^{i\varphi}\eta(b,a) \right) \right| dt$$

$$\leq \left[e^{i\varphi}\eta(b,a) \right]^{p+q+1} \left(\int_0^1 |(1-t)^p t^q|\, dt \right)^{1-\frac{1}{l}} \left(\int_0^1 |(1-t)^p t^q| \left| f\left(a + (1-t) e^{i\varphi}\eta(b,a) \right) \right|^l dt \right)^{\frac{1}{l}}$$

$$\leq \left[e^{i\varphi}\eta(b,a) \right]^{p+q+1} (\beta(p+1, q+1))^{1-\frac{1}{l}} \left(\int_0^1 |(1-t)^p t^q| \left(|f(a)|^l + |f(b)|^l \right) dt \right)^{\frac{1}{l}}$$

$$\leq \left(e^{i\varphi}\eta(b,a) \right)^{p+q+1} \beta(p+1, q+1) \left[|f(a)|^l + |f(b)|^l \right]^{\frac{1}{l}},$$

the proof is done.

In this section some new integral inequalities for functions of several variables on preinvex subsets of $\mathbb{R}^n$ will be given. First we recall the notion of P_φ-preinvex convexity for functions on a preinvex subset U of $\mathbb{R}^n$.   □

Remark 2.7. If we consider $\eta(b,a) = b - a$ and $\varphi = 0$ in Theorem 6, we obtain Theorem 6 in [27],

$$\int_a^b (x-a)^p (b-x)^q f(x)\, dx$$

$$\leq (b-a)^{p+q+1} [\beta(p+1, q+1)] \left(|f(a)|^l + |f(b)|^l \right)^{\frac{1}{l}}.$$

Definition 2.1. The functions $f : U \to \mathbb{R}$ is said to be P_φ-preinvex convexity on U if it is nonnegative and, for all $x, y \in U$ and $\lambda \in [0,1]$, satisfies the inequality

$$f\left(x + (1-\lambda) e^{i\varphi}\eta(y,x) \right) \leq f(x) + f(y).$$

The following proposition will be used throughout this section.

Proposition 2.1. *Let* $U \subseteq \mathbb{R}$ *be a preinvex subset of* $\mathbb{R}$ *and* $f : U \to \mathbb{R}$ *be a function. Then* f *is* P_φ-*preinvex on* U *if and only if, for every* $x, y \in U$, *the function* $\varphi : [0,1] \to \mathbb{R}$, *defined by*

$$\varphi(t) := f\left(x + t e^{i\varphi}\eta(y,x) \right),$$

is P_φ-convex on I with $I = [0,1]$.

Theorem 2.7. *Let $U \subseteq \mathbb{R}$ be a preinvex subset of $\mathbb{R}$. Assume that $f : U \to \mathbb{R}^+$ is a P_φ-preinvex function on U. Then, for every $x, y \in U$ and every $[a,b] \in [0,1]$ with $a < b$, the following inequality holds:*

$$\int_a^{a+e^{i\varphi}\eta(b,a)} (t-a)^p \left(a + e^{i\varphi}\eta(b,a) - t\right)^q f\left(x + te^{i\varphi}\eta(y,x)\right) dt$$

$$\leq \left(e^{i\varphi}\eta(b,a)\right)^{p+q+1} \beta(p+1, q+1) \left[f\left(x + ae^{i\varphi}\eta(y,x)\right) + f\left(x + be^{i\varphi}\eta(y,x)\right)\right].$$

Proof. Let $x, y \in U$ and every $[a,b] \in [0,1]$ with $a < b$. Since $f : U \to \mathbb{R}^+$ is a P_φ-preinvex function, by Proposition 1 the function $\varphi : [0,1] \to \mathbb{R}^+$ defined by

$$\varphi(t) := f\left(x + te^{i\varphi}\eta(y,x)\right),$$

is P_φ-preinvex on I with $I = [0,1]$. Appling Theorem 4 to the function φ implies that

$$\int_a^{a+e^{i\varphi}\eta(b,a)} (t-a)^p \left(a + e^{i\varphi}\eta(b,a) - t\right)^q \varphi(t) dt$$

$$\leq \left(e^{i\varphi}\eta(b,a)\right)^{p+q+1} \beta(p+1, q+1) \left[|\varphi(a)| + |\varphi(b)|\right]$$
$$\leq \left(e^{i\varphi}\eta(b,a)\right)^{p+q+1} \beta(p+1, q+1) \left[f\left(x + ae^{i\varphi}\eta(y,x)\right) + f\left(x + be^{i\varphi}\eta(y,x)\right)\right],$$

the proof is done. $\qquad\qquad\square$

Remark 2.8. If we consider $\eta(b,a) = b - a$ and $\varphi = 0$ in Theorem 7, we obtain Theorem 7 in [27],

$$\int_a^b (t-a)^p (b-t)^q f((1-t)x + ty) dt$$

$$(b-a)^{p+q+1} \beta(p+1, q+1) \left[f((1-a)x + ay) + f((1-b)x + by)\right].$$

Theorem 2.8. *Let $U \subseteq \mathbb{R}$ be a preinvex subset of $\mathbb{R}$ and let $k > 1$. Assume that $f^{\frac{k}{k-1}} : U \to \mathbb{R}^+$ is a P_φ-preinvex function on U. Then, for every $x, y \in U$ and every $[a,b] \in [0,1]$ with $a < b$, the following inequality holds:*

$$\int_a^{a+e^{i\varphi}\eta(b,a)} (x-a)^p \left(a + e^{i\varphi}\eta(b,a) - x\right)^q f\left(x + te^{i\varphi}\eta(y,x)\right) dx$$

$$\leq \left(e^{i\varphi}\eta(b,a)\right)^{p+q+1} \left[\beta(kp+1, kq+1)\right]^{\frac{1}{k}} \left(f^{\frac{k}{k-1}}\left(x + ae^{i\varphi}\eta(y,x)\right) + f^{\frac{k}{k-1}}\left(x + be^{i\varphi}\eta(y,x)\right)\right)^{\frac{k-1}{k}}.$$

Remark 2.9. If we consider $\eta(b,a) = b - a$ and $\varphi = 0$ in Theorem 8, we obtain Theorem 8 in [27],

$$\int_a^b (t-a)^p (b-t)^q f((1-t)x + ty) dt$$

$$\leq (b-a)^{p+q+1} \left[\beta(kp+1, kq+1)\right]^{\frac{1}{k}} \left(f^{\frac{k}{k-1}}((1-a)x + ay) + f^{\frac{k}{k-1}}((1-b)x + by)\right)^{\frac{k-1}{k}}.$$

Theorem 2.9. *Let $U \subseteq \mathbb{R}$ be a preinvex subset of $\mathbb{R}$ and let $k > 1$. Assume that $f^l : U \to \mathbb{R}^+$ is a P_φ-preinvex function on U. Then, for every $x, y \in U$ and every*

$[a, b] \in [0, 1]$ *with* $a < b$, *the following inequality holds:*

$$\int_{a}^{a+e^{i\varphi}\eta(b,a)} (x-a)^{p} \left(a + e^{i\varphi}\eta(b,a) - x\right)^{q} f\left(x + te^{i\varphi}\eta(y,x)\right) dx$$

$$\leq \left(e^{i\varphi}\eta(b,a)\right)^{p+q+1} \beta(p+1, q+1) \left(f^{l}\left(x + ae^{i\varphi}\eta(y,x)\right) + f^{l}\left(x + be^{i\varphi}\eta(y,x)\right)\right)^{\frac{1}{l}}.$$

Remark 2.10. If we consider $\eta(b,a) = b - a$ and $\varphi = 0$ in Theorem 9, we obtain Theorem 9 in [27],

$$\int_{a}^{b} (t-a)^{p} (b-t)^{q} f\left((1-t)x + ty\right) dt$$

$$\leq (b-a)^{p+q+1} \beta(p+1, q+1) \left(f^{l}\left((1-a)x + ay\right) + f^{l}\left((1-b)x + by\right)\right)^{\frac{1}{l}}.$$

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BOUNDEDNESS OF THE SUBLINEAR OPERATORS WITH ROUGH KERNEL GENERATED BY FRACTIONAL INTEGRALS AND THEIR COMMUTATORS ON GENERALIZED VANISHING MORREY SPACES II

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ABSTRACT. In this paper, we consider the norm inequalities for sublinear operators with rough kernel generated by fractional integrals and their commutators on generalized Morrey spaces and on generalized vanishing Morrey spaces including their weak versions under generic size conditions which are satisfied by most of the operators in harmonic analysis, respectively. In all the cases the conditions for the boundedness of sublinear operators with rough kernel and their commutators are given in terms of Zygmund-type integral inequalities on (φ_1, φ_2), where there is no assumption on monotonicity of φ_1, φ_2 in r. As an example to the conditions of these theorems are satisfied, we can consider the Marcinkiewicz operator.

1. INTRODUCTION

The classical Morrey spaces $M_{p,\lambda}$ have been introduced by Morrey in [32] to study the local behavior of solutions of second order elliptic partial differential equations(PDEs). In recent years there has been an explosion of interest in the study of the boundedness of operators on Morrey-type spaces. It has been obtained that many properties of solutions to PDEs are concerned with the boundedness of some operators on Morrey-type spaces. In fact, better inclusion between Morrey and Hölder spaces allows to obtain higher regularity of the solutions to different elliptic and parabolic boundary problems (see [14, 36, 41, 43] for details).

Let $B = B(x_0, r_B)$ denote the ball with the center x_0 and radius r_B. For a given measurable set E, we also denote the Lebesgue measure of E by $|E|$. For any given $\Omega_0 \subseteq \mathbb{R}^n$ and $0 < p < \infty$, denote by $L_p(\Omega_0)$ the spaces of all functions f satisfying

$$\|f\|_{L_p(\Omega_0)} = \left(\int_{\Omega_0} |f(x)|^p \, dx\right)^{\frac{1}{p}} < \infty.$$

We recall the definition of classical Morrey spaces $M_{p,\lambda}$ as

$$M_{p,\lambda}\left(\mathbb{R}^n\right) = \left\{ f : \|f\|_{M_{p,\lambda}(\mathbb{R}^n)} = \sup_{x \in \mathbb{R}^n, r > 0} r^{-\frac{\lambda}{p}} \|f\|_{L_p(B(x,r))} < \infty \right\},$$

where $f \in L_p^{loc}(\mathbb{R}^n)$, $0 \le \lambda \le n$ and $1 \le p < \infty$.

Note that $M_{p,0} = L_p(\mathbb{R}^n)$ and $M_{p,n} = L_\infty(\mathbb{R}^n)$. If $\lambda < 0$ or $\lambda > n$, then $M_{p,\lambda} = \Theta$, where Θ is the set of all functions equivalent to 0 on $\mathbb{R}^n$.

We also denote by $WM_{p,\lambda} \equiv WM_{p,\lambda}(\mathbb{R}^n)$ the weak Morrey space of all functions $f \in WL_p^{loc}(\mathbb{R}^n)$ for which

$$\|f\|_{WM_{p,\lambda}} \equiv \|f\|_{WM_{p,\lambda}(\mathbb{R}^n)} = \sup_{x \in \mathbb{R}^n,\ r > 0} r^{-\frac{\lambda}{p}} \|f\|_{WL_p(B(x,r))} < \infty,$$

where $WL_p(B(x,r))$ denotes the weak L_p-space of measurable functions f for which

$$
\begin{aligned}
\|f\|_{WL_p(B(x,r))} &\equiv \|f\chi_{B(x,r)}\|_{WL_p(\mathbb{R}^n)} \\
&= \sup_{t>0} t\,|\{y \in B(x,r) : |f(y)| > t\}|^{1/p} \\
&= \sup_{0<t\le|B(x,r)|} t^{1/p} \left(f\chi_{B(x,r)}\right)^*(t) < \infty,
\end{aligned}
$$

where g^* denotes the non-increasing rearrangement of a function g.

Throughout the paper we assume that $x \in \mathbb{R}^n$ and $r > 0$ and also let $B(x,r)$ denotes the open ball centered at x of radius r, $B^C(x,r)$ denotes its complement and $|B(x,r)|$ is the Lebesgue measure of the ball $B(x,r)$ and $|B(x,r)| = v_n r^n$, where $v_n = |B(0,1)|$. It is known that $M_{p,\lambda}(\mathbb{R}^n)$ is an extension (a generalization) of $L_p(\mathbb{R}^n)$ in the sense that $M_{p,0} = L_p(\mathbb{R}^n)$.

Morrey has stated that many properties of solutions to PDEs can be attributed to the boundedness of some operators on Morrey spaces. For the boundedness of the Hardy–Littlewood maximal operator, the fractional integral operator and the Calderón–Zygmund singular integral operator on these spaces, we refer the readers to [1, 6, 38]. For the properties and applications of classical Morrey spaces, see [7, 8, 14, 36, 41, 43] and references therein. The generalized Morrey spaces $M_{p,\varphi}$ are obtained by replacing r^λ with a function $\varphi(r)$ in the definition of the Morrey space. During the last decades various classical operators, such as maximal, singular and potential operators have been widely investigated in classical and generalized Morrey spaces.

The study of the operators of harmonic analysis in vanishing Morrey space, in fact has been almost not touched. A version of the classical Morrey space $M_{p,\lambda}(\mathbb{R}^n)$ where it is possible to approximate by "nice" functions is the so called vanishing Morrey space $VM_{p,\lambda}(\mathbb{R}^n)$ has been introduced by Vitanza in [50] and has been applied there to obtain a regularity result for elliptic PDEs. This is a subspace of functions in $M_{p,\lambda}(\mathbb{R}^n)$, which satisfies the condition

$$\lim_{r \to 0} \sup_{\substack{x \in \mathbb{R}^n \\ 0 < t < r}} t^{-\frac{\lambda}{p}} \|f\|_{L_p(B(x,t))} = 0.$$

Later in [51] Vitanza has proved an existence theorem for a Dirichlet problem, under weaker assumptions than in [30] and a $W^{3,2}$ regularity result assuming that the partial derivatives of the coefficients of the highest and lower order terms belong to vanishing Morrey spaces depending on the dimension. Also Ragusa has proved a sufficient condition for commutators of fractional integral operators to belong to

vanishing Morrey spaces $VM_{p,\lambda}(\mathbb{R}^n)$ (see [39, 40]). For the properties and applications of vanishing Morrey spaces, see also [3]. It is known that, there is no research regarding boundedness of the sublinear operators with rough kernel on vanishing Morrey spaces.

Maximal functions and singular integrals play a key role in harmonic analysis since maximal functions could control crucial quantitative information concerning the given functions, despite their larger size, while singular integrals, Hilbert transform as it's prototype, recently intimately connected with PDEs, operator theory and other fields.

Let $f \in L^{loc}(\mathbb{R}^n)$. The Hardy-Littlewood(H–L) maximal operator M is defined by

$$Mf(x) = \sup_{t>0} |B(x,t)|^{-1} \int\limits_{B(x,t)} |f(y)|dy.$$

Let $\overline{T}$ be a standard Calderón-Zygmund(C–Z) singular integral operator, briefly a C–Z operator, i.e., a linear operator bounded from $L_2(\mathbb{R}^n)$ to $L_2(\mathbb{R}^n)$ taking all infinitely continuously differentiable functions f with compact support to the functions $f \in L_1^{loc}(\mathbb{R}^n)$ represented by

$$\overline{T}f(x) = p.v. \int\limits_{\mathbb{R}^n} k(x-y)f(y)\,dy \qquad x \notin suppf.$$

Such operators have been introduced in [11]. Here k is a C–Z kernel [16]. Chiarenza and Frasca [6] have obtained the boundedness of H–L maximal operator M and C–Z operator $\overline{T}$ on $M_{p,\lambda}(\mathbb{R}^n)$. It is also well known that H–L maximal operator M and C–Z operator $\overline{T}$ play an important role in harmonic analysis (see [15, 29, 46, 47, 48]). Also, the theory of the C–Z operator is one of the important achievements of classical analysis in the last century, which has many important applications in Fourier analysis, complex analysis, operator theory and so on.

Let $f \in L^{loc}(\mathbb{R}^n)$. The fractional maximal operator M_α and the fractional integral operator (also known as the Riesz potential) $\overline{T}_\alpha$ are defined by

$$M_\alpha f(x) = \sup_{t>0} |B(x,t)|^{-1+\frac{\alpha}{n}} \int\limits_{B(x,t)} |f(y)|dy \qquad 0 \le \alpha < n$$

$$\overline{T}_\alpha f(x) = \int\limits_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}} dy \qquad 0 < \alpha < n.$$

It is well known that M_α and $\overline{T}_\alpha$ play an important role in harmonic analysis (see [47, 48]).

An early impetus to the study of fractional integrals originated from the problem of fractional derivation, see e.g. [35]. Besides its contributions to harmonic analysis, fractional integrals also play an essential role in many other fields. The H-L Sobolev inequality about fractional integral is still an indispensable tool to establish time-space estimates for the heat semigroup of nonlinear evolution equations, for some of this work, see e.g. [24]. In recent times, the applications to Chaos and Fractal have become another motivation to study fractional integrals, see e.g. [26]. It is well known that $\overline{T}_\alpha$ is bounded from L_p to L_q, where $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{n}$ and $1 < p < \frac{n}{\alpha}$.

Spanne (published by Peetre [38]) and Adams [1] have studied boundedness of the fractional integral operator $\overline{T}_\alpha$ on $M_{p,\lambda}(\mathbb{R}^n)$. Their results, can be summarized as follows.

Theorem 1.1. *(Spanne, but published by Peetre [38]) Let $0 < \alpha < n$, $1 < p < \frac{n}{\alpha}$, $0 < \lambda < n - \alpha p$. Moreover, let $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{n}$ and $\frac{\lambda}{p} = \frac{\mu}{q}$. Then for $p > 1$ the operator $\overline{T}_\alpha$ is bounded from $M_{p,\lambda}$ to $M_{q,\lambda}$ and for $p = 1$ the operator $\overline{T}_\alpha$ is bounded from $M_{1,\lambda}$ to $WM_{q,\lambda}$.*

Theorem 1.2. *(Adams [1]) Let $0 < \alpha < n$, $1 < p < \frac{n}{\alpha}$, $0 < \lambda < n - \alpha p$ and $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{n-\lambda}$. Then for $p > 1$ the operator $\overline{T}_\alpha$ is bounded from $M_{p,\lambda}$ to $M_{q,\lambda}$ and for $p = 1$ the operator $\overline{T}_\alpha$ is bounded from $M_{1,\lambda}$ to $WM_{q,\lambda}$.*

Recall that, for $0 < \alpha < n$,

$$M_\alpha f(x) \leq \nu_n^{\frac{\alpha}{n}-1} \overline{T}_\alpha(|f|)(x)$$

holds (see [25], Remark 2.1). Hence Theorems 1.1 and 1.2 also imply boundedness of the fractional maximal operator M_α, where υ_n is the volume of the unit ball on $\mathbb{R}^n$.

Suppose that S^{n-1} is the unit sphere in $\mathbb{R}^n$ ($n \geq 2$) equipped with the normalized Lebesgue measure $d\sigma$. Let $\Omega \in L_s(S^{n-1})$ with $1 < s \leq \infty$ be homogeneous of degree zero. We define $s' = \frac{s}{s-1}$ for any $s > 1$. Suppose that $T_{\Omega,\alpha}$, $\alpha \in (0, n)$ represents a linear or a sublinear operator, which satisfies that for any $f \in L_1(\mathbb{R}^n)$ with compact support and $x \notin suppf$

$$(1.1) \qquad |T_{\Omega,\alpha}f(x)| \leq c_0 \int_{\mathbb{R}^n} \frac{|\Omega(x-y)|}{|x-y|^{n-\alpha}} |f(y)| \, dy,$$

where c_0 is independent of f and x.

For a locally integrable function b on $\mathbb{R}^n$, suppose that the commutator operator $T_{\Omega,b,\alpha}$, $\alpha \in (0, n)$ represents a linear or a sublinear operator, which satisfies that for any $f \in L_1(\mathbb{R}^n)$ with compact support and $x \notin suppf$

$$(1.2) \qquad |T_{\Omega,b,\alpha}f(x)| \leq c_0 \int_{\mathbb{R}^n} |b(x) - b(y)| \frac{|\Omega(x-y)|}{|x-y|^{n-\alpha}} |f(y)| \, dy,$$

where c_0 is independent of f and x.

We point out that the condition (1.1) in the case of $\Omega \equiv 1$, $\alpha = 0$ has been introduced by Soria and Weiss in [44]. The conditions (1.1) and (1.2) are satisfied by many interesting operators in harmonic analysis, such as fractional Marcinkiewicz operator, fractional maximal operator, fractional integral operator (Riesz potential) and so on (see [27], [44] for details).

In 1971, Muckenhoupt and Wheeden [34] defined the fractional integral operator with rough kernel $\overline{T}_{\Omega,\alpha}$ by

$$\overline{T}_{\Omega,\alpha}f(x) = \int_{\mathbb{R}^n} \frac{\Omega(x-y)}{|x-y|^{n-\alpha}} f(y)dy \qquad 0 < \alpha < n$$

and a related fractional maximal operator with rough kernel $M_{\Omega,\alpha}$ is given by

$$M_{\Omega,\alpha}f(x) = \sup_{t>0} |B(x,t)|^{-1+\frac{\alpha}{n}} \int\limits_{B(x,t)} |\Omega(x-y)| |f(y)|dy \qquad 0 \leq \alpha < n,$$

where $\Omega \in L_s(S^{n-1})$ with $1 < s \leq \infty$ is homogeneous of degree zero on $\mathbb{R}^n$ and $\overline{T}_{\Omega,\alpha}$ satisfies the condition (1.1).

If $\alpha = 0$, then $M_{\Omega,0} \equiv M_\Omega$ H-L maximal operator with rough kernel. It is obvious that when $\Omega \equiv 1$, $M_{1,\alpha} \equiv M_\alpha$ and $\overline{T}_{1,\alpha} \equiv \overline{T}_\alpha$ are the fractional maximal operator and the fractional integral operator, respectively.

In recent years, the mapping properties of $\overline{T}_{\Omega,\alpha}$ on some kinds of function spaces have been studied in many papers (see [5], [12], [13], [34] for details). In particular, the boundedness of $\overline{T}_{\Omega,\alpha}$ in Lebesgue spaces has been obtained.

Lemma 1.1. [5, 12, 33] *Let* $0 < \alpha < n$, $1 < p < \frac{n}{\alpha}$ *and* $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$. *If* $\Omega \in L_s(S^{n-1})$, $s > \frac{n}{n-\alpha}$, *then we have*

$$\left\|\overline{T}_{\Omega,\alpha}f\right\|_{L_q} \leq C \left\|f\right\|_{L_p}.$$

Corollary 1.1. *Under the assumptions of Lemma 1.1, the operator* $M_{\Omega,\alpha}$ *is bounded from* $L_p(\mathbb{R}^n)$ *to* $L_q(\mathbb{R}^n)$. *Moreover, we have*

$$\left\|M_{\Omega,\alpha}f\right\|_{L_q} \leq C \left\|f\right\|_{L_p}.$$

Proof. Set

$$\widetilde{T}_{|\Omega|,\alpha}(|f|)(x) = \int\limits_{\mathbb{R}^n} \frac{|\Omega(x-y)|}{|x-y|^{n-\alpha}} |f(y)| \, dy \qquad 0 < \alpha < n,$$

where $\Omega \in L_s(S^{n-1})\,(s > 1)$ is homogeneous of degree zero on $\mathbb{R}^n$. It is easy to see that, for $\widetilde{T}_{|\Omega|,\alpha}$, Lemma 1.1 is also hold. On the other hand, for any $t > 0$, we have

$$\widetilde{T}_{|\Omega|,\alpha}(|f|)(x) \geq \int\limits_{B(x,t)} \frac{|\Omega(x-y)|}{|x-y|^{n-\alpha}} |f(y)| \, dy$$

$$\geq \frac{1}{t^{n-\alpha}} \int\limits_{B(x,t)} |\Omega(x-y)| |f(y)| \, dy.$$

Taking the supremum for $t > 0$ on the inequality above, we get

$$M_{\Omega,\alpha}f(x) \leq C_{n,\alpha}^{-1} \widetilde{T}_{|\Omega|,\alpha}(|f|)(x) \qquad C_{n,\alpha} = |B(0,1)|^{\frac{n-\alpha}{n}}.$$

$\square$

In 1976, Coifman, Rocherberg and Weiss [9] introduced the commutator generated by $\overline{T}_\Omega$ and a local integrable function b:

$$(1.3) \quad [b, \overline{T}_\Omega]f(x) \equiv b(x)\overline{T}_\Omega f(x) - \overline{T}_\Omega(bf)(x) = p.v. \int\limits_{\mathbb{R}^n} [b(x)-b(y)]\frac{\Omega(x-y)}{|x-y|^n} f(y)dy.$$

Sometimes, the commutator defined by (1.3) is also called the commutator in Coifman-Rocherberg-Weiss's sense, which has its root in the complex analysis and harmonic analysis (see [9]).

Let b be a locally integrable function on $\mathbb{R}^n$, then for $0 < \alpha < n$ and f is a suitable function, we define the commutators generated by fractional integral and maximal operators with rough kernel and b as follows, respectively:

$$[b, \overline{T}_{\Omega,\alpha}]f(x) \equiv b(x)\overline{T}_{\Omega,\alpha}f(x) - \overline{T}_{\Omega,\alpha}(bf)(x) = \int_{\mathbb{R}^n} [b(x) - b(y)]\frac{\Omega(x-y)}{|x-y|^{n-\alpha}}f(y)dy,$$

$$M_{\Omega,b,\alpha}(f)(x) = \sup_{t>0} |B(x,t)|^{-1+\frac{\alpha}{n}} \int_{B(x,t)} |b(x) - b(y)| |\Omega(x-y)| |f(y)|dy$$

satisfy condition (1.2). The proof of boundedness of $[b, \overline{T}_{\Omega,\alpha}]$ in Lebesgue spaces can be found in [12] (by taking $w = 1$ there).

Theorem 1.3. [12] *Suppose that $\Omega \in L_s(S^{n-1})$, $1 < s \leq \infty$, is homogeneous of degree zero and has mean value zero on S^{n-1}. Let $0 < \alpha < n$, $1 \leq p < \frac{n}{\alpha}$, and $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$ and $b \in BMO(\mathbb{R}^n)$. If $s' < p$ or $q < s$, then the operator $[b, \overline{T}_{\Omega,\alpha}]$ is bounded from $L_p(\mathbb{R}^n)$ to $L_q(\mathbb{R}^n)$.*

Remark 1.1. Using the method in the proof of Corollary 1.1 we have that

$$(1.4) \qquad M_{\Omega,b,\alpha}f(x) \leq C_{n,\alpha}^{-1}[b, \overline{T}_{|\Omega|,\alpha}](|f|)(x) \qquad C_{n,\alpha} = |B(0,1)|^{\frac{n-\alpha}{n}}.$$

By (1.4) we see that under the conditions of Theorem 1.3, the consequences of (L_p, L_q)-boundedness still hold for $M_{\Omega,b,\alpha}$.

Remark 1.2. [41, 42] When Ω satisfies the specified size conditions, the kernel of the operator $\overline{T}_{\Omega,\alpha}$ has no regularity, so the operator $\overline{T}_{\Omega,\alpha}$ is called a rough fractional integral operator. In recent years, a variety of operators related to the fractional integrals, but lacking the smoothness required in the classical theory, have been studied. These include the operator $[b, \overline{T}_{\Omega,\alpha}]$. For more results, we refer the reader to [2, 4, 12, 13, 18, 19, 20, 28].

Finally, we present a relationship between essential supremum and essential infimum.

Lemma 1.2. *(see [52] page 143) Let f be a real-valued nonnegative function and measurable on E. Then*

$$(1.5) \qquad \left(\operatorname*{essinf}_{x \in E} f(x)\right)^{-1} = \operatorname*{esssup}_{x \in E} \frac{1}{f(x)}.$$

Throughout the paper we use the letter C for a positive constant, independent of appropriate parameters and not necessarily the same at each occurrence. By $A \lesssim B$ we mean that $A \leq CB$ with some positive constant C independent of appropriate quantities. If $A \lesssim B$ and $B \lesssim A$, we write $A \approx B$ and say that A and B are equivalent.

2. GENERALIZED VANISHING MORREY SPACES

After studying Morrey spaces in detail, researchers have passed to generalized Morrey spaces. Mizuhara [31] has given generalized Morrey spaces $M_{p,\varphi}$ considering $\varphi = \varphi(r)$ instead of r^λ in the above definition of the Morrey space. Later, Guliyev [17] has defined the generalized Morrey spaces $M_{p,\varphi}$ with normalized norm as follows:

Definition 2.1. [17] (**generalized Morrey space**) Let $\varphi(x, r)$ be a positive measurable function on $\mathbb{R}^n \times (0, \infty)$ and $1 \leq p < \infty$. We denote by $M_{p,\varphi} \equiv M_{p,\varphi}(\mathbb{R}^n)$ the generalized Morrey space, the space of all functions $f \in L_p^{loc}(\mathbb{R}^n)$ with finite quasinorm

$$\|f\|_{M_{p,\varphi}} = \sup_{x \in \mathbb{R}^n, r > 0} \varphi(x, r)^{-1} |B(x, r)|^{-\frac{1}{p}} \|f\|_{L_p(B(x,r))}.$$

Also by $WM_{p,\varphi} \equiv WM_{p,\varphi}(\mathbb{R}^n)$ we denote the weak generalized Morrey space of all functions $f \in WL_p^{loc}(\mathbb{R}^n)$ for which

$$\|f\|_{WM_{p,\varphi}} = \sup_{x \in \mathbb{R}^n, r > 0} \varphi(x, r)^{-1} |B(x, r)|^{-\frac{1}{p}} \|f\|_{WL_p(B(x,r))} < \infty.$$

According to this definition, we recover the Morrey space $M_{p,\lambda}$ and weak Morrey space $WM_{p,\lambda}$ under the choice $\varphi(x, r) = r^{\frac{\lambda - n}{p}}$:

$$M_{p,\lambda} = M_{p,\varphi} \big|_{\varphi(x,r) = r^{\frac{\lambda - n}{p}}}, \ \ WM_{p,\lambda} = WM_{p,\varphi} \big|_{\varphi(x,r) = r^{\frac{\lambda - n}{p}}} .$$

For brevity, in the sequel we use the notations

$$\mathfrak{M}_{p,\varphi}(f; x, r) := \frac{|B(x, r)|^{-\frac{1}{p}} \|f\|_{L_p(B(x,r))}}{\varphi(x, r)}$$

and

$$\mathfrak{M}_{p,\varphi}^W(f; x, r) := \frac{|B(x, r)|^{-\frac{1}{p}} \|f\|_{WL_p(B(x,r))}}{\varphi(x, r)}.$$

In this paper, extending the definition of vanishing Morrey spaces [50], we introduce the generalized vanishing Morrey spaces $VM_{p,\varphi}(\mathbb{R}^n)$, including their weak versions and studies the boundedness of the sublinear operators with rough kernel generated by fractional integrals and their commutators in these spaces. Indeed, we find it convenient to define generalized vanishing Morrey spaces in the form as follows.

Definition 2.2. (**generalized vanishing Morrey space**) The generalized vanishing Morrey space $VM_{p,\varphi}(\mathbb{R}^n)$ is defined as the spaces of functions $f \in M_{p,\varphi}(\mathbb{R}^n)$ such that

$$(2.1) \qquad\qquad \lim_{r \to 0} \sup_{x \in \mathbb{R}^n} \mathfrak{M}_{p,\varphi}(f; x, r) = 0.$$

Definition 2.3. (**weak generalized vanishing Morrey space**) The weak generalized vanishing Morrey space $WVM_{p,\varphi}(\mathbb{R}^n)$ is defined as the spaces of functions $f \in WM_{p,\varphi}(\mathbb{R}^n)$ such that

$$(2.2) \qquad\qquad \lim_{r \to 0} \sup_{x \in \mathbb{R}^n} \mathfrak{M}_{p,\varphi}^W(f; x, r) = 0.$$

Everywhere in the sequel we assume that

$$(2.3) \qquad\qquad \lim_{r \to 0} \frac{1}{\inf\limits_{x \in \mathbb{R}^n} \varphi(x, r)} = 0,$$

and

$$(2.4) \qquad\qquad \sup_{0 < r < \infty} \frac{1}{\inf\limits_{x \in \mathbb{R}^n} \varphi(x, r)} < \infty,$$

which make the spaces $VM_{p,\varphi}(\mathbb{R}^n)$ and $WVM_{p,\varphi}(\mathbb{R}^n)$ non-trivial, because bounded functions with compact support belong to this space. The spaces $VM_{p,\varphi}(\mathbb{R}^n)$ and $WVM_{p,\varphi}(\mathbb{R}^n)$ are Banach spaces with respect to the norm

$$(2.5) \qquad \|f\|_{VM_{p,\varphi}} \equiv \|f\|_{M_{p,\varphi}} = \sup_{x\in\mathbb{R}^n, r>0} \mathfrak{M}_{p,\varphi}(f;x,r),$$

$$(2.6) \qquad \|f\|_{WVM_{p,\varphi}} = \|f\|_{WM_{p,\varphi}} = \sup_{x\in\mathbb{R}^n, r>0} \mathfrak{M}_{p,\varphi}^W(f;x,r),$$

respectively.

3. Sublinear operators with rough kernel $T_{\Omega,\alpha}$ on the spaces $M_{p,\varphi}$ and $VM_{p,\varphi}$

In this section, we will first prove the boundedness of the operator $T_{\Omega,\alpha}$ satisfying (1.1) on the generalized Morrey spaces $M_{p,\varphi}(\mathbb{R}^n)$ by using Lemma 1.2 and the following Lemma 3.1. Then, we will also give the boundedness of $T_{\Omega,\alpha}$ satisfying (1.1) on generalized vanishing Morrey spaces $VM_{p,\varphi}(\mathbb{R}^n)$.

We first prove the following lemma (our main lemma).

Lemma 3.1. *(Our main lemma) Suppose that $\Omega \in L_s(S^{n-1})$, $1 < s \le \infty$, is homogeneous of degree zero. Let $0 < \alpha < n$, $1 \le p < \frac{n}{\alpha}$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$. Let $T_{\Omega,\alpha}$ be a sublinear operator satisfying condition (1.1), bounded from $L_p(\mathbb{R}^n)$ to $L_q(\mathbb{R}^n)$ for $p > 1$, and bounded from $L_1(\mathbb{R}^n)$ to $WL_q(\mathbb{R}^n)$ for $p = 1$.*

If $p > 1$ and $s' \le p$, then the inequality

$$(3.1) \qquad \|T_{\Omega,\alpha}f\|_{L_q(B(x_0,r))} \lesssim r^{\frac{n}{q}} \int_{2r}^{\infty} t^{-\frac{n}{q}-1} \|f\|_{L_p(B(x_0,t))}\, dt$$

holds for any ball $B(x_0,r)$ and for all $f \in L_p^{loc}(\mathbb{R}^n)$.

If $p > 1$ and $q < s$, then the inequality

$$\|T_{\Omega,\alpha}f\|_{L_q(B(x_0,r))} \lesssim r^{\frac{n}{q}-\frac{n}{s}} \int_{2r}^{\infty} t^{\frac{n}{s}-\frac{n}{q}-1} \|f\|_{L_p(B(x_0,t))}\, dt$$

holds for any ball $B(x_0,r)$ and for all $f \in L_p^{loc}(\mathbb{R}^n)$.

Moreover, for $p = 1 < q < s$ the inequality

$$(3.2) \qquad \|T_{\Omega,\alpha}f\|_{WL_q(B(x_0,r))} \lesssim r^{\frac{n}{q}} \int_{2r}^{\infty} t^{-\frac{n}{q}-1} \|f\|_{L_1(B(x_0,t))}\, dt$$

holds for any ball $B(x_0,r)$ and for all $f \in L_1^{loc}(\mathbb{R}^n)$.

Proof. Let $0 < \alpha < n$, $1 \le s' < p < \frac{n}{\alpha}$ and $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$. Set $B = B(x_0,r)$ for the ball centered at x_0 and of radius r and $2B = B(x_0, 2r)$. We represent f as
(3.3)
$$f = f_1 + f_2, \qquad f_1(y) = f(y)\chi_{2B}(y), \qquad f_2(y) = f(y)\chi_{(2B)^C}(y), \qquad r > 0$$

and have

$$\|T_{\Omega,\alpha}f\|_{L_q(B)} \le \|T_{\Omega,\alpha}f_1\|_{L_q(B)} + \|T_{\Omega,\alpha}f_2\|_{L_q(B)}.$$

Since $f_1 \in L_p(\mathbb{R}^n)$, $T_{\Omega,\alpha} f_1 \in L_q(\mathbb{R}^n)$ and from the boundedness of $T_{\Omega,\alpha}$ from $L_p(\mathbb{R}^n)$ to $L_q(\mathbb{R}^n)$ (see Lemma 1.1) it follows that:

$$\|T_{\Omega,\alpha} f_1\|_{L_q(B)} \leq \|T_{\Omega,\alpha} f_1\|_{L_q(\mathbb{R}^n)} \leq C \|f_1\|_{L_p(\mathbb{R}^n)} = C \|f\|_{L_p(2B)},$$

where constant $C > 0$ is independent of f.

It is clear that $x \in B$, $y \in (2B)^C$ implies $\frac{1}{2}|x_0 - y| \leq |x - y| \leq \frac{3}{2}|x_0 - y|$. We get

$$|T_{\Omega,\alpha} f_2(x)| \leq 2^{n-\alpha} c_1 \int\limits_{(2B)^C} \frac{|f(y)| \, |\Omega(x-y)|}{|x_0 - y|^{n-\alpha}} dy.$$

By the Fubini's theorem, we have

$$\int\limits_{(2B)^C} \frac{|f(y)| \, |\Omega(x-y)|}{|x_0 - y|^{n-\alpha}} dy \approx \int\limits_{(2B)^C} |f(y)| \, |\Omega(x-y)| \int\limits_{|x_0 - y|}^{\infty} \frac{dt}{t^{n+1-\alpha}} dy$$

$$\approx \int\limits_{2r}^{\infty} \int\limits_{2r \leq |x_0 - y| \leq t} |f(y)| \, |\Omega(x-y)| \, dy \, \frac{dt}{t^{n+1-\alpha}}$$

$$\lesssim \int\limits_{2r}^{\infty} \int\limits_{B(x_0,t)} |f(y)| \, |\Omega(x-y)| \, dy \, \frac{dt}{t^{n+1-\alpha}}.$$

Applying the Hölder's inequality, we get

$$\int\limits_{(2B)^C} \frac{|f(y)| \, |\Omega(x-y)|}{|x_0 - y|^{n-\alpha}} dy$$

(3.4)
$$\lesssim \int\limits_{2r}^{\infty} \|f\|_{L_p(B(x_0,t))} \|\Omega(x-\cdot)\|_{L_s(B(x_0,t))} |B(x_0,t)|^{1-\frac{1}{p}-\frac{1}{s}} \frac{dt}{t^{n+1-\alpha}}.$$

For $x \in B(x_0, t)$, notice that Ω is homogenous of degree zero and $\Omega \in L_s(S^{n-1})$, $s > 1$. Then, we obtain

$$\left(\int_{B(x_0,t)} |\Omega(x-y)|^s \, dy \right)^{\frac{1}{s}} = \left(\int_{B(x-x_0,t)} |\Omega(z)|^s \, dz \right)^{\frac{1}{s}}$$

$$\leq \left(\int_{B(0,t+|x-x_0|)} |\Omega(z)|^s \, dz \right)^{\frac{1}{s}}$$

$$\leq \left(\int_{B(0,2t)} |\Omega(z)|^s \, dz \right)^{\frac{1}{s}}$$

$$= \left(\int_{S^{n-1}} \int_0^{2t} |\Omega(z')|^s \, d\sigma(z') \, r^{n-1} dr \right)^{\frac{1}{s}}$$

$$(3.5) \qquad = C \|\Omega\|_{L_s(S^{n-1})} |B(x_0, 2t)|^{\frac{1}{s}} .$$

Thus, by (3.5), it follows that:

$$|T_{\Omega,\alpha} f_2(x)| \lesssim \int_{2r}^{\infty} \|f\|_{L_p(B(x_0,t))} \frac{dt}{t^{\frac{n}{q}+1}}.$$

Moreover, for all $p \in [1, \infty)$ the inequality

$$(3.6) \qquad \|T_{\Omega,\alpha} f_2\|_{L_q(B)} \lesssim r^{\frac{n}{q}} \int_{2r}^{\infty} \|f\|_{L_p(B(x_0,t))} \frac{dt}{t^{\frac{n}{q}+1}}$$

is valid. Thus, we obtain

$$\|T_{\Omega,\alpha} f\|_{L_q(B)} \lesssim \|f\|_{L_p(2B)} + r^{\frac{n}{q}} \int_{2r}^{\infty} \|f\|_{L_p(B(x_0,t))} \frac{dt}{t^{\frac{n}{q}+1}}.$$

On the other hand, we have

$$\|f\|_{L_p(2B)} \approx r^{\frac{n}{q}} \|f\|_{L_p(2B)} \int_{2r}^{\infty} \frac{dt}{t^{\frac{n}{q}+1}}$$

$$(3.7) \qquad \leq r^{\frac{n}{q}} \int_{2r}^{\infty} \|f\|_{L_p(B(x_0,t))} \frac{dt}{t^{\frac{n}{q}+1}}.$$

By combining the above inequalities, we obtain

$$\|T_{\Omega,\alpha} f\|_{L_q(B)} \lesssim r^{\frac{n}{q}} \int_{2r}^{\infty} \|f\|_{L_p(B(x_0,t))} \frac{dt}{t^{\frac{n}{q}+1}}.$$

Let $1 < q < s$. Similarly to (3.5), when $y \in B(x_0, t)$, it is true that

$$(3.8) \qquad \left(\int\limits_{B(x_0,r)} |\Omega(x - y)|^s \, dy \right)^{\frac{1}{s}} \leq C \|\Omega\|_{L_s(S^{n-1})} \left| B\left(x_0, \frac{3}{2}t\right) \right|^{\frac{1}{s}}.$$

By the Fubini's theorem, the Minkowski inequality and (3.8) , we get

$$\|T_{\Omega,\alpha} f_2\|_{L_q(B)} \leq \left(\int\limits_B \left| \int\limits_{2r}^{\infty} \int\limits_{B(x_0,t)} |f(y)| |\Omega(x - y)| \, dy \frac{dt}{t^{n+1-\alpha}} \right|^q dx \right)^{\frac{1}{q}}$$

$$\leq \int\limits_{2r}^{\infty} \int\limits_{B(x_0,t)} |f(y)| \|\Omega(\cdot - y)\|_{L_q(B)} \, dy \frac{dt}{t^{n+1-\alpha}}$$

$$\leq |B(x_0,r)|^{\frac{1}{q} - \frac{1}{s}} \int\limits_{2r}^{\infty} \int\limits_{B(x_0,t)} |f(y)| \|\Omega(\cdot - y)\|_{L_s(B)} \, dy \frac{dt}{t^{n+1-\alpha}}$$

$$\lesssim r^{\frac{n}{q} - \frac{n}{s}} \int\limits_{2r}^{\infty} \|f\|_{L_1(B(x_0,t))} \left| B\left(x_0, \frac{3}{2}t\right) \right|^{\frac{1}{s}} \frac{dt}{t^{n+1-\alpha}}$$

$$\lesssim r^{\frac{n}{q} - \frac{n}{s}} \int\limits_{2r}^{\infty} \|f\|_{L_p(B(x_0,t))} \, t^{\frac{n}{s} - \frac{n}{q} - 1} dt.$$

Let $p = 1 < q < s \leq \infty$. From the weak $(1, q)$ boundedness of $T_{\Omega,\alpha}$ and (3.7) it follows that:

$$\|T_{\Omega,\alpha} f_1\|_{WL_q(B)} \leq \|T_{\Omega,\alpha} f_1\|_{WL_q(\mathbb{R}^n)} \lesssim \|f_1\|_{L_1(\mathbb{R}^n)}$$

$$(3.9) \qquad = \|f\|_{L_1(2B)} \lesssim r^{\frac{n}{q}} \int\limits_{2r}^{\infty} \|f\|_{L_1(B(x_0,t))} \frac{dt}{t^{\frac{n}{q}+1}}.$$

Then from (3.6) and (3.9) we get the inequality (3.2), which completes the proof. $\square$

In the following theorem (our main result), we get the boundedness of the operator $T_{\Omega,\alpha}$ on the generalized Morrey spaces $M_{p,\varphi}$.

Theorem 3.1. *(Our main result) Suppose that $\Omega \in L_s(S^{n-1})$, $1 < s \leq \infty$, is homogeneous of degree zero. Let $0 < \alpha < n$, $1 \leq p < \frac{n}{\alpha}$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$. Let $T_{\Omega,\alpha}$ be a sublinear operator satisfying condition (1.1), bounded from $L_p(\mathbb{R}^n)$ to $L_q(\mathbb{R}^n)$ for $p > 1$, and bounded from $L_1(\mathbb{R}^n)$ to $WL_q(\mathbb{R}^n)$ for $p = 1$. Let also, for $s' \leq p < q$, $p \neq 1$, the pair (φ_1, φ_2) satisfies the condition*

$$(3.10) \qquad \int\limits_r^{\infty} \frac{\operatorname*{essinf}\limits_{t<\tau<\infty} \varphi_1(x,\tau)\tau^{\frac{n}{p}}}{t^{\frac{n}{q}+1}} dt \leq C\varphi_2(x,r),$$

and for $q < s$ the pair (φ_1, φ_2) satisfies the condition

$$(3.11) \qquad \int\limits_r^\infty \frac{\operatorname*{essinf}\limits_{t<\tau<\infty} \varphi_1(x,\tau)\tau^{\frac{n}{p}}}{t^{\frac{n}{q}-\frac{n}{s}+1}} \, dt \leq C\, \varphi_2(x,r)r^{\frac{n}{s}},$$

where C does not depend on x and r.

Then the operator $T_{\Omega,\alpha}$ is bounded from M_{p,φ_1} to M_{q,φ_2} for $p > 1$ and from M_{1,φ_1} to WM_{q,φ_2} for $p = 1$. Moreover, we have for $p > 1$

$$\|T_{\Omega,\alpha}f\|_{M_{q,\varphi_2}} \lesssim \|f\|_{M_{p,\varphi_1}},$$

and for $p = 1$

$$\|T_{\Omega,\alpha}f\|_{WM_{q,\varphi_2}} \lesssim \|f\|_{M_{1,\varphi_1}}.$$

Proof. Since $f \in M_{p,\varphi_1}$, by (2.6) and the non-decreasing, with respect to t, of the norm $\|f\|_{L_p(B(x_0,t))}$, we get

$$\frac{\|f\|_{L_p(B(x_0,t))}}{\operatorname*{essinf}\limits_{0<t<\tau<\infty} \varphi_1(x_0,\tau)\tau^{\frac{n}{p}}}$$

$$\leq \operatorname*{esssup}\limits_{0<t<\tau<\infty} \frac{\|f\|_{L_p(B(x_0,t))}}{\varphi_1(x_0,\tau)\tau^{\frac{n}{p}}}$$

$$\leq \operatorname*{esssup}\limits_{0<\tau<\infty} \frac{\|f\|_{L_p(B(x_0,\tau))}}{\varphi_1(x_0,\tau)\tau^{\frac{n}{p}}}$$

$$\leq \|f\|_{M_{p,\varphi_1}}.$$

For $s' \leq p < \infty$, since (φ_1, φ_2) satisfies (3.10), we have

$$\int\limits_r^\infty \|f\|_{L_p(B(x_0,t))} \, t^{-\frac{n}{q}} \frac{dt}{t}$$

$$\leq \int\limits_r^\infty \frac{\|f\|_{L_p(B(x_0,t))}}{\operatorname*{essinf}\limits_{t<\tau<\infty} \varphi_1(x_0,\tau)\tau^{\frac{n}{p}}} \frac{\operatorname*{essinf}\limits_{t<\tau<\infty} \varphi_1(x_0,\tau)\tau^{\frac{n}{p}}}{t^{\frac{n}{q}}} \frac{dt}{t}$$

$$\leq C \|f\|_{M_{p,\varphi_1}} \int\limits_r^\infty \frac{\operatorname*{essinf}\limits_{t<\tau<\infty} \varphi_1(x_0,\tau)\tau^{\frac{n}{p}}}{t^{\frac{n}{q}}} \frac{dt}{t}$$

$$\leq C \|f\|_{M_{p,\varphi_1}} \varphi_2(x_0,r).$$

Then by (3.1), we get

$$\|T_{\Omega,\alpha}f\|_{M_{q,\varphi_2}} = \sup_{x_0\in\mathbb{R}^n, r>0} \varphi_2(x_0,r)^{-1} |B(x_0,r)|^{-\frac{1}{q}} \|T_{\Omega,\alpha}f\|_{L_q(B(x_0,r))}$$

$$\leq C \sup_{x_0\in\mathbb{R}^n, r>0} \varphi_2(x_0,r)^{-1} \int\limits_r^\infty \|f\|_{L_p(B(x_0,t))} \, t^{-\frac{n}{q}} \frac{dt}{t}$$

$$\leq C \|f\|_{M_{p,\varphi_1}}.$$

For the case of $p = 1 < q < s$, we can also use the same method, so we omit the details. This completes the proof of Theorem 3.1. $\qquad\square$

In the case of $q = \infty$ by Theorem 3.1, we get

Corollary 3.1. *Let $1 \leq p < \infty$, $0 < \alpha < \frac{n}{p}$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$ and the pair (φ_1, φ_2) satisfies condition (3.10). Then the operators M_α and $\overline{T}_\alpha$ are bounded from M_{p,φ_1} to M_{q,φ_2} for $p > 1$ and from M_{1,φ_1} to WM_{q,φ_2} for $p = 1$.*

Corollary 3.2. *Suppose that $\Omega \in L_s(S^{n-1})$, $1 < s \leq \infty$, is homogeneous of degree zero. Let $0 < \alpha < n$, $1 \leq p < \frac{n}{\alpha}$ and $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$. Let also for $s' \leq p$ the pair (φ_1, φ_2) satisfies condition (3.10) and for $q < s$ the pair (φ_1, φ_2) satisfies condition (3.11). Then the operators $M_{\Omega,\alpha}$ and $\overline{T}_{\Omega,\alpha}$ are bounded from M_{p,φ_1} to M_{q,φ_2} for $p > 1$ and from M_{1,φ_1} to WM_{q,φ_2} for $p = 1$.*

Now using above results, we get the boundedness of the operator $T_{\Omega,\alpha}$ on the generalized vanishing Morrey spaces $VM_{p,\varphi}$.

Theorem 3.2. *(Our main result) Let $\Omega \in L_s(S^{n-1})$, $1 < s \leq \infty$, be homogeneous of degree zero. Let $0 < \alpha < n$, $1 \leq p < \frac{n}{\alpha}$ and $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$. Let $T_{\Omega,\alpha}$ be a sublinear operator satisfying condition (1.1), bounded on $L_p(\mathbb{R}^n)$ for $p > 1$, and bounded from $L_1(\mathbb{R}^n)$ to $WL_1(\mathbb{R}^n)$. Let for $s' \leq p$, $p \neq 1$, the pair (φ_1, φ_2) satisfies conditions (2.3)-(2.4) and*

$$(3.12) \qquad c_\delta := \int\limits_{\delta}^{\infty} \sup_{x \in \mathbb{R}^n} \varphi_1(x,t) \frac{t^{\frac{n}{p}}}{t^{\frac{n}{q}+1}} dt < \infty$$

for every $\delta > 0$, and

$$(3.13) \qquad \int\limits_{r}^{\infty} \varphi_1(x,t) \frac{t^{\frac{n}{p}}}{t^{\frac{n}{q}+1}} dt \leq C_0 \varphi_2(x,r),$$

and for $q < s$ the pair (φ_1, φ_2) satisfies conditions (2.3)-(2.4) and also

$$(3.14) \qquad c_{\delta'} := \int\limits_{\delta'}^{\infty} \sup_{x \in \mathbb{R}^n} \varphi_1(x,t) \frac{t^{\frac{n}{p}}}{t^{\frac{n}{q}-\frac{n}{s}+1}} dt < \infty$$

for every $\delta' > 0$, and

$$(3.15) \qquad \int\limits_{r}^{\infty} \varphi_1(x,t) \frac{t^{\frac{n}{p}}}{t^{\frac{n}{q}-\frac{n}{s}+1}} dt \leq C_0 \varphi_2(x,r) r^{\frac{n}{s}},$$

where C_0 does not depend on $x \in \mathbb{R}^n$ and $r > 0$.

Then the operator $T_{\Omega,\alpha}$ is bounded from VM_{p,φ_1} to VM_{q,φ_2} for $p > 1$ and from M_{1,φ_1} to WVM_{q,φ_2} for $p = 1$. Moreover, we have for $p > 1$

$$(3.16) \qquad \|T_{\Omega,\alpha}f\|_{VM_{q,\varphi_2}} \lesssim \|f\|_{VM_{p,\varphi_1}},$$

and for $p = 1$

$$(3.17) \qquad \|T_{\Omega,\alpha}f\|_{WVM_{q,\varphi_2}} \lesssim \|f\|_{VM_{1,\varphi_1}}.$$

Proof. The norm inequalities follow from Theorem 3.1. Thus we only have to prove that

$$(3.18) \qquad \lim_{r \to 0} \sup_{x \in \mathbb{R}^n} \mathfrak{M}_{p,\varphi_1}(f; x, r) = 0 \text{ implies } \lim_{r \to 0} \sup_{x \in \mathbb{R}^n} \mathfrak{M}_{q,\varphi_2}(T_{\Omega,\alpha}f; x, r) = 0$$

and

(3.19) $\quad \lim\limits_{r \to 0} \sup\limits_{x \in \mathbb{R}^n} \mathfrak{M}_{p,\varphi_1}(f;x,r) = 0 \text{ implies } \lim\limits_{r \to 0} \sup\limits_{x \in \mathbb{R}^n} \mathfrak{M}_{q,\varphi_2}^{W}(T_{\Omega,\alpha}f;x,r) = 0.$

To show that $\sup\limits_{x \in \mathbb{R}^n} \dfrac{r^{-\frac{n}{q}} \|T_{\Omega,\alpha}f\|_{L_q(B(x,r))}}{\varphi_2(x,r)} < \epsilon$ for small r, we split the right-hand side of (3.1):

(3.20) $\qquad \dfrac{r^{-\frac{n}{q}} \|T_{\Omega,\alpha}f\|_{L_q(B(x,r))}}{\varphi_2(x,r)} \le C \left[I_{\delta_0}(x,r) + J_{\delta_0}(x,r) \right],$

where $\delta_0 > 0$ (we may take $\delta_0 < 1$), and

$$I_{\delta_0}(x,r) := \frac{1}{\varphi_2(x,r)} \int\limits_{r}^{\delta_0} t^{-\frac{n}{q}-1} \|f\|_{L_p(B(x,t))} \, dt,$$

and

$$J_{\delta_0}(x,r) := \frac{1}{\varphi_2(x,r)} \int\limits_{\delta_0}^{\infty} t^{-\frac{n}{q}-1} \|f\|_{L_p(B(x,t))} \, dt,$$

and $r < \delta_0$. Now we use the fact that $f \in VM_{p,\varphi_1}$ and we choose any fixed $\delta_0 > 0$ such that

$$\sup\limits_{x \in \mathbb{R}^n} \frac{t^{-\frac{n}{p}} \|f\|_{L_p(B(x,t))}}{\varphi_1(x,t)} < \frac{\epsilon}{2CC_0}, \qquad t \le \delta_0,$$

where C and C_0 are constants from (3.13) and (3.20). This allows to estimate the first term uniformly in $r \in (0, \delta_0)$:

$$\sup\limits_{x \in \mathbb{R}^n} C I_{\delta_0}(x,r) < \frac{\epsilon}{2}, \qquad 0 < r < \delta_0.$$

The estimation of the second term may be obtained by choosing r sufficiently small. Indeed, we have

$$J_{\delta_0}(x,r) \le c_{\delta_0} \frac{\|f\|_{M_{p,\varphi_1}}}{\varphi_2(x,r)},$$

where c_{δ_0} is the constant from (3.12) with $\delta = \delta_0$. Then, by (2.3) it suffices to choose r small enough such that

$$\sup\limits_{x \in \mathbb{R}^n} \frac{1}{\varphi_2(x,r)} \le \frac{\epsilon}{2c_{\delta_0} \|f\|_{M_{p,\varphi_1}}},$$

which completes the proof of (3.18).

The proof of (3.19) is similar to the proof of (3.18). For the case of $q < s$, we can also use the same method, so we omit the details. $\qquad \square$

Remark 3.1. Conditions (3.12) and (3.14) are not needed in the case when $\varphi(x,r)$ does not depend on x, since (3.12) follows from (3.13) and similarly, (3.14) follows from (3.15) in this case.

Corollary 3.3. *Let $\Omega \in L_s(S^{n-1})$, $1 < s \le \infty$, be homogeneous of degree zero. Let $0 < \alpha < n$, $1 \le p < \frac{n}{\alpha}$ and $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$. Let also for $s' \le p$, $p \ne 1$, the pair (φ_1, φ_2) satisfies conditions (2.3)-(2.4) and (3.12)-(3.13) and for $q < s$ the pair (φ_1, φ_2) satisfies conditions (2.3)-(2.4) and (3.14)-(3.15). Then the operators*

$M_{\Omega,\alpha}$ and $\overline{T}_{\Omega,\alpha}$ are bounded from VM_{p,φ_1} to VM_{q,φ_2} for $p > 1$ and from VM_{1,φ_1} to WVM_{q,φ_2} for $p = 1$.

In the case of $q = \infty$ by Theorem 3.2, we get

Corollary 3.4. *Let $1 \le p < \infty$ and the pair (φ_1, φ_2) satisfies conditions (2.3)-(2.4) and (3.12)-(3.13). Then the operators M_α and $\overline{T}_\alpha$ are bounded from VM_{p,φ_1} to VM_{q,φ_2} for $p > 1$ and from VM_{1,φ_1} to WVM_{q,φ_2} for $p = 1$.*

4. Commutators of the sublinear operators with rough kernel $T_{\Omega,\alpha}$ on the spaces $M_{p,\varphi}$ and $VM_{p,\varphi}$

In this section, we will first prove the boundedness of the operator $T_{\Omega,b,\alpha}$ satisfying (1.2) with $b \in BMO(\mathbb{R}^n)$ on the generalized Morrey spaces $M_{p,\varphi}$ by using Lemma 1.2 and the following Lemma 4.1. Then, we will also obtain the boundedness of $T_{\Omega,b,\alpha}$ satisfying (1.2) with $b \in BMO(\mathbb{R}^n)$ on generalized vanishing Morrey spaces $VM_{p,\varphi}$.

Let T be a linear operator. For a locally integrable function b on $\mathbb{R}^n$, we define the commutator $[b, T]$ by

$$[b, T]f(x) = b(x)Tf(x) - T(bf)(x)$$

for any suitable function f. Let $\overline{T}$ be a C–Z operator. A well known result of Coifman et al. [9] states that when $K(x) = \frac{\Omega(x')}{|x|^n}$ and Ω is smooth, the commutator $[b, \overline{T}]f = b\overline{T}f - \overline{T}(bf)$ is bounded on $L_p(\mathbb{R}^n)$, $1 < p < \infty$, if and only if $b \in BMO(\mathbb{R}^n)$. The commutator of C–Z operators plays an important role in studying the regularity of solutions of elliptic partial differential equations of second order (see, for example, [7, 8, **?**]). The boundedness of the commutator has been generalized to other contexts and important applications to some non-linear PDEs have been given by Coifman et al. [10]. On the other hand, For $b \in L_1^{loc}(\mathbb{R}^n)$, the commutator $[b, \overline{T}_\alpha]$ of fractional integral operator (also known as the Riesz potential) is defined by

$$[b, \overline{T}_\alpha]f(x) = b(x)\overline{T}_\alpha f(x) - \overline{T}_\alpha(bf)(x) = \int\limits_{\mathbb{R}^n} \frac{b(x) - b(y)}{|x-y|^{n-\alpha}} f(y)dy \qquad 0 < \alpha < n$$

for any suitable function f.

The function b is also called the symbol function of $[b, \overline{T}_\alpha]$. The characterization of (L_p, L_q)-boundedness of the commutator $[b, \overline{T}_\alpha]$ of fractional integral operator has been given by Chanillo [4]. A well known result of Chanillo [4] states that the commutator $[b, \overline{T}_\alpha]$ is bounded from $L_p(\mathbb{R}^n)$ to $L_q(\mathbb{R}^n)$, $1 < p < q < \infty$, $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{n}$ if and only if $b \in BMO(\mathbb{R}^n)$. There are two major reasons for considering the problem of commutators. The first one is that the boundedness of commutators can produce some characterizations of function spaces (see [2, 4, 18, 19, 20, 21, 37, 42]). The other one is that the theory of commutators plays an important role in the study of the regularity of solutions to elliptic and parabolic PDEs of the second order (see [7, 8, 14, 41, 43]).

Let us recall the defination of the space of $BMO(\mathbb{R}^n)$.

Definition 4.1. Suppose that $b \in L_1^{loc}(\mathbb{R}^n)$, let

$$\|b\|_* = \sup_{x \in \mathbb{R}^n, r > 0} \frac{1}{|B(x,r)|} \int\limits_{B(x,r)} |b(y) - b_{B(x,r)}|dy < \infty,$$

where

$$b_{B(x,r)} = \frac{1}{|B(x,r)|} \int\limits_{B(x,r)} b(y)dy.$$

Define

$$BMO(\mathbb{R}^n) = \{b \in L_1^{loc}(\mathbb{R}^n) \ : \ \|b\|_* < \infty\}.$$

If one regards two functions whose difference is a constant as one, then the space $BMO(\mathbb{R}^n)$ is a Banach space with respect to norm $\|\cdot\|_*$.

Remark 4.1. [23] (1) The John-Nirenberg inequality [22]: there are constants C_1, $C_2 > 0$, such that for all $b \in BMO(\mathbb{R}^n)$ and $\beta > 0$

$$|\{x \in B \ : \ |b(x) - b_B| > \beta\}| \leq C_1|B|e^{-C_2\beta/\|b\|_*}, \ \forall B \subset \mathbb{R}^n.$$

(2) The John-Nirenberg inequality implies that

$$(4.1) \qquad \|b\|_* \approx \sup_{x \in \mathbb{R}^n, r>0} \left(\frac{1}{|B(x,r)|} \int\limits_{B(x,r)} |b(y) - b_{B(x,r)}|^p dy \right)^{\frac{1}{p}}$$

for $1 < p < \infty$.

(3) Let $b \in BMO(\mathbb{R}^n)$. Then there is a constant $C > 0$ such that

$$(4.2) \qquad \left|b_{B(x,r)} - b_{B(x,t)}\right| \leq C\|b\|_* \ln\frac{t}{r} \text{ for } 0 < 2r < t,$$

where C is independent of b, x, r and t.

As in the proof of Theorem 3.1, it suffices to prove the following Lemma (our main lemma).

Lemma 4.1. *(Our main lemma) Let* $\Omega \in L_s(S^{n-1})$, $1 < s \leq \infty$, *be homogeneous of degree zero. Let* $1 < p < \infty$, $0 < \alpha < \frac{n}{p}$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$, $b \in BMO(\mathbb{R}^n)$, *and* $T_{\Omega,b,\alpha}$ *is a sublinear operator satisfying condition (1.2) and bounded from* $L_p(\mathbb{R}^n)$ *to* $L_q(\mathbb{R}^n)$. *Then, for* $s' \leq p$ *the inequality*

$$(4.3) \qquad \|T_{\Omega,b,\alpha}f\|_{L_q(B(x_0,r))} \lesssim \|b\|_* r^{\frac{n}{q}} \int\limits_{2r}^{\infty} \left(1 + \ln\frac{t}{r}\right) t^{-\frac{n}{q}-1} \|f\|_{L_p(B(x_0,t))} dt$$

holds for any ball $B(x_0,r)$ *and for all* $f \in L_p^{loc}(\mathbb{R}^n)$.

Also, for $q < s$ *the inequality*

$$\|T_{\Omega,b,\alpha}f\|_{L_q(B(x_0,r))} \lesssim \|b\|_* r^{\frac{n}{q}-\frac{n}{s}} \int\limits_{2r}^{\infty} \left(1 + \ln\frac{t}{r}\right) t^{\frac{n}{s}-\frac{n}{q}-1} \|f\|_{L_p(B(x_0,t))} dt$$

holds for any ball $B(x_0,r)$ *and for all* $f \in L_p^{loc}(\mathbb{R}^n)$.

Proof. Let $1 < p < \infty$, $0 < \alpha < \frac{n}{p}$ and $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$. As in the proof of Lemma 3.1, we represent f in form (3.3) and have

$$\|T_{\Omega,b,\alpha}f\|_{L_q(B)} \leq \|T_{\Omega,b,\alpha}f_1\|_{L_q(B)} + \|T_{\Omega,b,\alpha}f_2\|_{L_q(B)}.$$

From the boundedness of $T_{\Omega,b,\alpha}$ from $L_p(\mathbb{R}^n)$ to $L_q(\mathbb{R}^n)$ (see Theorem 1.3) it follows that:

$$\|T_{\Omega,b,\alpha}f_1\|_{L_q(B)} \leq \|T_{\Omega,b,\alpha}f_1\|_{L_q(\mathbb{R}^n)}$$
$$\lesssim \|b\|_* \|f_1\|_{L_p(\mathbb{R}^n)} = \|b\|_* \|f\|_{L_p(2B)}.$$

It is known that $x \in B$, $y \in (2B)^C$, which implies $\frac{1}{2}|x_0 - y| \leq |x - y| \leq \frac{3}{2}|x_0 - y|$. Then for $x \in B$, we have

$$|T_{\Omega,b,\alpha}f_2(x)| \lesssim \int\limits_{\mathbb{R}^n} \frac{|\Omega(x-y)|}{|x-y|^{n-\alpha}}|b(y) - b(x)||f(y)|\,dy$$
$$\approx \int\limits_{(2B)^C} \frac{|\Omega(x-y)|}{|x_0-y|^{n-\alpha}}|b(y) - b(x)||f(y)|\,dy.$$

Hence we get

$$\|T_{\Omega,b,\alpha}f_2\|_{L_q(B)} \lesssim \left(\int\limits_B \left(\int\limits_{(2B)^C} \frac{|\Omega(x-y)|}{|x_0-y|^{n-\alpha}}|b(y) - b(x)||f(y)|\,dy\right)^q dx\right)^{\frac{1}{q}}$$

$$\lesssim \left(\int\limits_B \left(\int\limits_{(2B)^C} \frac{|\Omega(x-y)|}{|x_0-y|^{n-\alpha}}|b(y) - b_B||f(y)|\,dy\right)^q dx\right)^{\frac{1}{q}}$$

$$+ \left(\int\limits_B \left(\int\limits_{(2B)^C} \frac{|\Omega(x-y)|}{|x_0-y|^{n-\alpha}}|b(x) - b_B||f(y)|\,dy\right)^q dx\right)^{\frac{1}{q}}$$

$$= J_1 + J_2.$$

We have the following estimation of J_1. When $s' \leq p$ and $\frac{1}{\mu} + \frac{1}{p} + \frac{1}{s} = 1$, by the Fubini's theorem

$$J_1 \approx r^{\frac{n}{q}} \int\limits_{(2B)^C} \frac{|\Omega(x-y)|}{|x_0-y|^{n-\alpha}}|b(y) - b_B||f(y)|\,dy$$

$$\approx r^{\frac{n}{q}} \int\limits_{(2B)^C} |\Omega(x-y)||b(y) - b_B||f(y)| \int\limits_{|x_0-y|}^{\infty} \frac{dt}{t^{n+1-\alpha}}\,dy$$

$$\approx r^{\frac{n}{q}} \int\limits_{2r}^{\infty} \int\limits_{2r \leq |x_0-y| \leq t} |\Omega(x-y)||b(y) - b_B||f(y)|\,dy \frac{dt}{t^{n+1-\alpha}}$$

$$\lesssim r^{\frac{n}{q}} \int\limits_{2r}^{\infty} \int\limits_{B(x_0,t)} |\Omega(x-y)||b(y) - b_B||f(y)|\,dy \frac{dt}{t^{n+1-\alpha}} \text{ holds.}$$

Applying the Hölder's inequality and by (3.8), (4.1), (4.2), we get

$$J_1 \lesssim r^{\frac{n}{q}} \int\limits_{2r}^{\infty} \int\limits_{B(x_0,t)} |\Omega(x-y)| \, |b(y) - b_{B(x_0,t)}| \, |f(y)| \, dy \frac{dt}{t^{n+1-\alpha}}$$

$$+ r^{\frac{n}{q}} \int\limits_{2r}^{\infty} |b_{B(x_0,r)} - b_{B(x_0,t)}| \int\limits_{B(x_0,t)} |\Omega(x-y)| \, |f(y)| \, dy \frac{dt}{t^{n+1-\alpha}}$$

$$\lesssim r^{\frac{n}{q}} \int\limits_{2r}^{\infty} \|\Omega(\cdot - y)\|_{L_s(B(x_0,t))} \|(b(\cdot) - b_{B(x_0,t)})\|_{L_\mu(B(x_0,t))} \|f\|_{L_p(B(x_0,t))} \frac{dt}{t^{n+1-\alpha}}$$

$$+ r^{\frac{n}{q}} \int\limits_{2r}^{\infty} |b_{B(x_0,r)} - b_{B(x_0,t)}| \, \|\Omega(\cdot - y)\|_{L_s(B(x_0,t))} \|f\|_{L_p(B(x_0,t))} |B(x_0,t)|^{1-\frac{1}{p}-\frac{1}{s}} \frac{dt}{t^{n+1-\alpha}}$$

$$\lesssim \|b\|_* \, r^{\frac{n}{q}} \int\limits_{2r}^{\infty} \left(1 + \ln \frac{t}{r}\right) \|f\|_{L_p(B(x_0,t))} \frac{dt}{t^{\frac{n}{q}+1}}.$$

In order to estimate J_2 note that

$$J_2 = \|(b(\cdot) - b_{B(x_0,t)})\|_{L_q(B(x_0,t))} \int\limits_{(2B)^C} \frac{|\Omega(x-y)|}{|x_0-y|^{n-\alpha}} |f(y)| \, dy.$$

By (4.1), we get

$$J_2 \lesssim \|b\|_* \, r^{\frac{n}{q}} \int\limits_{(2B)^C} \frac{|\Omega(x-y)|}{|x_0-y|^{n-\alpha}} |f(y)| \, dy.$$

Thus, by (3.4) and (3.5)

$$J_2 \lesssim \|b\|_* \, r^{\frac{n}{q}} \int\limits_{2r}^{\infty} \|f\|_{L_p(B(x_0,t))} \frac{dt}{t^{\frac{n}{q}+1}}.$$

Summing up J_1 and J_2, for all $p \in (1, \infty)$ we get

$$(4.4) \qquad \|T_{\Omega,b,\alpha} f_2\|_{L_q(B)} \lesssim \|b\|_* \, r^{\frac{n}{q}} \int\limits_{2r}^{\infty} \left(1 + \ln \frac{t}{r}\right) \|f\|_{L_p(B(x_0,t))} \frac{dt}{t^{\frac{n}{q}+1}}.$$

Finally, we have the following

$$\|T_{\Omega,b,\alpha} f\|_{L_q(B)} \lesssim \|b\|_* \, \|f\|_{L_p(2B)} + \|b\|_* \, r^{\frac{n}{q}} \int\limits_{2r}^{\infty} \left(1 + \ln \frac{t}{r}\right) \|f\|_{L_p(B(x_0,t))} \frac{dt}{t^{\frac{n}{q}+1}},$$

which completes the proof of first statement by (3.7).

On the other hand when $q < s$, by the Fubini's theorem and the Minkowski inequality, we get

$$
\begin{aligned}
J_1 &\lesssim \left(\int_B \left| \int_{2r}^{\infty} \int_{B(x_0,t)} \left| b\left(y\right) - b_{B(x_0,t)} \right| \left| f\left(y\right) \right| \left| \Omega\left(x - y\right) \right| dy \frac{dt}{t^{n+1-\alpha}} \right|^q dx \right)^{\frac{1}{q}} \\
&\quad + \left(\int_B \left| \int_{2r}^{\infty} \left| b_{B(x_0,r)} - b_{B(x_0,t)} \right| \int_{B(x_0,t)} \left| f\left(y\right) \right| \left| \Omega\left(x - y\right) \right| dy \frac{dt}{t^{n+1-\alpha}} \right|^q dx \right)^{\frac{1}{q}} \\
&\lesssim \int_{2r}^{\infty} \int_{B(x_0,t)} \left| b\left(y\right) - b_{B(x_0,t)} \right| \left| f\left(y\right) \right| \left\| \Omega\left(\cdot - y\right) \right\|_{L_q(B(x_0,t))} dy \frac{dt}{t^{n+1-\alpha}} \\
&\quad + \int_{2r}^{\infty} \left| b_{B(x_0,r)} - b_{B(x_0,t)} \right| \int_{B(x_0,t)} \left| f\left(y\right) \right| \left\| \Omega\left(\cdot - y\right) \right\|_{L_q(B(x_0,t))} dy \frac{dt}{t^{n+1-\alpha}} \\
&\lesssim |B|^{\frac{1}{q} - \frac{1}{s}} \int_{2r}^{\infty} \int_{B(x_0,t)} \left| b\left(y\right) - b_{B(x_0,t)} \right| \left| f\left(y\right) \right| \left\| \Omega\left(\cdot - y\right) \right\|_{L_s(B(x_0,t))} dy \frac{dt}{t^{n+1-\alpha}} \\
&\quad + |B|^{\frac{1}{q} - \frac{1}{s}} \int_{2r}^{\infty} \left| b_{B(x_0,r)} - b_{B(x_0,t)} \right| \int_{B(x_0,t)} \left| f\left(y\right) \right| \left\| \Omega\left(\cdot - y\right) \right\|_{L_s(B(x_0,t))} dy \frac{dt}{t^{n+1-\alpha}}.
\end{aligned}
$$

Applying the Hölder's inequality and by (3.8), (4.1), (4.2), we get

$$
\begin{aligned}
J_1 &\lesssim r^{\frac{n}{q} - \frac{n}{s}} \int_{2r}^{\infty} \left\| \left(b\left(\cdot\right) - b_{B(x_0,t)}\right) f \right\|_{L_1(B(x_0,t))} \left| B\left(x_0, \frac{3}{2}t\right) \right|^{\frac{1}{s}} \frac{dt}{t^{n+1-\alpha}} \\
&\quad + r^{\frac{n}{q} - \frac{n}{s}} \int_{2r}^{\infty} \left| b_{B(x_0,r)} - b_{B(x_0,t)} \right| \left\| f \right\|_{L_p(B(x_0,t))} \left| B\left(x_0, \frac{3}{2}t\right) \right|^{\frac{1}{s}} \frac{dt}{t^{\frac{n}{q}+1}} \\
&\lesssim r^{\frac{n}{q} - \frac{n}{s}} \int_{2r}^{\infty} \left\| \left(b\left(\cdot\right) - b_{B(x_0,t)}\right) \right\|_{L_{p'}(B(x_0,t))} \left\| f \right\|_{L_p(B(x_0,t))} t^{\frac{n}{s}} \frac{dt}{t^{n+1}} \\
&\quad + r^{\frac{n}{q} - \frac{n}{s}} \int_{2r}^{\infty} \left| b_{B(x_0,r)} - b_{B(x_0,t)} \right| \left\| f \right\|_{L_p(B(x_0,t))} t^{\frac{n}{s}} \frac{dt}{t^{\frac{n}{q}+1}} \\
&\lesssim \left\| b \right\|_* r^{\frac{n}{q} - \frac{n}{s}} \int_{2r}^{\infty} \left(1 + \ln\frac{t}{r} \right) t^{\frac{n}{s} - \frac{n}{q} - 1} \left\| f \right\|_{L_p(B(x_0,t))} dt.
\end{aligned}
$$

Let $\frac{1}{p} = \frac{1}{\nu} + \frac{1}{s}$, then for J_2, by the Fubini's theorem, the Minkowski inequality, the Hölder's inequality and from (3.8), we get

$$J_2 \lesssim \left(\int_B \left| \int_{2r}^{\infty} \int_{B(x_0,t)} |f(y)| |b(x) - b_B| |\Omega(x-y)| \, dy \frac{dt}{t^{n+1-\alpha}} \right|^q dx \right)^{\frac{1}{q}}$$

$$\lesssim \int_{2r}^{\infty} \int_{B(x_0,t)} |f(y)| \|(b(\cdot) - b_B)\Omega(\cdot - y)\|_{L_q(B)} \, dy \frac{dt}{t^{n+1-\alpha}}$$

$$\lesssim \int_{2r}^{\infty} \int_{B(x_0,t)} |f(y)| \|b(\cdot) - b_B\|_{L_\nu(B)} \|\Omega(\cdot - y)\|_{L_s(B)} \, dy \frac{dt}{t^{n+1-\alpha}}$$

$$\lesssim \|b\|_* |B|^{\frac{1}{q} - \frac{1}{s}} \int_{2r}^{\infty} \int_{B(x_0,t)} |f(y)| \|\Omega(\cdot - y)\|_{L_s(B)} \, dy \frac{dt}{t^{n+1-\alpha}}$$

$$\lesssim \|b\|_* r^{\frac{n}{q} - \frac{n}{s}} \int_{2r}^{\infty} \|f\|_{L_1(B(x_0,t))} \left| B\left(x_0, \frac{3}{2}t\right) \right|^{\frac{1}{s}} \frac{dt}{t^{n+1-\alpha}}$$

$$\lesssim \|b\|_* r^{\frac{n}{q} - \frac{n}{s}} \int_{2r}^{\infty} \left(1 + \ln \frac{t}{r} \right) t^{\frac{n}{s} - \frac{n}{q} - 1} \|f\|_{L_p(B(x_0,t))} dt.$$

By combining the above estimates, we complete the proof of Lemma 4.1. $\qquad \square$

Now we can give the following theorem (our main result).

Theorem 4.1. *(Our main result) Suppose that $\Omega \in L_s(S^{n-1})$, $1 < s \le \infty$, is homogeneous of degree zero and $T_{\Omega,b,\alpha}$ is a sublinear operator satisfying condition (1.2) and bounded from $L_p(\mathbb{R}^n)$ to $L_q(\mathbb{R}^n)$. Let $1 < p < \infty$ $0 < \alpha < \frac{n}{p}$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$ and $b \in BMO(\mathbb{R}^n)$.*

Let also, for $s' \le p$ the pair (φ_1, φ_2) satisfies the condition

$$(4.5) \qquad \int_r^{\infty} \left(1 + \ln \frac{t}{r} \right) \frac{\essinf_{t < \tau < \infty} \varphi_1(x, \tau) \tau^{\frac{n}{p}}}{t^{\frac{n}{q}+1}} dt \le C \varphi_2(x, r),$$

and for $q < s$ the pair (φ_1, φ_2) satisfies the condition

$$(4.6) \qquad \int_r^{\infty} \left(1 + \ln \frac{t}{r} \right) \frac{\essinf_{t < \tau < \infty} \varphi_1(x, \tau) \tau^{\frac{n}{p}}}{t^{\frac{n}{q} - \frac{n}{s}+1}} dt \le C \varphi_2(x, r) r^{\frac{n}{s}},$$

where C does not depend on x and r.

Then, the operator $T_{\Omega,b,\alpha}$ is bounded from M_{p,φ_1} to M_{q,φ_2}. Moreover

$$\|T_{\Omega,b,\alpha} f\|_{M_{q,\varphi_2}} \lesssim \|b\|_* \|f\|_{M_{p,\varphi_1}}.$$

Proof. The statement of Theorem 4.1 follows by Lemma 1.2 and Lemma 4.1 in the same manner as in the proof of Theorem 3.1. $\qquad \square$

By Theorem 4.1, we get the following new result.

Corollary 4.1. *Suppose that $\Omega \in L_s(S^{n-1})$, $1 < s \leq \infty$, is homogeneous of degree zero. Let $1 < p < \infty$ $0 < \alpha < \frac{n}{p}$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$ and $b \in BMO(\mathbb{R}^n)$. If for $s' \leq p$ the pair (φ_1, φ_2) satisfies the condition (4.5) and for $q < s$ the pair (φ_1, φ_2) satisfies the condition (4.6). Then, the operators $M_{\Omega,b,\alpha}$ and $[b, \overline{T}_{\Omega,\alpha}]$ are bounded from M_{p,φ_1} to M_{q,φ_2}.*

For the sublinear commutator of the fractional maximal operator is defined as follows

$$M_{b,\alpha}(f)(x) = \sup_{t>0} |B(x,t)|^{-1+\frac{\alpha}{n}} \int\limits_{B(x,t)} |b(x) - b(y)|\, |f(y)| dy$$

by Theorem 4.1 we get the following new result.

Corollary 4.2. *Let $0 < \alpha < n$, $1 < p < \frac{n}{\alpha}$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$, $b \in BMO(\mathbb{R}^n)$ and the pair (φ_1, φ_2) satisfies the condition (4.5). Then, the operators $M_{b,\alpha}$ and $[b, \overline{T}_\alpha]$ are bounded from M_{p,φ_1} to M_{q,φ_2}.*

Now using above results, we also obtain the boundedness of the operator $T_{\Omega,b,\alpha}$ on the generalized vanishing Morrey spaces $VM_{p,\varphi}$.

Theorem 4.2. *(Our main result) Let $\Omega \in L_s(S^{n-1})$, $1 < s \leq \infty$, be homogeneous of degree zero. Let $1 < p < \infty$, $0 < \alpha < \frac{n}{p}$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$, $b \in BMO(\mathbb{R}^n)$, and $T_{\Omega,b,\alpha}$ is a sublinear operator satisfying condition (1.2) and bounded from $L_p(\mathbb{R}^n)$ to $L_q(\mathbb{R}^n)$. Let for $s' \leq p$ the pair (φ_1, φ_2) satisfies conditions (2.3)-(2.4) and*

$$(4.7) \qquad \int\limits_r^\infty \left(1 + \ln \frac{t}{r}\right) \varphi_1(x,t) \frac{t^{\frac{n}{p}}}{t^{\frac{n}{q}+1}} dt \leq C_0 \varphi_2(x,r),$$

where C_0 does not depend on $x \in \mathbb{R}^n$ and $r > 0$,

$$(4.8) \qquad \lim_{r\to 0} \frac{\ln \frac{1}{r}}{\inf\limits_{x\in\mathbb{R}^n} \varphi_2(x,r)} = 0$$

and

$$(4.9) \qquad c_\delta := \int\limits_\delta^\infty (1 + \ln|t|) \sup_{x\in\mathbb{R}^n} \varphi_1(x,t) \frac{t^{\frac{n}{p}}}{t^{\frac{n}{q}+1}} dt < \infty$$

for every $\delta > 0$, and for $q < s$ the pair (φ_1, φ_2) satisfies conditions (2.3)-(2.4) and also

$$(4.10) \qquad \int\limits_r^\infty \left(1 + \ln \frac{t}{r}\right) \varphi_1(x,t) \frac{t^{\frac{n}{p}}}{t^{\frac{n}{q}-\frac{n}{s}+1}} dt \leq C_0 \varphi_2(x,r) r^{\frac{n}{s}},$$

where C_0 does not depend on $x \in \mathbb{R}^n$ and $r > 0$,

$$\lim_{r\to 0} \frac{\ln \frac{1}{r}}{\inf\limits_{x\in\mathbb{R}^n} \varphi_2(x,r)} = 0$$

and

$$(4.11) \qquad c_{\delta'} := \int\limits_{\delta'}^\infty (1 + \ln|t|) \sup_{x\in\mathbb{R}^n} \varphi_1(x,t) \frac{t^{\frac{n}{p}}}{t^{\frac{n}{q}-\frac{n}{s}+1}} dt < \infty$$

for every $\delta' > 0$.

Then the operator $T_{\Omega,b,\alpha}$ *is bounded from* VM_{p,φ_1} *to* VM_{q,φ_2}. *Moreover,*

$$(4.12) \qquad \|T_{\Omega,b,\alpha}f\|_{VM_{q,\varphi_2}} \lesssim \|b\|_* \|f\|_{VM_{p,\varphi_1}} .$$

Proof. The norm inequality having already been provided by Theorem 4.1, we only have to prove the implication
(4.13)

$$\lim_{r \to 0} \sup_{x \in \mathbb{R}^n} \frac{r^{-\frac{n}{p}} \|f\|_{L_p(B(x,r))}}{\varphi_1(x,r)} = 0 \text{ implies } \lim_{r \to 0} \sup_{x \in \mathbb{R}^n} \frac{r^{-\frac{n}{q}} \|T_{\Omega,b,\alpha}f\|_{L_q(B(x,r))}}{\varphi_2(x,r)} = 0.$$

To show that

$$\sup_{x \in \mathbb{R}^n} \frac{r^{-\frac{n}{q}} \|T_{\Omega,b,\alpha}f\|_{L_q(B(x,r))}}{\varphi_2(x,r)} < \epsilon \text{ for small } r,$$

we use the estimate (4.3):

$$\sup_{x \in \mathbb{R}^n} \frac{r^{-\frac{n}{q}} \|T_{\Omega,b,\alpha}f\|_{L_q(B(x,r))}}{\varphi_2(x,r)} \lesssim \frac{\|b\|_*}{\varphi_2(x,r)} \int_r^\infty \left(1 + \ln \frac{t}{r}\right) t^{-\frac{n}{q}-1} \|f\|_{L_p(B(x_0,t))} dt.$$

We take $r < \delta_0$, where δ_0 will be chosen small enough and split the integration:

$$(4.14) \qquad \frac{r^{-\frac{n}{q}} \|T_{\Omega,b,\alpha}f\|_{L_q(B(x,r))}}{\varphi_2(x,r)} \leq C \left[I_{\delta_0}(x,r) + J_{\delta_0}(x,r) \right],$$

where $\delta_0 > 0$ (we may take $\delta_0 < 1$), and

$$I_{\delta_0}(x,r) := \frac{1}{\varphi_2(x,r)} \int_r^{\delta_0} \left(1 + \ln \frac{t}{r}\right) t^{-\frac{n}{q}-1} \|f\|_{L_p(B(x,t))} dt,$$

and

$$J_{\delta_0}(x,r) := \frac{1}{\varphi_2(x,r)} \int_{\delta_0}^\infty \left(1 + \ln \frac{t}{r}\right) t^{-\frac{n}{q}-1} \|f\|_{L_p(B(x,t))} dt$$

Now we choose any fixed $\delta_0 > 0$ such that

$$\sup_{x \in \mathbb{R}^n} \frac{t^{-\frac{n}{p}} \|f\|_{L_p(B(x,t))}}{\varphi_1(x,t)} < \frac{\epsilon}{2CC_0}, \qquad t \leq \delta_0,$$

where C and C_0 are constants from (4.7) and (4.14). This allows to estimate the first term uniformly in $r \in (0, \delta_0)$:

$$\sup_{x \in \mathbb{R}^n} C I_{\delta_0}(x,r) < \frac{\epsilon}{2}, \qquad 0 < r < \delta_0.$$

For the second term, writing $1 + \ln \frac{t}{r} \leq 1 + |\ln t| + \ln \frac{1}{r}$, we obtain

$$J_{\delta_0}(x,r) \leq \frac{c_{\delta_0} + \widetilde{c_{\delta_0}} \ln \frac{1}{r}}{\varphi_2(x,r)} \|f\|_{M_{p,\varphi}},$$

where c_{δ_0} is the constant from (4.9) with $\delta = \delta_0$ and $\widetilde{c_{\delta_0}}$ is a similar constant with omitted logarithmic factor in the integrand. Then, by (4.8) we can choose small enough r such that

$$\sup_{x \in \mathbb{R}^n} J_{\delta_0}(x,r) < \frac{\epsilon}{2},$$

which completes the proof of (4.13).

For the case of $q < s$, we can also use the same method, so we omit the details. $\quad\square$

Remark 4.2. Conditions (4.9) and (4.11) are not needed in the case when $\varphi(x, r)$ does not depend on x, since (4.9) follows from (4.7) and similarly, (4.11) follows from (4.10) in this case.

Corollary 4.3. *Suppose that $\Omega \in L_s(S^{n-1})$, $1 < s \le \infty$, is homogeneous of degree zero. Let $1 < p < \infty$, $0 < \alpha < \frac{n}{p}$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$ and $b \in BMO(\mathbb{R}^n)$. If for $s' \le p$ the pair (φ_1, φ_2) satisfies conditions (2.3)-(2.4)-(4.8) and (4.9)-(4.7) and for $p < q$ the pair (φ_1, φ_2) satisfies conditions (2.3)-(2.4)-(4.8) and (4.11)-(4.10). Then, the operators $M_{\Omega,b,\alpha}$ and $[b, \overline{T}_{\Omega,\alpha}]$ are bounded from $VM_{p,\varphi_1}(\mathbb{R}^n)$ to $VM_{q,\varphi_2}(\mathbb{R}^n)$.*

In the case of $q = \infty$ by Theorem 4.2, we get

Corollary 4.4. *Let $1 < p < \infty$, $0 < \alpha < \frac{n}{p}$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$ and $b \in BMO(\mathbb{R}^n)$ and the pair (φ_1, φ_2) satisfies conditions (2.3)-(2.4)-(4.8) and (4.9)-(4.7). Then the operators $M_{b,\alpha}$ and $[b, \overline{T}_\alpha]$ are bounded from $VM_{p,\varphi_1}(\mathbb{R}^n)$ to $VM_{q,\varphi_2}(\mathbb{R}^n)$.*

5. SOME APPLICATIONS

In this section, we give the applications of Theorem 3.1, Theorem 3.2, Theorem 4.1, Theorem 4.2 for the Marcinkiewicz operator.

5.1. **Marcinkiewicz Operator.** Let $S^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$ be the unit sphere in $\mathbb{R}^n$ equipped with the Lebesgue measure $d\sigma$. Suppose that Ω satisfies the following conditions.

(a) Ω is the homogeneous function of degree zero on $\mathbb{R}^n \setminus \{0\}$, that is,

$$\Omega(\mu x) = \Omega(x), \text{ for any } \mu > 0, x \in \mathbb{R}^n \setminus \{0\}.$$

(b) Ω has mean zero on S^{n-1}, that is,

$$\int_{S^{n-1}} \Omega(x')d\sigma(x') = 0,$$

where $x' = \frac{x}{|x|}$ for any $x \neq 0$.

(c) $\Omega \in Lip_\gamma(S^{n-1})$, $0 < \gamma \le 1$, that is there exists a constant $M > 0$ such that,

$$|\Omega(x') - \Omega(y')| \le M|x' - y'|^\gamma \text{ for any } x', y' \in S^{n-1}.$$

In 1958, Stein [45] defined the Marcinkiewicz integral of higher dimension μ_Ω as

$$\mu_\Omega(f)(x) = \left(\int_0^\infty |F_{\Omega,t}(f)(x)|^2 \frac{dt}{t^3} \right)^{1/2},$$

where

$$F_{\Omega,t}(f)(x) = \int_{|x-y|\le t} \frac{\Omega(x - y)}{|x - y|^{n-1}} f(y)dy.$$

Since Stein's work in 1958, the continuity of Marcinkiewicz integral has been extensively studied as a research topic and also provides useful tools in harmonic analysis [29, 46, 47, 48].

The Marcinkiewicz operator is defined by (see [49])

$$\mu_{\Omega,\alpha}(f)(x) = \left(\int\limits_0^\infty |F_{\Omega,\alpha,t}(f)(x)|^2 \frac{dt}{t^3} \right)^{1/2},$$

where

$$F_{\Omega,\alpha,t}(f)(x) = \int\limits_{|x-y|\leq t} \frac{\Omega(x-y)}{|x-y|^{n-1-\alpha}} f(y)dy.$$

Note that $\mu_\Omega f = \mu_{\Omega,0} f$.

The sublinear commutator of the operator $\mu_{\Omega,\alpha}$ is defined by

$$[b, \mu_{\Omega,\alpha}](f)(x) = \left(\int\limits_0^\infty |F_{\Omega,\alpha,t,b}(f)(x)|^2 \frac{dt}{t^3} \right)^{1/2},$$

where

$$F_{\Omega,\alpha,t,b}(f)(x) = \int\limits_{|x-y|\leq t} \frac{\Omega(x-y)}{|x-y|^{n-1-\alpha}} [b(x)-b(y)]f(y)dy.$$

We consider the space $H = \{h : \|h\| = (\int\limits_0^\infty |h(t)|^2 \frac{dt}{t^3})^{1/2} < \infty\}$. Then, it is clear that $\mu_{\Omega,\alpha}(f)(x) = \|F_{\Omega,\alpha,t}(x)\|$.

By the Minkowski inequality, we get

$$\mu_{\Omega,\alpha}(f)(x) \leq \int\limits_{\mathbb{R}^n} \frac{|\Omega(x-y)|}{|x-y|^{n-1-\alpha}} |f(y)| \left(\int\limits_{|x-y|}^\infty \frac{dt}{t^3} \right)^{1/2} dy \leq C \int\limits_{\mathbb{R}^n} \frac{|\Omega(x-y)|}{|x-y|^{n-\alpha}} |f(y)|dy.$$

Thus, $\mu_{\Omega,\alpha}$ satisfies the condition (1.1). It is known that for $b \in BMO(\mathbb{R}^n)$ the operators $\mu_{\Omega,\alpha}$ and $[b, \mu_{\Omega,\alpha}]$ are bounded from $L_p(\mathbb{R}^n)$ to $L_q(\mathbb{R}^n)$ for $p > 1$, and bounded from $L_1(\mathbb{R}^n)$ to $WL_q(\mathbb{R}^n)$ for $p = 1$ (see [49]), then by Theorems 3.1, 3.2, 4.1 and 4.2 we get

Corollary 5.1. *Suppose that $\Omega \in L_s(S^{n-1})$, $1 < s \leq \infty$, is homogeneous of degree zero. Let $0 < \alpha < n$, $1 \leq p < \frac{n}{\alpha}$ and $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$. Let also, for $s' \leq p$, $p \neq 1$, the pair (φ_1, φ_2) satisfies condition (3.10) and for $q < s$ the pair (φ_1, φ_2) satisfies condition (3.11) and Ω satisfies conditions (a)–(c). Then the operator $\mu_{\Omega,\alpha}$ is bounded from M_{p,φ_1} to M_{q,φ_2} for $p > 1$ and from M_{1,φ_1} to WM_{q,φ_2} for $p = 1$.*

Corollary 5.2. *Suppose that $\Omega \in L_s(S^{n-1})$, $1 < s \leq \infty$, is homogeneous of degree zero. Let $0 < \alpha < n$, $1 \leq p < \frac{n}{\alpha}$ and $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$. Let also, for $s' \leq p$, $p \neq 1$, the pair (φ_1, φ_2) satisfies conditions (2.3)-(2.4) and (3.12)-(3.13) and for $q < s$ the pair (φ_1, φ_2) satisfies conditions (2.3)-(2.4) and (3.14)-(3.15) and Ω satisfies conditions (a)–(c). Then the operator $\mu_{\Omega,\alpha}$ is bounded from VM_{p,φ_1} to VM_{q,φ_2} for $p > 1$ and from VM_{1,φ_1} to WVM_{q,φ_2} for $p = 1$.*

Corollary 5.3. *Suppose that $\Omega \in L_s(S^{n-1})$, $1 < s \leq \infty$, is homogeneous of degree zero. Let $1 < p < \infty$, $0 < \alpha < \frac{n}{p}$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$ and $b \in BMO(\mathbb{R}^n)$. Let also, for $s' \leq p$ the pair (φ_1, φ_2) satisfies condition (4.5) and for $q < s$ the pair (φ_1, φ_2) satisfies condition (4.6) and Ω satisfies conditions (a)–(c). Then, the operator $[b, \mu_{\Omega,\alpha}]$ is bounded from M_{p,φ_1} to M_{q,φ_2}.*

Corollary 5.4. *Suppose that $\Omega \in L_s(S^{n-1})$, $1 < s \leq \infty$, is homogeneous of degree zero. Let $1 < p < \infty$, $0 < \alpha < \frac{n}{p}$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$ and $b \in BMO(\mathbb{R}^n)$. Let also, for $s' \leq p$ the pair (φ_1, φ_2) satisfies conditions (2.3)-(2.4)-(4.8) and (4.9)-(4.7) and for $q < s$ the pair (φ_1, φ_2) satisfies conditions (2.3)-(2.4)-(4.8) and (4.11)-(4.10) and Ω satisfies conditions (a)–(c). Then, the operator $[b, \mu_{\Omega,\alpha}]$ is bounded from VM_{p,φ_1} to VM_{q,φ_2}.*

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TEACHERS' OPINIONS ABOUT MATHEMATIC PROGRAM REVISED WITH 4+4+4 EDUCATION SYSTEM

ADEM BAŞKAYA AND ORKUN ÇOŞKUNTUNCEL

Abstract. The purpose of this research is taking the teachers' opinions about the re-prepared secondary school (5, 6, 7 and 8 grades) mathematics program with gradually changing education system in 2013-2014 academic year. The universe of research is secondary school mathematics teachers working in Mersin province, and the sample of research contains 149 teachers working in city center, districts and villages and all agree to participate on a voluntary basis to the research.

The scale of the research containing 5 factors (General features, activity, and applications of mathematics course, new education system, and textbook) with 28 questions has been developed by the researchers and the Cronbach Alfa rate of the scale is 0,886. For the analysis of the data, descriptive statistics, t-test and one-way ANOVA have been used.

As a result of the research, while the subdimension of achievement and content has come out at middle level, and also the applications of mathematics course has come out at medium level, the teaching period has been at good level but course books and education system has been at low level. It has been indicated that course books are not enough and the opinions of teachers hasn't been considered in research's renewing stage. Some has mentioned that the subjects has become simple regarding the omitted ones from curriculum while some mentioned that the curriculum has been completely cleared. Besides, it has also been mentioned that the curriculum has been able to be taught at fundamental level.

1. Introduction

In our world maintaining to change continuously from past to present countries have tried to keep up with this change. They are still placing great emphasize on education in order to succeed it. Because reaching the way of the target indicated by Mustafa Kemal Ataturk as "Reach and pass the level of contemporary civilization" is provided with the education. The education programs and curriculums prepared for our today's conditions need to be improved most effectively and the programs related to all of the courses need to be structured regarding this purpose so that this education system can be performed and the individuals can succeed to get necessary knowledge and skills (Karagöz,2010; Olkun and Toluk, 2007). Our world has been in a complicated and quantitative status with the last technologic improvements; therefore, mathematical thinking has been more important and the need of teaching

mathematics has increased (Willoughby, 1990). In this period, mathematics as a course has a great importance and our world's need of people understanding and interpreting the course, likely arises. As for considering either TEOG exam applied in our country or TIMMS and PISA exams applied in international area, we see which level we are at in terms of mathematics. So, it has been a must to change the education system and mathematics programs regarding these results. Because the education programs should follow the improvements all the time depending on the ones in science and technology. Besides, the teaching ones should also change according to time and conditions (Kemertaş, 1999). With these changes, redefining and reviewing of mathematics and it's education in accordance with the identified needs need to be performed (MEB, 2005). The reason of the aforesaid is that mathematics has always been the supporter of the forward but it hasn't been able to exceed the traditional status without it (Savaş and others, 2006). Considering these reasons, mathematics programs regarding 5.,6.,7.,8. grades of secondary school was renewed gradually in 2013-2014 school year in accordance with the decision dated 01.02.2013 prepared by the Ministry of National Education, the Board of Education and Discipline. This research has also been for analyzing teacher's opinions concerning the renewed mathematics program.

Consequently, this research has tried to seek answers for the question of "What are the opinions of secondary school mathematics teacher about the reprepared mathematics program with 4+4+4 education system?" and the following subproblems.

2. Subproblems

1. How do the Secondary school mathematics teachers think about 4+4+4 education system, it's achievement and teaching period degree, the course of mathematical applications and the course books?

2. Is there a significant difference between the opinions of secondary school mathematics teachers about sex, seniority, place of duty, whether they take seminar or not, and whether the school buildings are the same or not when considered the program's general and subdimensions (System, achievement, teaching period, mathematical applications, and course books) ?

3. How do the mathematics teachers think about the omission of some subjects from the curriculum of secondary school mathematics program reprepared with 4+4+4 education system?

3. Method

In this study, the survey research design has been used. Survey models are the approaches aiming to describe the situations as either it is in the past or it is still continued (Karasar, 1995). Also, the study deals with that the different groups are compared in terms of some variables thereby performing a relational research. The data collection tool is a kind of likert scale developed by a researcher and the rate of Cronbach alpha reliability coefficient has obtained as 0,886. The scale has 5 factors and contains 28 articles.

4. Findings and Results

According to findings obtained from the first subproblem of the research regarding the question "How do the teachers think about 4+4+4 education system,

achievement dimension, teaching methods and techniques' dimension, the course of mathematical applications and course books?", teachers have positively reacted 4+4+4 education system's being compulsory and gradual with the arithmetic mean 2,62 in middle level. According to teachers' opinions related with the research of Aybek ve Aslan (2015), teachers have positively reacted the 12 years and discontinuous education regarding 4+4+4 discontinuous compulsory education. In 4 interviews of 6, they have positively reacted the 12 years compulsory education in the study of Doğan, Demir ve Pınar (2014). But teachers have negatively reacted to the article "5 grades have been accepted as secondary school" with arithmetic mean 3,32 in middle level in terms of mathematics teaching. Teachers have expressed opinions to the article "The opinions of teachers working in the field have been received while passing to system" with the arithmetic mean 1,85 in weak level. Considering the general of the articles related with the system, they have expressed opinion with the arithmetic mean in weak level.

According to opinions obtained from the research regarding the achievements of the program, suitability has been identified in middle level that the achievement to the mental development is has arithmetic mean 3,24; the achievement to multi-directional thinking has arithmetic mean 2,98; the achievement to their readiness level has arithmetic mean 3,16. The study of Mercan (2011) done for the article about readiness in the past years has the quality to support. Also, the achievement associated with daily life has arithmetic mean 3,20; it's suitability to Turkish National Education and the general features of mathematics education has arithmetic mean 3,32 in middle level. The ordering of subject from concrete to abstract has arithmetic mean 3,42 in good level. The program organized from simple to complicated and it's having cyclical structure has arithmetic mean 3,53 in good level. Teachers have expressed opinions that the article "the content of program is consistent with the general features of mathematics education" has arithmetic mean 3,38; the article "the subject in the program's content is appropriate for the cognitive development of students" has arithmetic mean 3,26; the article "the achievement of the program are clearly understandable and applicable" has arithmetic mean 3,40; the article "the program have had the students like the mathematics course" has arithmetic mean 2,71 in middle level. The results obtained also from the same studies have quality to support the study (Mercan, 2011; Karagöz 2010;İyiol 2011).

In this research, the findings regarding the teaching period have been obtained are the followings: Teachers have expressed opinions in good level to the articles "Students are participating actively in course with the help of my applied activities" with arithmetic mean 3,54; "I am applying learning activities for increasing problems solving skills of students" with arithmetic mean 3,76; "I am benefiting from materials effectively while using learning methods" with arithmetic mean 3,50; "Learning and teaching activities I am using have quality to like mathematics" with arithmetic mean 3,69; "Teaching methods and techniques I am using addresses varied intelligence areas" with arithmetic mean 3,49; "Activities I am getting to be performed are at level which students can apply" with arithmetic mean 3,93; "I am trying to increase the interaction between students in learning and teaching period" with arithmetic mean 3,93.

When looked at the opinions of teachers about the mathematical applications course, the rate of arithmetic mean has come out at 3,17 with middle level regarding the article "Mathematics course has been more funny with the mathematical

applications". Another article "I am using the mathematical applications most appropriately" has come out at arithmetic mean 3.20 and the other article "The expectations of both mathematical applications and parents are coincided" has come out at arithmetic mean 2,65 in middle level. The information missing of parents about elected courses and the sufficient information can't be transferred are the problems come across by the teachers that have been identified also in the study of Aslan ve Aybek (2015).

When looked at the opinions of the research regarding the course books, our teachers has expressed opinions to the article "no assistant source are needed as course books are enough for learning-teaching period" in weak level with the arithmetic mean 1,60. They have expressed opinions with arithmetic mean 1,97 to the article "The teaching style of course in course books is sufficiently clear", 1,90 to the article "The exercises developing operation capabilities are sufficiently included in books" in weak level. Teachers has also mentioned in the research of Mutu (2008) done about the same subject in the past years that the content of 6. and 7. grade books is totally weak and the subject ordering is inconsistent; so they need assistant books because of this, and the sample questions are missing.

When looked at the answers to open ended questions we have asked regarding the omissions of some subjects, some teacher look positively at program's becoming simple but others has a number of concerns about that. They have considered that the subjects fully omitted will cause problems in high school and next education life, and they have defended that these subjects should be mentioned even a little. Furthermore, the positive contribution of the subjects for the students to understand and focus on other subjects is their another opinion and they have also considered that it has a positive impact in terms of time because of the mathematics course's heavy subjects.

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THE DIRECT AND INVERSE SPECTRAL PROBLEM FOR STURM-LIOUVILLE OPERATOR WITH DISCONTINUOUS COEFFICIENT

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Abstract. In this study, for Sturm-Liouville operator with discontinuous coefficient encountered in the non-homogeneous materials, direct and inverse problems are investigated. The spectral properties of the Sturm-Liouville problem with discontinuous coefficient such as the orthogonality of its eigenfunctions and simplicity of its eigenvalues are examined. Asymptotic formula is found for eigenvalues, and resolvent operator is constructed. The expansion formula with respect to eigenfunctions is obtained. It is shown that its eigenvalues are in the form of a complete system. Also, the Weyl solution and Weyl function are defined. Uniqueness theorems for the solution of the inverse problem according to spectral date are proved.

1. Introduction

In this study, the heat problem of a rod that consists of two parts with fixed cross section is examined. The side surfaces of the rod have been isolated and have different physical features [1]-[4]. When initial temperature is given arbitrary and the temperature at the ending points is not equal to zero, the heat problem of the rod takes the following form:

$$\rho(x)U_t = U_{xx} + q(x)U, \quad 0 \le x \le \pi,$$

$$U(x,0) = \phi(x), \quad U_t(x,0) = \psi(x), \quad 0 \le x\pi$$

$$U_x(0,t) = 0, \quad U_x(\pi,t) = 0, \quad t > 0$$

where the function $U(x,t)$ is the temperature in the bar at the time t, $\rho(x)$ is a piecewise constant function and refers to the density of the rod and $\phi(x)$, $\psi(x)$ are enough smooth functions. By the method of separation of variables, the preceding equation is reduced to a boundary value problem for Sturm-Liouville equation:

$$(1.1) \qquad -y'' + q(x)y = \lambda^2 \rho(x)y, \quad 0 \le x \le \pi,$$

$$(1.2) \qquad y'(0) = y'(\pi) = 0,$$

where in particular, $\rho(x)$ is chosen as

(1.3) $$\rho(x) = \begin{cases} 1 & 0 \le x < a \\ \alpha^2 & a \le x \le \pi. \end{cases}$$

$q(x) \in L_2(0, \pi)$ is a real valued function and λ is a complex parameter. Then, in finding the solution of the above diffusion problem, spectral problem (1.1), (1.2) must be examined [1]-[5]. The spectral problems with discontinuous coefficient on the bounded interval are investigated in [6]-[15]. The similar problems on the half line by different authors have been studied (see [16]-[18]). Let $\varphi(x)$ and $\psi(x)$ be solutions of (1.1), (1.2) boundary value problem satisfying the initial conditions

(1.4) $$\varphi(0, \lambda) = 1, \varphi'(0, \lambda) = 0$$

and

(1.5) $$\psi(\pi, \lambda) = 1, \psi'(\pi, \lambda) = 0.$$

Denote

(1.6) $$\Delta(\lambda) = W[\varphi(x, \lambda), \psi(x, \lambda)] = \varphi'(x, \lambda)\psi(x, \lambda) - \varphi(x, \lambda)\psi'(x, \lambda).$$

The function $\Delta(\lambda)$ is called the characteristic function of the problem (1.1), (1.2), and substituting $x = 0$ and $x = \pi$ into (1.6), we get

(1.7) $$\Delta(\lambda) = \varphi'(\pi, \lambda) = -\psi'(0, \lambda).$$

Lemma 1.1. *The eigenfunctions $y_1(x, \lambda_1)$ and $y_2(x, \lambda_2)$ corresponding to different eigenvalues $\lambda_1 \neq \lambda_2$ are orthogonal.*

Proof. Since $y_1(x, \lambda_1)$ and $y_2(x, \lambda_2)$ are eigenfunctions of problem (1.1), (1.2), we get

$$-y_1''(x, \lambda_1) + q(x)y_1(x, \lambda_1) = \lambda_1^2 \rho(x)y_1(x, \lambda_1),$$
$$-y_2''(x, \lambda_2) + q(x)y_2(x, \lambda_2) = \lambda_2^2 \rho(x)y_2(x, \lambda_2).$$

Multiplying these equalities by $y_1(x, \lambda_1)$ and $-y_2(x, \lambda_2)$, respectively, and adding together,

$$\frac{d}{dx}\{< y_2(x, \lambda_2), y_1(x, \lambda_1) >\} = (\lambda_1^2 - \lambda_2^2)\rho(x)y_1(x, \lambda_1)y_2(x, \lambda_2)$$

is found. Integrating from 0 to π and using the condition (1.2), we have

$$(\lambda_1^2 - \lambda_2^2) \int_0^\pi \rho(x)y_1(x, \lambda_1)y_2(x, \lambda_2)dx = 0.$$

Since $\lambda_1 \neq \lambda_2$,

$$\int_0^\pi \rho(x)y_1(x, \lambda_1)y_2(x, \lambda_2)dx = 0.$$

$\square$

Corollary 1.1. *The eigenvalues of the boundary value problem (1.1), (1.2) are real.*

Lemma 1.2. *The zeros λ_n of characteristic function $\Delta(\lambda)$ coincide with the eigenvalues of the boundary value problem (1.1), (1.2). The functions $\varphi(x, \lambda_n)$ and $\psi(x, \lambda_n)$ are eigenfunctions and there exists a sequence β_n such that*

(1.8) $$\psi(x, \lambda_n) = \beta_n \varphi(x, \lambda_n), \quad \beta_n \neq 0.$$

Proof. 1) Let λ_0 be zero of $\Delta(\lambda)$. Then, because of (1.6), $\psi(x, \lambda_0) = \beta_0 \varphi(x, \lambda_0)$ and the function $\psi(x, \lambda_0)$ and $\varphi(x, \lambda_0)$ satisfy the boundary condition (1.2). Thus, λ_0 is an eigenvalue and $\psi(x, \lambda_0)$, $\varphi(x, \lambda_0)$ are corresponding eigenfunctions.

2) Let λ_0 be an eigenvalue of the problem (1.1), (1.2) and let $y_0(x)$ be a corresponding eigenfunction. Then, $y_0(x)$ satisfies the boundary condition (1.2). Clearly, $y_0(x) \neq 0$. Without loss of generality, we put $y_0(0) = 1$. Then $y_0'(0) = 0$,and consequently, $y_0(x) \equiv \varphi(x, \lambda)$. Hence, from (1.7), $\Delta_0(\lambda) = 0$. We have proved that for each eigenvalue there exists only one eigenfunction. $\qquad \square$

Lemma 1.3. *The eigenvalues of the boundary value problem (1.1), (1.2) are simple and*

$$(1.9) \qquad \qquad \dot{\Delta}(\lambda_n) = 2\lambda_n \alpha_n \beta_n,$$

where

$$\alpha_n := \int_0^\pi \rho(x) \varphi^2(x, \lambda_n) dx.$$

is the normalizing number of (1.1), (1.2).

Proof. Since $\varphi(x, \lambda_n)$ and $\psi(x, \lambda)$ are the solutions of this problem,

$$-\varphi''(x, \lambda_n) + q(x)\varphi(x, \lambda_n) = \lambda_n^2 \rho(x)\varphi(x, \lambda_n),$$

$$-\psi''(x, \lambda) + q(x)\psi(x, \lambda) = \lambda^2 \rho(x)\psi(x, \lambda),$$

are valid. Multiplying these equations by $\psi(x, \lambda)$ and $-\varphi(x, \lambda_n)$, respectively, and adding them together, we get

$$\frac{d}{dx}\{< \psi(x, \lambda), \varphi(x, \lambda_n) >\} = (\lambda_n^2 - \lambda^2)\rho(x)\varphi(x, \lambda_n)\psi(x, \lambda).$$

Integrating from 0 to π and using the condition (1.2),

$$\int_0^\pi \rho(x)\varphi(x, \lambda_n)\psi(x, \lambda) = \frac{\Delta(\lambda_n) - \Delta(\lambda)}{\lambda_n^2 - \lambda^2}$$

is found. From Lemma 2, since $\psi(x, \lambda_n) = \beta_n \varphi(x, \lambda_n)$, as $\lambda \to \lambda_n$, we obtain

$$\dot{\Delta}(\lambda_n) = 2\lambda_n \alpha_n \beta_n$$

where $\beta_n = \psi(0, \lambda_n)$. Thus, it follows that $\dot{\Delta}(\lambda_n) \neq 0$. $\qquad \square$

2. On the Eigenvalues of Problem (1.1), (1.2) at $q(x) \equiv 0$

Denote by $\varphi_0(x, \lambda)$ the solution equation $-y'' = \lambda^2 \rho(x)y$, satisfying the condition (1.4). It has the following form:

$$(2.1) \qquad \varphi_0(x, \lambda) = \frac{1}{2}\left(1 + \frac{1}{\sqrt{\rho(x)}}\right)\cos \lambda \mu^+(x) + \frac{1}{2}\left(1 - \frac{1}{\sqrt{\rho(x)}}\right)\cos \lambda \mu^-(x),$$

where $\mu^\pm(x) = \pm x\sqrt{\rho(x)} + a(1 \mp \sqrt{\rho(x)})$.

It is easy show that if $(\lambda_n^0)^2$ are eigenvalues of problem (1.1), (1.2) at $q(x) \equiv 0$, then λ_n^0 can be found from the equation $\varphi_0'(\pi, \lambda) = 0$,that is, from the equation

$$\Delta_0(\lambda) = -\frac{1}{2}\lambda(\alpha + 1)\sin \lambda \mu^+(\pi) + \frac{1}{2}(\alpha - 1)\lambda \sin \lambda \mu^-(\pi) = 0$$

$$(2.2) \qquad \qquad \sin \lambda \mu^+(\pi) - \frac{\alpha - 1}{\alpha + 1}\sin \lambda \mu^-(\pi) = 0.$$

At last, from (2.2), it follows that

$$(2.3) \qquad \lambda_n^0 = \frac{1}{\mu^+(\pi)} \left(n\pi + \epsilon_n \right)$$

where

$$\epsilon_n = (-1)^n \frac{\alpha - 1}{\alpha + 1} sin \left(\frac{\mu^+(\pi)}{\mu^-(\pi)} n\pi \right) + O\left(\frac{1}{n} \right).$$

Lemma 2.1. *Roots of the function $\Delta_0(\lambda)$ are isolated, that is,*

$$\inf_{n \neq k} \left| \lambda_n^0 - \lambda_k^0 \right| = \beta > 0.$$

3. Asymptotic Formulas of Eigenvalues

Using representation for solution $e(x, \lambda)$ of equation (1.1) satisfying the initial conditions $e(0, \lambda) = 1$, $e'(0, \lambda) = i\lambda$ (see[8]), it is easy to obtain the following integral representation for the solution $\varphi(x, \lambda)$:

$$\varphi(x, \lambda) = \varphi_0(x, \lambda) + \int_0^{\mu^+(x)} A(x, t) \cos \lambda t dt$$

where $K(x, .) \in L_1(-\mu^+(x), \mu^+(x))$ and $A(x, t) = K(x, t) - K(x, -t)$. The kernel $A(x, t)$ processes the following properties:

i) $A(\pi, \mu^+(\pi)) = \frac{1}{4} \int_0^\pi \frac{1}{\sqrt{\rho(t)}} \left(1 + \frac{1}{\sqrt{\rho(t)}} \right) q(t) dt$,

ii) $A(\pi, \mu^-(\pi) + 0) - A(\pi, \mu^-(\pi) - 0) = \frac{1}{4} \int_0^\pi \frac{1}{\sqrt{\rho(t)}} \left(1 - \frac{1}{\sqrt{\rho(t)}} \right) q(t) dt$.

Theorem 3.1. *Boundary value problem (1.1), (1.2) has a countable set of simple eigenvalues $\{\lambda_n^2\}_{n \geq 1}$, where*

$$(3.1) \qquad \lambda_n = \lambda_n^0 + \frac{d_n}{\lambda_n^0} + \frac{k_n}{n}, \quad (\lambda_n > 0), \quad k_n \in l_2$$

where λ_n^0 are zeros of the function

$$(3.2) \qquad \Delta_0(\lambda) = -\frac{1}{2}\lambda(\alpha + 1) \sin \lambda\mu^+(\pi) + \frac{1}{2}(\alpha - 1)\lambda \sin \lambda\mu^-(\pi).$$

$\{\lambda_n^0\}^2$ are the eigenvalues of problem (1.1), (1.2), when $q(x) \equiv 0$, d_n is a bounded sequence

$$(3.3) \qquad d_n = \frac{h^+ \cos \lambda_n^0 \mu^+(\pi) + h^- \cos \lambda_n^0 \mu^-(\pi)}{\frac{1}{2}(\alpha + 1)\mu^+(\pi) \cos \lambda_n^0 \mu^+(\pi) - \frac{1}{2}(\alpha - 1)\mu^-(\pi) \cos \lambda_n^0 \mu^-(\pi)}.$$

Proof. Let $\varphi(x, \lambda)$ be the solution of equation (1.1) at initial conditions $\varphi(0, \lambda) = 1$, $\varphi'(0, \lambda) = 0$. Then the characteristic function $\Delta(\lambda) = \varphi'(\pi, \lambda)$ is entire with respect to λ and it has the most countable set of zeros λ_n and numbers λ_n^2 are eigenvalues of boundary value problem (1.1), (1.2). The standard method of variations of an arbitrary constants leads to the following integral equation for the solution $\varphi(x, \lambda)$

$$(3.4) \qquad \varphi(x, \lambda) = \varphi_0(x, \lambda) + \int_0^x g(x, t; \lambda)q(t)\varphi(t, \lambda)dt$$

where

$$g(x,t;\lambda) = \frac{1}{2}\left(\frac{1}{\sqrt{\rho(x)}} + \frac{1}{\sqrt{\rho(t)}}\right)\frac{sin\lambda\left(\mu^+(x) - \mu^+(t)\right)}{\lambda} +$$

(3.5)
$$+\frac{1}{2}\left(\frac{1}{\sqrt{\rho(x)}} - \frac{1}{\sqrt{\rho(t)}}\right)\frac{sin\lambda\left(\mu^-(x) - \mu^+(t)\right)}{\lambda}$$

and $\varphi_0(x,\lambda)$ is the solution of equation (1.1) at q(x) = 0, satisfying the conditions $\varphi(0,\lambda) = 1, \quad \varphi'(0,\lambda) = 0$. From (3.4), after differentiating, we find

(3.6)
$$\varphi'(x,\lambda) = \varphi'_0(x,\lambda) + \int_0^x g_x(x,t;\lambda)q(t)\varphi(t,\lambda)dt$$

where

$$g_x(x,t;\lambda) = \sqrt{\rho(x)}\frac{1}{2}\left(\frac{1}{\sqrt{\rho(x)}} + \frac{1}{\sqrt{\rho(t)}}\right)\frac{cos\lambda\left(\mu^+(x) - \mu^+(t)\right)}{\lambda} +$$

(3.7)
$$+\sqrt{\rho(x)}\frac{1}{2}\left(\frac{1}{\sqrt{\rho(x)}} - \frac{1}{\sqrt{\rho(t)}}\right)\frac{cos\lambda\left(\mu^-(x) - \mu^+(t)\right)}{\lambda}$$

Consequently, if we put here $x = \pi$, we have

(3.8)
$$\Delta(\lambda) = \Delta_0(\lambda) + \int_0^\pi g'_\pi(x,t;\lambda)q(t)\varphi(t,\lambda)dt.$$

Now, from

(3.9)
$$\varphi(x,\lambda) = \varphi_0(x,\lambda) + O\left(\frac{e^{|Im\lambda|\mu^+(x)}}{|\lambda|}\right), \quad |\lambda| \to +\infty$$

we obtain

(3.10)
$$\Delta(\lambda) = \Delta_0(\lambda) + h^+ \cos\lambda_n^0\mu^+(\pi) + h^- \cos\lambda_n^0\mu^-(\pi) + K_0(\lambda),$$

where

(3.11)
$$h^\pm = \frac{1}{4}(1 \pm \alpha)\int_0^a q(t)dt + \frac{1}{4}\left(1 \pm \frac{1}{\alpha}\right)\int_a^\pi q(t)dt,$$

and

$$K_0(\lambda) = \frac{1}{4}\int_0^a \left[(1 + \alpha)cos\lambda\left(2\mu^+(t) - \mu^+(\pi)\right) + (1 - \alpha)cos\lambda\left(2\mu^+(t) - \mu^-(\pi)\right)\right]q(t)dt$$

$$+\frac{1}{4}\int_a^\pi \left[\left(1 + \frac{1}{\alpha}\right)cos\lambda\left(2\mu^+(t) - \mu^+(\pi)\right)\right]q(t)dt$$

$$+\frac{1}{4}\int_a^\pi \left[\left(1 - \frac{1}{\alpha}\right)cos\lambda\left(\mu^+(\pi) + \mu^-(t) - \mu^+(t)\right)\right]q(t)dt$$

(3.12)
$$+0\left(\frac{e^{|Im\lambda|\mu^+(\pi)}}{|\lambda|}\right).$$

Let us denote $G_\delta = \{\lambda : \left|\lambda - \lambda_n^0\right| \geq \delta\}$, where δ is a sufficiently small positive number $\delta < \frac{\beta}{2}$ (see lemma 4). It is easy to show that (see [3])

(3.13)
$$|\Delta_0(\lambda)| \geq |\lambda|C_\delta e^{|Im\lambda|\mu^+(\pi)}, \quad \lambda \in G_\delta, \quad C_\delta > 0.$$

On the other hand, we obtain

$$(3.14) \qquad \Delta(\lambda) - \Delta_0(\lambda) \leq O\left(e^{|Im\lambda|\mu^+(\pi)}\right), \quad |\lambda| \to \infty.$$

Consider the contour $\Gamma_n = \{\lambda : |\lambda| = \left|\lambda_n^0\right| + \frac{\beta}{2}\}$, $(n = 1, 2, ...)$. We have from (3.10)

$$(3.15) \qquad |\Delta(\lambda) - \Delta_0(\lambda)| \leq \tilde{C}e^{|Im\lambda|\mu^+(\pi)}, \quad \lambda \in \Gamma_n,$$

for sufficiently large n, where $\tilde{C} > 0$. Applying now Rouche's theorem, we have that the number of zeros of $\Delta_0(\lambda)$ inside Γ_n coincides with the number of zeros of $\Delta(\lambda) = \{\Delta(\lambda) - \Delta_0(\lambda)\} + \Delta_0(\lambda)$. Further applying the Rouche's theorem to the circle $\gamma_n(\delta) = \{\lambda : \left|\lambda - \lambda_n^0\right| \leq \delta\}$, we conclude that for sufficiently large n, there exist only one zero λ_n of the function $\Delta(\lambda)$ in $\gamma_n(\delta)$. By virtue of the arbitrariness of $\delta > 0$ we have

$$(3.16) \qquad \lambda_n = \lambda_n^0 + \varepsilon_n, \quad \varepsilon_n = o(1), \quad n \to \infty.$$

Substituting (3.16) into (3.10), we obtain and taking into our account the relations

$$\Delta_0(\lambda_n^0) = -\frac{1}{2}\lambda_n^0(\alpha + 1)\sin\lambda_n^0\mu^+(\pi) + \frac{1}{2}\lambda_n^0(\alpha - 1)\sin\lambda_n^0\mu^-(\pi) = 0,$$

$$\sin\varepsilon_n\mu^+(\pi) \sim \varepsilon_n\mu^+(\pi), \quad \cos\varepsilon_n\mu^+(\pi) \sim 1, \quad n \to \infty$$

we get

$$(3.17) \qquad \varepsilon_n = \frac{d_n}{\lambda_n^0 + \varepsilon_n} + \frac{\varepsilon_n}{\lambda_n^0 + \varepsilon_n}\widetilde{d_n} + \frac{\widetilde{K_n}}{\lambda_n^0 + \varepsilon_n}$$

where

$$d_n = \frac{h^+\cos\lambda_n^0\mu^+(\pi) + h^-\cos\lambda_n^0\mu^-(\pi)}{\frac{1}{2}(\alpha + 1)\mu^+(\pi)\cos\lambda_n^0\mu^+(\pi) - \frac{1}{2}(\alpha - 1)\mu^-(\pi)\cos\lambda_n^0\mu^-(\pi)},$$

$\widetilde{K_n} = K_0(\lambda_n^0 + \varepsilon_n)$ and

$$\widetilde{d_n} = \frac{h^+\mu^+(\pi)\sin\lambda_n^0\mu^+(\pi) + h^-\mu^-(\pi)\sin\lambda_n^0\mu^-(\pi)}{\frac{1}{2}(\alpha + 1)\mu^+(\pi)\cos\lambda_n^0\mu^+(\pi) - \frac{1}{2}(\alpha - 1)\mu^-(\pi)\cos\lambda_n^0\mu^-(\pi)}.$$

Since $\frac{1}{\lambda_n^0 + \varepsilon_n} = O(\frac{1}{n})$, $\frac{\varepsilon_n}{\lambda_n^0 + \varepsilon_n} = o(\frac{1}{n})$, $n \to \infty$ we have that d_n, $\widetilde{d_n}$ are bounded and (3.17) implies

$$\varepsilon_n = O(\frac{1}{n}), \quad n \to \infty.$$

Using (3.17) once more, we can obtain more precisely as $n \to \infty$

$$(3.18) \qquad \varepsilon_n = \frac{d_n}{\lambda_n^0} + \frac{k_n}{n}, \quad k_n \in l_2$$

where $k_n = \frac{\mu^+(\pi)}{\pi}\widetilde{K_n} + O(\frac{1}{n})$, $n \to \infty$ .The theorem is proved. $\qquad\square$

4. Spectral Expansion Formula

Theorem 4.1. *1) The system of eigenfunctions $\{\varphi(x, \lambda_n)\}_{n \geq 1}$ of boundary value problem (1.1), (1.2) is complete in $L_2(0, \pi; \rho)$;*

2) If $f(x)$ is an absolutely continuous function on the segment $[0, \pi]$, and $f'(0) = f'(\pi) = 0$, then

$$(4.1) \qquad f(x) = \sum_{n=1}^{\infty} a_n \varphi(x, \lambda_n),$$

where

$$(4.2) \qquad a_n = \frac{1}{\alpha_n} \int_0^{\pi} f(t) \varphi(t, \lambda_n) \rho(t) dt,$$

and the series (4.1) converges uniformly on $[0, \pi]$;

3) For $f(x) \in L_2(0, \pi; \rho)$ the series (4.1) converges in $L_2(0, \pi; \rho)$, moreover the Parseval equality

$$(4.3) \qquad \int_0^{\pi} |f(x)|^2 \rho(x) dx = \sum_{n=1}^{\infty} \alpha_n |a_n|^2$$

holds.

Proof. Let $\psi(x, \lambda)$ be a solution of equation (1.1) under the initial conditions $\psi(\pi, \lambda) = 1$, $\psi'(\pi, \lambda) = 0$. Denote

$$(4.4) \qquad G(x, t; \lambda) = -\frac{1}{\Delta(\lambda)} \begin{cases} \psi(x, \lambda) \varphi(t, \lambda), & x \geq t \\ \varphi(x, \lambda) \psi(t, \lambda), & t \geq x \end{cases}$$

and let us consider the function

$$(4.5) \qquad Y(x, \lambda) = \int_0^{\pi} \rho(t) f(t) G(x, t; \lambda) dt$$

which is a solution of the boundary value problem

$$(4.6) \qquad -Y''(x, \lambda) + q(x) Y(x, \lambda) = \lambda^2 \rho(x) Y(x, \lambda) - f(x) \rho(x),$$

$$Y'(0, \lambda) = 0, Y'(\pi, \lambda) = 0.$$

Using (1.9), we obtain

$$(4.7) \qquad \operatorname*{Res}_{\lambda = \lambda_n} Y(x, \lambda) = \frac{1}{2\alpha_n \lambda_n} \varphi(x, \lambda_n) \int_0^{\pi} \rho(t) f(t) \varphi(t, \lambda_n) dt.$$

Let $f(x) \in L_2(0, \pi; \rho)$ be such that

$$\int_0^{\pi} \rho(t) f(t) \varphi(t, \lambda_n) dt = 0 \quad n = 1, 2, 3, \dots.$$

Then, from (4.7), we have $\operatorname*{Res}_{\lambda = \lambda_n} Y(x, \lambda) = 0$; consequently, for fixed $x \in [0, \pi]$, the function $Y(x, \lambda)$ is entire with respect to λ. On the other hand, since

$$(4.8) \qquad \Delta(\lambda) \geq |\lambda| \tilde{C}_\delta e^{|Im\lambda| \mu^+(\pi)}, \quad \lambda \in G_\delta, \quad \tilde{C}_\delta > 0.$$

(4.9)
$$\varphi(x, \lambda) = O\left(e^{|Im\lambda| \mu^+(x)}\right), \quad \psi(x, \lambda) = O\left(e^{|Im\lambda|(\mu^+(\pi) - \mu^+(x))}\right), \quad |\lambda| \to \infty,$$

from (4.5), it follows that for fixed $\delta > 0$ and sufficiently large $\lambda^* > 0$:

$$|Y(x,\lambda)| \le \frac{C_\delta}{|\lambda|}, \quad \lambda \in G_\delta, \quad |\lambda| \ge \lambda^*.$$

Using the maximum principle for module of analytic functions and Liouville theorem, we conclude that $Y(x,\lambda) \equiv 0$. This fact and (4.6) imply that $f(x) = 0$ a.e. on $[0, \pi]$. Thus, statement (1) of theorem is proved.

Let $f(x) \in AC[0, \pi]$ be an arbitrary absolutely continuous function. Let us transform the function $Y(x, \lambda)$ to the form

$$Y(x, \lambda) = -\frac{1}{\lambda^2 \Delta(\lambda)} \left\{ \psi(x, \lambda) \int_0^x \left(-\varphi''(t, \lambda) + q(t)\varphi(t, \lambda)\right) f(t) dt - \right.$$
$$\left. -\varphi(x, \lambda) \int_x^\pi \left(-\psi''(t, \lambda) + q(t)\psi(t, \lambda)\right) f(t) dt \right\}.$$

Integrating by parts the addends with the second-order derivatives and taking into account conditions $f'(0) = 0$, $f'(\pi) = 0$, we have

(4.10) $$Y(x, \lambda) = \frac{f(x)}{\lambda^2} - \frac{1}{\lambda^2} \left(Z_1(x, \lambda) + Z_2(x, \lambda)\right),$$

where

$$Z_1(x, \lambda) = \frac{1}{\Delta(\lambda)} \left[\psi(x, \lambda) \int_0^x g(t)\varphi'(t, \lambda) dt + \varphi(x, \lambda) \int_x^\pi g(t)\psi'(t, \lambda) dt \right],$$

$$Z_2(x, \lambda) = \frac{1}{\Delta(\lambda)} \left[\psi(x, \lambda) \int_0^x \varphi(t, \lambda)) f(t)\rho(t) dt + \right.$$
$$\left. + \varphi(x, \lambda) \int_x^\pi \psi(t, \lambda)) f(t)\rho(t) dt \right].$$

Here $g(t) = f'(t)$. Now consider the contour integral

(4.11) $$I_N(x) = 2 \sum_{n=1}^N \operatorname*{Res}_{\lambda=\lambda_n} Y(x, \lambda) = \sum_{n=1}^N a_n \varphi(x, \lambda_n)$$

where

$$a_n = \frac{1}{\alpha_n} \int_0^\pi \rho(t) f(t) \varphi(t, \lambda_n) dt.$$

On the other hand taking into account (4.10), we have

(4.12) $$I_N(x) = f(x) - \frac{1}{2\pi i} \int_{\Gamma_n} \frac{1}{\lambda} \left(Z_1(x, \lambda) + Z_2(x, \lambda)\right) d\lambda.$$

Comparing (4.11) and (4.12), we obtain

$$f(x) = \sum_{n=1}^\infty a_n \varphi(x, \lambda_n) + \xi_N(x),$$

where

$$\xi_N(x) = \frac{1}{2\pi i} \int_{\Gamma_N} \frac{1}{\lambda} \left(Z_1(x, \lambda) + Z_2(x, \lambda)\right) d\lambda.$$

Therefore, in order to prove the item (2) of the theorem, it suffices to show that

(4.13) $$\lim_{N \to \infty} \max_{0 \le x \le \pi} |\xi_N(x)| = 0.$$

From the estimates of solution $\varphi(x, \lambda)$, $\quad \psi(x, \lambda)$ and the function $\Delta(\lambda)$, it follows that for fixed $\delta > 0$ and sufficiently large $\lambda^* > 0$,

$$(4.14) \qquad \max_{0 \leq x \leq \pi} |Z_2(x, \lambda)| \leq \frac{C_2}{|\lambda|}, \quad \lambda \in G_\delta, \quad |\lambda| \geq \lambda^*, C_2 > 0.$$

Let us show that

$$(4.15) \qquad \lim_{\substack{|\lambda| \to \infty \\ \lambda \in G_\delta}} \max_{0 \leq x \leq \pi} |Z_1(x, \lambda)| = 0.$$

At first, it was supposed that $g(t)$ is absolutely continuous on $[0, \pi]$. In this case, integration by parts gives

$$Z_1(x, \lambda) = -\frac{1}{\Delta(\lambda)} \left\{ \psi(x, \lambda) \int_0^x \varphi(t, \lambda) g'(t) dt + \varphi(x, \lambda) \int_x^\pi \psi(t, \lambda) g'(t) dt \right\},$$

therefore, similarly to $Z_2(x, \lambda)$, we have

$$\max_{0 \leq x \leq \pi} |Z_1(x, \lambda)| \leq \frac{C_1}{|\lambda|}, \quad \lambda \in G_\delta, \quad |\lambda| \geq \lambda^*, C_1 > 0.$$

In the general case, we fix $\varepsilon > 0$ and choose absolutely continuous function $g_\varepsilon(t)$ such that

$$\int_0^\pi |g_\varepsilon(t) - g(t)| \, dt < \varepsilon.$$

Then, using the estimates $\varphi(x, \lambda)$, $\quad \psi(x, \lambda) \quad \Delta(\lambda)$, one can find $\lambda^{**} > 0$ such that when $\lambda \in G_\delta, |\lambda| \geq \lambda^{**}$, from the relation

$$Z_1(x, \lambda) = \frac{1}{\Delta(\lambda)} \left[\psi(x, \lambda) \int_0^x \varphi'(t, \lambda)(g_\varepsilon(t) - g(t)) dt + \right.$$

$$\left. + \varphi(x, \lambda) \int_x^\pi \psi'(t, \lambda)(g_\varepsilon(t) - g(t)) dt \right] +$$

$$+ \frac{1}{\Delta(\lambda)} \left[\psi(x, \lambda) \int_0^x \varphi(t, \lambda) g'_\varepsilon(t) dt - \varphi(x, \lambda) \int_x^\pi \psi(t, \lambda) g'_\varepsilon(t) dt \right],$$

we have

$$\max_{0 \leq x \leq \pi} |Z_1(x, \lambda)| \leq C \int_0^\pi |g_\varepsilon(t) - g(t)| \, dt + \frac{\widetilde{C}(\epsilon)}{|\lambda|} < C_\epsilon + \frac{\widetilde{C}(\epsilon)}{|\lambda|}, \quad \lambda \in G_\delta, \quad |\lambda| \geq \lambda^{**}.$$

Consequently,

$$\overline{\lim_{\substack{|\lambda| \to \infty \\ \lambda \in G_\delta}}} \max_{0 \leq x \leq \pi} |Z_1(x, \lambda)| \leq C_\epsilon.$$

Since ε is an arbitrary positive number, we obtain the validity of equality (4.15). Relations (4.14), (4.15) immediately imply (4.13), thus, statement (2) of theorem is proved.

System of eigenfunction $\{\varphi(x, \lambda_n)\}_{n \geq 1}$ is complete and orthogonal in $L_2(0, \pi; \rho)$. Therefore, it forms the orthogonal basis in $L_{2,\rho}(0, \pi)$ and Parseval equality from theorem is valid. □

5. WEYL SOLUTION, WEYL FUNCTION

Let $\Phi(x, \lambda)$ be the solution of equation (1.1) that satisfied the conditions $\Phi'(0, \lambda) = 1$, $\Phi'(\pi, \lambda) = 0$. Denote by $C(x, \lambda)$ the solution of equation (1.1), which satisfied the initial conditions $C(0, \lambda) = 0$, $C'(0, \lambda) = 1$. Then, the solution $\psi(x, \lambda)$ can be represented as follows

$$(5.1) \qquad \psi(x, \lambda) = \psi(0, \lambda)\varphi(x, \lambda) - \Delta(\lambda)C(x, \lambda)$$

or

$$(5.2) \qquad -\frac{\psi(x, \lambda)}{\Delta(\lambda)} = C(x, \lambda) - \frac{\psi(0, \lambda)}{\Delta(\lambda)}\varphi(x, \lambda).$$

Denote

$$(5.3) \qquad M(\lambda) := -\frac{\psi(0, \lambda)}{\Delta(\lambda)}.$$

It is clear that

$$(5.4) \qquad \Phi(x, \lambda) = C(x, \lambda) + M(\lambda)\varphi(x, \lambda).$$

The function $\Phi(x, \lambda)$ and $M(\lambda) = \Phi(0, \lambda)$ are respectively called the Weyl solution and the Weyl function of the boundary value problem (1.1), (1.2). The Weyl function is a meromorphic function having simple poles at points λ_n eigenvalues of boundary value problem (1.1), (1.2). Relations (5.2), (5.4) yield

$$(5.5) \qquad \Phi(x, \lambda) = -\frac{\psi(x, \lambda)}{\Delta(\lambda)}.$$

It can be shown that

$$(5.6) \qquad < \varphi(x, \lambda), \Phi(x, \lambda) >= 1.$$

Theorem 5.1. *If $M(\lambda) = \widetilde{M}(\lambda)$, then $L = \widetilde{L}$; that is, the boundary value problem (1.1), (1.2), is unique by the Weyl function.*

Proof. We describe the matrix $P(x, \lambda) = [P_{ij}(x, \lambda)]_{i,j=1,2}$ with the formula

$$(5.7) \qquad P(x, \lambda)\begin{pmatrix} \widetilde{\varphi}(x, \lambda) & \widetilde{\Phi}(x, \lambda) \\ \widetilde{\varphi}'(x, \lambda) & \widetilde{\Phi}'(x, \lambda) \end{pmatrix} = \begin{pmatrix} \varphi(x, \lambda) & \Phi(x, \lambda) \\ \varphi'(x, \lambda) & \Phi'(x, \lambda) \end{pmatrix}.$$

From (5.7), we have

$$(5.8) \qquad \begin{aligned} \varphi(x, \lambda) &= P_{11}(x, \lambda)\widetilde{\varphi}(x, \lambda) + P_{12}(x, \lambda)\widetilde{\varphi}'(x, \lambda), \\ \Phi(x, \lambda) &= P_{11}(x, \lambda)\widetilde{\Phi}(x, \lambda) + P_{12}(x, \lambda)\widetilde{\Phi}'(x, \lambda), \end{aligned}$$

or

$$(5.9) \qquad \begin{aligned} P_{11}(x, \lambda) &= \varphi(x, \lambda)\widetilde{\Phi}'(x, \lambda) - \Phi(x, \lambda)\widetilde{\varphi}'(x, \lambda), \\ P_{12}(x, \lambda) &= -\varphi(x, \lambda)\widetilde{\Phi}(x, \lambda) + \Phi(x, \lambda)\widetilde{\varphi}(x, \lambda). \end{aligned}$$

Taking equation (5.5) into consideration in (5.9), we get (5.4) into (5.9), then we get

(5.10)

$$P_{11}(x, \lambda) = 1 + \tfrac{1}{\Delta(\lambda)}\left[\varphi(x, \lambda)(\widetilde{\psi}'(x, \lambda) - \psi'(x, \lambda)) - \psi(x, \lambda)(\widetilde{\varphi}'(x, \lambda) - \varphi'(x, \lambda))\right]$$

$$P_{12}(x, \lambda) = \tfrac{1}{\Delta(\lambda)}\left[\psi(x, \lambda)\widetilde{\varphi}(x, \lambda) - \varphi(x, \lambda)\widetilde{\psi}(x, \lambda)\right].$$

Now, from (4.8)and (4.9), we have from equation (5.10)

$$(5.11) \qquad \lim_{|\lambda|\to\infty} \max_{0\leq x\leq\pi} |P_{11}(x,\lambda) - 1| = \lim_{|\lambda|\to\infty} \max_{0\leq x\leq\pi} |P_{12}(x,\lambda)| = 0.$$

Now, if we take into consideration equation (5.4) into (5.9), we get

$$P_{11}(x,\lambda) = \varphi(x,\lambda)\widetilde{C}'(x,\lambda) - C(x,\lambda)\widetilde{\varphi}'(x,\lambda) + (\widetilde{M}(\lambda) - M(\lambda))\varphi(x,\lambda)\widetilde{\varphi}'(x,\lambda)$$
$$P_{12}(x,\lambda) = C(x,\lambda)\widetilde{\varphi}(x,\lambda) - \widetilde{C}(x,\lambda)\varphi(x,\lambda) - (\widetilde{M}(\lambda) - M(\lambda))\varphi(x,\lambda)\widetilde{\varphi}(x,\lambda).$$

Therefore if $M(\lambda) = \widetilde{M}(\lambda)$, then $P_{11}(x,\lambda)$ and $P_{12}(x,\lambda)$ are entire functions for every fixed x. It can be easily seen from (5.11) that $P_{11}(x,\lambda) = 1$ and $P_{12}(x,\lambda) = 0$. Substituting into (5.8), we get $\varphi(x,\lambda) \equiv \widetilde{\varphi}(x,\lambda)$ and $\Phi(x,\lambda) \equiv \widetilde{\Phi}(x,\lambda)$ for every x and λ. Hence, we arrive at $q(x) \equiv \widetilde{q}(x)$. $\qquad\square$

Theorem 5.2. *The expression*

$$(5.12) \qquad M(\lambda) = -\sum_{n=0}^{\infty} \frac{1}{2\alpha_n\lambda_n(\lambda - \lambda_n)}$$

holds.

Proof. Using (5.3), we get for sufficiently large $\lambda^* > 0$,

$$(5.13) \qquad M(\lambda) \leq \frac{C_\delta}{|\lambda|}, \quad \lambda \in G_\delta, \quad |\lambda| > \lambda^*.$$

Further using (1.9) and (5.4) we calculate:

$$(5.14) \qquad \operatorname*{Res}_{\lambda=\lambda_n} M(\lambda) = -\frac{\psi(0,\lambda_n)}{\dot{\Delta}(\lambda_n)} = -\frac{\beta_n}{\dot{\Delta}(\lambda_n)} = -\frac{1}{2\lambda_n\alpha_n}.$$

Now, let's consider the contour integral

$$J_N(\lambda) = \frac{1}{2\pi i} \int_{\Gamma_N} \frac{M(\mu)}{\lambda - \mu} d\mu, \quad \lambda \in Int\Gamma_N,$$

where $\Gamma_N = \left\{ \mu : |\mu| = \left|\lambda_N^0\right| + \frac{\gamma}{2} \right\}$ is a contour of counter-clockwise by pass.

By virtue of (5.13) we have $\lim_{N\to\infty} J_N(\lambda) = 0$. On the other hand, by residue theorem and (5.14) yield

$$J_N(\lambda) = -M(\lambda) - \sum_{n=-N}^{N} \frac{1}{2\lambda_n\alpha_n(\lambda - \lambda_n)}$$

and when $N \to \infty$ we arrive at (5.12). $\qquad\square$

Theorem 5.3. *If $\lambda_n = \widetilde{\lambda}_n, \alpha_n = \widetilde{\alpha}_n$ for all $n \in Z$ then $L = \widetilde{L}$. That is, the problem (1.1), (1.2) is uniquely determined by spectral date.*

Proof. Since $\lambda_n = \widetilde{\lambda}_n, \alpha_n = \widetilde{\alpha}_n$ for all $n \in Z$ and considering the formula (5.12), we have $M(\lambda) = \widetilde{M(\lambda)}$. Using Theorem 4, $L = \widetilde{L}$ is obtained. $\qquad\square$

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ABOUT CERTAIN HOMOLOGICAL PROPERTIES OF SYMMETRIC DERIVATIONS OF KÄHLER MODULES

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ABSTRACT. In this study, we express more informations concerning with homological properties of symmetric derivations of Kahler modules acquainted by H. Osborn in [1].

1. INTRODUCTION

The definition of n-th order symmetric derivations of Kähler modules were given by H.Osborn at 1965 in [1]. J.Johnson made known the structures of differential module on certain modules of Kähler differentials in [3]. Then, advanced principal theories about the calculus of high order derivations and a few functorial features of high order differential modules were presented by Y. Nakai in [4]. Higher derivations and universal differential operators of Kähler modules were studied by R. Hart in [2]. Olgun defined generalized symmetric derivations on high order Kähler modules in [6].Komatsu presented right differential operators on a noncommutative ring extension in [10]. The more informations about these subjects were found in [5,7,8,9,11].The aim of this study is to investigate these homological structures and is to give more knowldege about them.

2. PRELIMINARY

Throughout this paper we assume R be a commutative algebra over an algebraically closed field k with characteristic zero. When R is a k-algebra , $J_n(R)$ denotes the universal module of n-th order differentials of R over k and $\Omega_n(R)$ be the module of n-th order Kähler differentials of R over k and d_n be the canonical $n-th$ order k-derivation $R \longrightarrow \Omega_n(R)$ of R.The pair $\{\Omega_n(R), d_n\}$ has the universal mapping property with regard to the $n-th$ order k-derivations of R. $\Omega_n(R)$ is generated by the set $\{d_n(r) : r \in R\}$.

Definition 2.1. Let R be a commutative algebra over a field k of characteristic zero,A be an R-module, $A \otimes_R A$ be the tensor product of A with itself and let K be the submodule of $A \otimes_R A$ generated by the elements of the form $a \otimes b - b \otimes a$ where $a, b \in A$. Consider the factor module $\vee^2 A = A \otimes A/K$. The module $\vee^2 A$ is said to be the second symmetric power of A. The canonic balanced map is defined such that

$$\otimes : A \times A \longrightarrow A \otimes A$$
$$\otimes(a,b) = a \otimes b$$

and a natural surjective map defined such that $\gamma : A \otimes A \longrightarrow \vee^2 A$. Then the composite map is bilinear and called $\gamma\otimes = \vee$.

Lemma 2.1. *Let A and B be R-modules and let $\zeta : A \times A \longrightarrow B$ be a bilinear alternating map. Then there exists an R-module homomorphism $f : \vee^2 A \longrightarrow B$ such that the diagram*

$$\begin{array}{ccc} A \times A & \xrightarrow{\zeta} & B \\ \vee \searrow & & \nearrow f \\ & \vee^2 A & \end{array}$$

commutes.

Definition 2.2. Let R be any k-algebra (commutative with unit), $R \to \Omega_1(R)$ be first order Kähler derivation of R and let $\vee(\Omega_1(R))$ be the symmetric algebra $\bigoplus_{p \geq 0} \vee^p(\Omega_1(R))$ generated over R by $\Omega_1(R)$.

A symmetric derivation is any linear map D of $\vee(\Omega_1(R))$ into itself such that
i) $D(\vee^p(\Omega_1(R))) \subset \vee^{p+1}(\Omega_1(R))$
ii) D is a first order derivation over k and
iii) the restriction of D to R $(R \simeq \vee^0(\Omega_1(R)))$ is the Kähler derivation
$d_1 : R \to \Omega_1(R)$.

Definition 2.3. Let R be any k-algebra (commutative with unit), $R \to \Omega_n(R)$ be n-th order Kähler derivation of R and let $\vee(\Omega_n(R))$ be the symmetric algebra $\bigoplus_{p \geq 0} \vee^p(\Omega_n(R))$ generated over R by $\Omega_n(R)$.

A generalized symmetric derivation is any k-linear map D of $\vee(\Omega_n(R))$ into itself such that
i) $D(\vee^p(\Omega_n(R))) \subset \vee^{p+1}(\Omega_n(R))$
ii) D is a n-th order derivation over k and
iii) the restriction of D to R $(R \simeq \vee^0(\Omega_n(R)))$ is the Kähler derivation
$d_n : R \to \Omega_n(R)$.

Proposition 2.1. *Let $R = k[x_1, ..., x_s]$ be a polynomial algebra of dimension s. Then $\Omega_n(R)$ is a free R-module of rank $\binom{n+s}{s} - 1$ with basis*

$\{ d_n(x_1^{i_1}.....x_s^{i_s}) : i_1 + ... + i_s \leq n\}$

$\vee^2(\Omega_n(R))$ *is a free R-module of rank $\binom{t+1}{t-1}$*

where $t = \binom{n+s}{s} - 1$ with basis $\{ d_n(x_1^{i_1}.....x_s^{i_s})\otimes d_n(x_1^{i_1}.....x_s^{i_s}) : i_1 + ... + i_s \leq n\}$

3. Symmetric Powers of Kähler Modules

In this section, we consider the tensor, exterior and symmetric algebras of Kähler modules and define the symmetric powers of a given module A over a k-algebra and a few elementary properties.

Definition 3.1. Let A be a R-module.

i) By $\otimes^n A$ we shall denote the R-module with a universal R-bilinear map of $A^n \longrightarrow \otimes^n A$ written $(x_1, ..., x_n) \longrightarrow x_1 \otimes ... \otimes x_n$. This module is called the n-fold tensor power of A.

ii) By $\Lambda^n A$ we shall denote the R-module with a universal alternating R-bilinear map of $A^n \longrightarrow \Lambda^n$ written $(x_1, ...x_n) \longrightarrow x_1 \wedge ... \wedge x_n$. This module is called the n-fold exterior power of A.

iii) By $\vee^n A$ we shall denote the R-module with a universal symmetric R-bilinear map of $A^n \longrightarrow \vee^n A$ written $(x_1, ..., x_n) \longrightarrow x_1...x_n$. This module is called the n-fold symmetric power of A.

Let us the conventin that $\otimes^1 A, \Lambda^1 A$ and $\vee^1 A$ are all identified with A, while $\otimes^0 A$, $\Lambda^0 A$ and $\vee^0 A$ are all identified with R.

Theorem 3.1. *Let A be a free R-module on a basis $X = \{x_1, ..., x_d\}$. If A is generated by $x_1, ..., x_d$ then $A^{\otimes k}$ is generated by $x_{i_1} \otimes ... \otimes x_{i_k}$ as an R-module. Where $1 \leq i_1, ..., i_k \leq d$ and $dim_R(\otimes^k A) = d^k$.*

Since $\Lambda^k(A)$ is factor module of $A^{\otimes k}$, so $\Lambda^k(A)$ is generated by $x_{i_1} \wedge ... \wedge x_{i_k}$ as an R-module where $1 \leq i_1, ..., i_k \leq d$. For any $x_1, x_2 \in X$, it satisfied $x_1 \wedge x_1 = 0$ and $x_1 \wedge x_2 + x_2 \wedge x_1 = 0$. If $0 \neq A$ is affine free with $x_1, ..., x_d$ then $x_{i_1} \wedge ... \wedge x_{i_k}$ is basis for $\Lambda^k A$ and $dim_R(\Lambda^k A) = \binom{d}{k}$.

$\vee A$ may be presented by the generating set X, and relation $xy = yx$ $(x, y \in X)$ and is the (commutative) polynomial algebra $R[X]$. Then an R-module basis for $\vee A$ is given by those products $x_1...x_n$ with $x_1 \leq ... \leq x_n \in X$. If X is a finite set $\{x_1, ..., x_r\}$, then the elements of this basis can be written $x_1^{i_1}...x_r^{i_r}$ with $i_1, ..., i_r \geq 0$, and for each n, $dim(\vee^n A) = \binom{r+n-1}{r-1}$.

4. Homological Properties of Symmetric Derivations

Theorem 4.1. *Let R be an affine k-algebra. Then we have a long exact sequence of R-modules*

$$0 \longrightarrow ker\eta \longrightarrow \Omega_{(2n)}(R) \xrightarrow{\eta} J_n(\Omega_n(R)) \longrightarrow coker\eta \longrightarrow 0.$$

for all $n \geq 0$.

Example 4.1. $R = k[a, b]$ be a polynomial algebra of dimension 2.Then $\Omega_1(R)$ is a free R-module of rank 2 with basis $\{d_1(a), d_1(b)\}$ and $\Omega_2(R))$ is a free R-module of rank 5 with basis $\{d_2(a), d_2(b)), d_2(a^2), d_2(ab), d_2(b^2)\}$.

$J_1(\Omega_1(R))$ is a free R-module generated by $\{\Delta_1(d_1(a)), \Delta_1(d_1(b)), \Delta_1(ad_1(a)), \Delta_1(ad_1(b)), \Delta_1(bd_1(a)), \Delta_1(bd_1(b))\}$

Theorem 4.2. *Let R be an affine k-algebra. Then we have a long exact sequence of R-modules*

$$0 \longrightarrow ker\gamma \longrightarrow J_n(\Omega_n(R)) \xrightarrow{\gamma} \vee^2(\Omega_n(R)) \longrightarrow coker\gamma \longrightarrow 0.$$

for all $n \geq 0$.

Lemma 4.1. *Let R be an affine domain.Then $\Omega_n(R)$ is a free R-module if and only if $\vee^2(\Omega_n(R))$ is a free R-module.*

Theorem 4.3. *Let R be an affine k-algebra and $\vee(\Omega_1(R))$ has at least one symmetric derivations. $\Omega_1(R)$ is a projective R-module if and only if $\Omega_2(R)$ is a projective R-module.*

Corollary 4.1. *Let R be an affine local k-algebra and $\vee(\Omega_1(R))$ has at least one symmetric derivation. $\Omega_1(R)$ is a free R-module if and only if $\Omega_2(R)$ is a free R-module.*

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ON A NONLOCAL BOUNDARY VALUE PROBLEM

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Abstract. In this study, parabolic partial differential equation with two integral boundary conditions are considered for distribution of family savings for a family set. By seperation of variables method, eigenvalues and eigenfunctions of the problem are obtained and solution is written. Moreover, Method of limes method and Crank Nicolson method are applied to the problem and errors of numerical methods are presented.

1. Introduction

Integral boundary conditions for parabolic equations are well known problem in applications (see, for example, Cannon[1], Ionkin[7], Kamynin[8], Day[3], Erofeeko and Kozlovski[6]). Such a boundary condition are called nonlocal boundary condition or nonclassical boundary condition. Similar problems are also used for hyperbolic equations.

In this study, we deal with a family saving model which can be represented by Kolmogorov equation with two integral boundary conditions.

Suppose that $x(t)$ denotes the saving of a family at time t and satisfy the differential equation

$$(1.1) \qquad dx = F\left(x, t\right) dt + G\left(x, t\right) dX, \quad G \geq 0$$

where X is the Markov process, $F(x, t)$ is the rate of the change for the family saving and $G\left(x, t\right) dX$ is the random change of the family income.

For a family set let us assume that equation (1.1) describes the saving of all families by ignoring the dynamic of individual family saving. The density distribution of the saving of families $u(x, t)$ satisfies

$$(1.2) \qquad \frac{\partial u}{\partial t} = -\frac{\partial}{\partial x}\left((c(x, t) + F(x, t))\ u\right) + \frac{1}{2}\frac{\partial^2}{\partial x^2}\left(b(x, t)u\right) + f(x, t)$$

with initial condition

$$(1.3) \qquad u(x, 0) = \varphi(x),\ 0 \leq x \leq l$$

and boundary conditions

$$(1.4) \qquad \int_0^l u\left(x, t\right) dx = N\left(t\right), t \geq 0$$

$$(1.5) \qquad \int_0^l x\, u\,(x,t)\, dx = K\,(t)\,, t \geq 0$$

where $c(x,t)$, $b(x,t)$, $K(t)$, $N(t)$, $\varphi(x)$ anf $f(x,t)$ are continuously differentiable functions. $N(t)$, $K(t)$ denote total number of families and total amount of family saving in $[0,l]$ respectively [6].

2. Special Case of The Model

We will consider special case of problem (1.2)-(1.5) on region $D = (0 < t < \infty) \times (0 < x < l)$

$$(2.1) \qquad \frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2} + f(x,t),$$

$$(2.2) \qquad u(x,0) = \varphi(x), \ 0 \leq t \leq T,$$

$$(2.3) \qquad \int_0^1 u\,(x,t)\, dx = N\,(t)\,, t \geq 0,$$

$$(2.4) \qquad \int_0^1 x\, u\,(x,t)\, dx = K\,(t)\,, t \geq 0,$$

where $f(x,t)$, $K(t)$, $N(t)$, $\varphi(x)$ are continuously differentiable functions on region D. Compatibility conditions of this problem are
$\int_0^1 x\varphi(x)dx = N(0)$ and $\int_0^1 \varphi(x)dx = K(0)$.

Using the transform

$$u(x,t) = v(x,t) + (12K(t) - 6N(t))x + 4N(t) - 6K(t)$$

boundary conditions of equation (2.1)-(2.4) become homogenous:

$$(2.5) \qquad \frac{\partial v}{\partial t} = a^2 \frac{\partial^2 v}{\partial x^2} + F(x,t),$$

$$(2.6) \qquad v(x,0) = \psi(x),$$

$$(2.7) \qquad \int_0^1 v(x,t)dx = 0,$$

$$(2.8) \qquad \int_0^1 xv(x,t)dx = 0,$$

where

$$F(x,t) = f(x,t) - (12K'(t) - 6N'(t))x + 4N'(t) - 6K'(t)$$

and

$$\psi(x) = \varphi(x) - (12K(0) - 6N(0))x + 4N(0) - 6K(0).$$

Equations (2.5)-(2.8) are linear with respect to v(x,t), then this problem can split into two auxiliary problems:

i)

$$(2.9) \qquad \frac{\partial v}{\partial t} = a^2 \frac{\partial^2 v}{\partial x^2},$$

$$(2.10) \qquad v(x,0) = \psi(x),$$

$$(2.11) \qquad \int_0^1 v(x,t)dx = 0$$

$$(2.12) \qquad \int_0^1 xv(x,t)dx = 0$$

ii)

$$(2.13) \qquad \frac{\partial v}{\partial t} = a^2 \frac{\partial^2 v}{\partial x^2} + F(x,t),$$

$$(2.14) \qquad v(x,0) = 0,$$

$$(2.15) \qquad \int_0^1 v(x,t)dx = 0$$

$$(2.16) \qquad \int_0^1 xv(x,t)dx = 0$$

If solution of the problem (i) is $v_1(x,t)$ and solution of the problem (ii) is $v_2(x,t)$ then solution of the problem (2.5)-(2.8) is $v(x,t) = v_1(x,t) + v_2(x,t)$.

Integrating both sides of (2.9) with respect to x from 0 to 1 and using integration by parts, integral boundary conditions in (2.11) and (2.12) become, respectively,

$$v_x(1,t) - v_x(0,t) = 0,$$

$$v_x(1,t) - v(1,t) + v(0,t) = 0.$$

Substituting these equations in (2.9)-(2.12), we have

$$(2.17) \qquad \frac{\partial v}{\partial t} = a^2 \frac{\partial^2 v}{\partial x^2},$$

$$(2.18) \qquad v(x,0) = \psi(x),$$

$$(2.19) \qquad v_x(1,t) - v_x(0,t) = 0,$$

$$(2.20) \qquad v_x(1,t) - v(1,t) + v(0,t) = 0.$$

By the separation of variables, a Sturm-Liouville problem and an ODE are, respectively, obtained as

$$(2.21) \qquad X''(x) + \lambda X(x) = 0,$$

$$(2.22) \qquad X'(1) - X'(0) = 0,$$

$$(2.23) \qquad X'(1) - X(1) + X(0) = 0,$$

and

$$(2.24) \qquad T'(t) + \lambda a^2 T(t) = 0.$$

Sturm-Liouville problem (2.21)-(2.23) is self adjoint and boundary conditions are regular, and also strongly regular. Therefore, the eigenfunctions of the Sturm-Liouville problem are the Riesz basis on $L^2[0,1]$ (Naimark[11], Kesselman[9], Mikhailov [10]).

Characteristic equation of the Sturm-Liouville problem is

$$(2.25) \qquad 2 - 2\cos k - k \sin k = 0,$$

where $\sqrt{\lambda} = k$.

It is easily seen that $k_0 = 0$ and $k_{2n} = 2n\pi, (n = 1, 2, \cdots)$ are roots of the equation (2.25). There is also another root of equation (2.25) in $\left[n\pi, \frac{(2n+1)}{2}\pi\right]$. By using Langrange-Burmann formula root is calculated asymptotically as

$$
\begin{aligned}
k_{2n+1} &= (2n+1)\pi - 4((2n+1)\pi)^{-1} - \frac{32}{3}((2n+1)\pi)^{-3} - \frac{832}{15}((2n+1)\pi)^{-5} \\
&\quad + O(\frac{1}{n^7}).
\end{aligned}
$$

Corresponding eigenfunctions are obtained by

$$X_0(x) = 1,$$

$$X_{2n} = \cos(2\pi n)x, \; n = 1, 2, \cdots .$$

$$X_{2n+1} = \frac{-k_n}{2}\cos(k_n x) + \sin(k_n x), \; n = 1, 2, \cdots .$$

Therefore, solution of the problem (2.17)-(2.20) is

$$v_1(x,t) = \sum_{n=0}^{\infty} A_{2n}\cos(2\pi nx)e^{-a^2 4\pi^2 n^2 t} + \sum_{n=1}^{\infty} B_n\left(\frac{-k_n}{2}\cos(k_n x) + \sin(k_n x)\right)e^{-a^2 k_n^2 t},$$

where

$$A_0 = \int_0^1 \psi(x)dx,$$

$$A_n = 2\int_0^1 \psi(x)\cos(2\pi n)x, , \; n = 1, 2, \cdots .$$

$$B_n = \frac{1}{\|X_{2n+1}(x)\|^2}\int_0^1 \psi(x)\left(\frac{-k_n}{2}\cos(k_n x) + \sin(k_n x)\right), , \; n = 1, 2, \cdots .$$

Solution of the problem (2.13)-(2.16) can be easily obtained by

$$
v_2(x,t) = \sum_{n=0}^{\infty} \left[\int_0^t F_{2n}(\tau) e^{-k_n^2(t-\tau)} d\tau \right] X_{2n}(x)
$$

$$
+ \left[\int_0^t F_{2n+1}(\tau) e^{-k_n^2(t-\tau)} d\tau \right] X_{2n+1}(x),
$$

where

$$
F_0(\tau) = \int_0^1 F(x,\tau) x \, dx,
$$

$$
F_{2n}(\tau) = \int_0^1 F(x,\tau) X_{2n}(x) dx, \ n = 1, 2, \cdots.
$$

$$
F_{2n+1}(\tau) = \int_0^1 F(x,\tau) X_{2n+1}(x) dx. \ n = 1, 2, \cdots.
$$

3. Numerical Solution

Method of Lines [12] and the Crank-Nicolson method [13] are used for numerical solution of problem (2.1)-(2-4). In both methods, the Simpson's rule is used to approximate the integral in (2.3) and (2.4) numerically. We display here a few of numerical results.

Example 3.1.

$$
\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + (x^2 - 2)e^t,
$$

$$
u(x,0) = x^2,
$$

$$
\int_0^1 u(x,t) dx = (1/6) - 2t,
$$

$$
\int_0^1 x u(x,t) dx = (1/12) - t.
$$

Exact solution of example 1 is $u(x,t) = x - x^2 - 2t$. The absolute relative errors at various spatial lengths for $u(0.5, 0.5)$ are shown in Table 1.

Relative Error at u(0.5,0.5) in Example 1

Spatial Length	MOL Method	Crank-Nicolson Method
h=0.1	1.3471E-14	2.9606E-16
h=0.05	8.4510E-13	2.9606E-16
h=0.025	1.7494E-12	6.8094E-15
h=0.0125	4.6876E-12	1.9244E-15

Example 3.2.

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2},$$

$$u(x,0) = \sin(\pi x),$$

$$\int_0^1 u(x,t)dx = \frac{2}{\pi}\exp(-\pi^2 t),$$

$$\int_0^1 xu(x,t)dx = (1/12) - t.$$

Exact solution of example 2 is $u(x,t) = \sin(\pi x)\exp(-\pi^2 t)$. The absolute relative errors at various spatial lengths for $u(0.5, 0.5)$ are shown in Table 2.

Relative Error at u(0.5,0.5) in Example 2

Spatial Length	MOL Method	Crank-Nicolson Method
h=0.1	0.0029	0.0075
h=0.05	4.5074E-4	0.0023
h=0.025	6.4370E-5	5.6888E-4
h=0.0125	4.4595E-6	9.1160E-5

4. Conclusion

Diffusion equation with two integral boundary conditions is studied. Integral boundary conditions are transformed to local one and by separation of variables, analytic solution of this problem is found. In addition, by applying the Method of Lines [12] and Crank Nicolson method [13], numerical solution of the problem is found.

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FUZZY RIGHT FRACTIONAL OSTROWSKI INEQUALITIES

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ABSTRACT. In this paper, firstly fuzzy basic concept is studied.We investigated other Ostrowski type inequalities in literature. We obtained the very general fuzzy fractional Ostrowski type inequality with right fractional Caputo derivative using the Hölder inequality in this type.

1. INTRODUCTION

Mathematical inequalities take an important place among mathematical concepts.These enable us to find the values of these quantities approximately. Mathematical inequalities have also important applications in functional analysis. For example when building norms on some linear spaces.

The following result is known in the literature as an Ostrowski's inequality. In 1938, the classical integral inequality was proved by A.M. Ostrowski [9] .

The inequality of Ostrowski gives us an estimation for the deviation of the values of a smooth function from its mean value. More precisely, if $f : [a, b] \to \mathbb{R}$ is a differentiable function with bounded derivative, then

$$\left| f(x) - \frac{1}{b-a} \int\limits_a^b f(t)dt \right| \leq \left[\frac{1}{4} + \frac{(x - \frac{a+b}{2})^2}{(b-a)^2} \right] (b-a) \left\| f' \right\|_\infty$$

for every $x \in [a, b]$. Moreover the constant $1/4$ in the right side of the inequality is the best possible value for the better result.

The theory of fractional calculus has known an intensive development over the last few decades. It is shown that derivatives and integrals of fractional type provide an adequate mathematical modelling of real objects and processes see $[7] - [8]$.

We notice that the first generalization of Ostrowski's inequality was given by Milanovic and Pecaric in [2].

In [10] Pachpatte has proved the Ostrowski inequality in three independent variables. In the past few years, many authors have obtained various generalizations of this type of inequality and many researchers worked on a fractional form of it as well as on time scale calculus [11] .

Univariate right fractional Ostrowski inequalities has been shown by Anatassiou [12].

Fuzzy sets were defined in [1] A standard fuzzy set in X is characterized by a membership function $\mu : X \to [0,1]$ A standard fuzzy set is called normalized if $\sup_{x \in X} \mu(x) = 1$

Fuzzy fractional calculus and the Ostrowski inequalities have been studied by Anatassiou [5].

The main purpose of this manuscript is to establish Ostrowski-type inequality involving right Caputo differentiability. First of all; we give basic information about the fuzzy set .Then ,we introduce the very general univariate fuzzy fractional Ostrowski type inequality.We show this inequality in fuzzy space.

2. Background

We need the following basic concepts

Definition 2.1. [5] Let $\mu : \mathbb{R} \to [0,1]$ with the following properties

$i)$ is normal ,i.e.,$\exists x_0 \in \mathbb{R};\ \mu(x_0) = 1$

$ii)$ $\mu(\lambda x + (1-\lambda) y) \geq \min\{\mu(x), \mu(y)\}$,$\forall x, y \in \mathbb{R}$, $\forall \lambda \in [0,1]$ (μ is called a convexs fuzzy subset).

$iii)$ μ is upper semicontinouns on $\mathbb{R}$, i.e. $\forall\ x_0 \in \mathbb{R}$ and $\forall \epsilon > 0, \exists$ neighborhood $V(x_0) : \mu(x) \leq \mu(x_0) + \epsilon, \forall x \in V(x_0)$

$iv)$ The set $\overline{supp(\mu)}$ is compact in $\mathbb{R}$. where $(supp(\mu) = \{x \in \mathbb{R}\ :\ \mu(x) > 0\})$

We call μ a fuzzy real number.Denote the set of all μ with $\mathbb{R}_F$ .E.g., $\chi_{\{x_0\}} \in \mathbb{R}_F$, for any $x_0 \in \mathbb{R}$, where $\chi_{\{x_0\}}$ is the characteristic function at x_0.

For $0 < r \leq 1$ and $\mu \in \mathbb{R}_F$ define

$$[\mu]^r = \{x \in \mathbb{R} : \mu(x) \geq r\}$$

and

$$[\mu]^0 = \overline{\{x \in \mathbb{R} : \mu(x) \geq r\}}$$

Then it is well known that for each $r \in [0,1]$, $[\mu]^r$ is a closed and bounded interval of $\mathbb{R}$.For $u, v \in \mathbb{R}_F$ and $\lambda \in \mathbb{R}$,we define uniquely the sum $u \oplus v$ and the product $\lambda \odot u$ by

$$[u \oplus v]^r = [u]^r + [v]^r,\ [\lambda \odot u]^r = \lambda [u]^r, \forall r \in [0,1]$$

Notice $1 \odot u = u$ and its holds

$$u \oplus v = v \oplus u, \lambda \odot u = u \odot \lambda$$

İf $0 \leq r_1 \leq r_2 \leq 1$ then $[\mu]^{r_2} \subseteq [\mu]^{r_1}$ .Actually $[u]^r = \left[u_-^r, u_+^r\right]$, $u_-^r \leq u_+^r$, $u_-^r, u_+^r \in \mathbb{R}, \forall r \in [0,1]$

For $\lambda > 0$ one has $\lambda_\pm^r = (\lambda \odot u)_\pm^r$, respectively.

Definition 2.2. [5] $D : \mathbb{R}_F \times \mathbb{R}_F \to \mathbb{R}_+ \cup \{0\}$

$$D(u,v) = \sup \max \left\{\left|\ u_-^r - v_-^r\right|, \left|\ u_+^r - v_+^r\right|\right\}_{r \in [0,1]}$$

where $[u]^r = [u_-^r, u_+^r]$; $u, v \in \mathbb{R}_F$. We have that D is a metric on $\mathbb{R}_F$.
Then $(\mathbb{R}_F, D)$ is a complete metric space with the following properties:

$i) \; D(u \oplus w, v \oplus w) = D(u, v) \; \forall u, v, w \in \mathbb{R}_F$
$ii) \; D(\lambda \odot u, \lambda \odot v) = |\lambda| \, D(u, v) \; \; \forall \lambda \in \mathbb{R}, \forall u, v \in \mathbb{R}_F$
$iii) \; D(u \oplus v, w \oplus e) \leq D(u \oplus w) + D(v \oplus e), \forall u, v, w, e \in \mathbb{R}_F$

Here $\sum^*$ is stands for fuzzy summation and $\widetilde{0} : \chi_{\{0\}} \in \mathbb{R}_F$ is the neutral element with respect to $\oplus$,i.e.,

$$u \oplus \widetilde{0} = \widetilde{0} \oplus u = u \; , \forall u \in \mathbb{R}_F$$

Denote

$$D^*(f, g) = \sup D(f, g)_{x \in [a,b]}$$

Where $f, g : [a, b] \to \mathbb{R}_F$.

We define $C_F^U([a, b])$ the space of uniformly continuous functions from $[a, b] \to \mathbb{R}_F$,also $C_F([a, b])$ the space of fuzzy continuous functions on $[a, b]$.İt is clear that

$$C_F^U([a, b]) = C_F([a, b])$$

and $L_F([a, b])$ is the space of Lebesque integrable functions.

Definition 2.3. [13] Let $u, v \in \mathbb{R}_F$ .If there exists $w \in \mathbb{R}_F$ such that $u = v + w$, the w is called the Hukuhara difference of u and v, and it is denoted by $u \ominus v$.

Definition 2.4. [13] Let $u, v \in \mathbb{R}_F$ .If there exists $w \in \mathbb{R}_F$ such that

$$u \ominus_{gH} v = w \Longleftrightarrow \begin{cases} (i) & u = v + w \\ or \;\; (ii) & u = v + (-1)w \end{cases}$$

Then w is called the generalized Hukuhara difference of u and v.
Please note that a function $f : [a, b] \to \mathbb{R}_F$ so called fuzzy-valued function.The $r-$level representation of fuzzy-valued function f is expressed by

$$f_r(t) = \left[f_r^-(t), f_r^+(t) \right], t \in [a, b], r \in [0, 1]$$

Here, $f^r(t) = f_r(t)$

Definition 2.5. [5] Let $f : [a, b] \to \mathbb{R}_F$ We say that f is Fuzzy-Riemann integrable to $I \in \mathbb{R}_F$ if for any $\epsilon > 0$,there exits $\delta > 0$ such that for any division $P = \{[u, v] ; \xi\}$

of $[a, b]$ with the norms $\Delta(P) < \delta$,we have

$$D \left(\sum_P^* (v - u) \odot f(\xi), I \right) < \epsilon$$

We write

$$I = (FR) \int_a^b f(x) \, dx$$

Theorem 2.1. [9] *Let* $f : [a, b] \to \mathbb{R}_F$ *be fuzzy continuous. Then* $(FR) \int_a^b f(x) \, dx$
exists and belongs to $\mathbb{R}_F$, *furthermore it holds*

$$\left[(FR) \int_a^b f(x) \, dx \right]^r = \left[\int_a^b f_-^{(r)}(x) \, dx, \int_a^b f_+^{(r)}(x) \, dx \right], \ r \in [0, 1]$$

Theorem 2.2. [5] *Let* $f \in C_F([a, b])$ *and* $c \in [a, b]$. *Then*

$$(FR) \int_a^b f(x) \, dx = (FR) \int_a^c f(x) \, dx + (FR) \int_c^b f(x) \, dx$$

Theorem 2.3. [5] *Let* $f, g \in C_F([a, b])$ *and* $c_1, c_2 \in \mathbb{R}$. *Then*

$$(FR) \int_a^b (c_1 f(x) + c_2 g(x)) \, dx = c_1 (FR) \int_a^b f(x) \, dx + c_2 (FR) \int_a^b g(x) \, dx$$

also we need

Lemma 2.1. [5] *If* $f, g : [a, b] \subseteq \mathbb{R} \to \mathbb{R}_F$ *are fuzzy continuous functions, then the function* $F : [a, b] \to \mathbb{R}_+$ *defined by* $F(x) = D(f(x), g(x))$ *is continuous on* $[a, b]$:

$$D \left((FR) \int_a^b f(x) \, dx, (FR) \int_a^b g(x) \, dx \right) \leq (FR) \int_a^b D(f(x), g(x)) \, dx$$

Definition 2.6. [4] *Let* $f \in C_F([a, b]) \cap L_F([a, b])$, $0 < v \leq 1$.

The fuzzy Riemann-Liouville integral of fuzzy-valued function f is defined as following:

$$(I_{\alpha+}^v f)(x) = \frac{1}{\Gamma(v)} \odot \int_a^x (x - t)^{v-1} \odot f(t) dt, x \in [a, b]$$

$$I_{\alpha+}^0 f(x) = f$$

Let us consider the $r - level$ representation of fuzzy-valued function f as $f_r(t) = [f_r^-(t), f_r^+(t)], t \in [a, b], r \in [0, 1]$

Also, we define the fuzzy fractional right Riemann-Liouville operator by

$$I_{b-}^v f(x) = \frac{1}{\Gamma(v)} \odot \int_x^b (t - x)^{v-1} \odot f(t) dt, x \in [a, b]$$

$$I_{b-}^0 f(x) = f$$

Above, Γ denotes the gamma function:

$$\Gamma(v) = \int_0^\infty e^{-t} t^{v-1} dt$$

Definition 2.7. [4] Let $f \in C_F\left([a,b]\right) \cap L_F\left([a,b]\right)$, x_0 in (a,b) and $\Phi(x) = \frac{1}{\Gamma(1-v)} \int\limits_a^x \frac{f(t)}{(x-t)^v} dt$ .We say that f is Riemann-Liouville H-differentiable about order $0 < v < 1$ at x_0 , if there exists an element $\left({}^{RL}D_{\alpha+}^v\right)(x_0) \in \mathbb{R}_F$,such that for $h > 0$ sufficiently small

   $i)$ $\left({}^{RL}D_{\alpha+}^v\right)(x_0) = \lim\limits_{h\to 0^+} \frac{\Phi(x_0+h)\ominus\Phi(x_0)}{h} = \lim\limits_{h\to 0^+} \frac{\Phi(x_0)\ominus\Phi(x_0-h)}{h}$
   or
   $ii)$ $\left({}^{RL}D_{\alpha+}^v\right)(x_0) = \lim\limits_{h\to 0^+} \frac{\Phi(x_0)\ominus\Phi(x_0+h)}{-h} = \lim\limits_{h\to 0^+} \frac{\Phi(x_0-h)\ominus\Phi(x_0)}{-h}$
   or
   $iii)$ $\left({}^{RL}D_{\alpha+}^v\right)(x_0) = \lim\limits_{h\to 0^+} \frac{\Phi(x_0+h)\ominus\Phi(x_0)}{h} = \lim\limits_{h\to 0^+} \frac{\Phi(x_0-h)\ominus\Phi(x_0)}{-h}$
   or
   $iv)$ $\left({}^{RL}D_{\alpha+}^v\right)(x_0) = \lim\limits_{h\to 0^+} \frac{\Phi(x_0)\ominus\Phi(x_0+h)}{-h} = \lim\limits_{h\to 0^+} \frac{\Phi(x_0)\ominus\Phi(x_0-h)}{h}$

3. Main results

Definition 3.1. [14] Let $f \in C_F\left([a,b]\right) \cap L_F\left([a,b]\right)$ be a fuzzy set-valued function.Then f is said to be Caputo's H-differentiable at x when

$$\left(D_{\alpha+}^v\, f(x)\right) = \frac{1}{\Gamma(1-v)} \int\limits_a^x \frac{f'(t)}{(x-t)^v} dt$$

where $0 < \alpha < 1$ and $0 < v < 1$.

Also,we adopt the some procedure to present Caputo's H-differentiability,we say f is

$[(i)-v]$-differentiable if Eq. (8) holds while f is (i)-differentiable,and f is $[(ii)-v]$-differentiable if Eq. (21) holds while f is (ii)-differentiable.

Definition 3.2. [15] Let $f \in C_F\left([a,b]\right) \cap L_F\left([a,b]\right)$, f^n is integrable.Then the right fuzzy Caputo derivate of f for $n-1 < v < n$, and $x \in [a,b]$, $D_{b-}^v f(x) \in \mathbb{R}_F$ and defined by

$$D_{b-}^v f(x) = \frac{(-1)^n}{\Gamma(n-v)} \odot \int\limits_x^b (t-x)^{-v+n-1} \odot f^n(t)dt$$

and for $n = 1$

$$D_{b-}^v f(x) = \frac{(-1)}{\Gamma(1-v)} \odot \int\limits_x^b (t-x)^{-v} \odot f'(t)dt$$

Theorem 3.1. [14] *Let* $f \in C_F\left([a,b]\right) \cap L_F\left([a,b]\right)$, $0 < v < 1, 0 \leq r \leq 1$,

   $i)$ Let f be (ii)-differentiable, then we have $[(i)-v]$ differentiable right fuzzy Caputo derivative and

$$\left(D_{b-}^v f\right)(x,r) = \left[\left(D_{b-}^v f_-\right)(x,r),\ \left(D_{b-}^v f_+\right)(x,r)\right]$$

ii)   Let f be (i)-differentiable, then we have $[(ii) - v]$ differentiable right fuzzy Caputo derivative and

$$\left(D_{b-}^{v} f \right)(x, r) = \left[\left(D_{b-}^{v} f_{+} \right)(x, r), \ \left(D_{b-}^{v} f_{-} \right)(x, r) \right]$$

Theorem 3.2. [14]   *Let $0 < V < 1$, $D_{b-}^{v} f(x) = g(x, f(x))$ with the fuzzy initial condition $f_0 = f(b)$,the fuzzy fractional differential equation is equivalent to one of the following integral equations:*

i) if f is a $[(i) - v]$ differentiable fuzzy-valued function,then

$$f(x) = f(b) \oplus \frac{1}{\Gamma(v)} \odot \int\limits_{x}^{b} (t - x)^{v-1} \odot (D_{b-}^{v} f)(t) dt$$

ii) if f is a $[(ii) - v]$ differentiable fuzzy-valued function,then

$$f(x) = f(b) \oplus \frac{-1}{\Gamma(v)} \odot \int\limits_{x}^{b} (t - x)^{v-1} \odot (D_{b-}^{v} f)(t) dt$$

Theorem 3.3. *Let $f \in C_F([a, b]) \cap L_F([a, b])$, $0 < v < 1$, $p, q > 0$ such that $\frac{1}{p} + \frac{1}{q} = 1$,and $(D_{b-}^{v} f)(x) \in \mathbb{R}_F$; $(t \in [a, b])$*

$$D\left(\frac{1}{b-a} \odot (FR) \int\limits_{a}^{b} f(x)\, dx, f(b) \right) \leq \frac{\sup D((D_{b-}^{v} f)(t), \widetilde{0})}{\Gamma(v)(p(v-1)+1)^{\frac{1}{p}} (v + \frac{1}{p})} (b-a)^{v-1+\frac{1}{p}}$$

Proof. We have

$$D\left(\tfrac{1}{b-a} \odot (FR) \int\limits_{a}^{b} f(x)\, dx, f(b) \right) = D\left(\tfrac{1}{b-a} \odot (FR) \int\limits_{a}^{b} f(x)\, dx, \tfrac{f(b)}{b-a} \int\limits_{a}^{b} dx \right)$$

$$= D\left(\tfrac{1}{b-a} \odot (FR) \int\limits_{a}^{b} f(x)\, dx, \tfrac{1}{b-a} \odot (FR) \int\limits_{a}^{b} f(b)\, dx \right)$$

$$= \tfrac{1}{b-a} D\left((FR) \int\limits_{a}^{b} f(x)\, dx, (FR) \int\limits_{a}^{b} f(b)\, dx \right)$$

$$\leq \tfrac{1}{b-a} \int\limits_{a}^{b} D(f(x), f(b))\, dx \qquad\qquad (*)$$

Here $[(i) - v]$ differentiable .
We notice that $f \in C_F([a, b]) \cap L_F([a, b])$, $0 < v < 1$,

$$f(x) = f(b) \oplus \frac{1}{\Gamma(v)} \odot \int\limits_{x}^{b} (t - x)^{v-1} \odot (D_{b-}^{v} f)(t) dt$$

For $a \leq x \leq b$, we have

$$
\begin{aligned}
D\left(f\left(x\right), f(b)\right) \ &= D\left(f(b) \oplus \tfrac{1}{\Gamma(v)} \odot \int\limits_{x}^{b}(t-x)^{v-1} \odot (D_{b-}^{v}f)(t)dt, f(b)\right) \\
&= D\left(\tfrac{1}{\Gamma(v)} \odot \int\limits_{x}^{b}(t-x)^{v-1} \odot (D_{b-}^{v}f)(t)dt, \widetilde{0}\right) \\
&\leq \tfrac{1}{\Gamma(v)}D\left(\int\limits_{x}^{b}(t-x)^{v-1} \odot (D_{b-}^{v}f)(t)dt, \widetilde{0}\right) \\
&\leq \tfrac{1}{\Gamma(v)}D\left(\int\limits_{x}^{b}(t-x)^{v-1} \odot (D_{b-}^{v}f)(t)dt, \int\limits_{x}^{b}\widetilde{0}dt\right) \\
&\leq \tfrac{1}{\Gamma(v)}\int\limits_{x}^{b}(t-x)^{v-1}\left(D\left((D_{b-}^{v}f)(t), \widetilde{0}\right)\right)dt \\
&\leq \tfrac{1}{\Gamma(v)}\left(\int\limits_{x}^{b}(t-x)^{p(v-1)}\right)^{\frac{1}{p}}\left(\int\limits_{x}^{b}\left(D\left((D_{b-}^{v}f)(t), \widetilde{0}\right)\right)^{q}dt\right)^{\frac{1}{q}} \\
&\leq \tfrac{1}{\Gamma(v)}\tfrac{(b-x)^{v-1+\frac{1}{p}}}{(p(v-1)+1)^{\frac{1}{p}}}\left(\int\limits_{x}^{b}\left(D\left((D_{b-}^{v}f)(t), \widetilde{0}\right)\right)^{q}dt\right)^{\frac{1}{q}}
\end{aligned}
$$

Now , $\forall x \in [a, b]$ and for $(*)$

$$
\begin{aligned}
D\left(\tfrac{1}{b-a} \odot (FR)\int\limits_{a}^{b}f\left(x\right)dx, f(b)\right) \ &\leq \tfrac{1}{b-a}\int\limits_{a}^{b}D\left(f\left(x\right), f(b)\right)dx \\
&\leq \tfrac{sup\left(D\left((D_{b-}^{v}f)(t), \widetilde{0}\right)\right)}{(b-a)\Gamma(v)(p(v-1)+1)^{\frac{1}{p}}}\left(\int\limits_{a}^{b}\left((b-x)^{v-1+\frac{1}{p}}\right)dx\right) \\
&= \tfrac{sup\left(D\left((D_{b-}^{v}f)(t), \widetilde{0}\right)\right)}{\Gamma(v)(p(v-1)+1)^{\frac{1}{p}}}\tfrac{(b-a)^{v-1+\frac{1}{p}}}{v+\frac{1}{p}} \quad (t \in [a, b])
\end{aligned}
$$

$\square$

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NEW INTUITIONISTIC FUZZY LEVEL SETS

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ABSTRACT. The concept of Intuitionistic Fuzzy Sheet $t-$Cut Set and Intuitionistic Fuzzy $\alpha - t$ Block Cut Set are introduced. The differences between $C_{\alpha,\beta}$ level set and new intuitionistic fuzzy sets is shown.

1. INTRODUCTION

The function $\mu : X \to [0,1]$ is called a fuzzy set over $X(FS(X))$[**?**]. For $x \in X$, $\mu(x)$ is the membership degree of x and the non-membership degree is $1 - \mu(x)$.Intuitionistic fuzzy sets have been introduced by Atanassov [2], as an extension of fuzzy sets. If X is a universal then a intuitionistic fuzzy set A, the membership and non-membership degree for each $x \in X$ respectively, $\mu_A(x)(\mu_A : X \to [0,1])$ and $\nu_A(x)(\ \nu_A:X \to [0,1])$ such that $0 \le \mu_A(x) + \nu_A(x) \le 1$. The class of intuitionistic fuzzy sets on X is denoted by $IFS(X)$.

Definition 1.1. [2] An intuitionistic fuzzy set (shortly IFS) on a set X is an object of the form

$$A = \{< x, \mu_A(x), \nu_A(x) >: x \in X\}$$

where $\mu_A(x), (\mu_A : X \to [0,1])$ is called the "degree of membership of x in A ", $\nu_A(x), (\nu_A : X \to [0,1])$is called the " degree of non- membership of x in A ",and where μ_A and ν_A satisfy the following condition:

$$\mu_A(x) + \nu_A(x) \le 1, \text{ for all } x \in X.$$

Definition 1.2. [1] An intuitionistic fuzzy set A is said to be contained in an intuitionistic fuzzy set B if and only if, for all $x \in X : \mu_A(x) \le \mu_B(x)$ and $\nu_A(x) \ge \nu_B(x)$. If fuzzy set B contains fuzz set A then it is shown by $A \sqsubseteq B$.

It is clear that $A = B$ if and only if $A \sqsubseteq B$ and $B \sqsubseteq A$.

Definition 1.3. [2]Let $A \in IFS(X)$ and let $A = \{< x, \mu_A(x), \nu_A(x) >: x \in X\}$ then the set

$$A^c = \{< x, \nu_A(x), \mu_A(x) >: x \in X\}$$

is called the complement of A.

FIGURE 1

The intersection and the union of two IFSs A and B on X are defined by

$$A \sqcap B = \{< x, \mu_A(x) \wedge \mu_B(x), \nu_A(x) \vee \nu_B(x) >: x \in X\}$$

$$A \sqcup B = \{< x, \mu_A(x) \vee \mu_B(x), \nu_A(x) \wedge \nu_B(x) >: x \in X\}$$

Some special Intuitionistic Fuzzy Sets on X are defined as following;

$$\begin{aligned} O^* &= \{\langle x, 0, 1 \rangle : x \in X\} \\ U^* &= \{\langle x, 0, 0 \rangle : x \in X\} \end{aligned}$$

Definition 1.4. [4] Let $A \in IFS(X)$. Then $(\alpha, \beta)-$cut of A is a crisp subset $C_{\alpha,\beta}(A)$ of the IFS A is given by

$$C_{\alpha,\beta}(A) = \{x : x \in X \text{ such that } \mu_A(x) \geq \alpha, \nu_A(x) \leq \beta\}$$

where $\alpha, \beta \in [0, 1]$ with $\alpha + \beta \leq 1$.

2. Sheet and Block Cut Intuitionistic Fuzzy Level Sets

Definition 2.1. Let X be a set and $A = \{< x, \mu_A(x), \nu_A(x) >: x \in X\} \in IFS(X)$. If $t \in [0, 1]$ then sheet $t-$cut of A defined as following

$$A(t) = \{\langle x, \mu_A(x), \nu_A(x) \rangle : \mu_A(x) + \nu_A(x) = t, x \in X\}$$

Proposition 2.1. *Let X be a set and $A, B \in IFS(X)$. For every $t \in [0, 1]$,*

(1) $(A \sqcup B)(t) = A(t) \sqcup B(t)$
(2) $A(t) \sqcap B(t) = (A \sqcap B)(t)$
(3) $(A^c(t))^c = A(t)$

Proof. (1)

$$A(t) \sqcup B(t) = \left\{ \begin{array}{c} \langle x, \max(\mu_A\left(x\right), \mu_B\left(x\right)), \min(\nu_A\left(x\right), \nu_B\left(x\right)) \rangle : \\ \mu_A\left(x\right) + \nu_A\left(x\right) = t \wedge \mu_B\left(x\right) + \nu_B\left(x\right) = t, x \in X \end{array} \right\}$$

If $\mu_A\left(x\right) \geq \mu_B\left(x\right)$ then from $\mu_A\left(x\right) + \nu_A\left(x\right) = t = \mu_B\left(x\right) + \nu_B\left(x\right)$ we obtain $\nu_A\left(x\right) \leq \nu_B\left(x\right)$.

$\max(\mu_A\left(x\right), \mu_B\left(x\right)) + \min(\nu_A\left(x\right), \nu_B\left(x\right)) = \mu_A\left(x\right) + \nu_A\left(x\right) = t$

If $\mu_A\left(x\right) \leq \mu_B\left(x\right)$ then from $\mu_A\left(x\right) + \nu_A\left(x\right) = t = \mu_B\left(x\right) + \nu_B\left(x\right)$ we obtain $\nu_A\left(x\right) \leq \nu_B\left(x\right)$.

$\max(\mu_A\left(x\right), \mu_B\left(x\right)) + \min(\nu_A\left(x\right), \nu_B\left(x\right)) = \mu_B\left(x\right) + \nu_B\left(x\right) = t$

Thence, $(A \sqcup B)(t) = A(t) \sqcup B(t)$.

(2)

$$A(t) \sqcap B(t) = \left\{ \begin{array}{c} \langle x, \min(\mu_A\left(x\right), \mu_B\left(x\right)), \max(\nu_A\left(x\right), \nu_B\left(x\right)) \rangle : \\ \mu_A\left(x\right) + \nu_A\left(x\right) = t \wedge \mu_B\left(x\right) + \nu_B\left(x\right) = t, x \in X \end{array} \right\}$$

If $\mu_A\left(x\right) \geq \mu_B\left(x\right)$ then from $\mu_A\left(x\right) + \nu_A\left(x\right) = t = \mu_B\left(x\right) + \nu_B\left(x\right)$ we obtain $\nu_A\left(x\right) \leq \nu_B\left(x\right)$.

$\min(\mu_A\left(x\right), \mu_B\left(x\right)) + \max(\nu_A\left(x\right), \nu_B\left(x\right)) = \mu_B\left(x\right) + \nu_B\left(x\right) = t$

If $\mu_A\left(x\right) \leq \mu_B\left(x\right)$ then from $\mu_A\left(x\right) + \nu_A\left(x\right) = t = \mu_B\left(x\right) + \nu_B\left(x\right)$ we obtain $\nu_A\left(x\right) \leq \nu_B\left(x\right)$.

$\min(\mu_A\left(x\right), \mu_B\left(x\right)) + \max(\nu_A\left(x\right), \nu_B\left(x\right)) = \mu_A\left(x\right) + \nu_A\left(x\right) = t$

Therefore, we obtain that $A(t) \sqcap B(t) = (A \sqcap B)(t)$.

(3) It is clear. $\square$

Remark 2.1. Let X be a set and $A \in IFS(X).A(t)$ is a fuzzy set on $[0, t]$.

Proposition 2.2. *Let X be a set and $A \in IFS(X)$. If $t, s \in [0, 1]$ then*

$$\text{Either } A(t) \sqcap A(s) = O^* \text{ or } t = s$$

Proof. If $A(t) \sqcap A(s) \neq O^*$ and $t \neq s$ then there exists $x \in X$,

$$\begin{aligned} \mu_A\left(x\right) + \nu_A\left(x\right) &= t \text{ and } \mu_A\left(x\right) + \nu_A\left(x\right) = s \\ \Rightarrow t &= s \end{aligned}$$

$\square$

Corollary 2.1. *There exist an equivalence relation on X such that the sheet $t-$cuts are equivalence class of that relation.*

Definition 2.2. Let X be a set and $A \in IFS(X)$. If $t \in [0, 1]$ and $\alpha \in [0, t]$ then

$$A(t)_\alpha = \{x : x \in X, \ A(t)(x) \geq (\alpha, t - \alpha)\}$$

is called $\alpha - t$ block cut of A.

From definitions, it is easily seen that for every $t \in [0, 1], A(t) \in FS(X)$. Because $A(t) : X \to [0, t]$ and $[0, t] \sim [0, 1]$. For short notation, if $A(t) : X \to [0, t]$ then $A(t)$ will be called $t-$fuzzy set on $X(A(t) \in FS_t(X))$. It is clear that $A(t)_\alpha$ is a crisp set.

Proposition 2.3. *Let X be a set and $A \in IFS(X)$. If $t \in [0, 1]$ then*

(1) $A(t)_t = \{x : x \in X, \ \mu_A\left(x\right) = t \wedge \nu_A\left(x\right) = 0\}$

(2) $A(t)_0 = \{x : x \in X, \ \mu_A\left(x\right) \geq 0 \wedge \nu_A\left(x\right) \leq t\}$
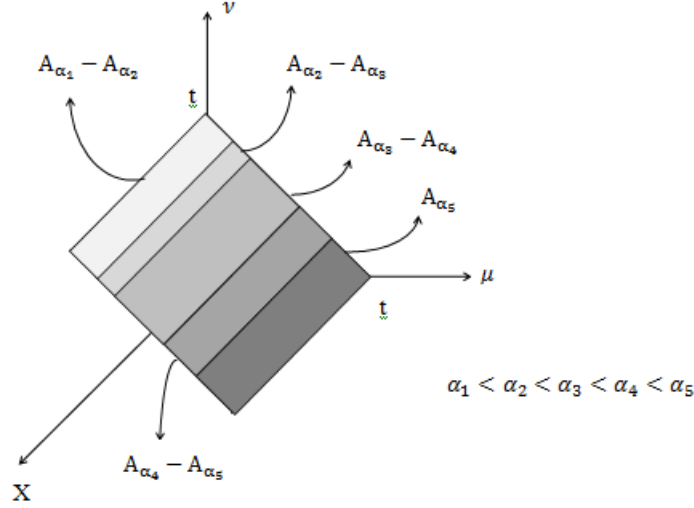
FIGURE 2

(3) $A(0) = U^*$

Example 2.1. Let $X = \{a, b, c, d, e\}$ and
$A = \{(a, 0.5, 0.4), (b, 0.2, 0.3), (c, 0.5, 0.3), (d, 0.4, 0.4), (e, 0.4, 0.1)\}$.

(1) $A(0.5)_{0.3} = \{e\}$ but $C_{0.3,0.5}(A) = \{a, c, d, e\}$ and $C_{0.5,0.3}(A) = \{c\}$.
(2) $A(0.8)_{0.5} = \{c\}$ but $0.8 + 0.5 > 1$ so, we can not obtain $C_{0.5,0.8}(A)$ or $C_{0.8,0.5}(A)$.

Example 2.2. Let $X = \{a, b, c, d, e\}$ and
$A = \{(a, 0.1, 0.2), (b, 0.4, 0.3), (c, 0.6, 0.2), (d, 0.7, 0.1), (e, 0.2, 0.5)\}$.

$A(0.3)_{0.2} = \varnothing$ but $C_{0.2,0.3}(A) = \{b, c, d\}$ and $C_{0.3,0.2}(A) = \{c, d\}$.

That is seen from the examples, $(\alpha, \beta)-$cut of an intuitionistic fuzzy set A and $\alpha - t$ block cut of A are different sets. For all $t \in [0, 1]$ and $\alpha \in [0, t]$, we can determine $\alpha - t$ block cut of A, if $\alpha + t > 1$ then we can not determine $(\alpha, \beta)-$cut of A. Consequently, $\alpha - t$ block cut of an intuitionistic fuzzy set allows a more extensive studying area.

Proposition 2.4. Let X be a set and $A \in IFS(X)$. If $t \in [0, 1]$ and $\alpha, \beta \in [0, t]$ such that $\alpha \leq \beta$ then $A(t)_\beta \subseteq A(t)_\alpha$.

Proof. Let $\alpha \leq \beta$. If $x \in A(t)_\beta$ then

$$A(t)(x) \geq (\beta, t - \beta) \geq (\alpha, t - \alpha)$$

Therefore $x \in A(t)_\alpha$. □

Proposition 2.5. Let X be a set and $A, B \in IFS(X)$. If $t \in [0, 1]$ and $\alpha \in [0, t]$ then

(1) $A(t)_\alpha \cup B(t)_\alpha = (A(t) \sqcup B(t))_\alpha$
(2) $A(t)_\alpha \cap B(t)_\alpha = (A(t) \sqcap B(t))_\alpha$

(3) $(A(t)^c)_\alpha = t - A(t)_\alpha, (t(x) = t)$

(4) $(A^c(t)^c)_\alpha = t - A(t)_\alpha = t - A^c(t)_\alpha$

Proof. (1)

$$
\begin{aligned}
x \quad &\in \quad A(t)_\alpha \cup B(t)_\alpha \Leftrightarrow A(t)(x) \geq (\alpha, t - \alpha) \vee B(t)(x) \geq (\alpha, t - \alpha) \\
&\Leftrightarrow \quad (\mu_{A(t)}(x) \geq \alpha \wedge \nu_{A(t)}(x) \leq t - \alpha) \vee (\mu_{B(t)}(x) \geq \alpha \wedge \nu_{B(t)}(x) \leq t - \alpha) \\
&\Leftrightarrow \quad (\mu_{A(t)}(x) \geq \alpha \vee \mu_{B(t)}(x) \geq \alpha) \wedge (\nu_{A(t)}(x) \leq t - \alpha \vee \nu_{B(t)}(x) \leq t - \alpha) \\
&\Leftrightarrow \quad (\mu_{A(t)}(x) \vee \mu_{B(t)}(x)) \geq \alpha \wedge (\nu_{A(t)}(x) \wedge \nu_{B(t)}(x)) \leq t - \alpha \\
&\Leftrightarrow \quad \mu_{A(t) \sqcup B(t)}(x) \geq \alpha \wedge \nu_{A(t) \sqcup B(t)}(x) \leq t - \alpha \\
&\Leftrightarrow \quad x \in (A(t) \sqcup B(t))_\alpha
\end{aligned}
$$

(2)

$$
\begin{aligned}
&A(t)_\alpha \cap B(t)_\alpha = \{x \in X : A(t)(x) \geq (\alpha, t - \alpha) \wedge B(t)(x) \geq (\alpha, t - \alpha)\} \\
=\quad &\{x \in X : (\mu_{A(t)}(x) \geq \alpha \wedge \nu_{A(t)}(x) \leq t - \alpha) \wedge (\mu_{B(t)}(x) \geq \alpha \wedge \nu_{B(t)}(x) \leq t - \alpha)\} \\
=\quad &\{x \in X : (\mu_{A(t)}(x) \geq \alpha \wedge \mu_{B(t)}(x) \geq \alpha) \wedge (\nu_{A(t)}(x) \leq t - \alpha \wedge \nu_{B(t)}(x) \leq t - \alpha)\} \\
=\quad &\{x \in X : (\mu_{A(t)}(x) \wedge \mu_{B(t)}(x)) \geq \alpha \wedge (\nu_{A(t)}(x) \vee \nu_{B(t)}(x)) \leq t - \alpha)\} \\
=\quad &\{x \in X : \mu_{A(t) \sqcap B(t)}(x) \geq \alpha \wedge \nu_{A(t) \sqcap B(t)}(x) \leq t - \alpha)\} \\
=\quad &(A(t) \sqcap B(t))_\alpha
\end{aligned}
$$

(3)

$$
\begin{aligned}
(A(t)^c)_\alpha \quad &= \quad \{x \in X : A(t)^c(x) \geq (\alpha, t - \alpha)\} \\
&= \quad \{x \in X : \nu_{A(t)}(x) \geq \alpha \wedge \mu_{A(t)}(x) \leq t - \alpha \} \\
&= \quad \{x \in X : t - \nu_{A(t)}(x) \leq t - \alpha \wedge t - \mu_{A(t)}(x) \geq \alpha\} \\
&= \quad \{x \in X : (t - A(t))(x) \geq (\alpha, t - \alpha) \} \\
&= \quad t - A(t)_\alpha
\end{aligned}
$$

$\square$

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ENERGY SPECTRUM OF SPINLESS PARTICLES IN ELECTROMAGNETIC FIELDS

ÖZGÜR MIZRAK AND KENAN SOGUT

Abstract. Dynamics of the non-relativistic and relativistic charged spinless particles subjected to space-dependent parallel and orthogonal electromagnetic fields is investigated by solving Schrödinger and Klein-Gordon equations. Exact solutions of the motion are used to obtain the quantized energy spectrum and momentum of the particles. Some numerical results for the first few quantum levels are determined with the help of MATHEMATICA software.

1. Introduction

Finding the exact solutions of the wave equations for the external fields is one of the old problems. Among these equations Schrödinger and Klein-Gordon are the most studied ones. Besides by the increase in the applications of the electric and magnetic fields in fundamental areas of technology, especially in electromechanics, health physics and so forth ,a significant interest has been given to these solutions. Such efforts have been performed for different configurations of the external fields [1-3].

These studies provide remarkable information regarding the quantum mechanical systems. Some of these attempts are the interpretation of the physical processes. The most important ones are Compton scattering by a laser source, Brownian motion, coherent states, and energy levels of electrons.

There are very few studies in the literature on the solution of the wave equation of the spinless particles in the presence of both electric and magnetic fields. The aim of this study is to move this attempt one step further by obtaining the exact solutions of the spinless particles for two orientations of decaying electric and magnetic fields given by $Case(i)$ $A_0 = \frac{E_0}{z}$, $A_1 = \frac{B_0}{y}$ and $Case(ii)$ $A_0 = \frac{E_0}{y}$, $A_1 = \frac{B_0}{y}$, where E_0 and B_0 are constants. The first and second cases belong to the parallel and orthogonal fields, respectively. We note that y and z variables are defined in the region $(0, \infty)$ to keep the finite external fields. Such kind of varying electromagnetic field is encountered in semiconductor heterostructures.

In the following sections, the exact solutions for nonrelativistic and relativistic cases will be obtained, respectively. By comparing the solutions of the nonrelativistic and relativistic wave equations of the spinless particles, contributions coming

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from the relativistic effects will be considered and by using the mathematical properties of the wave functions we will obtain the energy spectrum and exact solutions for both cases.

2. Solution of the Schrodinger Equation

Motion of the nonrelativistic spinless particles is described by the Schrödinger equation and in the existence of the external electromagnetic fields, it is given by (we take $\hbar = 1$)

$$(2.1) \qquad \left[\frac{\left(\vec{P} - e\vec{A} \right)^2}{2m} \right] \Phi = (i\frac{\partial}{\partial t} - eA_0)\Phi$$

where e is charge, m is mass ot the particle, $\vec{A}$ is the vector electromagnetic potential. In the following steps we solve the Schrödinger equation for the cases where electric and magnetic fields are parallel and perpendicular to each other.

2.1. **Case (i) Parallel EM Fields.** For the choice of $A_0 = \frac{E_0}{z}$, $A_1 = \frac{B_0}{y}$, $\vec{E} \parallel \vec{B}$. We define the solution of (2.1) by

$$(2.2) \qquad \Phi_\parallel = e^{i(P_x - \epsilon t)} H(y) K(z)$$

Plugging this solution into (2.1) we obtain,

$$\left[\underbrace{\left(P_x - \frac{eB_0}{y} \right)^2 + P_y^2}_{\hat{Q}(y)} + \underbrace{P_z^2 - 2m \left(\epsilon - \frac{eE_0}{z} \right)}_{\hat{D}(z)} \right] H(y) K(z) = 0$$

In short we can write

$$\left[\hat{Q}(y) + \hat{D}(z) \right] H(y) K(z) = 0$$

Separating this equation with respect to y and z, we obtain

$$(2.3) \qquad \left[\hat{Q}(y) + b \right] H(y) = 0$$

$$(2.4) \qquad \left[\hat{D}(z) - b \right] K(z) = 0$$

where b is the constant of separation.

Let $\gamma^2 = (P_x^2 + b)$, and making $\rho = 2\gamma y$ change of variable (2.3) becomes Whittaker equation [4]

$$(2.5) \qquad \left[\frac{d^2}{d\rho^2} - \frac{e^2 B_0^2}{\rho^2} + \frac{2eB_0 P_x}{\gamma \rho} - \frac{1}{4} \right] H(\rho) = 0$$

So exact solution of (2.3) is

$$(2.6) \qquad H(y) = W_{\lambda,\mu}(2\gamma y)$$

where $\mu^2 = \frac{1}{4} + e^2 B_0^2$, and $\lambda = \frac{eB_0 P_x}{\gamma}$.

In order Whittaker function to be bounded [4]

$$\mu - \lambda = -(n + \frac{1}{2}) = -N, \quad n = 0, 1, 2, \cdots$$

So from this equality we find

$$b = P_x^2 \left(\frac{1}{1 + \frac{(1/4 + N^2)}{e^2 B_o^2} + 2NeB_o\sqrt{1 + \frac{1}{4e^2 B_0^2}}} - 1 \right)$$

Now for the solution of (2.4), this equation is written as

$$\left[P_z^2 - 2m\left(\epsilon - \frac{eE_0}{z}\right) - b \right] K(z) = 0$$

(2.7)
$$\left[\frac{d^2}{dz^2} - \frac{2meE_0}{z} + (2m\epsilon + b) \right] K(z) = 0$$

(2.7) is similar to the below equation

Definition 2.1.
$$xy'' + (ax + b)y' + (cx + d)y = 0$$

For $a^2 > 4c$ solution is given by [5]

$$y = x^{-\frac{b}{2}} e^{-\frac{ax}{2}} {}_1F_1\left(\frac{2d - ab}{2\sqrt{a^2 - 4c}}, \frac{b-1}{2}, x\sqrt{a^2 - 4c} \right)$$

Returning to the equation (2.7),

$$zK''(z) + [(2m\epsilon + b)z - 2meE_0] K(z) = 0$$

for $0 > 2m\epsilon + b$

$$K(z) = {}_1F_1\left(\frac{-meE_0}{\sqrt{-(2m\epsilon + b)}}, -\frac{1}{2}, z\sqrt{-4(2m\epsilon + b)} \right)$$

From the requirement of Hypergeometric functions to be finite

$$\frac{-2meE_0}{\sqrt{-4(2m\epsilon + b)}} = -n$$

where $n = 0, 1, 2, \dots$ we obtain the energy spectrum of Schrödinger equation for the parallel case as

$$\epsilon_\parallel = \frac{P_x^2}{2m} \left[1 - \frac{1}{1 + \frac{(1/4 + N^2)}{e^2 B_o^2} + 2NeB_o\sqrt{1 + \frac{1}{4e^2 B_0^2}}} \right] - \frac{me^2 E_0^2}{2n^2}$$

So the exact solution of (2.1) for parallel case is written as

$$\Phi_\parallel = e^{i(xP_x - \epsilon t)} W_{\lambda,\mu}(2\gamma y) {}_1F_1(z)$$

2.2. **Case (ii) Orthogonal EM Fields.** For the choice of $A_0 = \frac{E_0}{y}$, $A_1 = \frac{B_0}{y}$, $\vec{E} \perp \vec{B}$. In this case, we will look for the solution of (2.1) as

$$\Phi_\perp = e^{i(xP_x + zP_z - \epsilon t)} M(y)$$

Writing this in (2.1), we obtain

$$\left[\frac{d^2}{dy^2} - \frac{e^2 B_0^2}{y^2} + \frac{2e(P_x B_0 - mE_0)}{y} + (2m\epsilon - P_x^2 - P_z^2) \right] M(y) = 0$$

Again solution of this equation is given by Whittaker function as

$$M(y) = W_{\kappa,\sigma}(2uy)$$

where $\kappa = \frac{e(P_x - mE_0)}{iu}$, $\sigma^2 = \frac{1}{4} - e^2 B_0^2$, $u^2 = (2m\epsilon - P_x^2 - P_z^2)$. For Whittaker functions,

$$\sigma - \kappa = -(n + 1/2)$$

should be satisfied. From this condition, we obtain the energy spectrum for the Schrödinger equation for the orthogonal case

$$\epsilon_\perp = \frac{1}{2m} \left[P_x^2 + P_z^2 - \frac{e^2 P_x^2 B_0^2 + e^2 m^2 E_0^2 - 2e^2 m P_x B_0 E_0}{\frac{1}{4} + e^2 B_0^2 + N^2 + 2N\sqrt{\frac{1}{4} + e^2 B_0^2}} \right]$$

So the exact solution of (2.1) in orthogonal case is written as

$$\Phi_\perp = e^{i(xP_x + zP_z - \epsilon t)} W_{\kappa,\sigma}(2uy)$$

3. Solution of the Klein-Gordon Equation

The Klein-Gordon equation for the relativistic spinless particles is given by (we take $\hbar = 1$)

$$(3.1) \qquad \left[(\vec{P} - e\vec{A})^2 + m^2 \right] \phi = (P_0 - eA_0)^2 \phi$$

3.1. **Case (i) Parallel EM Fields.** Again we will look for the solution as

$$\phi_\parallel = e^{i(xP_x - \epsilon t)} F(y) G(z)$$

writing this in (3.1) we obtain

$$\left[\underbrace{-\frac{d^2}{dy^2} + (P_x - \frac{eB_0}{y})^2}_{\hat{Q}(y)} \underbrace{- \frac{d^2}{dz^2} - (\epsilon - \frac{eE_0}{z})^2}_{\hat{R}(z)} + m^2 \right] F(y) G(z) = 0$$

In short we can write

$$\left[\hat{Q}(y) + \hat{R}(z) + m^2 \right] F(y) G(z) = 0$$

Separating this equation with respect to y and z, we obtain

$$(3.2) \qquad \left[\hat{Q}(y) + s \right] F(y) = 0$$

$$(3.3) \qquad \left[\hat{R}(z) + m^2 - s \right] G(z) = 0$$

where s is the separation constant.

Equation (3.2) is written as

$$(3.4) \qquad \left[-\frac{d^2}{dy^2} + (P_x - \frac{eB_0}{y})^2 + s \right] F(y) = 0$$

This equation is the same equation obtained in the Schrödinger case. So the solution is

$$F(y) = W_{\lambda,\mu}(2\gamma y)$$

where $\mu = \pm\sqrt{\frac{1}{4} + e^2 B_0^2}$, $\lambda = \frac{eB_0}{\sqrt{1 + \frac{s}{P_x^2}}}$ and $\gamma = \sqrt{P_x^2 + s}$

As before

$$s = P_x^2 \left(\frac{1}{1 + \frac{(1/4 + N^2)}{e^2 B_o^2} + 2NeB_o \sqrt{1 + \frac{1}{4e^2 B_0^2}}} - 1 \right)$$

Equation (3.3) is written as

(3.5)
$$\left[-\frac{d^2}{dz^2} - \left(\epsilon - \frac{eE_0}{z} \right)^2 + m^2 - s \right] G(z) = 0$$

Again solution of this equation is given by Whittaker functions

$$G(z) = W_{\tilde{\lambda}, \tilde{\mu}}(2\alpha z)$$

where $\tilde{\lambda} = \frac{ieE_0\epsilon}{\alpha}$, $\tilde{\mu} = \pm\sqrt{\frac{1}{4} - e^2 E_0^2}$, and $\alpha^2 = \epsilon^2 - m^2 + s$ and the energy spectrum for the parallel case is given by

$$\epsilon_\parallel = \pm \left[\frac{(m^2 - s)\left(\frac{1}{4} - e^2 E_0^2 + N^2 + 2\tilde{N}\sqrt{\frac{1}{4} - e^2 E_0^2} \right)}{\frac{1}{4} + \tilde{N}^2 + 2\tilde{N}\sqrt{\frac{1}{4} - e^2 E_0^2}} \right]^{\frac{1}{2}}$$

and the exact solution of Klein-Gordon equation for the parallel case is given by

$$\phi_\parallel = e^{i(xP_x - \epsilon t)} W_{\lambda,\mu}(2\gamma y) W_{\tilde{\lambda},\tilde{\mu}}(2\alpha z)$$

3.2. **Case (ii) Orthogonal EM Fields.** For the choice of $A_0 = \frac{E_0}{y}$, $A_1 = \frac{B_0}{y}$, $\vec{E} \perp \vec{B}$.

Again we will look for the solution of (3.1) as

$$\phi_\perp = e^{i(xP_x + zP_z - \epsilon t)} N(y)$$

Writing this in (3.1), we obtain

$$\left[\frac{d^2}{dy^2} + \frac{e^2(E_0^2 - B_0^2)}{y^2} + \frac{2e(P_x B_0 - \epsilon E_0)}{y} + (\epsilon^2 - m^2 - P_x^2 - P_z^2) \right] N(y) = 0$$

Again solution of this equation is given by Whittaker function as

$$N(y) = W_{\tilde{\kappa}, \tilde{\sigma}}(2vy)$$

where $\tilde{\kappa} = \frac{e(P_x - \epsilon E_0)}{iv}$, $\tilde{\sigma}^2 = \frac{1}{4} - e^2(E_0^2 - B_0^2)$, $v^2 = (\epsilon^2 - m^2 - P_x^2 - P_z^2)$. For Whittaker functions,

$$\tilde{\sigma} - \tilde{\kappa} = -(\tilde{n} + 1/2)$$

should be satisfied. From this condition, we obtain the energy spectrum for the Klein-Gordon equation for the orthogonal case from below quadratic equation

$$\epsilon^2 \underbrace{(w^2 + e^2 E_0^2)}_{a} + \epsilon \underbrace{(-2e^2 P_x B_0 E_0)}_{b} + \underbrace{e^2 P_x^2 B_0^2 - w^2(m^2 + P_x^2 + P_z^2)}_{c} = 0$$

where $w = \sqrt{\frac{1}{4} - e^2(E_0^2 - B_0^2)} + \tilde{N}$.

$$\epsilon_\perp = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Exact solution of the (3.1) for the orthogonal case is

$$\phi_\perp = e^{i(xP_x + zP_z - \epsilon t)} W_{\tilde{\kappa}, \tilde{\sigma}}(2vy)$$
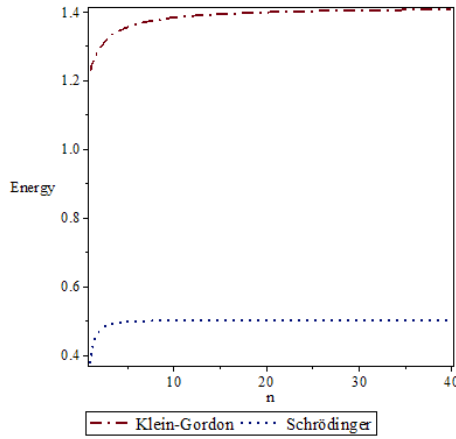
FIGURE 1. Parallel Case
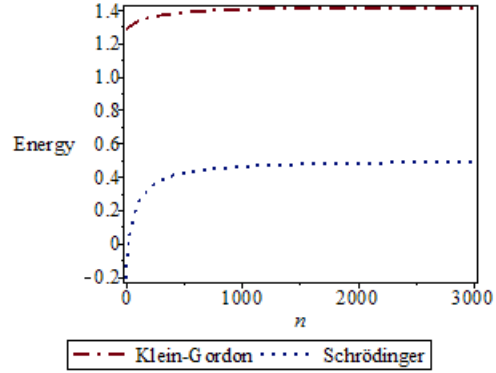
FIGURE 2. Orthogonal Case

4. CONCLUSION

We investigated the motion of the spin-0 particles in electromagnetic fields for parallel and orthogonal orientations. Analysis is performed for Schrödinger and Klein-Gordon cases and that present us the contribution of the relativistic effects. In the case of $\overrightarrow{E} \parallel \overrightarrow{B}$, the relativistic effects arise only for the motion in the z-direction. In that case the Whittaker functions that occurred in the relativistic solutions are replaced by the confluent hypergeometric function for nonrelativistic solutions. In case of the orthogonal fields $\overrightarrow{E} \perp \overrightarrow{B}$, exact solutions of the Schrödinger and Klein-Gordon equations are found in terms of the Whittaker functions.

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SOME INTUITIONISTIC FUZZY MODAL OPERATORS OVER INTUITIONISTIC FUZZY IDEALS AND GROUPS

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ABSTRACT. K.T. Atanassov generalized fuzzy sets in to Intuitionistic Fuzzy Sets in 1983[1]. Intuitionistic Fuzzy Modal Operator was firstly defined by same author and the other operators were defined by several researchers[2, 3, 4]. Intuitionistic fuzzy algebraic stuctures and their properties were studied in[5, 6, 7].

In this paper, we studied some intuitionistic fuzzy operators on intuitionistic fuzzy ideals and groups.

1. INTRODUCTION

The original concept of fuzzy sets in Zadeh [9] was introduced as an extension of crisp sets by enlarging the truth value set to the real unit interval $[0, 1]$. In fuzzy set theory, if the membership degree of an element x is $\mu(x)$ then the nonmembership degree is $1 - \mu(x)$ and thus it is fixed. Intuitionistic fuzzy sets have been introduced by Atanassov in 1983 [1] and form an extension of fuzzy sets by enlarging the truth value set to the lattice $[0, 1] \times [0, 1]$.

Definition 1.1. [1] An intuitionistic fuzzy set (shortly IFS) on a set X is an object of the form

$$A = \{< x, \mu_A(x), \nu_A(x) >: x \in X\}$$

where $\mu_A(x), (\mu_A : X \to [0,1])$ is called the "degree of membership of x in A ", $\nu_A(x), (\nu_A : X \to [0,1])$ is called the " degree of non- membership of x in A ",and where μ_A and ν_A satisfy the following condition:

$$\mu_A(x) + \nu_A(x) \leq 1, \text{ for all } x \in X.$$

The hesitation degree of x is defined by $\pi_A(x) = 1 - \mu_A(x) - \nu_A(x)$

Definition 1.2. [1]An IFS A is said to be contained in an IFS B (notation $A \sqsubseteq B$) if and only if, for all $x \in X : \mu_A(x) \leq \mu_B(x)$ and $\nu_A(x) \geq \nu_B(x)$.

It is clear that $A = B$ if and only if $A \sqsubseteq B$ and $B \sqsubseteq A$.

Definition 1.3. [1]Let $A \in IFS$ and let $A = \{< x, \mu_A(x), \nu_A(x) >: x \in X\}$ then the above set is callede the complement of A

$$A^c = \{< x, \nu_A(x), \mu_A(x) >: x \in X\}$$

Definition 1.4. [2] Let X be a set and $A = \{< x, \mu_A(x), \nu_A(x) >: x \in X\} \in$

$IFS(X)$.

(1) $\square A = \{< x, \mu_A(x), 1 - \mu_A(x) >: x \in X\}$

(2) $\lozenge A = \{< x, 1 - \nu_A(x), \nu_A(x) >: x \in X\}$

Definition 1.5. [3] Let X be a set and $A = \{\langle x, \mu_A(x), \nu_A(x) \rangle : x \in X\} \in IFS(X)$, for $\alpha, \beta \in I$

(1) $\boxplus(A) = \left\{ \left\langle x. \frac{\mu_A(x)}{2}, \frac{\nu_A(x)+1}{2} \right\rangle : x \in X \right\}$

(2) $\boxtimes(A) = \left\{ \left\langle x, \frac{\mu_A(x)+1}{2}, \frac{\nu_A(x)}{2} \right\rangle : x \in X \right\}$

(3) $\boxplus_\alpha(A) = \{\langle x, \alpha\mu_A(x), \alpha\nu_A(x) + 1 - \alpha\rangle : x \in X\}$

(4) $\boxtimes_\alpha(A) = \{\langle x, \alpha\mu_A(x) + 1 - \alpha, \alpha\nu_A(x)\rangle : x \in X\}$

(5) for $\max\{\alpha, \beta\} + \gamma \in I$, $\boxplus_{\alpha,\beta,\gamma}(A) = \{< x, \alpha\mu_A(x), \beta\nu_A(x) + \gamma >: x \in X\}$

(6) for $\max\{\alpha, \beta\} + \gamma \in I$, $\boxtimes_{\alpha,\beta,\gamma}(A) = \{< x, \alpha\mu_A(x) + \gamma, \beta\nu_A(x) >: x \in X\}$

Definition 1.6. [8] Let X be a set and $A = \{< x, \mu_A(x), \nu_A(x) >: x \in X\} \in IFS(X)$, $\alpha, \beta, \alpha + \beta \in I$

(1) $\boxplus_{\alpha,\beta}(A) = \{\langle x, \alpha\mu_A(x), \alpha\nu_A(x) + \beta\rangle : x \in X\}$

(2) $\boxtimes_{\alpha,\beta}(A) = \{\langle x, \alpha\mu_A(x) + \beta, \alpha\nu_A(x)\rangle : x \in X\}$

The operators $\boxplus_{\alpha,\beta,\gamma}, \boxtimes_{\alpha,\beta,\gamma}$ are an extensions of $\boxplus_{\alpha,\beta}, \boxtimes_{\alpha,\beta}$ (resp.).

In 2007, the author[4] defined a new operator and studied some of its properties. This operator is named $E_{\alpha,\beta}$ and defined as follows:

Definition 1.7. [4]Let X be a set and $A = \{< x, \mu_A(x), \nu_A(x) >: x \in X\} \in IFS(X)$, $\alpha, \beta \in [0, 1]$. We define the following operator:

$$E_{\alpha,\beta}(A) = \{< x, \beta(\alpha\mu_A(x) + 1 - \alpha), \alpha(\beta\nu_A(x) + 1 - \beta) >: x \in X\}$$

If we choose $\alpha = 1$ and write α instead of β we get the operator $\boxplus$. Similarly, if $\beta = 1$ is chosen and writen instead of β, we get the operator $\boxtimes_\alpha$.

In2007, Atanassov introduced the operator $\boxdot_{\alpha,\beta,\gamma,\delta}$ which is a natural extension of all these operators in [3].

Definition 1.8. [3]Let X be a set, $A \in IFS(X)$, $\alpha, \beta, \gamma, \delta \in [0, 1]$ such that

$$\max(\alpha, \beta) + \gamma + \delta \leqslant 1$$

then the operator $\boxdot_{\alpha,\beta,\gamma,\delta}$ defined by

$$\boxdot_{\alpha,\beta,\gamma,\delta}(A) = \{< x, \alpha\mu_A(x) + \gamma, \beta\nu_A(x) + \delta >: x \in X\}$$

In 2010, the author [4] defined a new operator which is a generalization of $E_{\alpha,\beta}$.

Definition 1.9. [4]Let X be a set and $A \in IFS(X)$, $\alpha, \beta, \omega \in [0,1]$. We define the following operator:

$$Z_{\alpha,\beta}^{\omega}(A) = \{< x, \beta(\alpha\mu_A(x) + \omega - \omega.\alpha), \alpha(\beta\nu_A(x) + \omega - \omega.\beta) >: x \in X\}$$

We have defined a new OTMO on IFS, that is generalization of the some OTMOs. $Z_{\alpha,\beta}^{\omega,\theta}$ defined as follows:

Definition 1.10. [4]Let X be a set and $A \in IFS(X)$, $\alpha, \beta, \omega, \theta \in [0,1]$. We define the following operator:

$$Z_{\alpha,\beta}^{\omega,\theta}(A) = \{< x, \beta(\alpha\mu_A(x) + \omega - \omega.\alpha), \alpha(\beta\nu_A(x) + \theta - \theta.\beta) >: x \in X\}$$

The operator $Z_{\alpha,\beta}^{\omega,\theta}$ is a generalization of $Z_{\alpha,\beta}^{\omega}$,and also, $E_{\alpha,\beta}, \boxplus_{\alpha,\beta}, \boxtimes_{\alpha,\beta}$.

Definition 1.11. [5]Let G be a groupoid, $A \in IFS(G)$.If for all $x, y \in G$,

$$A(xy) \geq \min(A(x), A(y))$$

then A called an intuitionistic fuzzy subgrupoid over G.

Definition 1.12. [6]Let G be a grupoid, $A \in IFS(G)$. If for all $x, y \in G$,

$$A(xy) \geq \max(A(x), A(y))$$

then A called an intuitionistic fuzzy ideal over G, shortly $IFI(G)$.

Definition 1.13. [6]Let G be a grup and $A \in IFS(G)$ a grupoid. If for all $x \in G$,

$$A(x^{-1}) \geq A(x)$$

then A called an intuitionistic fuzzy subgroup over G,shortly $IFG(G)$.

2. MAIN RESULTS

Theorem 2.1. *Let G be a groupoid and $A \in IFS(G)$.*

(1) If $A \in IFI(G)$ then $\Box A \in IFI(G)$
(2) If $A \in IFI(G)$ then $\Diamond A \in IFI(G)$

Proof. (1)For $x, y \in G$,

$$\mu_{\Box A}(xy) = \mu_A(xy) \geq \mu_A(x) \vee \mu_A(y)$$

and

$$\begin{aligned}\nu_{\Box A}(xy) &= 1 - \mu_A(xy) \leq (1 - \mu_A(x)) \wedge (1 - \mu_A(y)) \\ &= \nu_{\Box A}(x) \wedge \nu_{\Box A}(y)\end{aligned}$$

So,

$$\Box A(xy) \geq \Box A(x) \vee \Box A(y)$$

$\Box$

Theorem 2.2. *Let G be a groupoid and $A \in IFS(G)$.*

(1) If $A \in IFI(G)$ then $\boxplus(A) \in IFI(G)$
(2) If $A \in IFI(G)$ then $\boxtimes(A) \in IFI(G)$

Proof. (1)For $x, y \in G$,

$$\begin{aligned} \mu_{\boxplus(A)}(xy) &= \frac{\mu_A(xy)}{2} \geq \frac{\mu_A(x)}{2} \vee \frac{\mu_A(y)}{2} \\ &= \mu_{\boxplus(A)}(x) \vee \mu_{\boxplus(A)}(y) \end{aligned}$$

and

$$\begin{aligned} \nu_{\boxplus(A)}(xy) &= \frac{\nu_A(xy)+1}{2} \leq \frac{\nu_A(x)+1}{2} \wedge \frac{\nu_A(y)+1}{2} \\ &= \nu_{\boxplus(A)}(x) \wedge \nu_{\boxplus(A)}(y) \end{aligned}$$

So,

$$\boxplus(A)(xy) \geq \boxplus(A)(x) \vee \boxplus(A)(y)$$

$\square$

Theorem 2.3. *Let G be a groupoid and $A \in IFS(G)$.*

   (1) If $A \in IFI(G)$ then $\boxplus_\alpha(A) \in IFI(G)$
   (2) If $A \in IFI(G)$ then $\boxtimes_\alpha(A) \in IFI(G)$

Proof. (1)For $x, y \in G$,

$$\begin{aligned} \mu_{\boxtimes_\alpha(A)}(xy) &= \alpha\mu_A(xy)+1-\alpha \geq (\alpha\mu_A(x)+1-\alpha) \vee (\alpha\mu_A(y)+1-\alpha) \\ &= \mu_{\boxtimes_\alpha(A)}(x) \vee \mu_{\boxtimes_\alpha(A)}(y) \end{aligned}$$

and

$$\begin{aligned} \nu_{\boxtimes_\alpha(A)}(xy) &= \alpha\nu_A(xy) \leq (\alpha\nu_A(x)) \wedge (\alpha\nu_A(y)) \\ &= \nu_{\boxtimes_\alpha(A)}(x) \vee \nu_{\boxtimes_\alpha(A)}(y) \end{aligned}$$

So,

$$\boxtimes_\alpha(A)(xy) \geq \boxtimes_\alpha(A)(x) \vee \boxtimes_\alpha(A)(y)$$

$\square$

Theorem 2.4. *Let G be a groupoid and $A \in IFS(G)$.*

   (1) If $A \in IFI(G)$ then $\boxplus_{\alpha,\beta}(A) \in IFI(G)$
   (2) If $A \in IFI(G)$ then $\boxtimes_{\alpha,\beta}(A) \in IFSI(G)$
   (3) If $A \in IFI(G)$ then $\boxplus_{\alpha,\beta,\gamma}(A) \in IFI(G)$
   (4) If $A \in IFI(G)$ then $\boxtimes_{\alpha,\beta,\gamma}(A) \in IFI(G)$

Proof. For $x, y \in G$,

$$\begin{aligned} \mu_{\boxplus_{\alpha,\beta,\gamma}(A)}(xy) &= \alpha\mu_A(xy) \geq \alpha\mu_A(x) \vee \alpha\mu_A(y) \\ &= \mu_{\boxplus_{\alpha,\beta,\gamma}(A)}(x) \vee \mu_{\boxplus_{\alpha,\beta,\gamma}(A)}(y) \end{aligned}$$

and

$$\begin{aligned} \nu_{\boxplus_{\alpha,\beta,\gamma}(A)}(xy) &= \beta\nu_A(xy)+\gamma \leq (\beta\nu_A(x)+\gamma) \wedge (\beta\nu_A(y)+\gamma) \\ &= \nu_{\boxplus_{\alpha,\beta,\gamma}(A)}(x) \wedge \nu_{\boxplus_{\alpha,\beta,\gamma}(A)}(y) \end{aligned}$$

So,

$$\boxplus_{\alpha,\beta,\gamma}(A)(xy) \geq \boxplus_{\alpha,\beta,\gamma}(A)(x) \vee \boxplus_{\alpha,\beta,\gamma}(A)(y)$$

The other properties can proof with same way. $\square$

Theorem 2.5. *Let G be a groupoid and $A \in IFS(G)$ an ideal then $E_{\alpha,\beta}(A) \in IFS(G)$ is an ideal.*

Proof. For $x, y \in G$,

$$
\begin{aligned}
\mu_{E_{\alpha,\beta}(A)}(xy) &= \beta(\alpha\mu_A(xy) + 1 - \alpha) \geq \beta(\alpha\mu_A(x) + 1 - \alpha) \vee \beta(\alpha\mu_A(y) + 1 - \alpha) \\
&= \mu_{E_{\alpha,\beta}(A)}(x) \vee \mu_{E_{\alpha,\beta}(A)}(y)
\end{aligned}
$$

and

$$
\begin{aligned}
\nu_{E_{\alpha,\beta}(A)}(xy) &= \alpha(\beta\nu_A(xy) + 1 - \beta) \leq \alpha(\beta\nu_A(x) + 1 - \beta) \wedge \alpha(\beta\nu_A(y) + 1 - \beta) \\
&= \nu_{E_{\alpha,\beta}(A)}(x) \wedge \nu_{E_{\alpha,\beta}(A)}(y)
\end{aligned}
$$

So,

$$
E_{\alpha,\beta}(A)(xy) \geq E_{\alpha,\beta}(A)(x) \vee E_{\alpha,\beta}(A)(y)
$$

$\square$

Theorem 2.6. *Let G be a groupoid and $A \in IFS(G)$ an ideal then $\boxdot_{\alpha,\beta,\gamma,\delta}(A) \in IFS(G)$ is an ideal.*

Proof. For $x, y \in G$,

$$
\begin{aligned}
\mu_{\boxdot_{\alpha,\beta,\gamma,\delta}(A)}(xy) &= \alpha\mu_A(xy) + \gamma \geq (\alpha\mu_A(x) + \gamma) \vee (\alpha\mu_A(y) + \gamma) \\
&= \mu_{\boxdot_{\alpha,\beta,\gamma,\delta}(A)}(x) \vee \mu_{\boxdot_{\alpha,\beta,\gamma,\delta}(A)}(y)
\end{aligned}
$$

and

$$
\begin{aligned}
\nu_{\boxdot_{\alpha,\beta,\gamma,\delta}(A)}(xy) &= \beta\nu_A(xy) + \delta \leq (\beta\nu_A(x) + \delta) \wedge (\beta\nu_A(y) + \delta) \\
&= \nu_{\boxdot_{\alpha,\beta,\gamma,\delta}(A)}(x) \wedge \nu_{\boxdot_{\alpha,\beta,\gamma,\delta}(A)}(y)
\end{aligned}
$$

So,$\boxdot_{\alpha,\beta,\gamma,\delta}(A)(xy) \geq \boxdot_{\alpha,\beta,\gamma,\delta}(A)(x) \vee \boxdot_{\alpha,\beta,\gamma,\delta}(A)(y)$

$\square$

Theorem 2.7. *Let G be a groupoid and $A \in IFS(G)$ an ideal then $Z_{\alpha,\beta}^{\omega,\theta}(A) \in IFS(G)$ is an ideal.*

Proof. For $x, y \in G$,

$$
\begin{aligned}
\mu_{Z_{\alpha,\beta}^{\omega,\theta}(A)}(xy) &= \beta(\alpha\mu_A(xy) + \omega - \omega.\alpha) \geq \beta(\alpha\mu_A(x) + \omega - \omega.\alpha) \vee \beta(\alpha\mu_A(y) + \omega - \omega.\alpha) \\
&= \mu_{Z_{\alpha,\beta}^{\omega,\theta}(A)}(x) \vee \mu_{Z_{\alpha,\beta}^{\omega,\theta}(A)}(y)
\end{aligned}
$$

and

$$
\begin{aligned}
\nu_{Z_{\alpha,\beta}^{\omega,\theta}(A)}(xy) &= \alpha(\beta\nu_A(xy) + \theta - \theta.\beta) \leq \alpha(\beta\nu_A(x) + \theta - \theta.\beta) \wedge \alpha(\beta\nu_A(y) + \theta - \theta.\beta) \\
&= \nu_{Z_{\alpha,\beta}^{\omega,\theta}(A)}(x) \wedge \nu_{Z_{\alpha,\beta}^{\omega,\theta}(A)}(y)
\end{aligned}
$$

Therefore, we obtain $Z_{\alpha,\beta}^{\omega,\theta}(A)(xy) \geq Z_{\alpha,\beta}^{\omega,\theta}(A)(x) \vee Z_{\alpha,\beta}^{\omega,\theta}(A)(y)$.

$\square$

Theorem 2.8. *Let G be a group and $A \in IFS(G)$.*

(1) If $A \in IFG(G)$ then $\Box A \in IFG(G)$.
(2) If $A \in IFG(G)$ then $\Diamond A \in IFG(G)$.

Proof. It is clear that, if $A \in IFG(G)$ then it means $A \in IFI(G)$ and for all $x \in G$, $A(x^{-1}) \geq A(x)$.

So, it will be enough to prove the correctness of the second condition.

(2)For $x \in G$

$$
\mu_{\Diamond A}(x^{-1}) = 1 - \nu_A(x^{-1}) \geq 1 - \nu_A(x) = \mu_{\Diamond A}(x)
$$

and

$$
\nu_{\Diamond A}(x^{-1}) = \nu_A(x^{-1}) \leq \nu_A(x) = \nu_{\Diamond A}(x)
$$

The other property can be proved same way. $\square$

Theorem 2.9. *Let G be a group and $A \in IFS(G)$.*

(1) If $A \in IFG(G)$ then $\boxplus(A) \in IFG(G)$
(2) If $A \in IFG(G)$ then $\boxtimes(A) \in IFG(G)$

Proof. (2)For $x, y \in G$,If $A \in IFG(G)$ then $\boxtimes(A) \in IFI(G)$. So, $\boxtimes(A)(xy) \geq \boxtimes(A)(x) \wedge \boxtimes(A)(y)$.

Now,

$$\mu_{\boxtimes(A)}(x^{-1}) = \frac{\mu_A(x^{-1}) + 1}{2} \geq \frac{\mu_A(x) + 1}{2} = \mu_{\boxtimes(A)}(x)$$

and

$$\nu_{\boxtimes(A)}(x^{-1}) = \frac{\nu_A(x^{-1})}{2} \leq \frac{\nu_A(x)}{2} = \nu_{\boxtimes(A)}(x)$$

Therefore,

$$\boxtimes(A)(x^{-1}) \geq \boxtimes(A)(x)$$

$\square$

Theorem 2.10. *Let G be a group and $A \in IFS(G)$.*

(1) If $A \in IFG(G)$ then $\boxplus_\alpha(A) \in IFG(G)$
(2) If $A \in IFG(G)$ then $\boxtimes_\alpha(A) \in IFG(G)$

Proof. (1)For $x, y \in G$, it is clear that $\boxplus_\alpha(A)(xy) \geq \boxplus_\alpha(A)(x) \wedge \boxplus_\alpha(A)(y)$. On the other hand,

$$\mu_{\boxplus_\alpha(A)}(x^{-1}) = \alpha\mu_A(x^{-1}) \geq \alpha\mu_A(x) = \mu_{\boxplus_\alpha(A)}(x)$$

and

$$\nu_{\boxplus_\alpha(A)}(x^{-1}) = \alpha\nu_A(x^{-1}) + 1 - \alpha \leq \alpha\nu_A(x) + 1 - \alpha = \nu_{\boxplus_\alpha(A)}(x)$$

So,

$$\boxplus_\alpha(A)(x^{-1}) \geq \boxplus_\alpha(A)(x)$$

$\square$

Theorem 2.11. *Let G be a group and $A \in IFS(G)$.*

(1) If $A \in IFG(G)$ then $\boxplus_{\alpha,\beta}(A) \in IFG(G)$
(2) If $A \in IFG(G)$ then $\boxtimes_{\alpha,\beta}(A) \in IFG(G)$
(3) If $A \in IFG(G)$ then $\boxplus_{\alpha,\beta,\gamma}(A) \in IFG(G)$
(4) If $A \in IFG(G)$ then $\boxtimes_{\alpha,\beta,\gamma}(A) \in IFG(G)$

Proof. For $x, y \in G$,

$$\mu_{\boxtimes_{\alpha,\beta,\gamma}(A)}(x^{-1}) = \alpha\mu_A(x^{-1}) + \gamma \geq \alpha\mu_A(x) + \gamma = \mu_{\boxtimes_{\alpha,\beta,\gamma}(A)}(x)$$

and

$$\nu_{\boxtimes_{\alpha,\beta,\gamma}(A)}(x^{-1}) = \beta\nu_A(x^{-1}) \leq \beta\nu_A(x) = \nu_{\boxtimes_{\alpha,\beta,\gamma}(A)}(x)$$

So,

$$\boxtimes_{\alpha,\beta,\gamma}(A)(x^{-1}) \geq \boxtimes_{\alpha,\beta,\gamma}(A)(x)$$

The other properties can proof with same way. $\square$

Theorem 2.12. *Let G be a group and $A \in IFS(G)$. If A is an intuitionistic fuzzy subgroup on G then $E_{\alpha,\beta}(A) \in IFG(G)$.*

Proof. It is clear that for $x, y \in G$, $E_{\alpha,\beta}(A)(xy) \geq E_{\alpha,\beta}(A)(x) \wedge E_{\alpha,\beta}(A)(y)$.

$$
\begin{aligned}
\mu_{E_{\alpha,\beta}(A)}(x^{-1}) &= \beta(\alpha\mu_A(x^{-1}) + 1 - \alpha) \geq \beta(\alpha\mu_A(x) + 1 - \alpha) \\
&= \mu_{E_{\alpha,\beta}(A)}(x)
\end{aligned}
$$

and

$$
\begin{aligned}
\nu_{E_{\alpha,\beta}(A)}(x^{-1}) &= \alpha(\beta\nu_A(x^{-1}) + 1 - \beta) \leq \alpha(\beta\nu_A(x) + 1 - \beta) \\
&= \nu_{E_{\alpha,\beta}(A)}(x)
\end{aligned}
$$

So, $E_{\alpha,\beta}(A) \in IFG(G)$. $\hfill\square$

Theorem 2.13. *Let G be a group and $A \in IFS(G)$ an intuitionistic fuzzy group then $\boxdot_{\alpha,\beta,\gamma,\delta}(A) \in IFS(G)$ is an intuitionistic fuzzy subgroup.*

Proof. For $x \in G$,

$$
\begin{aligned}
\mu_{\boxdot_{\alpha,\beta,\gamma,\delta}(A)}(x^{-1}) &= \alpha\mu_A(x^{-1}) + \gamma \geq \alpha\mu_A(x) + \gamma \\
&= \mu_{\boxdot_{\alpha,\beta,\gamma,\delta}(A)}(x)
\end{aligned}
$$

and

$$
\begin{aligned}
\nu_{\boxdot_{\alpha,\beta,\gamma,\delta}(A)}(x^{-1}) &= \beta\nu_A(x^{-1}) + \delta \leq \beta\nu_A(x) + \delta \\
&= \nu_{\boxdot_{\alpha,\beta,\gamma,\delta}(A)}(x)
\end{aligned}
$$

Therefore $\boxdot_{\alpha,\beta,\gamma,\delta}(A) \in IFG(G)$. $\hfill\square$

Theorem 2.14. *Let G be a group and $A \in IFS(G)$. If A is an intuitionistic fuzzy subgroup on G then $Z_{\alpha,\beta}^{\omega,\theta}(A) \in IFG(G)$.*

Proof. It can shown easily. $\hfill\square$

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STABILITY ANALYSIS ON EFFECT OF SYSTEM RESTORE ON EPIDEMIC MODEL FOR COMPUTER VIRUSES

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Abstract. More than 317 million new pieces of malware computer viruses or other malicious software were created last year. That means nearly one million new threats were released each day. Every year computer viruses cost homes and businesses billions of dollars in lost time and equipment. Computer viruses are continually evolving and their structures increasingly becoming more complex and transmission capabilities are becoming more powerful. So, we consider a SEIS model to demonstrate the system restore has a more effective role than antivirus softwares on virus defense. Also we have investigated the global behavior of the endemic equilibrium and we have supported our results with numerical simulation.

1. Introduction

Epidemiology is an area of medicine concerned with the identification of factors and conditions associated with the spread of an infectious process in a community. Because of a virus programs behavior is similar to the infectious process, this areas detects and strategies that may be useful for us [1]. Biological viruses enter their host through an opening after passively being breathed in, swallowed or via direct contact. Virtual viruses also enter their host passively when you insert an infected disk or open an infected e-mail attachment. Similarly to a biological virus which has to have the correct host and tissue specicity to gain a foothold a horse virus would not make a human being sick a computer virus has to be compatible with the system to gain a foothold. The damage these viruses do is also similar. Biological viruses replicate at the cost of the host damage can include pain, suffering and even death. Computer viruses slow down the computer files can become inaccessible and even lost, and sometimes the complete hard disk gets damaged [2]. Community, population, carrier, portal of entry, vector, symptom, modes of transmission, extra-host survival, immunity, susceptibility, sub-clinical, indicator, effective transfer rate, quarantine, isolation, infection, medium and culture are all terms from epidemiology that are useful in understanding and fighting computer viruses [1].

A computer virus is a program that can infect other programs by modifying them to include a possibly evolved version of itself. With this infection property, a virus can spread to the transitive closure of information flow, corrupting the integrity of information as it spreads. Given the widespread use of sharing in current computer systems, the threat of a virus causing widespread integrity corruption is significant [3]. Viruses, worms and trojans are all part of a class of software called malware. Malware or malicious code (malcode) is short for malicious software. It is code or software that is specifically designed to damage, disrupt, steal, or in general inflict some other bad or illegitimate action on data, hosts, or networks. A computer virus is a type of malware that propagates by inserting a copy of itself into and becoming part of another program. It spreads from one computer to another, leaving infections as it travels. Viruses can range in severity from causing mildly annoying effects to damaging data or software and causing denial-of-service conditions. Almost all viruses are attached to an executable file, which means the virus may exist on a system but will not be active or able to spread until a user runs or opens the malicious host file or program. When the host code is executed, the viral code is executed as well. Normally, the host program keeps functioning after it is infected by the virus. However, some viruses overwrite other programs with copies of themselves, which destroys the host program altogether. Viruses spread when the software or document they are attached to is transferred from one computer to another using the network, a disk, file sharing, or infected e-mail attachments.

The proper assessment of computer viruses in the management of information security and integrity depends on estimates of the risk and impact of computer virus incidents and an analysis of how they are influenced by various factors in the computing environment. Mathematical or computer simulation models of the transmission and control of computer viruses can be useful in synthesizing available information and providing a theoretical basis for control strategies [4]. Predicting virus outbreaks is extremely difficult due to human natre of the attacks but more importantly, detecting outbreaks early with a low probability of false alarms seems quiet difficult [5]. By developing models it is possible to characterize essential properties of the attacks [6]. Consequently, anti-virus software has been developed to take precautions. In order to understanding the effectiveness of the antivirus technologies, numbers of mathematical models were suggested to investigate the epidemic behaviors of computer virus. Due to analogical similarity between the computer viruses and infectious diseases biological counterparts, several propagation models of computer viruses have been proposed, and the obtained results indicate that the long-term behavior of computer virus can be predicted. One of the early triumphs of mathematical epidemiology was the formulation of a simple model that predicted behaviour very similar to the behaviour observed in countless epidemics [7]. The Kermack McKendrick model is a compartmental model based on relatively simple assumptions on the rates of flow between different classes of members of the population [8]. Thenceforward many computer viruses modelling studies have been made. These are; SI models [9-11], SIS models [12-14], SIR models [15-20], SIRS models [21-26], SAIC models [27], SEIR models [28], SEIQR-SEIQRS models [29-32], SLBS models [33-36], and some other models [37-41], have been proposed that every compartment which are Susceptible computers, Infected computers including the latent and breaking-out computers based on some

models, Recovered computers including the quarantine computers based on some models are considered to have the same connecting and disconnecting constants, on some models for the latent and breaking-out computers have the same infecting, corrupting and recovered constants When considering today's conditions, it is clear that this situation is how inadequate. That's what we have done in this study, eliminating this deficiency we tried to develop an advanced epidemic model by thinking total of 14 separate constants for connecting, disconnecting, infecting, recovering and corrupting to the each compartment, instead of previous models have been constructed by 4-5 parametres. Furthermore, we optimized our model for the present day including the effect of system restore which is not considered so far.

So, we consider a SEIS model (see also Fig.1), where the compartments are: $S(t)$ susceptible, $E(t)$ exposed, $I(t)$ infective at time t, respectively. The parameters of the model are defined as:

D1.: Every computer connects to the Internet with constant rate $\alpha > 0$. Respectively positive constant rates for each compartments are: α_1 for Susceptible, α_2 for Exposed, α_3 for Infective. Let $\alpha = \alpha_1 + \alpha_2 + \alpha_3$.

D2.: Every computer disconnects from the Internet with constant rate $\delta > 0$. Respectively positive constant rates for each compartments are: δ_1 for Susceptible,δ_2 for Exposed and δ_3 for Infective. Let $\delta = \delta_1 + \delta_2 + \delta_3$.

D3.: Every susceptible computer is infected with constant rate $\beta > 0$ by infected removable storage media.

D4.: Every susceptible computer is infected by exposed computers with constant rate $\gamma_1 > 0$ and infective computers with constant rate $\gamma_2 > 0$, where $\gamma = \gamma_1 + \gamma_2$.

D5.: Every exposed computer is transformed into infective computer with constant rate $\eta > 0$, and recovered with constant rate $\theta_1 > 0$.

D6.: Every infective computer is recovered with constant rate $\theta_2 > 0$.

D7.: Every infective computer is returned to susceptible computer with constant rate $\mu_1 > 0$, and returned to exposed computer with constant rate $\mu_2 > 0$ by using system restore. Let $\mu = \mu_1 + \mu_2$.
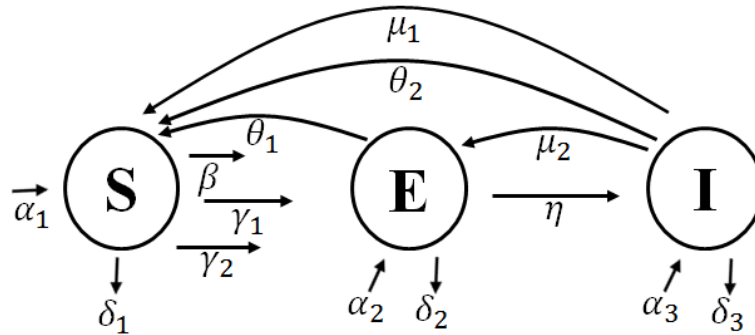
FIGURE 1. The SEIS Model

From the definitions, model can be shown by:

$$S'(t) = \alpha_1 + \theta_1 E(t) + (\theta_2 + \mu_1) I(t) - (\beta + \gamma_1 E(t) + \gamma_2 I(t) + \delta_1) S(t)$$

(1.1) $$E'(t) = \alpha_2 + (\beta + \gamma_1 E(t) + \gamma_2 I(t)) S(t) - (\theta_1 + \eta + \delta_2) E(t) + \mu_2 I(t)$$

$$I'(t) = \alpha_3 + \eta E(t) - (\theta_2 + \mu_1 + \mu_2 + \delta_3) I(t)$$

with the initial conditions $(S(0), E(0), I(0)) \in R_+^3$. Let $N(t) = S(t) + E(t) + I(t)$. Summing the system and simplifying, we get $\frac{dN(t)}{dt} = \alpha - \delta N(t)$, it is easy to get $\lim_{t \to \infty} N(t) = \frac{\alpha}{\delta}$. Hence, the following system would be obtained by system (1.1):

(1.2)
$$E' = \alpha_2 + (\beta + \gamma_1 E + \gamma_2 I)(N - E - I) - (\theta_1 + \eta + \delta_2) E + \mu_2 I$$
$$I' = \alpha_3 + \eta E - (\theta_2 + \mu_1 + \mu_2 + \delta_3) I$$

with initial conditions $(E(0), I(0)) \in R_+^2$. It is easy to verify that $\Delta = \{(E, I) | E, I \geq 0, N \geq E + I\}$ is positively invariant for the system (1.2). As thus, we will consider the global stability of (1.2) on the set Δ. Let us shortly overview the theory of asymptotically autonomous systems. An ordinary differential equation in R^n,

(1.3) $$x = f(t, x)$$

is called asymptotically autonomous with limit equation

(1.4) $$y = g(y)$$

if $f(t, x) \to g(x)$, $t \to \infty$, locally uniformly in $x \in R^n$, i.e. for x in any compact subset of R^n. For simplicity we assume that $f(t, x), g(x)$ are continuous functions and locally Lipschitz in x. The $\omega-$limit sets, $\omega(t_0, x_0)$, of forward bounded solutions x to (1.3), subject to $x(t_0) = x_0$ $x \in \omega(t_0, x_0) \Leftrightarrow x = \lim_{j \to \infty} x(t_j)$ for some sequence $t_j \to \infty$.

Theorem 1.1. *Let $n = 2$ and ω be the $\omega-$ limit set of a forward bounded solution x of the asymptotically autonomous system (1.3). Assume that there exists a neighborhood of ω which contains at most finitely many equilibria of system (1.4). Then the following trichotomy holds:*

- *ω consists of an equilibrium of system (1.4).*
- *ω is the union of periodic orbits of system (1.4) and possibly of centers of system (1.4) that are surrounded by periodic orbits of system (1.4) lying in ω.*
- *ω contains equilibria of system (1.4) that are cyclically chained to each other in ω orbits of system (1.4).*

It follows from the Thieme's Theorem and the subsequent study that, for the purpose of understanding the behavior of the original system (1.1), it suffices to investigate its limit system (1.2) [42].

2. MODEL ANALYSIS

2.1. **Equilibrium Point.**

Theorem 2.1. *Assume* $(\gamma_1 + \gamma_2 b)\left(\frac{\alpha}{\delta} - a\right) + \mu_2 b > (b+1)(\beta + \gamma_2 a) + (\theta_1 + \eta + \delta_2).$

Then the system (1.1) has a unique equilibrium point $\bar{E} = (S_*, E_*, I_*)$*, where*

$$\text{(2.1)} \qquad S_* = N_* - (E_* + I_*)$$

$$\text{(2.2)} \qquad E_* = \frac{-m_1 - \sqrt{m_1^2 - 4m_0 m_2}}{2m_0}$$

$$\text{(2.3)} \qquad I_* = \frac{\alpha_3 + \eta E_*}{\theta_2 + \mu + \delta_3}$$

$$\text{(2.4)} \qquad a = \frac{\alpha_3}{\theta_2 + \mu + \delta_3}$$

$$\text{(2.5)} \qquad b = \frac{\eta}{\theta_2 + \mu + \delta_3}$$

$$\text{(2.6)} \qquad m_0 = -(b+1)(\gamma_1 + \gamma_2 b)$$

$$\text{(2.7)} \qquad m_1 = (\gamma_1 + \gamma_2 b)\left(\frac{\alpha}{\delta} - a\right) - (b+1)(\beta + \gamma_2 a) - (\theta_1 + \eta + \delta_2) + \mu_2 b$$

$$\text{(2.8)} \qquad m_2 = \alpha_2 + (\beta + \gamma_2 a)\left(\frac{\alpha}{\delta} - a\right) + \mu_2 a$$

Furthermore, $S_* + E_* + I_* > 0.$

Proof. 1. *If we solve the system,*

$$\alpha - \delta(S + E + I) = 0$$

$$\text{(2.9)} \qquad \alpha_2 + (\beta + \gamma_1 E + \gamma_2 I)\left(\frac{\alpha}{\delta} - E - I\right) - (\theta_1 + \eta + \delta_2) E + \mu_2 I = 0$$

$$\alpha_3 + \eta E - (\theta_2 + \mu + \delta_3) I = 0$$

From the third equation of system (2.9), we can get $I_* = \dfrac{\alpha_3 + \eta E_*}{\theta_2 + \mu + \delta_3}$. *Substituting this equation into the second equation of system (2.9) and rearranging terms we have,*

$$[-(b+1)(\gamma_1 + \gamma_2 b)] E_*^2 + \left[(\gamma_1 + \gamma_2 b)\left(\frac{\alpha}{\delta} - a\right) - (b+1)(\beta + \gamma_2 a)\right.$$

$$\left. - (\theta_1 + \eta + \delta_2) + \mu_2 b\right] E_* + \alpha_2 + (\beta + \gamma_2 a)\left(\frac{\alpha}{\delta} - a\right) + \mu_2 a = 0$$

where $a = \dfrac{\alpha_3}{\theta_2 + \mu + \delta_3} > 0,\ b = \dfrac{\eta}{\theta_2 + \mu + \delta_3} > 0.$ *If we substituting equations* $(2.6), (2.7), (2.8)$ *into this equation, we have*

$$\text{(2.10)} \qquad\qquad m_0 E_*^2 + m_1 E_* + m_2 = 0$$

where $m_0 = -(b+1)(\gamma_1 + \gamma_2 b) < 0,$ *and we can get,*

$$m_1 = (\gamma_1 + \gamma_2 b)\left(\frac{\alpha}{\delta} - a\right) - (b+1)(\beta + \gamma_2 a) - (\theta_1 + \eta + \delta_2) + \mu_2 b > 0,$$

from our assuming

$$m_2 = \alpha_2 + (\beta + \gamma_2 a)\left(\frac{\alpha}{\delta} - a\right)\mu_2 a > 0,$$

Let the discriminant of (2.10) be $\Delta = m_1^2 - 4m_0 m_2 > 0.$ *So,* $E_* = \frac{-m_1 - \sqrt{\Delta}}{2m_0} > 0.$

That is shows us $\bar{E} = (S_*, E_*, I_*)$ *is the unique equilibrium of the system (1.1). It is trivial to verify that* $E_* + I_* > 0.$

2.2. **Stability Analysis.**

Lemma 1. *The equilibrium point $\bar{E}$ is locally asymptotically stable.*

Proof. 2. *According to Theorem* (1.1),

$$N' = \alpha - \delta N,$$

(2.11) $\quad E' = \alpha_2 + (\beta + \gamma_1 E + \gamma_2 I)(N - E - I) - (\theta_1 + \eta + \delta_2) E + \mu_2 I$

$$I' = \alpha_3 + \eta E - (\theta_2 + \mu_1 + \mu_2 + \delta_3) I$$

The dynamical system (2.11) *has a unique equilibrium point $\bar{E} = (N_*, E_*, I_*)$ and locally asymptotically stable at $\bar{E}$. The Jacobian matrix of the linearized system of system* (2.11) *evaluated at $\bar{E}$ is*

$$J = \begin{pmatrix} -\delta & 0 & 0 \\ j_{21} & j_{22} & j_{23} \\ 0 & -\eta & -(\theta_2 + \mu + \delta_3) \end{pmatrix}$$

where

(2.12)
$$\begin{aligned}
j_{21} &= \beta + \gamma_1 E_* + \gamma_2 I_*, \\
j_{22} &= -(\beta + \gamma_1 E_* + \gamma_2 I_*) + \gamma_1 (N_* - E_* - I_*) - (\theta_1 + \eta + \delta_2), \\
j_{23} &= -(\beta + \gamma_1 E_* + \gamma_2 I_*) + \gamma_2 (N_* - E_* - I_*) + \mu_2,
\end{aligned}$$

in order to determine the characteristic equation,

$$\det |\lambda I_n - J| = \begin{vmatrix} \lambda + \delta & 0 & 0 \\ j_{21} & \lambda - j_{22} & i_{23} \\ 0 & \eta & \lambda + (\theta_2 + \mu + \delta_3) \end{vmatrix},$$

$$(\lambda + \delta)\left[(\lambda - j_{22})(\lambda + \theta_2 + \mu + \delta_3) - \eta j_{23}\right] = 0$$

Thus the characteristic polynomial is

(2.13) $\qquad h(\lambda) = (\lambda + \delta)\left(n_2 \lambda^2 + n_1 \lambda + n_0\right),$

where

$$\begin{aligned}
n_2 &= 1, \\
n_1 &= \theta_2 + \mu + \delta_3 + (\beta + \gamma_1 E_* + \gamma_2 I_*) - \gamma_1 (N_* - E_* - I_*) + (\theta_1 + \eta + \delta_2), \\
n_0 &= (\theta_2 + \mu + \delta_3 + \eta)(\beta + \gamma_1 E_* + \gamma_2 I_*) - (\gamma_1 (\theta_2 + \mu + \delta_3) + \eta \gamma_2)(N_* - E_* - I_*) \\
&\quad + (\theta_2 + \mu + \delta_3)(\theta_1 + \eta + \delta_2) - \eta \mu_2,
\end{aligned}$$

it is obvious that

$$\alpha_2 + (\beta + \gamma_1 E_* + \gamma_2 I_*) S_* - (\theta_1 + \eta + \delta_2) E_* + \mu_2 I_* = 0$$

hence

$$\alpha_2 + \beta S_* + \gamma_1 S_* E_* + \gamma_2 S_* \frac{\alpha_3 + \eta E_*}{\theta_2 + \mu + \delta_3} - (\theta_1 + \eta + \delta_2) E_* + \mu_2 \frac{\alpha_3 + \eta E_*}{\theta_2 + \mu + \delta_3} = 0,$$

we have

$$\gamma_1 S_* + \gamma_2 S_* \frac{\eta}{\theta_2 + \mu + \delta_3} < \theta_1 + \eta + \delta_2$$

and

$$\gamma_1 S_* + \gamma_2 S_* \frac{\eta}{\theta_2 + \mu + \delta_3} + \mu_2 \frac{\eta}{\theta_2 + \mu + \delta_3} < \theta_1 + \eta + \delta_2$$

it is,

$$S_* < \frac{(\theta_1 + \eta + \delta_2)(\theta_2 + \mu + \delta_3) - \eta\mu_2}{\gamma_1(\theta_2 + \mu + \delta_3) + \gamma_2\eta}$$

from here,

$$
\begin{aligned}
n_1 &= \theta_2 + \mu + \delta_3 + (\beta + \gamma_1 E_* + \gamma_2 I_*) - \gamma_1 S_* + (\theta_1 + \eta + \delta_2) \\
&> \theta_2 + \mu + \delta_3 + (\beta + \gamma_1 E_* + \gamma_2 I_*) - \frac{\theta_1 + \eta + \delta_2}{1 + \frac{\gamma_2\eta}{\gamma_1(\theta_2 + \mu + \delta_3)}} + \eta\mu_2 + (\theta_1 + \eta + \delta_2) \\
&> \theta_2 + \mu + \delta_3 + (\beta + \gamma_1 E_* + \gamma_2 I_*) + \eta\mu_2 \\
&> 0 \\
n_0 &= (\theta_2 + \mu + \delta_3 + \eta)(\beta + \gamma_1 E_* + \gamma_2 I_*) - (\gamma_1(\theta_2 + \mu + \delta_3) + \eta\gamma_2) S_* \\
&\quad + (\theta_2 + \mu + \delta_3)(\theta_1 + \eta + \delta_2) - \eta\mu_2, \\
&> \eta\left(\beta + \gamma_1 E_* + \gamma_2 \frac{\alpha_3 + \eta E_*}{\theta_2 + \mu + \delta_3}\right) \\
&> 0
\end{aligned}
$$

it follows from the Hurwitz criterion [43], that three roots of (2.13) have negative reel parts. So, the claimed result follows by the Lyapunov theorem [43].

As a consequence of Theorem (2.1) to prove the global stability of the endemic equilibrium point $\bar{E}$ of system (1.1), it suffices to prove the global stability of $\bar{E} = (E_*, I_*)$ for system (1.2). For that purpose, let us establish two lemmas.

Lemma 2. (1.2) *allows no periodic solution in the interior of* Δ.

Proof. 3. *Let,*

$$
\begin{aligned}
f_1(E, I) &= \alpha_2 + (\beta + \gamma_1 E + \gamma_2 I)\left(\frac{\alpha}{\delta} - E - I\right) - (\theta_1 + \eta + \delta_2) E + \mu_2 I \\
f_2(E, I) &= \alpha_3 + \eta E - (\theta_2 + \mu_1 + \mu_2 + \delta_3) I \\
D(E, I) &= \frac{1}{I}.
\end{aligned}
$$

Then

$$
\begin{aligned}
\frac{\partial(Df_1)}{\partial E} + \frac{\partial(Df_2)}{\partial I} &= -\gamma_1\left(1 + \frac{E}{I} - \frac{\alpha}{\delta I}\right) - (\beta + \gamma_1 E + \gamma_2 I) - (\theta_1 + \eta + \delta_2) \\
&\quad - \frac{\alpha_3 + \eta E}{I^2} \\
&< 0.
\end{aligned}
$$

The claimed result follows from the Bendixson-Dulac criterion [43].

Lemma 3. *System* (1.2) *allows no periodic solution that passes through a point on the boundary of* Δ.

Proof. 4. *If there is a periodic solution that passing through a non-corner point on* $\partial\Delta$, *then it must be tangent to* $\partial\Delta$ *at this point. On the contrary, suppose there is a periodic solution* Γ *that passes through a non-corner point* (E, I) *on* $\partial\Delta$. *There are three cases to be considered.*

Case-1:: $0 < E < \frac{\alpha}{\delta}, I = 0$. Then, $I|_{E,I} = \alpha_3 + \eta E > 0$, implying that Γ is not tangent to $\partial \Delta$ at this point, which leads to a contradiction.

Case-2:: $0 < I < \frac{\alpha}{\delta}, E = 0$. Then, $E|_{E,I} = \alpha_2 + (\beta + \gamma_2 I)\left(\frac{\alpha}{\delta} - I\right) + \mu_2 I > 0$, implying that Γ is not tangent to $\partial \Delta$ at this point, which leads to a contradiction.

Case-3:: $E + I = \frac{\alpha}{\delta}, E \neq 0$ and $I \neq 0$. Then, $\frac{d(E+I)}{dt}|_{E,I} = -(\theta_1 + \delta_2) E - (\theta_2 + \mu_1 + \delta_3) I < 0$, implying that Γ is not tangent to $\partial \Delta$ at this point, also a contradiction.

The claimed result follows by combining the above discussions. Hence, the proof is complete.

On this basis, we present

Theorem 2.2. *The equilibrium point $\bar{E}$ is globally asymptotically stable for system (1.2).*

Proof. **5.** *The claimed result follows by combining the generalized Poincarè-Bendixson theorem [43] with lemmas 1-3.*

3. Numerical Examples

This section provides numerical examples for illustrating main result and the effects of System Restore and antivirus software on virus spread. In what follows, observe the asymptotic behavior of system (1.2) with varying $\alpha_1, \alpha_2, \alpha_3, \delta_1, \delta_2, \delta_3, \beta, \eta, \mu_1, \mu_2, \theta_1, \theta_2, \gamma_1$ and γ_2.

In Figs. 2, 3, 4, 5 Evolutions of $S(t), E(t), I(t)$ are performed with constant $\alpha_1 = 0.6, \alpha_2 = 0.2, \alpha_3 = 0.3, \beta = 0.32, \eta = 0.65, \delta_1 = 0.02, \delta_2 = 0.03, \delta_3 = 0.04, \gamma_1 = 0.28, \gamma_2 = 0.42$ and initial conditions $(S(0), E(0), I(0)) = (10, 5, 2)$, and varying μ_1, μ_2, θ_1 and θ_2 which are recover with antivirus software and system restore parameters.

Whether system restore and a scanning with an antivirus software will not be done, the number of infected computers will be equivalent to the number of the total computer in the system shown by Fig.2 with $\mu_1, \mu_2, \theta_1, \theta_2 = 0$. Also the system of computers equilibrium point $\bar{E}_1 = (S_1, E_1, I_1)$ equivalent to $\bar{E}_1 = (0.0574, 1.1753, 23.3037)$ in Fig.**??**.

If only a scanning with an antivirus software will be done, the system which is newly formed is shown in Fig.3 with $\mu_1, \mu_2 = 0$ and $\theta_1 = 0.65, \theta_2 = 0.35$. Also just system restore will be done, the system which is newly formed by new parameters is shown in Fig.4 with $\mu_1 = 0.7, \mu_2 = 0.2$ and $\theta_1, \theta_2 = 0$. And also the system of computers equilibrium point $\bar{E}_2 = (S_2, E_2, I_2)$ equivalent to $\bar{E}_2 = (1.2676, 9.1327, 15.7034)$ in Fig.3 and the system of computers equilibrium point $\bar{E}_3 = (S_3, E_3, I_3)$ equivalent to $\bar{E}_3 = (0.8915, 15.2111, 10.7503)$ in Fig.4.

If system restore and a scanning with an antivirus software will be done, one can easily see in the Fig.5 with $\mu_1 = 0.7, \mu_2 = 0.2$ and $\theta_1 = 0.65, \theta_2 = 0.35$ that
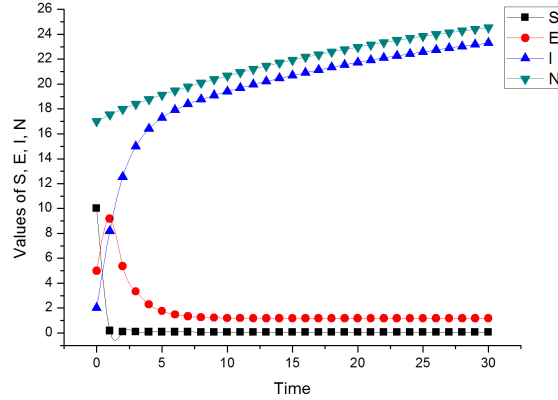
FIGURE 2. Evolutions of S, E, I, N with $\mu_1, \mu_2, \theta_1, \theta_2 = 0$.
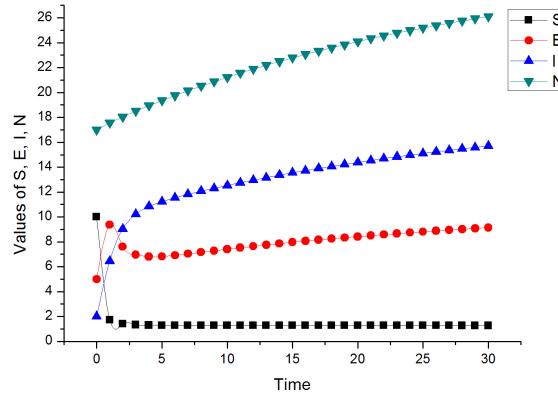
FIGURE 3. Evolutions of S, E, I, N with $\mu_1, \mu_2 = 0$ and $\theta_1 = 0.65, \theta_2 = 0.35$.

quite decreasing of the number of infected computers. And the system of computers equilibrium point $\bar{E}_4 = (S_4, E_4, I_4)$ equivalent to $\bar{E}_4 = (2.3740, 16.6080, 8.5448)$ in Fig.5.

Additionally one can see from Fig. 2, 3, 4, 5 that the equilibrium points $\bar{E}_{1,2,3,4}$ are globally asymptotically stable. From the figures which exhibits solutions $S(t), E(t), I(t)$ are converging to stable state, which is consistent with the main result.

If the System Restore will not done, infected computers will be dominate the system in time. Therefore, total number of computers will be equivalent to infected
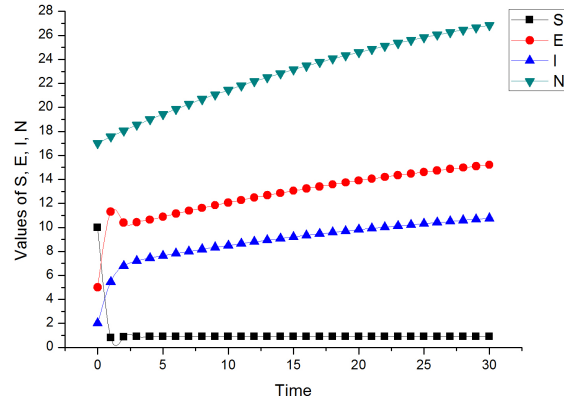
FIGURE 4. Evolutions of S, E, I, N with $\mu_1 = 0.7, \mu_2 = 0.2$ and $\theta_1, \theta_2 = 0$.
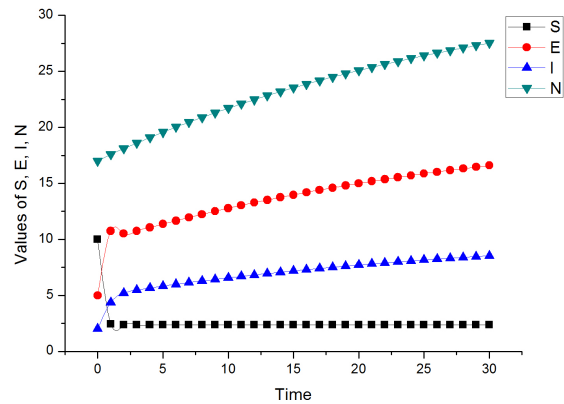
FIGURE 5. Evolutions of S, E, I, N with $\mu_1 = 0.7, \mu_2 = 0.2$ and $\theta_1 = 0.65, \theta_2 = 0.35$.

computers. (Fig. 2).

If the System Restore will done, we will recover non-copy datas in infected computers also total number of exposed and susceptible computers will be increase while infected computers number decreasing. (Fig. 4).

If we use system restore after have a comprehensive virus scanning, we will have prevented the pretty much deterioration of our computer. (Fig. 5).

4. Acknowledgements

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BOUNDEDNESS OF THE SUBLINEAR OPERATORS WITH ROUGH KERNEL GENERATED BY CALDERÓN–ZYGMUND OPERATORS AND THEIR COMMUTATORS ON GENERALIZED VANISHING MORREY SPACES

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Abstract. In this paper, we are interested in the boundedness of sublinear operators with rough kernel generated by Calderón–Zygmund operators on generalized vanishing Morrey spaces and give bounded mean oscillation space estimates for their commutators on these spaces.

1. Introduction

The classical Morrey spaces $M_{p,\lambda}$ have been introduced by Morrey in [32] to study the local behavior of solutions of second order elliptic partial differential equations(PDEs). Later, there are many applications of Morrey space to the Navier-Stokes equations (see [29]), the Schrödinger equations (see [40]) and the elliptic problems with discontinuous coefficients (see [3, 13, 35]).

Let $B = B(x_0, r_B)$ denote the ball with the center x_0 and radius r_B. For a given measurable set E, we also denote the Lebesgue measure of E by $|E|$. For any given $\Omega_0 \subseteq \mathbb{R}^n$ and $0 < p < \infty$, denote by $L_p(\Omega_0)$ the spaces of all functions f satisfying

$$\|f\|_{L_p(\Omega_0)} = \left(\int\limits_{\Omega_0} |f(x)|^p \, dx \right)^{\frac{1}{p}} < \infty.$$

We recall the definition of classical Morrey spaces $M_{p,\lambda}$ as

$$M_{p,\lambda}(\mathbb{R}^n) = \left\{ f : \|f\|_{M_{p,\lambda}(\mathbb{R}^n)} = \sup_{x \in \mathbb{R}^n, r > 0} r^{-\frac{\lambda}{p}} \|f\|_{L_p(B(x,r))} < \infty \right\},$$

where $f \in L_p^{loc}(\mathbb{R}^n)$, $0 \leq \lambda \leq n$ and $1 \leq p < \infty$.

Note that $M_{p,0} = L_p(\mathbb{R}^n)$ and $M_{p,n} = L_\infty(\mathbb{R}^n)$. If $\lambda < 0$ or $\lambda > n$, then $M_{p,\lambda} = \Theta$, where Θ is the set of all functions equivalent to 0 on $\mathbb{R}^n$.

We also denote by $WM_{p,\lambda} \equiv WM_{p,\lambda}(\mathbb{R}^n)$ the weak Morrey space of all functions $f \in WL_p^{loc}(\mathbb{R}^n)$ for which

$$\|f\|_{WM_{p,\lambda}} \equiv \|f\|_{WM_{p,\lambda}(\mathbb{R}^n)} = \sup_{x \in \mathbb{R}^n, r > 0} r^{-\frac{\lambda}{p}} \|f\|_{WL_p(B(x,r))} < \infty,$$

where $WL_p(B(x,r))$ denotes the weak L_p-space of measurable functions f for which

$$
\begin{aligned}
\|f\|_{WL_p(B(x,r))} &\equiv \|f\chi_{B(x,r)}\|_{WL_p(\mathbb{R}^n)} \\
&= \sup_{t > 0} t \, |\{y \in B(x,r) : |f(y)| > t\}|^{1/p} \\
&= \sup_{0 < t \leq |B(x,r)|} t^{1/p} \left(f\chi_{B(x,r)}\right)^* (t) < \infty,
\end{aligned}
$$

Where g^* denotes the non-increasing rearrangement of a function g.

Throughout the paper we assume that $x \in \mathbb{R}^n$ and $r > 0$ and also let $B(x,r)$ denotes the open ball centered at x of radius r, $B^C(x,r)$ denotes its complement and $|B(x,r)|$ is the Lebesgue measure of the ball $B(x,r)$ and $|B(x,r)| = v_n r^n$, where $v_n = |B(0,1)|$.

Morrey has stated that many properties of solutions to PDEs can be attributed to the boundedness of some operators on Morrey spaces. For the boundedness of the Hardy–Littlewood maximal operator, the fractional integral operator and the Calderón–Zygmund singular integral operator on these spaces, we refer the readers to [1, 5, 37]. For the properties and applications of classical Morrey spaces, see [6, 7, 12, 13] and references therein.

The study of the operators of harmonic analysis in vanishing Morrey space, in fact has been almost not touched. A version of the classical Morrey space $M_{p,\lambda}(\mathbb{R}^n)$ where it is possible to approximate by "nice" functions is the so called vanishing Morrey space $VM_{p,\lambda}(\mathbb{R}^n)$ has been introduced by Vitanza in [51] and has been applied there to obtain a regularity result for elliptic PDEs. This is a subspace of functions in $M_{p,\lambda}(\mathbb{R}^n)$, which satisfies the condition

$$\lim_{r \to 0} \sup_{\substack{x \in \mathbb{R}^n \\ 0 < t < r}} t^{-\frac{\lambda}{p}} \|f\|_{L_p(B(x,t))} = 0.$$

Later in [52] Vitanza has proved an existence theorem for a Dirichlet problem, under weaker assumptions than in [30] and a $W^{3,2}$ regularity result assuming that the partial derivatives of the coefficients of the highest and lower order terms belong to vanishing Morrey spaces depending on the dimension. Also Ragusa has proved a sufficient condition for commutators of fractional integral operators to belong to vanishing Morrey spaces $VM_{p,\lambda}(\mathbb{R}^n)$ (see [38, 39]). For the properties and applications of vanishing Morrey spaces, see also [4]. It is known that, there is no research regarding boundedness of the sublinear operators with rough kernel on vanishing Morrey spaces.

Maximal functions and singular integrals play a key role in harmonic analysis since maximal functions could control crucial quantitative information concerning the given functions, despite their larger size, while singular integrals, Hilbert transform as it's prototype, recently intimately connected with PDEs, operator theory and other fields.

Let $f \in L^{loc}(\mathbb{R}^n)$. The Hardy-Littlewood(H–L) maximal operator M is defined by

$$Mf(x) = \sup_{t>0} |B(x,t)|^{-1} \int_{B(x,t)} |f(y)|dy.$$

Let $\overline{T}$ be a standard Calderón-Zygmund(C–Z) singular integral operator, briefly a C–Z operator, i.e., a linear operator bounded from $L_2(\mathbb{R}^n)$ to $L_2(\mathbb{R}^n)$ taking all infinitely continuously differentiable functions f with compact support to the functions $f \in L_1^{loc}(\mathbb{R}^n)$ represented by

$$\overline{T}f(x) = p.v. \int_{\mathbb{R}^n} k(x-y)f(y)\,dy \qquad x \notin suppf.$$

Such operators have been introduced in [10]. Here k is a C–Z kernel [16]. Chiarenza and Frasca [5] have obtained the boundedness of H–L maximal operator M and C–Z operator $\overline{T}$ on $M_{p,\lambda}(\mathbb{R}^n)$. It is also well known that H–L maximal operator M and C–Z operator $\overline{T}$ play an important role in harmonic analysis (see [15, 27, 47, 48, 49]). Also, the theory of the C–Z operator is one of the important achievements of classical analysis in the last century, which has many important applications in Fourier analysis, complex analysis, operator theory and so on.

Suppose that S^{n-1} is the unit sphere in $\mathbb{R}^n$ ($n \geq 2$) equipped with the normalized Lebesgue measure $d\sigma$. Let $\Omega \in L_q(S^{n-1})$ with $1 < q \leq \infty$ be homogeneous of degree zero. Suppose that T_Ω represents a linear or a sublinear operator, which satisfies that for any $f \in L_1(\mathbb{R}^n)$ with compact support and $x \notin suppf$

(1.1) $$|T_\Omega f(x)| \leq c_0 \int_{\mathbb{R}^n} \frac{|\Omega(x-y)|}{|x-y|^n}\,|f(y)|\,dy,$$

where c_0 is independent of f and x.

For a locally integrable function b on $\mathbb{R}^n$, suppose that the commutator operator $T_{\Omega,b}$ represents a linear or a sublinear operator, which satisfies that for any $f \in L_1(\mathbb{R}^n)$ with compact support and $x \notin suppf$

(1.2) $$|T_{\Omega,b}f(x)| \leq c_0 \int_{\mathbb{R}^n} |b(x)-b(y)|\,\frac{|\Omega(x-y)|}{|x-y|^n}\,|f(y)|\,dy,$$

where c_0 is independent of f and x.

We point out that condition (1.1) in the case of $\Omega \equiv 1$ has been introduced by Soria and Weiss in [45] . Conditions (1.1) and (1.2) are satisfied by many interesting operators in harmonic analysis, such as Marcinkiewicz operator, the C–Z operators, Carleson's maximal operator, H–L maximal operator, C. Fefferman's singular multipliers, R. Fefferman's singular integrals, Ricci–Stein's oscillatory singular integrals, the Bochner–Riesz means and so on (see [25], [45] for details).

Let $\Omega \in L_q(S^{n-1})$ with $1 < q \leq \infty$ be homogeneous of degree zero and satisfies the cancellation condition

$$\int_{S^{n-1}} \Omega(x')d\sigma(x') = 0,$$

where $x' = \frac{x}{|x|}$ for any $x \neq 0$. The C–Z singular integral operator with rough kernel $\overline{T}_\Omega$ is defined by

$$\overline{T}_\Omega f(x) = p.v. \int\limits_{\mathbb{R}^n} \frac{\Omega(x-y)}{|x-y|^n} f(y) dy,$$

satisfies condition (1.1).

It is obvious that when $\Omega \equiv 1$, $\overline{T}_\Omega$ is the C–Z operator $\overline{T}$.

The case when Ω is a smooth kernel and $\overline{T}_\Omega$ a standard C–Z singular integral operator has been fully studied by many authors (see [16]).

In 1976, Coifman, Rocherberg and Weiss [8] introduced the commutator generated by $\overline{T}_\Omega$ and a local integrable function b as follows:

$$(1.3) \quad [b, \overline{T}_\Omega] f(x) \equiv b(x) \overline{T}_\Omega f(x) - \overline{T}_\Omega (bf)(x) = p.v. \int\limits_{\mathbb{R}^n} [b(x) - b(y)] \frac{\Omega(x-y)}{|x-y|^n} f(y) dy.$$

Sometimes, the commutator defined by (1.3) is also called the commutator in Coifman-Rocherberg-Weiss's sense, which has its root in the complex analysis and harmonic analysis (see [8]).

Remark 1.1. [43, 44] When Ω satisfies the specified size conditions, the kernel of the operator $\overline{T}_\Omega$ has no regularity, so the operator $\overline{T}_\Omega$ is called a rough C–Z singular integral operator. In recent years, a variety of operators related to the C–Z singular integral operators, but lacking the smoothness required in the classical theory, have been studied. These include the operator $[b, \overline{T}_\Omega]$. For more results, we refer the reader to [2, 18, 19, 20, 26, 27].

In this paper, we prove the boundedness of certain sublinear operators with rough kernel T_Ω satisfying condition (1.1), generated by C–Z singular integral operators on generalized vanishing Morrey spaces $VM_{p,\varphi}$ for $q' \leq p$, $p \neq 1$ or $p < q$, where $\Omega \in L_q(S^{n-1})$, $1 < q \leq \infty$ is a homogeneous of degree zero. The boundedness of the commutators of sublinear operators $T_{\Omega,b}$ satisfying condition (1.2) on generalized vanishing Morrey spaces are also obtained. Provided that $b \in BMO$ and $T_{\Omega,b}$ is a sublinear operator, we obtain the sufficient conditions on the pair (φ_1, φ_2) which ensures the boundedness of the operators $T_{\Omega,b}$, from one vanishing generalized Morrey space VM_{p,φ_1} to another VM_{p,φ_2}, where $1 < p < \infty$. In all the cases the conditions for the boundedness of T_Ω and $T_{\Omega,b}$ are given in terms of Zygmund-type integral inequalities on (φ_1, φ_2), where there is no assumption on monotonicity of φ_1, φ_2 in r. As an example to the conditions of these theorems are satisfied, we can consider the Marcinkiewicz operator.

Finally, we present a relationship between essential supremum and essential infimum.

Lemma 1.1. *(see* [53] *page 143) Let f be a real-valued nonnegative function and measurable on E. Then*

$$(1.4) \qquad\qquad \left(\operatorname*{essinf}_{x \in E} f(x) \right)^{-1} = \operatorname*{esssup}_{x \in E} \frac{1}{f(x)}.$$

By $A \lesssim B$ we mean that $A \leq CB$ with some positive constant C independent of appropriate quantities. If $A \lesssim B$ and $B \lesssim A$, we write $A \approx B$ and say that A and B are equivalent.

2. GENERALIZED VANISHING MORREY SPACES

After studying Morrey spaces in detail, researchers have passed to generalized Morrey spaces. Mizuhara [31] has given generalized Morrey spaces $M_{p,\varphi}$ considering $\varphi = \varphi(r)$ instead of r^λ in the above definition of the Morrey space. Later, Guliyev [14] has defined the generalized Morrey spaces $M_{p,\varphi}$ with normalized norm as follows:

Definition 2.1. [14] Let $\varphi(x, r)$ be a positive measurable function on $\mathbb{R}^n \times (0, \infty)$ and $1 \leq p < \infty$. We denote by $M_{p,\varphi} \equiv M_{p,\varphi}(\mathbb{R}^n)$ the generalized Morrey space, the space of all functions $f \in L_p^{loc}(\mathbb{R}^n)$ with finite quasinorm

$$\|f\|_{M_{p,\varphi}} = \sup_{x \in \mathbb{R}^n, r>0} \varphi(x,r)^{-1} |B(x,r)|^{-\frac{1}{p}} \|f\|_{L_p(B(x,r))}.$$

Also by $WM_{p,\varphi} \equiv WM_{p,\varphi}(\mathbb{R}^n)$ we denote the weak generalized Morrey space of all functions $f \in WL_p^{loc}(\mathbb{R}^n)$ for which

$$\|f\|_{WM_{p,\varphi}} = \sup_{x \in \mathbb{R}^n, r>0} \varphi(x,r)^{-1} |B(x,r)|^{-\frac{1}{p}} \|f\|_{WL_p(B(x,r))} < \infty.$$

According to this definition, we recover the Morrey space $M_{p,\lambda}$ and weak Morrey space $WM_{p,\lambda}$ under the choice $\varphi(x,r) = r^{\frac{\lambda-n}{p}}$:

$$M_{p,\lambda} = M_{p,\varphi}\Big|_{\varphi(x,r)=r^{\frac{\lambda-n}{p}}}, \ WM_{p,\lambda} = WM_{p,\varphi}\Big|_{\varphi(x,r)=r^{\frac{\lambda-n}{p}}}.$$

Everywhere in the sequel we assume that $\inf_{x \in \mathbb{R}^n, r>0} \varphi(x,r) > 0$ which makes the above spaces non-trivial, since the spaces of bounded functions are contained in these spaces.

In [14, 23, 24, 31, 34], the boundedness of the maximal operator and C–Z singular integral operator on the generalized Morrey spaces has been obtained. For generalized Morrey spaces with nondoubling measures see also [42].

For brevity, in the sequel we use the notations

$$\mathfrak{M}_{p,\varphi}(f; x, r) := \frac{|B(x,r)|^{-\frac{1}{p}} \|f\|_{L_p(B(x,r))}}{\varphi(x,r)}$$

and

$$\mathfrak{M}_{p,\varphi}^W(f; x, r) := \frac{|B(x,r)|^{-\frac{1}{p}} \|f\|_{WL_p(B(x,r))}}{\varphi(x,r)}.$$

In this paper, extending the definition of vanishing Morrey spaces [51], we introduce the generalized vanishing Morrey spaces $VM_{p,\varphi}(\mathbb{R}^n)$, including their weak versions and studies the boundedness of the sublinear operators with rough kernel generated by C–Z singular integral operators and their commutators in these spaces. Indeed, we find it convenient to define generalized vanishing Morrey spaces in the form as follows.

Definition 2.2. (generalized vanishing Morrey space) The generalized vanishing Morrey space $VM_{p,\varphi}(\mathbb{R}^n)$ is defined as the spaces of functions $f \in M_{p,\varphi}(\mathbb{R}^n)$ such that

$$(2.1) \qquad\qquad \lim_{r \to 0} \sup_{x \in \mathbb{R}^n} \mathfrak{M}_{p,\varphi}(f; x, r) = 0.$$

Definition 2.3. (weak generalized vanishing Morrey space) The weak generalized vanishing Morrey space $WVM_{p,\varphi}(\mathbb{R}^n)$ is defined as the spaces of functions $f \in WM_{p,\varphi}(\mathbb{R}^n)$ such that

$$(2.2) \qquad \lim_{r\to 0} \sup_{x\in\mathbb{R}^n} \mathfrak{M}_{p,\varphi}^{W}(f;x,r) = 0.$$

Everywhere in the sequel we assume that

$$(2.3) \qquad \lim_{r\to 0} \frac{1}{\inf\limits_{x\in\mathbb{R}^n} \varphi(x,r)} = 0,$$

and

$$(2.4) \qquad \sup_{0<r<\infty} \frac{1}{\inf\limits_{x\in\mathbb{R}^n} \varphi(x,r)} < \infty,$$

which make the spaces $VM_{p,\varphi}(\mathbb{R}^n)$ and $WVM_{p,\varphi}(\mathbb{R}^n)$ non-trivial, because bounded functions with compact support belong to this space. The spaces $VM_{p,\varphi}(\mathbb{R}^n)$ and $WVM_{p,\varphi}(\mathbb{R}^n)$ are Banach spaces with respect to the norm

$$(2.5) \qquad \|f\|_{VM_{p,\varphi}} \equiv \|f\|_{M_{p,\varphi}} = \sup_{x\in\mathbb{R}^n, r>0} \mathfrak{M}_{p,\varphi}(f;x,r),$$

$$(2.6) \qquad \|f\|_{WVM_{p,\varphi}} = \|f\|_{WM_{p,\varphi}} = \sup_{x\in\mathbb{R}^n, r>0} \mathfrak{M}_{p,\varphi}^{W}(f;x,r),$$

respectively.

3. Sublinear operators with rough kernel T_Ω on the spaces $M_{p,\varphi}$ and $VM_{p,\varphi}$

In this section, we will first prove the boundedness of the operator T_Ω satisfying (1.1) on the generalized Morrey spaces $M_{p,\varphi}$ by using Lemma 1.1 and the following Lemma 3.1. Then, We will also give the boundedness of T_Ω satisfying (1.1) on generalized vanishing Morrey spaces $VM_{p,\varphi}$.

Theorem 3.1. [11, 33] *Suppose that $1 \le p < \infty$, $\Omega \in L_q(S^{n-1})$, $q > 1$, is homogeneous of degree zero and has mean value zero on S^{n-1}. If $q' \le p$, $p \ne 1$ or $p < q$, then the operator $\overline{T}_\Omega$ is bounded on $L_p(\mathbb{R}^n)$. Also the operator $\overline{T}_\Omega$ is bounded from $L_1(\mathbb{R}^n)$ to $WL_1(\mathbb{R}^n)$. Moreover, we have for $p > 1$*

$$\left\|\overline{T}_\Omega f\right\|_{L_p} \le C \|f\|_{L_p},$$

and for $p = 1$

$$\left\|\overline{T}_\Omega f\right\|_{WL_1} \le C \|f\|_{L_1}.$$

Lemma 3.1. *(Our main lemma) Let $\Omega \in L_q(S^{n-1})$, $1 < q \le \infty$, be homogeneous of degree zero, and $1 \le p < \infty$. Let T_Ω be a sublinear operator satisfying condition (1.1), bounded on $L_p(\mathbb{R}^n)$ for $p > 1$, and bounded from $L_1(\mathbb{R}^n)$ to $WL_1(\mathbb{R}^n)$.*
    If $p > 1$ and $q' \le p$, then the inequality

$$(3.1) \qquad \|T_\Omega f\|_{L_p(B(x_0,r))} \lesssim r^{\frac{n}{p}} \int_{2r}^{\infty} t^{-\frac{n}{p}-1} \|f\|_{L_p(B(x_0,t))} \, dt$$

holds for any ball $B(x_0, r)$ and for all $f \in L_p^{loc}(\mathbb{R}^n)$.

If $p > 1$ and $p < q$, then the inequality

$$\|T_\Omega f\|_{L_p(B(x_0,r))} \lesssim r^{\frac{n}{p}-\frac{n}{q}} \int_{2r}^{\infty} t^{\frac{n}{q}-\frac{n}{p}-1} \|f\|_{L_p(B(x_0,t))} \, dt$$

holds for any ball $B(x_0, r)$ and for all $f \in L_p^{loc}(\mathbb{R}^n)$.

Moreover, for $q > 1$ the inequality

$$(3.2) \qquad \|T_\Omega f\|_{WL_1(B(x_0,r))} \lesssim r^n \int_{2r}^{\infty} t^{-n-1} \|f\|_{L_1(B(x_0,t))} \, dt$$

holds for any ball $B(x_0, r)$ and for all $f \in L_1^{loc}(\mathbb{R}^n)$.

Proof. Let $1 < p < \infty$ and $q' \leq p$. Set $B = B(x_0, r)$ for the ball centered at x_0 and of radius r and $2B = B(x_0, 2r)$. We represent f as

$$(3.3) \quad f = f_1 + f_2, \quad f_1(y) = f(y)\chi_{2B}(y), \quad f_2(y) = f(y)\chi_{(2B)^C}(y), \quad r > 0$$

and have

$$\|T_\Omega f\|_{L_p(B)} \leq \|T_\Omega f_1\|_{L_p(B)} + \|T_\Omega f_2\|_{L_p(B)}.$$

Since $f_1 \in L_p(\mathbb{R}^n)$, $T_\Omega f_1 \in L_p(\mathbb{R}^n)$ and from the boundedness of T_Ω on $L_p(\mathbb{R}^n)$ (see Theorem 3.1) it follows that:

$$\|T_\Omega f_1\|_{L_p(B)} \leq \|T_\Omega f_1\|_{L_p(\mathbb{R}^n)} \leq C\|f_1\|_{L_p(\mathbb{R}^n)} = C\|f\|_{L_p(2B)},$$

where constant $C > 0$ is independent of f.

It is clear that $x \in B$, $y \in (2B)^C$ implies $\frac{1}{2}|x_0 - y| \leq |x - y| \leq \frac{3}{2}|x_0 - y|$. We get

$$|T_\Omega f_2(x)| \leq 2^n c_1 \int_{(2B)^C} \frac{|f(y)|\,|\Omega(x-y)|}{|x_0 - y|^n} dy.$$

By the Fubini's theorem, we have

$$\int_{(2B)^C} \frac{|f(y)|\,|\Omega(x-y)|}{|x_0 - y|^n} dy \approx \int_{(2B)^C} |f(y)|\,|\Omega(x-y)| \int_{|x_0-y|}^{\infty} \frac{dt}{t^{n+1}} dy$$

$$\approx \int_{2r}^{\infty} \int_{2r \leq |x_0-y| \leq t} |f(y)|\,|\Omega(x-y)| \, dy \frac{dt}{t^{n+1}}$$

$$\lesssim \int_{2r}^{\infty} \int_{B(x_0,t)} |f(y)|\,|\Omega(x-y)| \, dy \frac{dt}{t^{n+1}}.$$

Applying the Hölder's inequality, we get

$$\int_{(2B)^C} \frac{|f(y)|\,|\Omega(x-y)|}{|x_0-y|^n} dy$$

$$(3.4) \qquad \lesssim \int_{2r}^{\infty} \|f\|_{L_p(B(x_0,t))} \|\Omega(x-\cdot)\|_{L_q(B(x_0,t))} |B(x_0,t)|^{1-\frac{1}{p}-\frac{1}{q}} \frac{dt}{t^{n+1}}.$$

For $x \in B(x_0, t)$, notice that Ω is homogenous of degree zero and $\Omega \in L_q(S^{n-1})$, $q > 1$. Then, we obtain

$$
\left(\int_{B(x_0,t)} |\Omega(x-y)|^q \, dy \right)^{\frac{1}{q}} = \left(\int_{B(x-x_0,t)} |\Omega(z)|^q \, dz \right)^{\frac{1}{q}}
$$

$$
\leq \left(\int_{B(0,t+|x-x_0|)} |\Omega(z)|^q \, dz \right)^{\frac{1}{q}}
$$

$$
\leq \left(\int_{B(0,2t)} |\Omega(z)|^q \, dz \right)^{\frac{1}{q}}
$$

$$
= \left(\int_0^{2t} \int_{S^{n-1}} |\Omega(z')|^q \, d\sigma(z') \, r^{n-1} dr \right)^{\frac{1}{q}}
$$

$$
(3.5) \qquad = C \, \|\Omega\|_{L_q(S^{n-1})} \, |B(x_0, 2t)|^{\frac{1}{q}}.
$$

Thus, by (3.5), it follows that:

$$
|T_\Omega f_2(x)| \lesssim \int_{2r}^{\infty} \|f\|_{L_p(B(x_0,t))} \, \frac{dt}{t^{\frac{n}{p}+1}}.
$$

Moreover, for all $p \in [1, \infty)$ the inequality

$$
(3.6) \qquad \|T_\Omega f_2\|_{L_p(B)} \lesssim r^{\frac{n}{p}} \int_{2r}^{\infty} \|f\|_{L_p(B(x_0,t))} \, \frac{dt}{t^{\frac{n}{p}+1}}
$$

holds. Thus

$$
\|T_\Omega f\|_{L_p(B)} \lesssim \|f\|_{L_p(2B)} + r^{\frac{n}{p}} \int_{2r}^{\infty} \|f\|_{L_p(B(x_0,t))} \, \frac{dt}{t^{\frac{n}{p}+1}}.
$$

On the other hand, we have

$$
\|f\|_{L_p(2B)} \approx r^{\frac{n}{p}} \|f\|_{L_p(2B)} \int_{2r}^{\infty} \frac{dt}{t^{\frac{n}{p}+1}}
$$

$$
(3.7) \qquad \leq r^{\frac{n}{p}} \int_{2r}^{\infty} \|f\|_{L_p(B(x_0,t))} \, \frac{dt}{t^{\frac{n}{p}+1}}.
$$

By combining the above inequalities, we obtain

$$
\|T_\Omega f\|_{L_p(B)} \lesssim r^{\frac{n}{p}} \int_{2r}^{\infty} \|f\|_{L_p(B(x_0,t))} \, \frac{dt}{t^{\frac{n}{p}+1}}.
$$

Let $1 < p < q$. Similarly to (3.5), when $y \in B(x_0, t)$, notice that

$$(3.8) \qquad \left(\int_{B(x_0,r)} |\Omega(x-y)|^q \, dy \right)^{\frac{1}{q}} \leq C \|\Omega\|_{L_q(S^{n-1})} \left| B\left(x_0, \frac{3}{2}t\right) \right|^{\frac{1}{q}}.$$

By the Fubini's theorem, the Minkowski inequality and (3.8) , we get

$$\|T_\Omega f_2\|_{L_p(B)} \leq \left(\int_B \left| \int_{2r}^{\infty} \int_{B(x_0,t)} |f(y)| \, |\Omega(x-y)| \, dy \frac{dt}{t^{n+1}} \right|^p dx \right)^{\frac{1}{p}}$$

$$\leq \int_{2r}^{\infty} \int_{B(x_0,t)} |f(y)| \, \|\Omega(\cdot - y)\|_{L_p(B)} \, dy \frac{dt}{t^{n+1}}$$

$$\leq |B(x_0,r)|^{\frac{1}{p} - \frac{1}{q}} \int_{2r}^{\infty} \int_{B(x_0,t)} |f(y)| \, \|\Omega(\cdot - y)\|_{L_q(B)} \, dy \frac{dt}{t^{n+1}}$$

$$\lesssim r^{\frac{n}{p} - \frac{n}{q}} \int_{2r}^{\infty} \|f\|_{L_1(B(x_0,t))} \left| B\left(x_0, \frac{3}{2}t\right) \right|^{\frac{1}{q}} \frac{dt}{t^{n+1}}$$

$$\lesssim r^{\frac{n}{p} - \frac{n}{q}} \int_{2r}^{\infty} \|f\|_{L_p(B(x_0,t))} \, t^{\frac{n}{q} - \frac{n}{p} - 1} dt.$$

Let $p = 1 < q \leq \infty$. From the weak $(1,1)$ boundedness of T_Ω and (3.7) it follows that:

$$\|T_\Omega f_1\|_{WL_1(B)} \leq \|T_\Omega f_1\|_{WL_1(\mathbb{R}^n)} \lesssim \|f_1\|_{L_1(\mathbb{R}^n)}$$

$$(3.9) \qquad\qquad = \|f\|_{L_1(2B)} \lesssim r^n \int_{2r}^{\infty} \|f\|_{L_1(B(x_0,t))} \frac{dt}{t^{n+1}}.$$

Then from (3.6) and (3.9) we get the inequality (3.2), which completes the proof. $\qquad \square$

In the following theorem, we get the boundedness of the operator T_Ω on the generalized Morrey spaces $M_{p,\varphi}$.

Theorem 3.2. *(Our main result) Let $\Omega \in L_q(S^{n-1})$, $1 < q \leq \infty$, be homogeneous of degree zero, and $1 \leq p < \infty$. Let T_Ω be a sublinear operator satisfying condition (1.1), bounded on $L_p(\mathbb{R}^n)$ for $p > 1$, and bounded from $L_1(\mathbb{R}^n)$ to $WL_1(\mathbb{R}^n)$. Let also, for $q' \leq p$, $p \neq 1$, the pair (φ_1, φ_2) satisfies the condition*

$$(3.10) \qquad \int_r^{\infty} \frac{\operatorname*{essinf}_{t < \tau < \infty} \varphi_1(x,\tau) \tau^{\frac{n}{p}}}{t^{\frac{n}{p}+1}} dt \leq C\varphi_2(x,r),$$

and for $1 < p < q$ the pair (φ_1, φ_2) satisfies the condition

$$(3.11) \qquad \int_r^{\infty} \frac{\operatorname*{essinf}_{t < \tau < \infty} \varphi_1(x,\tau) \tau^{\frac{n}{p}}}{t^{\frac{n}{p} - \frac{n}{q} + 1}} dt \leq C \, \varphi_2(x,r) r^{\frac{n}{q}},$$

where C does not depend on x and r.

Then the operator T_Ω is bounded from M_{p,φ_1} to M_{p,φ_2} for $p > 1$ and from M_{1,φ_1} to WM_{1,φ_2}. *Moreover, we have for $p > 1$*

(3.12)
$$\|T_\Omega f\|_{M_{p,\varphi_2}} \lesssim \|f\|_{M_{p,\varphi_1}},$$

and for $p = 1$

(3.13)
$$\|T_\Omega f\|_{WM_{1,\varphi_2}} \lesssim \|f\|_{M_{1,\varphi_1}}.$$

Proof. Since $f \in M_{p,\varphi_1}$, by (2.6) and the non-decreasing, with respect to t, of the norm $\|f\|_{L_p(B(x_0,t))}$, we get

$$\frac{\|f\|_{L_p(B(x_0,t))}}{\underset{0<t<\tau<\infty}{\mathrm{essinf}}\ \varphi_1(x_0,\tau)\tau^{\frac{n}{p}}}$$

$$\leq \underset{0<t<\tau<\infty}{\mathrm{esssup}}\ \frac{\|f\|_{L_p(B(x_0,t))}}{\varphi_1(x_0,\tau)\tau^{\frac{n}{p}}}$$

$$\leq \underset{0<\tau<\infty}{\mathrm{esssup}}\ \frac{\|f\|_{L_p(B(x_0,\tau))}}{\varphi_1(x_0,\tau)\tau^{\frac{n}{p}}}$$

$$\leq \|f\|_{M_{p,\varphi_1}}.$$

For $q' \leq p < \infty$, since (φ_1, φ_2) satisfies (3.10), we have

$$\int_r^\infty \|f\|_{L_p(B(x_0,t))}\, t^{-\frac{n}{p}}\frac{dt}{t}$$

$$\leq \int_r^\infty \frac{\|f\|_{L_p(B(x_0,t))}}{\underset{t<\tau<\infty}{\mathrm{essinf}}\ \varphi_1(x_0,\tau)\tau^{\frac{n}{p}}}\ \frac{\underset{t<\tau<\infty}{\mathrm{essinf}}\ \varphi_1(x_0,\tau)\tau^{\frac{n}{p}}}{t^{\frac{n}{p}}}\frac{dt}{t}$$

$$\leq C \|f\|_{M_{p,\varphi_1}} \int_r^\infty \frac{\underset{t<\tau<\infty}{\mathrm{essinf}}\ \varphi_1(x_0,\tau)\tau^{\frac{n}{p}}}{t^{\frac{n}{p}}}\frac{dt}{t}$$

$$\leq C \|f\|_{M_{p,\varphi_1}}\ \varphi_2(x_0,r).$$

Then by (3.1), we get

$$\|T_\Omega f\|_{M_{p,\varphi_2}} = \sup_{x_0 \in \mathbb{R}^n, r>0} \varphi_2(x_0,r)^{-1}|B(x_0,r)|^{-\frac{1}{p}}\|T_\Omega f\|_{L_p(B(x_0,r))}$$

$$\leq C \sup_{x_0 \in \mathbb{R}^n, r>0} \varphi_2(x_0,r)^{-1}\int_r^\infty \|f\|_{L_p(B(x_0,t))}\, t^{-\frac{n}{p}}\frac{dt}{t}$$

$$\leq C \|f\|_{M_{p,\varphi_1}}.$$

For the case of $1 \leq p < q$, we can also use the same method, so we omit the details. This completes the proof of Theorem 3.2. $\qquad\qquad\square$

In the case of $q = \infty$ by Theorem 3.2, we get

Corollary 3.1. *Let $1 \leq p < \infty$ and the pair (φ_1, φ_2) satisfies condition (3.10). Then the operators M and $\overline{T}$ are bounded from M_{p,φ_1} to M_{p,φ_2} for $p > 1$ and from M_{1,φ_1} to WM_{1,φ_2}.*

Let $f \in L_1^{loc}(\mathbb{R}^n)$. The rough H–L maximal operator M_Ω is defined by

$$M_\Omega f(x) = \sup_{t>0} \frac{1}{|B(x,t)|} \int_{B(x,t)} |\Omega(x-y)| \, |f(y)| \, dy.$$

Then we can get the following corollary.

Corollary 3.2. *Let $1 \le p < \infty$, $\Omega \in L_q(S^{n-1})$, $1 < q \le \infty$. For $q' \le p$, $p \ne 1$, the pair (φ_1, φ_2) satisfies condition (3.10) and for $1 < p < q$ the pair (φ_1, φ_2) satisfies condition (3.11). Then the operators M_Ω and $\overline{T}_\Omega$ are bounded from M_{p,φ_1} to M_{p,φ_2} for $p > 1$ and from M_{1,φ_1} to WM_{1,φ_2}.*

Now using above results, we get the boundedness of the operator T_Ω on the vanishing generalized Morrey spaces $VM_{p,\varphi}$.

Theorem 3.3. *(Our main result) Let $\Omega \in L_q(S^{n-1})$, $1 < q \le \infty$, be homogeneous of degree zero, and $1 \le p < \infty$. Let T_Ω be a sublinear operator satisfying condition (1.1), bounded on $L_p(\mathbb{R}^n)$ for $p > 1$, and bounded from $L_1(\mathbb{R}^n)$ to $WL_1(\mathbb{R}^n)$. Let for $q' \le p$, $p \ne 1$, the pair (φ_1, φ_2) satisfies conditions (2.3)-(2.4) and*

$$(3.14) \qquad c_\delta := \int_\delta^\infty \sup_{x \in \mathbb{R}^n} \varphi_1(x,t) \frac{t^{\frac{n}{p}}}{t^{\frac{n}{p}+1}} dt < \infty$$

for every $\delta > 0$, and

$$(3.15) \qquad \int_r^\infty \varphi_1(x,t) \frac{t^{\frac{n}{p}}}{t^{\frac{n}{p}+1}} dt \le C_0 \varphi_2(x,r),$$

and for $1 < p < q$ the pair (φ_1, φ_2) satisfies conditions (2.3)-(2.4) and also

$$(3.16) \qquad c_{\delta'} := \int_{\delta'}^\infty \sup_{x \in \mathbb{R}^n} \varphi_1(x,t) \frac{t^{\frac{n}{p}}}{t^{\frac{n}{p}-\frac{n}{q}+1}} dt < \infty$$

for every $\delta' > 0$, and

$$(3.17) \qquad \int_r^\infty \varphi_1(x,t) \frac{t^{\frac{n}{p}}}{t^{\frac{n}{p}-\frac{n}{q}+1}} dt \le C_0 \, \varphi_2(x,r) r^{\frac{n}{q}},$$

where C_0 does not depend on $x \in \mathbb{R}^n$ and $r > 0$.

Then the operator T_Ω is bounded from VM_{p,φ_1} to VM_{p,φ_2} for $p > 1$ and from M_{1,φ_1} to WVM_{1,φ_2}. Moreover, we have for $p > 1$

$$(3.18) \qquad \|T_\Omega f\|_{VM_{p,\varphi_2}} \lesssim \|f\|_{VM_{p,\varphi_1}},$$

and for $p = 1$

$$(3.19) \qquad \|T_\Omega f\|_{WVM_{1,\varphi_2}} \lesssim \|f\|_{VM_{1,\varphi_1}}.$$

Proof. The norm inequalities follow from Theorem 3.2. Thus we only have to prove that

$$(3.20) \qquad \lim_{r \to 0} \sup_{x \in \mathbb{R}^n} \mathfrak{M}_{p,\varphi_1}(f; x, r) = 0 \text{ implies } \lim_{r \to 0} \sup_{x \in \mathbb{R}^n} \mathfrak{M}_{p,\varphi_2}(T_\Omega f; x, r) = 0$$

and

(3.21) $\lim\limits_{r\to 0}\sup\limits_{x\in\mathbb{R}^n}\mathfrak{M}_{p,\varphi_1}(f;x,r)=0$ implies $\lim\limits_{r\to 0}\sup\limits_{x\in\mathbb{R}^n}\mathfrak{M}_{p,\varphi_2}^{W}(T_\Omega f;x,r)=0.$

To show that $\sup\limits_{x\in\mathbb{R}^n}\dfrac{r^{-\frac{n}{p}}\|T_\Omega f\|_{L_p(B(x,r))}}{\varphi_2(x,r)}<\epsilon$ for small r, we split the right-hand side of (3.1):

(3.22) $\dfrac{r^{-\frac{n}{p}}\|T_\Omega f\|_{L_p(B(x,r))}}{\varphi_2(x,r)}\leq C\left[I_{\delta_0}(x,r)+J_{\delta_0}(x,r)\right],$

where $\delta_0>0$ (we may take $\delta_0<1$), and

$$I_{\delta_0}(x,r):=\dfrac{1}{\varphi_2(x,r)}\int\limits_{r}^{\delta_0}t^{-\frac{n}{p}-1}\|f\|_{L_p(B(x,t))}\,dt,$$

and

$$J_{\delta_0}(x,r):=\dfrac{1}{\varphi_2(x,r)}\int\limits_{\delta_0}^{\infty}t^{-\frac{n}{p}-1}\|f\|_{L_p(B(x,t))}\,dt$$

and $r<\delta_0$. Now we choose any fixed $\delta_0>0$ such that

$$\sup\limits_{x\in\mathbb{R}^n}\dfrac{t^{-\frac{n}{p}}\|f\|_{L_p(B(x,t))}}{\varphi_1(x,t)}<\dfrac{\epsilon}{2CC_0},\qquad t\leq\delta_0,$$

where C and C_0 are constants from (3.15) and (3.22). This allows to estimate the first term uniformly in $r\in(0,\delta_0)$:

$$\sup\limits_{x\in\mathbb{R}^n}CI_{\delta_0}(x,r)<\dfrac{\epsilon}{2},\qquad 0<r<\delta_0.$$

The estimation of the second term may be obtained by choosing r sufficiently small. Indeed, we have

$$J_{\delta_0}(x,r)\leq c_{\delta_0}\dfrac{\|f\|_{M_{p,\varphi_1}}}{\varphi_2(x,r)},$$

where c_{δ_0} is the constant from (3.14) with $\delta=\delta_0$. Then, by (2.3) it suffices to choose r small enough such that

$$\sup\limits_{x\in\mathbb{R}^n}\dfrac{1}{\varphi_2(x,r)}\leq\dfrac{\epsilon}{2c_{\delta_0}\|f\|_{M_{p,\varphi_1}}},$$

which completes the proof of (3.20).

The proof of (3.21) is similar to the proof of (3.20). For the case of $1\leq p<q$, we can also use the same method, so we omit the details. $\square$

Remark 3.1. Conditions (3.14) and (3.16) are not needed in the case when $\varphi(x,r)$ does not depend on x, since (3.14) follows from (3.15) and similarly, (3.16) follows from (3.17) in this case.

Corollary 3.3. *Let $1\leq p<\infty$, $\Omega\in L_q\left(S^{n-1}\right)$, $1<q\leq\infty$. For $q'\leq p$, $p\neq 1$, the pair (φ_1,φ_2) satisfies conditions (2.3)-(2.4) and (3.14)-(3.15) and for $1<p<q$ the pair (φ_1,φ_2) satisfies conditions (2.3)-(2.4) and (3.16)-(3.17). Then the operators M_Ω and $\overline{T}_\Omega$ are bounded from VM_{p,φ_1} to VM_{p,φ_2} for $p>1$ and from VM_{1,φ_1} to WVM_{1,φ_2}.*

In the case of $q = \infty$ by Theorem 3.3, we get

Corollary 3.4. *Let $1 \leq p < \infty$ and the pair (φ_1, φ_2) satisfies conditions (2.3)-(2.4) and (3.14)-(3.15). Then the operators M and $\overline{T}$ are bounded from VM_{p,φ_1} to VM_{p,φ_2} for $p > 1$ and from VM_{1,φ_1} to WVM_{1,φ_2}.*

4. COMMUTATORS OF THE SUBLINEAR OPERATORS WITH ROUGH KERNEL T_Ω ON THE SPACES $M_{p,\varphi}$ AND $VM_{p,\varphi}$

In this section, we will first prove the boundedness of the operator $T_{\Omega,b}$ satisfying (1.2) with $b \in BMO(\mathbb{R}^n)$ on the generalized Morrey spaces $M_{p,\varphi}$ by using Lemma 1.1 and the following Lemma 4.1. Then, we will also obtain the boundedness of $T_{\Omega,b}$ satisfying (1.2) with $b \in BMO(\mathbb{R}^n)$ on generalized vanishing Morrey spaces $VM_{p,\varphi}$.

Let T be a linear operator. For a locally integrable function b on $\mathbb{R}^n$, we define the commutator $[b, T]$ by

$$[b, T]f(x) = b(x)\,Tf(x) - T(bf)(x)$$

for any suitable function f.

The function b is also called the symbol function of $[b, T]$. The investigation of the operator $[b, T]$ begins with Coifman-Rocherberg-Weiss pioneering study of the operator T (see [8]). Let $\overline{T}$ be a C–Z operator. A well known result of Coifman et al. [8] states that when $K(x) = \frac{\Omega(x')}{|x|^n}$ and Ω is smooth, the commutator $[b, \overline{T}]f = b\overline{T}f - \overline{T}(bf)$ is bounded on $L_p(\mathbb{R}^n)$, $1 < p < \infty$, if and only if $b \in BMO(\mathbb{R}^n)$. There are two major reasons for considering the problem of commutators. The first one is that the boundedness of commutators can produce some characterizations of function spaces (see [2, 18, 19, 21, 36, 44]). The other one is that the theory of commutators plays an important role in the study of the regularity of solutions to elliptic and parabolic PDEs of the second order (see [6, 7, 12, 13]).

Many authors are interested in the study of commutators for which the symbol functions belong to $BMO(\mathbb{R}^n)$ spaces (see [17, 21, 23, 28, 36] for example). The boundedness of the commutator has also been generalized to other contexts and important applications to some non-linear PDEs have been given by Coifman et al. [9].

Let us recall the defination of the space of $BMO(\mathbb{R}^n)$ (bounded mean oscillation).

Definition 4.1. Suppose that $b \in L_1^{loc}(\mathbb{R}^n)$, let

$$\|b\|_* = \sup_{x \in \mathbb{R}^n, r > 0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |b(y) - b_{B(x,r)}|\, dy < \infty,$$

where

$$b_{B(x,r)} = \frac{1}{|B(x,r)|} \int_{B(x,r)} b(y)\, dy.$$

Define

$$BMO(\mathbb{R}^n) = \{b \in L_1^{loc}(\mathbb{R}^n) \ : \ \|b\|_* < \infty\}.$$

If one regards two functions whose difference is a constant as one, then the space $BMO(\mathbb{R}^n)$ is a Banach space with respect to norm $\|\cdot\|_*$.

Remark 4.1. [23] (1) The John-Nirenberg inequality [22]: there are constants C_1, $C_2 > 0$, such that for all $b \in BMO(\mathbb{R}^n)$ and $\beta > 0$

$$|\{x \in B \,:\, |b(x) - b_B| > \beta\}| \le C_1 |B| e^{-C_2 \beta / \|b\|_*}, \ \forall B \subset \mathbb{R}^n.$$

(2) The John-Nirenberg inequality implies that

$$(4.1) \qquad \|b\|_* \approx \sup_{x \in \mathbb{R}^n, r > 0} \left(\frac{1}{|B(x,r)|} \int_{B(x,r)} |b(y) - b_{B(x,r)}|^p dy \right)^{\frac{1}{p}}$$

for $1 < p < \infty$.

(3) Let $b \in BMO(\mathbb{R}^n)$. Then there is a constant $C > 0$ such that

$$(4.2) \qquad \left| b_{B(x,r)} - b_{B(x,t)} \right| \le C \|b\|_* \ln \frac{t}{r} \text{ for } 0 < 2r < t,$$

where C is independent of b, x, r and t.

Theorem 4.1. [11, 27] *Suppose that $\Omega \in L_q(S^{n-1})$, $q > 1$, is homogeneous of degree zero and has mean value zero on S^{n-1}. Let $1 < p < \infty$ and $b \in BMO(\mathbb{R}^n)$. If $q' \le p$ or $p < q$, then the commutator operator $[b, \overline{T}_\Omega]$ is bounded on $L_p(\mathbb{R}^n)$.*

Lemma 4.1. *(Our main lemma) Let $\Omega \in L_q(S^{n-1})$, $1 < q \le \infty$, be homogeneous of degree zero. Let $1 < p < \infty$, $b \in BMO(\mathbb{R}^n)$ and $T_{\Omega,b}$ is a sublinear operator satisfying condition (1.2), bounded on $L_p(\mathbb{R}^n)$. Then, for $q' \le p$ the inequality*

$$(4.3) \qquad \|T_{\Omega,b} f\|_{L_p(B(x_0,r))} \lesssim \|b\|_* \, r^{\frac{n}{p}} \int_{2r}^{\infty} \left(1 + \ln \frac{t}{r} \right) t^{-\frac{n}{p}-1} \|f\|_{L_p(B(x_0,t))} dt$$

holds for any ball $B(x_0, r)$ and for all $f \in L_p^{loc}(\mathbb{R}^n)$.

Also, for $p < q$ the inequality

$$\|T_{\Omega,b} f\|_{L_p(B(x_0,r))} \lesssim \|b\|_* \, r^{\frac{n}{p}-\frac{n}{q}} \int_{2r}^{\infty} \left(1 + \ln \frac{t}{r} \right) t^{\frac{n}{q}-\frac{n}{p}-1} \|f\|_{L_p(B(x_0,t))} dt$$

holds for any ball $B(x_0, r)$ and for all $f \in L_p^{loc}(\mathbb{R}^n)$.

Proof. Let $1 < p < \infty$. As in the proof of Lemma 3.1, we represent f in form (3.3) and have

$$\|T_{\Omega,b} f\|_{L_p(B)} \le \|T_{\Omega,b} f_1\|_{L_p(B)} + \|T_{\Omega,b} f_2\|_{L_p(B)}.$$

From the boundedness of $T_{\Omega,b}$ on $L_p(\mathbb{R}^n)$ (see Theorem 4.1) it follows that:

$$\|T_{\Omega,b} f_1\|_{L_p(B)} \le \|T_{\Omega,b} f_1\|_{L_p(\mathbb{R}^n)}$$
$$\lesssim \|b\|_* \|f_1\|_{L_p(\mathbb{R}^n)} = \|b\|_* \|f\|_{L_p(2B)}.$$

It is known that $x \in B$, $y \in (2B)^C$, which implies $\frac{1}{2} |x_0 - y| \le |x - y| \le \frac{3}{2} |x_0 - y|$. Then for $x \in B$, we have

$$|T_{\Omega,b} f_2(x)| \lesssim \int_{\mathbb{R}^n} \frac{|\Omega(x-y)|}{|x-y|^n} |b(y) - b(x)| |f(y)| dy$$
$$\approx \int_{(2B)^C} \frac{|\Omega(x-y)|}{|x_0 - y|^n} |b(y) - b(x)| |f(y)| dy.$$

Hence we get

$$
\begin{aligned}
\|T_{\Omega,b}f_2\|_{L_p(B)} &\lesssim \left(\int_B \left(\int_{(2B)^C} \frac{|\Omega(x-y)|}{|x_0-y|^n} |b(y)-b(x)| |f(y)| \, dy \right)^p dx \right)^{\frac{1}{p}} \\
&\lesssim \left(\int_B \left(\int_{(2B)^C} \frac{|\Omega(x-y)|}{|x_0-y|^n} |b(y)-b_B| |f(y)| \, dy \right)^p dx \right)^{\frac{1}{p}} \\
&\quad + \left(\int_B \left(\int_{(2B)^C} \frac{|\Omega(x-y)|}{|x_0-y|^n} |b(x)-b_B| |f(y)| \, dy \right)^p dx \right)^{\frac{1}{p}} \\
&= J_1 + J_2.
\end{aligned}
$$

We have the following estimation of J_1. When $s' \leq p$ and $\frac{1}{\mu} + \frac{1}{p} + \frac{1}{q} = 1$, by the Fubini's theorem

$$
\begin{aligned}
J_1 &\approx r^{\frac{n}{p}} \int_{(2B)^C} \frac{|\Omega(x-y)|}{|x_0-y|^n} |b(y)-b_B| |f(y)| \, dy \\
&\approx r^{\frac{n}{p}} \int_{(2B)^C} |\Omega(x-y)| |b(y)-b_B| |f(y)| \int_{|x_0-y|}^{\infty} \frac{dt}{t^{n+1}} \, dy \\
&\approx r^{\frac{n}{p}} \int_{2r}^{\infty} \int_{2r<|x_0-y|<t} |\Omega(x-y)| |b(y)-b_B| |f(y)| \, dy \frac{dt}{t^{n+1}} \\
&\lesssim r^{\frac{n}{p}} \int_{2r}^{\infty} \int_{B(x_0,t)} |\Omega(x-y)| |b(y)-b_B| |f(y)| \, dy \frac{dt}{t^{n+1}} \quad \text{holds.}
\end{aligned}
$$

Applying the Hölder's inequality and by (3.8), (4.1), (4.2), we get

$$J_1 \lesssim r^{\frac{n}{p}} \int\limits_{2r}^{\infty} \int\limits_{B(x_0,t)} |\Omega(x-y)| \, |b(y) - b_{B(x_0,t)}| \, |f(y)| \, dy \frac{dt}{t^{n+1}}$$

$$+ r^{\frac{n}{p}} \int\limits_{2r}^{\infty} |b_{B(x_0,r)} - b_{B(x_0,t)}| \int\limits_{B(x_0,t)} |\Omega(x-y)| \, |f(y)| \, dy \frac{dt}{t^{n+1}}$$

$$\lesssim r^{\frac{n}{p}} \int\limits_{2r}^{\infty} \|\Omega(\cdot - y)\|_{L_q(B(x_0,t))} \, \|(b(\cdot) - b_{B(x_0,t)})\|_{L_\mu(B(x_0,t))} \, \|f\|_{L_p(B(x_0,t))} \frac{dt}{t^{n+1}}$$

$$+ r^{\frac{n}{p}} \int\limits_{2r}^{\infty} |b_{B(x_0,r)} - b_{B(x_0,t)}| \, \|\Omega(\cdot - y)\|_{L_q(B(x_0,t))} \, \|f\|_{L_p(B(x_0,t))} \, |B(x_0,t)|^{1-\frac{1}{p}-\frac{1}{q}} \frac{dt}{t^{n+1}}$$

$$\lesssim \|b\|_* \, r^{\frac{n}{p}} \int\limits_{2r}^{\infty} \left(1 + \ln \frac{t}{r}\right) \|f\|_{L_p(B(x_0,t))} \frac{dt}{t^{\frac{n}{p}+1}}.$$

In order to estimate J_2 note that

$$J_2 = \|(b(\cdot) - b_{B(x_0,t)})\|_{L_p(B(x_0,t))} \int\limits_{(2B)^C} \frac{|\Omega(x-y)|}{|x_0 - y|^n} |f(y)| \, dy.$$

By (4.1), we get

$$J_2 \lesssim \|b\|_* \, r^{\frac{n}{p}} \int\limits_{(2B)^C} \frac{|\Omega(x-y)|}{|x_0 - y|^n} |f(y)| \, dy.$$

Thus, by (3.4) and (3.5)

$$J_2 \lesssim \|b\|_* \, r^{\frac{n}{p}} \int\limits_{2r}^{\infty} \|f\|_{L_p(B(x_0,t))} \frac{dt}{t^{\frac{n}{p}+1}}.$$

Summing up J_1 and J_2, for all $p \in (1,\infty)$ we get

$$(4.4) \qquad \|T_{\Omega,b} f_2\|_{L_p(B)} \lesssim \|b\|_* \, r^{\frac{n}{p}} \int\limits_{2r}^{\infty} \left(1 + \ln \frac{t}{r}\right) \|f\|_{L_p(B(x_0,t))} \frac{dt}{t^{\frac{n}{p}+1}}.$$

Finally, we have the following

$$\|T_{\Omega,b} f\|_{L_p(B)} \lesssim \|b\|_* \, \|f\|_{L_p(2B)} + \|b\|_* \, r^{\frac{n}{p}} \int\limits_{2r}^{\infty} \left(1 + \ln \frac{t}{r}\right) \|f\|_{L_p(B(x_0,t))} \frac{dt}{t^{\frac{n}{p}+1}},$$

which completes the proof of first statement by (3.7).

On the other hand when $p < q$, by the Fubini's theorem and the Minkowski inequality, we get

$$
J_1 \lesssim \left(\int_B \left| \int_{2r}^\infty \int_{B(x_0,t)} |b(y) - b_{B(x_0,t)}| |f(y)| |\Omega(x-y)| \, dy \frac{dt}{t^{n+1}} \right|^p dx \right)^{\frac{1}{p}}
$$

$$
+ \left(\int_B \left| \int_{2r}^\infty |b_{B(x_0,r)} - b_{B(x_0,t)}| \int_{B(x_0,t)} |f(y)| |\Omega(x-y)| \, dy \frac{dt}{t^{n+1}} \right|^p dx \right)^{\frac{1}{p}}
$$

$$
\lesssim \int_{2r}^\infty \int_{B(x_0,t)} |b(y) - b_{B(x_0,t)}| |f(y)| \|\Omega(\cdot - y)\|_{L_p(B(x_0,t))} \, dy \frac{dt}{t^{n+1}}
$$

$$
+ \int_{2r}^\infty |b_{B(x_0,r)} - b_{B(x_0,t)}| \int_{B(x_0,t)} |f(y)| \|\Omega(\cdot - y)\|_{L_p(B(x_0,t))} \, dy \frac{dt}{t^{n+1}}
$$

$$
\lesssim |B|^{\frac{1}{p} - \frac{1}{q}} \int_{2r}^\infty \int_{B(x_0,t)} |b(y) - b_{B(x_0,t)}| |f(y)| \|\Omega(\cdot - y)\|_{L_q(B(x_0,t))} \, dy \frac{dt}{t^{n+1}}
$$

$$
+ |B|^{\frac{1}{p} - \frac{1}{q}} \int_{2r}^\infty |b_{B(x_0,r)} - b_{B(x_0,t)}| \int_{B(x_0,t)} |f(y)| \|\Omega(\cdot - y)\|_{L_q(B(x_0,t))} \, dy \frac{dt}{t^{n+1}}.
$$

Applying the Hölder's inequality and by (3.8), (4.1), (4.2), we get

$$
J_1 \lesssim r^{\frac{n}{p} - \frac{n}{q}} \int_{2r}^\infty \left\| (b(\cdot) - b_{B(x_0,t)}) f \right\|_{L_1(B(x_0,t))} \left| B\left(x_0, \frac{3}{2}t\right) \right|^{\frac{1}{q}} \frac{dt}{t^{n+1}}
$$

$$
+ r^{\frac{n}{p} - \frac{n}{q}} \int_{2r}^\infty |b_{B(x_0,r)} - b_{B(x_0,t)}| \|f\|_{L_p(B(x_0,t))} \left| B\left(x_0, \frac{3}{2}t\right) \right|^{\frac{1}{q}} \frac{dt}{t^{\frac{n}{p}+1}}
$$

$$
\lesssim r^{\frac{n}{p} - \frac{n}{q}} \int_{2r}^\infty \left\| (b(\cdot) - b_{B(x_0,t)}) \right\|_{L_{p'}(B(x_0,t))} \|f\|_{L_p(B(x_0,t))} t^{\frac{n}{q}} \frac{dt}{t^{n+1}}
$$

$$
+ r^{\frac{n}{p} - \frac{n}{q}} \int_{2r}^\infty |b_{B(x_0,r)} - b_{B(x_0,t)}| \|f\|_{L_p(B(x_0,t))} t^{\frac{n}{q}} \frac{dt}{t^{\frac{n}{p}+1}}
$$

$$
\lesssim \|b\|_* \, r^{\frac{n}{p} - \frac{n}{q}} \int_{2r}^\infty \left(1 + \ln \frac{t}{r}\right) t^{\frac{n}{q} - \frac{n}{p} - 1} \|f\|_{L_p(B(x_0,t))} \, dt.
$$

Let $\frac{1}{p} = \frac{1}{\nu} + \frac{1}{q}$, then for J_2, by the Fubini's theorem, the Minkowski inequality, the Hölder's inequality and from (3.8), we get

$$J_2 \lesssim \left(\int\limits_B \left| \int\limits_{2r}^{\infty} \int\limits_{B(x_0,t)} |f(y)| \, |b(x) - b_B| \, |\Omega(x-y)| \, dy \frac{dt}{t^{n+1}} \right|^p dx \right)^{\frac{1}{p}}$$

$$\lesssim \int\limits_{2r}^{\infty} \int\limits_{B(x_0,t)} |f(y)| \, \|(b(\cdot) - b_B) \, \Omega(\cdot - y)\|_{L_p(B)} \, dy \frac{dt}{t^{n+1}}$$

$$\lesssim \int\limits_{2r}^{\infty} \int\limits_{B(x_0,t)} |f(y)| \, \|b(\cdot) - b_B\|_{L_\nu(B)} \, \|\Omega(\cdot - y)\|_{L_q(B)} \, dy \frac{dt}{t^{n+1}}$$

$$\lesssim \|b\|_* \, |B|^{\frac{1}{p} - \frac{1}{q}} \int\limits_{2r}^{\infty} \int\limits_{B(x_0,t)} |f(y)| \, \|\Omega(\cdot - y)\|_{L_q(B)} \, dy \frac{dt}{t^{n+1}}$$

$$\lesssim \|b\|_* r^{\frac{n}{p} - \frac{n}{q}} \int\limits_{2r}^{\infty} \|f\|_{L_1(B(x_0,t))} \left| B\left(x_0, \frac{3}{2}t\right) \right|^{\frac{1}{q}} \frac{dt}{t^{n+1}}$$

$$\lesssim \|b\|_* \, r^{\frac{n}{p} - \frac{n}{q}} \int\limits_{2r}^{\infty} \left(1 + \ln \frac{t}{r}\right) t^{\frac{n}{q} - \frac{n}{p} - 1} \|f\|_{L_p(B(x_0,t))} dt.$$

By combining the above estimates, we complete the proof of Lemma 4.1. $\qquad\square$

Now we can give the following theorem (our main result).

Theorem 4.2. *(Our main result) Suppose that $\Omega \in L_q(S^{n-1})$, $1 < q \le \infty$, is homogeneous of degree zero and $T_{\Omega,b}$ is a sublinear operator satisfying condition (1.2), bounded on $L_p(\mathbb{R}^n)$. Let $1 < p < \infty$ and $b \in BMO(\mathbb{R}^n)$. Let also, for $q' \le p$ the pair (φ_1, φ_2) satisfies the condition*

$$(4.5) \qquad \int\limits_r^{\infty} \left(1 + \ln \frac{t}{r}\right) \frac{\operatorname*{essinf}_{t < \tau < \infty} \varphi_1(x, \tau) \tau^{\frac{n}{p}}}{t^{\frac{n}{p} + 1}} dt \le C\varphi_2(x, r),$$

and for $p < q$ the pair (φ_1, φ_2) satisfies the condition

$$(4.6) \qquad \int\limits_r^{\infty} \left(1 + \ln \frac{t}{r}\right) \frac{\operatorname*{essinf}_{t < \tau < \infty} \varphi_1(x, \tau) \tau^{\frac{n}{p}}}{t^{\frac{n}{p} - \frac{n}{q} + 1}} dt \le C\varphi_2(x, r) r^{\frac{n}{q}},$$

where C does not depend on x and r.
Then, the operator $T_{\Omega,b}$ is bounded from M_{p,φ_1} to M_{p,φ_2}. Moreover,

$$\|T_{\Omega,b}f\|_{M_{p,\varphi_2}} \lesssim \|b\|_* \|f\|_{M_{p,\varphi_1}}.$$

Proof. The statement of Theorem 4.2 follows by Lemma 1.1 and Lemma 4.1 in the same manner as in the proof of Theorem 3.2. $\qquad\square$

For the sublinear commutator of the fractional maximal operator with rough kernel which is defined as follows

$$M_{\Omega,b}\left(f\right)(x) = \sup_{t>0} |B(x,t)|^{-1} \int\limits_{B(x,t)} |b\left(x\right) - b\left(y\right)| \, |\Omega\left(x - y\right)| \, |f(y)| dy$$

and for the linear commutator of the singular integral $[b, \overline{T}_\Omega]$ by Theorem 4.2 we get the following new results.

Corollary 4.1. *Suppose that $\Omega \in L_q(S^{n-1})$, $1 < q \le \infty$, is homogeneous of degree zero, $1 < p < \infty$ and $b \in BMO\left(\mathbb{R}^n\right)$. If for $q' \le p$ the pair (φ_1, φ_2) satisfies condition (4.5) and for $p < q$ the pair (φ_1, φ_2) satisfies condition (4.6). Then, the operators $M_{\Omega,b}$ and $[b, \overline{T}_\Omega]$ are bounded from M_{p,φ_1} to M_{p,φ_2}.*

We get the following new results for the sublinear commutator of the maximal operator

$$M_b\left(f\right)(x) = \sup_{t>0} |B(x,t)|^{-1} \int\limits_{B(x,t)} |b\left(x\right) - b\left(y\right)| \, |f(y)| dy$$

and for the linear commutator of the singular integral $[b, \overline{T}]$ by Theorem 4.2.

Corollary 4.2. *Let $1 < p < \infty$, $b \in BMO\left(\mathbb{R}^n\right)$ and the pair (φ_1, φ_2) satisfies condition (4.5). Then, the operators M_b and $[b, \overline{T}]$ are bounded from M_{p,φ_1} to M_{p,φ_2}.*

Now using above results, we also obtain the boundedness of the operator $T_{\Omega,b}$ on the vanishing generalized Morrey spaces $VM_{p,\varphi}$.

Theorem 4.3. *(Our main result) Let $\Omega \in L_q(S^{n-1})$, $1 < q \le \infty$, be homogeneous of degree zero. Let $T_{\Omega,b}$ is a sublinear operator satisfying condition (1.2) bounded on $L_p(\mathbb{R}^n)$. Let $1 < p < \infty$ and $b \in BMO\left(\mathbb{R}^n\right)$. Let for $q' \le p$ the pair (φ_1, φ_2) satisfies conditions (2.3)-(2.4) and*

$$(4.7) \qquad \int\limits_r^\infty \left(1 + \ln \frac{t}{r}\right) \varphi_1\left(x, t\right) \frac{t^{\frac{n}{p}}}{t^{\frac{n}{p}+1}} dt \le C_0 \varphi_2\left(x, r\right),$$

where C_0 does not depend on $x \in \mathbb{R}^n$ and $r > 0$,

$$(4.8) \qquad \lim_{r\to 0} \frac{\ln \frac{1}{r}}{\inf\limits_{x\in\mathbb{R}^n} \varphi_2(x,r)} = 0$$

and

$$(4.9) \qquad c_\delta := \int\limits_\delta^\infty (1 + \ln |t|) \sup_{x\in\mathbb{R}^n} \varphi_1\left(x, t\right) \frac{t^{\frac{n}{p}}}{t^{\frac{n}{p}+1}} dt < \infty$$

for every $\delta > 0$, and for $p < q$ the pair (φ_1, φ_2) satisfies conditions (2.3)-(2.4) and also

$$(4.10) \qquad \int\limits_r^\infty \left(1 + \ln \frac{t}{r}\right) \varphi_1\left(x, t\right) \frac{t^{\frac{n}{p}}}{t^{\frac{n}{p} - \frac{n}{q}+1}} dt \le C_0 \varphi_2(x, r) r^{\frac{n}{q}},$$

where C_0 does not depend on $x \in \mathbb{R}^n$ and $r > 0$

$$\lim_{r\to 0} \frac{\ln \frac{1}{r}}{\inf\limits_{x\in\mathbb{R}^n} \varphi_2(x,r)} = 0$$

and

$$(4.11) \qquad c_{\delta'} := \int\limits_{\delta'}^{\infty} (1 + \ln|t|) \sup_{x \in \mathbb{R}^n} \varphi_1(x, t) \frac{t^{\frac{n}{p}}}{t^{\frac{n}{p} - \frac{n}{q} + 1}} dt < \infty$$

for every $\delta' > 0$.

Then the operator $T_{\Omega,b}$ is bounded from VM_{p,φ_1} to VM_{p,φ_2}. Moreover,

$$(4.12) \qquad \|T_{\Omega,b}f\|_{VM_{p,\varphi_2}} \lesssim \|b\|_* \|f\|_{VM_{p,\varphi_1}}.$$

Proof. The norm inequality having already been provided by Theorem 4.2, we only have to prove the implication

(4.13)

$$\lim_{r \to 0} \sup_{x \in \mathbb{R}^n} \frac{r^{-\frac{n}{p}} \|f\|_{L_p(B(x,r))}}{\varphi_1(x, r)} = 0 \text{ implies } \lim_{r \to 0} \sup_{x \in \mathbb{R}^n} \frac{r^{-\frac{n}{p}} \|T_{\Omega,b}f\|_{L_p(B(x,r))}}{\varphi_2(x, r)} = 0.$$

To show that

$$\sup_{x \in \mathbb{R}^n} \frac{r^{-\frac{n}{p}} \|T_{\Omega,b}f\|_{L_p(B(x,r))}}{\varphi_2(x, r)} < \epsilon \text{ for small } r,$$

we use the estimate (4.3):

$$\sup_{x \in \mathbb{R}^n} \frac{r^{-\frac{n}{p}} \|T_{\Omega,b}f\|_{L_p(B(x,r))}}{\varphi_2(x, r)} \lesssim \frac{\|b\|_*}{\varphi_2(x, r)} \int\limits_r^{\infty} \left(1 + \ln \frac{t}{r}\right) t^{-\frac{n}{p} - 1} \|f\|_{L_p(B(x_0, t))} dt.$$

We take $r < \delta_0$, where δ_0 will be chosen small enough and split the integration:

$$(4.14) \qquad \frac{r^{-\frac{n}{p}} \|T_{\Omega,b}f\|_{L_p(B(x,r))}}{\varphi_2(x, r)} \leq C \left[I_{\delta_0}(x, r) + J_{\delta_0}(x, r) \right],$$

where $\delta_0 > 0$ (we may take $\delta_0 < 1$), and

$$I_{\delta_0}(x, r) := \frac{1}{\varphi_2(x, r)} \int\limits_r^{\delta_0} \left(1 + \ln \frac{t}{r}\right) t^{-\frac{n}{p} - 1} \|f\|_{L_p(B(x, t))} dt,$$

and

$$J_{\delta_0}(x, r) := \frac{1}{\varphi_2(x, r)} \int\limits_{\delta_0}^{\infty} \left(1 + \ln \frac{t}{r}\right) t^{-\frac{n}{p} - 1} \|f\|_{L_p(B(x, t))} dt$$

Now we choose any fixed $\delta_0 > 0$ such that

$$\sup_{x \in \mathbb{R}^n} \frac{t^{-\frac{n}{p}} \|f\|_{L_p(B(x, t))}}{\varphi_1(x, t)} < \frac{\epsilon}{2CC_0}, \qquad t \leq \delta_0,$$

where C and C_0 are constants from (4.7) and (4.14). This allows to estimate the first term uniformly in $r \in (0, \delta_0)$:

$$\sup_{x \in \mathbb{R}^n} CI_{\delta_0}(x, r) < \frac{\epsilon}{2}, \qquad 0 < r < \delta_0.$$

For the second term, writing $1 + \ln \frac{t}{r} \leq 1 + |\ln t| + \ln \frac{1}{r}$, we obtain

$$J_{\delta_0}(x, r) \leq \frac{c_{\delta_0} + \widetilde{c_{\delta_0}} \ln \frac{1}{r}}{\varphi_2(x, r)} \|f\|_{M_{p,\varphi}},$$

where c_{δ_0} is the constant from (4.9) with $\delta = \delta_0$ and $\widetilde{c_{\delta_0}}$ is a similar constant with omitted logarithmic factor in the integrand. Then, by (4.8) we can choose small enough r such that

$$\sup_{x \in \mathbb{R}^n} J_{\delta_0}(x, r) < \frac{\epsilon}{2},$$

which completes the proof of (4.13).

For the case of $p < q$, we can also use the same method, so we omit the details.
$\square$

Remark 4.2. Conditions (4.9) and (4.11) are not needed in the case when $\varphi(x, r)$ does not depend on x, since (4.9) follows from (4.7) and similarly, (4.11) follows from (4.10) in this case.

Corollary 4.3. *Suppose that* $\Omega \in L_q(S^{n-1})$, $1 < q \leq \infty$, *is homogeneous of degree zero,* $1 < p < \infty$ *and* $b \in BMO(\mathbb{R}^n)$. *If for* $q' \leq p$ *the pair* (φ_1, φ_2) *satisfies conditions (2.3)-(2.4)-(4.8) and (4.9)-(4.7) and for* $p < q$ *the pair* (φ_1, φ_2) *satisfies conditions (2.3)-(2.4)-(4.8) and (4.11)-(4.10). Then, the operators* $M_{\Omega,b}$ *and* $[b, \overline{T}_\Omega]$ *are bounded from* VM_{p,φ_1} *to* VM_{p,φ_2}.

In the case of $q = \infty$ by Theorem 4.3, we get

Corollary 4.4. *Let* $1 < p < \infty$, $b \in BMO(\mathbb{R}^n)$ *and the pair* (φ_1, φ_2) *satisfies conditions (2.3)-(2.4)-(4.8) and (4.9)-(4.7). Then the operators* M_b *and* $[b, \overline{T}]$ *are bounded from* VM_{p,φ_1} *to* VM_{p,φ_2}.

5. SOME APPLICATIONS

In this section, we give the applications of Theorem 3.2, Theorem 3.3, Theorem 4.2, Theorem 4.3 for the Marcinkiewicz operator.

5.1. **Marcinkiewicz Operator.** Let $S^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$ be the unit sphere in $\mathbb{R}^n$ equipped with the Lebesgue measure $d\sigma$. Suppose that Ω satisfies the following conditions.

(a) Ω is the homogeneous function of degree zero on $\mathbb{R}^n \setminus \{0\}$, that is,

$$\Omega(\mu x) = \Omega(x), \text{ for any } \mu > 0, x \in \mathbb{R}^n \setminus \{0\}.$$

(b) Ω has mean zero on S^{n-1}, that is,

$$\int_{S^{n-1}} \Omega(x') d\sigma(x') = 0,$$

where $x' = \frac{x}{|x|}$ for any $x \neq 0$.

(c) $\Omega \in Lip_\gamma(S^{n-1})$, $0 < \gamma \leq 1$, that is there exists a constant $M > 0$ such that,

$$|\Omega(x') - \Omega(y')| \leq M|x' - y'|^\gamma \text{ for any } x', y' \in S^{n-1}.$$

In 1958, Stein [46] defined the Marcinkiewicz integral of higher dimension μ_Ω as

$$\mu_\Omega(f)(x) = \left(\int_0^\infty |F_{\Omega,t}(f)(x)|^2 \frac{dt}{t^3} \right)^{1/2},$$

where

$$F_{\Omega,t}(f)(x) = \int_{|x-y| \leq t} \frac{\Omega(x-y)}{|x-y|^{n-1}} f(y) dy.$$

Since Stein's work in 1958, the continuity of Marcinkiewicz integral has been extensively studied as a research topic and also provides useful tools in harmonic analysis [27, 47, 48, 49].

The sublinear commutator of the operator μ_Ω is defined by

$$[b, \mu_\Omega](f)(x) = \left(\int_0^\infty |F_{\Omega,t,b}(f)(x)|^2 \frac{dt}{t^3} \right)^{1/2},$$

where

$$F_{\Omega,t,b}(f)(x) = \int_{|x-y| \leq t} \frac{\Omega(x-y)}{|x-y|^{n-1}} [b(x) - b(y)] f(y) dy.$$

We consider the space $H = \{h : \|h\| = (\int_0^\infty |h(t)|^2 \frac{dt}{t^3})^{1/2} < \infty\}$. Then, it is clear that $\mu_\Omega(f)(x) = \|F_{\Omega,t}(x)\|$.

By the Minkowski inequality and the conditions on Ω, we get

$$\mu_\Omega(f)(x) \leq \int_{\mathbb{R}^n} \frac{|\Omega(x-y)|}{|x-y|^{n-1}} |f(y)| \left(\int_{|x-y|}^\infty \frac{dt}{t^3} \right)^{1/2} dy \leq C \int_{\mathbb{R}^n} \frac{|\Omega(x-y)|}{|x-y|^n} |f(y)| dy.$$

Thus, μ_Ω satisfies the condition (1.1). It is known that μ_Ω is bounded on $L_p(\mathbb{R}^n)$ for $p > 1$, and bounded from $L_1(\mathbb{R}^n)$ to $WL_1(\mathbb{R}^n)$ for $p = 1$ (see [50]), then from Theorems 3.2, 3.3, 4.2 and 4.3 we get

Corollary 5.1. *Let $1 \leq p < \infty$, $\Omega \in L_q(S^{n-1})$, $1 < q \leq \infty$. Let also, for $q' \leq p$, $p \neq 1$, the pair (φ_1, φ_2) satisfies condition (3.10) and for $1 < p < q$ the pair (φ_1, φ_2) satisfies condition (3.11) and Ω satisfies conditions (a)–(c). Then the operator μ_Ω is bounded from M_{p,φ_1} to M_{p,φ_2} for $p > 1$ and from M_{1,φ_1} to WM_{1,φ_2} for $p = 1$.*

Corollary 5.2. *Let $1 \leq p < \infty$, $\Omega \in L_q(S^{n-1})$, $1 < q \leq \infty$. Let also, for $q' \leq p$, $p \neq 1$, the pair (φ_1, φ_2) satisfies conditions (2.3)-(2.4) and (3.14)-(3.15) and for $1 < p < q$ the pair (φ_1, φ_2) satisfies conditions (2.3)-(2.4) and (3.16)-(3.17) and Ω satisfies conditions (a)–(c). Then the operator μ_Ω is bounded from VM_{p,φ_1} to VM_{p,φ_2} for $p > 1$ and from VM_{1,φ_1} to WVM_{1,φ_2}.*

Corollary 5.3. *Suppose that $\Omega \in L_q(S^{n-1})$, $1 < q \leq \infty$, is homogeneous of degree zero, $1 < p < \infty$ and $b \in BMO(\mathbb{R}^n)$. Let also, for $q' \leq p$ the pair (φ_1, φ_2) satisfies condition (4.5) and for $p < q$ the pair (φ_1, φ_2) satisfies condition (4.6) and Ω satisfies conditions (a)–(c). Then, the operator $[b, \mu_\Omega]$ is bounded from M_{p,φ_1} to M_{p,φ_2}.*

Corollary 5.4. *Suppose that $\Omega \in L_q(S^{n-1})$, $1 < q \leq \infty$, is homogeneous of degree zero, $1 < p < \infty$ and $b \in BMO(\mathbb{R}^n)$. Let also, for $q' \leq p$ the pair (φ_1, φ_2) satisfies conditions (2.3)-(2.4)-(4.8) and (4.9)-(4.7) and for $p < q$ the pair (φ_1, φ_2) satisfies conditions (2.3)-(2.4)-(4.8) and (4.11)-(4.10) and Ω satisfies conditions (a)–(c). Then, the operator $[b, \mu_\Omega]$ is bounded from VM_{p,φ_1} to VM_{p,φ_2}.*

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DISTANCE MEASURE, SIMILARITY MEASURE, ENTROPY AND INCLUSION MEASURE FOR TEMPORAL INTUITIONISTIC FUZZY SETS

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Abstract. In this study, we proposed distance measures similarity measure, entropy and inclusion measure for temporal intuitionistic fuzzy sets and investigated some properties of these measures. We defined these concepts in two different ways, namely temporal and overall and we examined the relationship between these definitions. Also we gave numerical examples for TIFS. We compared these measures defined with two and three parameters in terms of reliability and applicability.

1. Introduction

The concept of temporal intuitionistic fuzzy sets is defined by Atanassov in 1991. In this theory membership and non-membership degrees are defined depending on the time-moment and element. This idea leads to a rich field to be used in applications on dynamic fields such as weather, economy, image and video processing. On the other hand, the similarity and distance measures on fuzzy and intuitionistic fuzzy sets, as seen in present studies, are used in many different areas and obtained effective results. Temporal intuitionistic fuzzy measures which achieved by combining these two ideas are still not defined in the literature. This is one of major shortcoming of temporal intuitionistic fuzzy set theory. Temporal intuitionistic fuzzy measurement is a natural consequence of idea that making dynamic measurements used in the dynamic areas.

In this study, firstly we give definitions of temporal intuitionistic fuzzy distance and similarity measures. Then, we investigate some major properties of these measures. Also we investigate how to define these measurements. With more clearly, these measures will be examined by defining which parameters need to adhere to. Additionally, the concept of entropy and inclusion which are closely related to aforementioned measures are defined to the temporal intuitionistic fuzzy sets. Finally, some other basic concepts needed in this context will be defined in temporal space intuitionistic fuzzy sets.

2. Preliminaries

Definition 2.1. [1] An intuitionistic fuzzy set on a non-empty set X given by a set of ordered triples $A = \{(x, \mu_A(x), \eta_A(x)) : x \in X\}$ where $\mu_A(x) : X \to I = [0, 1]$, $\eta_A(x) : X \to I$, are functions such that $0 \leq \mu(x) + \eta(x) \leq 1$ for all $x \in X$. For $x \in X$, $\mu_A(x)$ and $\eta_A(x)$ represent the degree of membership and degree of non-membership of x to A respectively. For each $x \in X$; intuitionistic fuzzy index of x in A is defined as follows $\pi_A(x) = 1 - \mu_A(x) - \eta_A(x)$. π_A is the called degree of hesitation or indeterminacy.

Definition 2.2. [1] Let $A, B \in IFS(X)$. Then,
(i) $A \subseteq B \Leftrightarrow \mu_A(x) \leq \mu_B(x)$ and $\eta_A(x) \geq \eta_B(x)$ for $\forall x \in X$,
(ii) $A = B \Leftrightarrow A \subseteq B$ and $B \subseteq A$,
(iii) $\bar{A} = \{(x, \eta_A(x), \mu_A(x)) : x \in X\}$,
(iv) $\bigcap A_i = \{(x, \wedge \mu_{A_i}(x), \vee \eta_{A_i}(x)) : x \in X\}$,
(v) $\bigcup A_i = \{(x, \vee \mu_{A_i}(x), \wedge \eta_{A_i}(x)) : x \in X\}$,
(vi) $\underset{\sim}{0} = \{(x, 0, 1) : x \in X\}$ and $\underset{\sim}{1} = \{(x, 1, 0) : x \in X\}$..

Definition 2.3. [2] Let X be an universe and T be a non-empty time set. We call the elements of T as "time moments". Based on the definition of IFS, a temporal intuitionistic fuzzy set (TIFS) is defined as the following:

$$A(T) = \{(x, \mu_A(x, t), \eta_A(x, t)) : X \times T\}$$

where:
a. $A \subseteq X$ is a fixed set
b. $\mu_A(x, t) + \eta_A(x, t) \leq 1$ for every $(x, t) \in X \times T$
c. $\mu_A(x, t)$ and $\eta_A(x, t)$ are the degrees of membership and non-membership, respectively, of the element $x \in X$ at the time moment $t \in T$

For brevity, we write A instead of $A(T)$. The hesitation degree of an TIFS is defined as $\pi_A(x, t) = 1 - \mu_A(x, t) - \eta_A(x, t)$. Obviously, every ordinary IFS can be regarded as TIFS for which T is a singleton set. All operations and operators on IFS can be defined for TIFSs.

By $TIFS^{(X,T)}$, we denote to the set of all temporal intuitionistic fuzzy sets defined on X and time set T. Obviously, each intuitionistic fuzzy sets can be expressed as temporal intuitionistic fuzzy set via a singular time set. In additionally, all operations and operators defined for intuitionistic fuzzy sets can be defined for temporal intuitionistic fuzzy sets.

Definition 2.4. [2] Let
$$A(T') = \{(x, \mu_A(x, t), \eta_A(x, t)) : X \times T'\}$$
and
$$B(T'') = \{(x, \mu_B(x, t), \eta_B(x, t)) : X \times T''\}$$
where T' and T'' have finite number of distinct time-elements or they are time intervals. Then;
$A(T') \cap B(T'') =$
$\{(x, \ min(\bar{\mu}_A(x, t), \bar{\mu}_B(x, t)), \ max(\bar{\eta}_A(x, t), \bar{\eta}_B(x, t))) : \ (x, t) \in X \times (T' \cup T'')\}$
and

$A(T') \cup B(T'') =$

$\{(x, \max(\bar{\mu}_A(x,t), \bar{\mu}_B(x,t)), \min(\bar{\eta}_A(x,t), \bar{\eta}_B(x,t))) : (x,t) \in X \times (T' \cup T'')\}$

Also from definition of subset in IFS theory, Subsets of TIFS can be defined as the following:

$$A(T') \subseteq B(T'') \Leftrightarrow \bar{\mu}_A(x,t) \geq \bar{\mu}_B(x,t) \text{ and } \bar{\eta}_A(x,t) \leq \bar{\eta}_B(x,t)$$

for every $[(x,t) \in X \times (T' \cup T'')$ where

$$\bar{\mu}_A(x,t) = \begin{cases} \mu_A(x,t) & , \text{ if } \quad t \in T' \\ 0 & , \text{ if } \quad t \in T'' - T' \end{cases}$$

$$\bar{\mu}_B(x,t) = \begin{cases} \mu_B(x,t) & , \text{ if } \quad t \in T'' \\ 0 & , \text{ if } \quad t \in T' - T'' \end{cases}$$

$$\bar{\eta}_A(x,t) = \begin{cases} \eta_A(x,t) & , \text{ if } \quad t \in T' \\ 1 & , \text{ if } \quad t \in T'' - T' \end{cases}$$

$$\bar{\eta}_B(x,t) = \begin{cases} \eta_B(x,t) & , \text{ if } \quad t \in T'' \\ 1 & , \text{ if } \quad t \in T' - T'' \end{cases}$$

It is obviously seen that if $T' = T''$; $\bar{\mu}_A(x,t) = \mu_A(x,t)$, $\bar{\mu}_B(x,t) = \mu_B(x,t)$, $\bar{\eta}_A(x,t) = \eta_A(x,t)$, $\bar{\eta}_B(x,t) = \eta_B(x,t)$. [2]

Let J be an index set and T_i is a time set for each $i \in J$. Let define that $T = \bigcup_{i \in J} T_i$. Now we extend union and intersection of temporal intuitionistic fuzzy sets to the family $F = \{A_i(T_i) = (x, \mu_{A_i}(x,t), \eta_{A_i}(x,t)) : x \in X \times T_i, i \in J\}$ as:

$$\bigcup_{i \in J} A(T_i) = \left\{ \left(x, \max_{i \in J}(\bar{\mu}_{A_i}(x,t)), \min_{i \in J}(\bar{\eta}_{A_i}(x,t)) : (x,t) \in X \times T \right) \right\},$$

$$\bigcap_{i \in J} A(T_i) = \left\{ \left(x, \min_{i \in J}(\bar{\mu}_{A_i}(x,t)), \max_{i \in J}(\bar{\eta}_{A_i}(x,t)) : (x,t) \in X \times T \right) \right\}$$

where

$$\bar{\mu}_{A_i}(x,t) = \begin{cases} \mu_{A_i}(x,t) & , \text{ if } \quad t \in T_i \\ 0 & , \text{ if } \quad t \in T - T_i \end{cases}$$

$$\bar{\eta}_{A_i}(x,t) = \begin{cases} \eta_{A_i}(x,t) & , \text{ if } \quad t \in T_i \\ 1 & , \text{ if } \quad t \in T - T_i \end{cases}$$

3. Distance Measure, Similarity Measure, Entropy And Inclusion Measure For Temporal Intuitionistic Fuzzy Sets

Let X be a universe and T be a non-empty time-moment set. The definition of distance measure defined in [31] can be extended to $TIFS^{(X,T)}$ such as:

Definition 3.1. Let X be a universe, T be a non-empty time-moment set and $d^t : TIFS^{(X,T)} \times TIFS^{(X,T)} \to R^+ \cup \{0\}$ be a mapping for fixed $t \in T$. If d^t satisfies following properties for all $A, B \in TIFS^{(X,T)}$ and fixed time-moment $t \in T$, then d^t is called a temporal intuitionistic fuzzy distance measure on $TIFS^{(X,T)}$ at time-moment t :

D1. $d^t(A, B) = 0 \Leftrightarrow A = B$,

D2. $d^t(A, B) = d^t(B, A)$,

D3. If A is crisp set, $d^t(A, \bar{A}) = \max_{B,C \in TIFS^{(X,T)}} d^t(B, C)$

D4. If $A \subseteq B \subseteq C$ for A, B, $C \in$ $TIFS^{(X,T)}$, $d^t(A,C) \geq d^t(A,B)$ and $d^t(A,C) \geq d^t(B,C)$.

If there exist a mapping $d : TIFS^{(X,T)} \times TIFS^{(X,T)} \to R^+ \cup \{0\}$ which satisfies these conditions for every time moment $t \in T$, then d is called a overall intuitionistic fuzzy distance measure on $TIFS^{(X,T)}$. In order to distinguish between these two concepts from each other, the first measure is referred as temporal intuitionistic fuzzy distance measure at just only time moment t. But the second one measures overall distance between temporal intuitionistic fuzzy sets in full time range.

Distance measure between temporal intuitionistic fuzzy sets in terms of being discrete time set or interval time set can be defined in the following way (see for more information: [4], [13], [16], [18], [20], [22], [31], [33], [34], [37], [43]):

Theorem 3.1. *Let X be non-empty set and $T = \{t_1, t_2, t_3, ..., t_n\}$ be finite and distinct time set. Let define $A, B \in TIFS^{(X,T)}$ such as*

$$A(T) = \{(x, \mu_A(x,t), \eta_A(x,t)) : (x,t) \in X \times T\}$$

and

$$B(T) = \{(x, \mu_B(x,t), \eta_B(x,t)) : (x,t) \in X \times T\}$$

, respectively. Then we define $d^i{}_t(A,B)$ mappings for $t \in T$ and $i = 1, 2, 3, 4$ as following:

1. $d^1{}_{t_0}(A,B) = \sqrt{\sum\limits_{x \in X} \left((\mu_A(x,t_0) - \mu_B(x,t_0))^2 + (\eta_A(x,t_0) - \eta_B(x,t_0))^2 \right)},$

2. $d^2{}_{t_0}(A,B) = \sum\limits_{x \in X} (|\mu_A(x,t_0) - \mu_B(x,t_0)| + |\eta_A(x,t_0) - \eta_B(x,t_0)|),$

3. $d^3{}_{t_0}(A,B) = \sum\limits_{x \in E} (\max\{|\mu_A(x,t_0) - \mu_B(x,t_0)|, |\eta_A(x,t_0) - \eta_B(x,t_0)|\}),$

4. $d^4{}_{t_0}(A,B) = \sqrt{\sum\limits_{x \in E} \max\left\{ (\mu_A(x,t_0) - \mu_B(x,t_0))^2, \left(\eta_A(x,t_0) - \eta_B(x,t_0)^2\right) \right\}}.$

It is clear that each mapping $d^i{}_t(A,B)$ is a temporal intuitionistic fuzzy distance measures for $t \in T$. On the other hand, the following temporal distance measures are obtained by adding the degree of uncertainty to former ones.

5. $D^1{}_{t_0}(A,B) =$

$$\sqrt{\sum\limits_{x \in X} \left((\mu_A(x,t_0) - \mu_B(x,t_0))^2 + (\eta_A(x,t_0) - \eta_B(x,t_0))^2 + (\pi_A(x,t_0) - \pi_B(x,t_0))^2 \right)}$$

6. $D^2{}_{t_0}(A,B) =$

$$\sum\limits_{x \in X} (|\mu_A(x,t_0) - \mu_B(x,t_0)| + |\eta_A(x,t_0) - \eta_B(x,t_0)| + |\pi_A(x,t_0) - \pi_B(x,t_0)|)$$

7. $D^3{}_{t_0}(A,B) =$

$$\sum\limits_{x \in E} \max\{|\mu_A(x,t_0) - \mu_B(x,t_0)|, |\eta_A(x,t_0) - \eta_B(x,t_0)|, |\pi_A(x,t_0) - \pi_B(x,t_0)|\}$$

8. $D^4{}_{t_0}(A,B) =$

$$\sqrt{\sum\limits_{x \in E} \max\left\{ (\mu_A(x,t_0) - \mu_B(x,t_0))^2, \left(\eta_A(x,t_0) - \eta_B(x,t_0)^2\right), (\pi_A(x,t_0) - \pi_B(x,t_0))^2 \right\}}$$

Additionally, the mappings d_i (or D_i) defined as $d_i(A,B) = \max\limits_{t \in T}\left(d^i{}_t(A,B)\right)$ (or $D_i(A,B) = \max\limits_{t \in T}\left(D^i{}_t(A,B)\right)$) are overall temporal intuitionistic fuzzy distance measures for $i = 1,2,3,4$. It is clear that these temporal and overall intuitionistic fuzzy distance measures are intuitionistic fuzzy distance measure for a singleton time set.

Proof. As it can be seen in studies ([4], [13], [16], [18], [20], [22], [31], [33], [34], [37], [43], etc.) temporal intuitionistic fuzzy distance measures defined above is obtained by the addition of time parameters to the intuitionistic fuzzy distance measures. As stated previously, these measures are also intuitionistic fuzzy distance measure for each individual time moment. Now we prove that $d_1(A,B) = \max\limits_{t \in T}\left(d^1{}_t(A,B)\right)$ is an overall intuitionistic fuzzy distance measure.

D1: Since $d^1{}_t(A,A) = 0$ for all $t \in T$ and $A \in TIFS^{(X,T)}$, it is clearly obtained that $d_1(A,A) = 0$.

D2: Since $d^1{}_t(A,B) = d^1{}_t(B,A)$ for all $t \in T$ and $A,B \in TIFS^{(X,T)}$, Then it is clearly obtained that $d_1(A,B) = \max\limits_{t \in T}\left(d^1{}_t(A,B)\right) = \max\limits_{t \in T}\left(d^1{}_t(B,A)\right) = d_1(B,A)$.

D3: Since $d^1{}_t\left(A,\bar{A}\right) = \max\limits_{B,C \in TIFS^{(X,T)}} d^1{}_t(B,C)$ for all $t \in T$ and A crisp set.

Then, we can easily get $d^1\left(A,\bar{A}\right) = \max\limits_{t \in T}\left(d^1{}_t\left(A,\bar{A}\right)\right) = \max\limits_{t \in T}\left\{\max\limits_{B,C \in TIFS^{(X,T)}} d^1{}_t(B,C)\right\} = \max\limits_{B,C \in TIFS^{(X,T)}}\left\{\max\limits_{t \in T} d_t(B,C)\right\} = \max\limits_{B,C \in TIFS^{(X,T)}} d^1(B,C)$

D4: From the definition of being subset in concept of TIFS, When $A \subseteq B \subseteq C$ for A, B, $C \in TIFS^{(X,T)}$, the inequalities $\mu_A(x,t) \geq \mu_B(x,t) \geq \mu_C(x,t)$ and $\eta_A(x,t) \leq \eta_B(x,t) \leq \eta_C(x,t)$ are hold for each $(x,t) \in X \times T$. From the last two inequalities the inequalities $d_1^t(A,C) \geq d_1^t(A,B)$ and $d_1^t(A,C) \geq d_1^t(B,C)$ are obtained for $t \in T$. Then the inequalities $d_1(A,C) \geq d_1(A,B)$ and $d_1(A,C) \geq d_1(B,C)$ are clearly obtained from definition of d_1. The other situations can be proved similarly. $\qquad\square$

Each distance measures $d^i{}_t$ indicates temporal distance between the temporal intuitionistic fuzzy sets at time moment $t \in T$. On the other hand, d_i measurements which are obtained from the maximum of $d^i{}_t$ gives a overall distance measurement between temporal intuitionistic fuzzy sets. These two approaches gain different importance degrees depending on the applications. With more open expression, overall distance measure expresses inferential distance in the total situation, while since the temporal distance between the temporal intuitionistic fuzzy sets are sensitive to sudden changes at distance, temporal distance is a measure of instant change between cases represented by the temporal intuitionistic fuzzy set. This situation offers multiple ways to ensure expected benefits from the application.

We give definition of temporal intuitionistic distance measures on infinite and interval time set as follow:

Proposition 3.1. *Let X be a infinite set and $T = \{t_1, t_2, , ..., t_i, ...\}$ be a infinite time set (or time interval). Let suppose that $t_k \leq t_{k+1}$ for each $t_k, t_{k+1} \in T$. On the other hand, let define A, $B \in TIFS^{(X,T)}$ TIFSs as follows:*

$$A(T) = \{(x, \mu_A(x,t), \eta_A(x,t)) : (x,t) \in X \times T\}$$

and

$$B\left(T\right) = \left\{\left(x, \mu_B\left(x, t\right), \eta_B\left(x, t\right)\right) : \left(x, t\right) \in X \times T\right\},$$

respectively. With these definitions, the following statements are overall intuitionistic fuzzy distance measure on $TIFS^{(X,T)}$

1. $d^*{}_1\left(A, B\right) =$

$$\sup_{t \in T} \left\{\max \left\{\sup_{x \in X} \left|\mu_A\left(x, t\right) - \mu_B\left(x, t\right)\right|, \sup_{x \in X} \left|\eta_A\left(x, t\right) - \eta_B\left(x, t\right)\right|\right\}\right\}$$

2. $d^*{}_2\left(A, B\right) =$

$$\sup_{t \in T} \left\{\sup_{x \in X} \left(\left|\mu_A\left(x, t\right) - \mu_B\left(x, t\right)\right| + \left|\eta_A\left(x, t\right) - \eta_B\left(x, t\right)\right|\right)\right\}$$

3. $d^*{}_3\left(A, B\right) =$

$$\sup_{t \in T} \sqrt{\sup_{x \in X} \left(\left(\mu_A\left(x, t\right) - \mu_A\left(x, t\right)\right)^2 + \left(\eta_A\left(x, t\right) - \eta_A\left(x, t\right)\right)^2\right)}$$

4. $D^*{}_1\left(A, B\right) =$

$$\sup_{t \in T} \left\{\max \left\{\sup_{x \in X} \left|\mu_A\left(x, t\right) - \mu_B\left(x, t\right)\right|, \sup_{x \in X} \left|\eta_A\left(x, t\right) - \eta_B\left(x, t\right)\right|, \sup_{x \in X} \left|\pi_A\left(x, t\right) - \pi_B\left(x, t\right)\right|\right\}\right\}$$

5. $D^*{}_2\left(A, B\right) =$

$$\sup_{t \in T} \left(\sup_{x \in X} \left(\left|\mu_A\left(x, t\right) - \mu_B\left(x, t\right)\right| + \left|\eta_A\left(x, t\right) - \eta_B\left(x, t\right)\right| + \left|\pi_A\left(x, t\right) - \pi_B\left(x, t\right)\right|\right)\right)$$

6. $D^*{}_3\left(A, B\right) =$

$$\sup_{t \in T} \sqrt{\sup_{x \in X} \left(\left(\mu_A\left(x, t\right) - \mu_B\left(x, t\right)\right)^2 + \left(\eta_A\left(x, t\right) - \eta_B\left(x, t\right)\right)^2 + \left(\pi_A\left(x, t\right) - \pi_B\left(x, t\right)\right)^2\right)}$$

(see [4], [13], [16], [18], [20], [22], [31], [33], [34], [37], [43], etc.)

Proposition 3.2. *Let* X *be an infinite set and* $T = [t_1, t_2]$ *for* $t_1, t_2 \in R^+$ *and* $t_1 < t_2$ *. Then the following statements are overall intuitionistic c fuzzy distance measures on* $TIFS^{(X,T)}$

1. $D^{***}{}_1\left(A, B\right) = \sup_{t \in T} \left\{\int_X \left(\mu_A\left(x, t\right) - \mu_B\left(x, t\right)\right).dx + \left|\int_X \left(\eta_A\left(x, t\right) - \eta_A\left(x, t\right)\right).dx\right|\right\}$

2. $D^{***}{}_2\left(A, B\right) =$

$$\sup_{t \in T} \left\{\left|\int_X \left(\mu_A\left(x, t\right) - \mu_B\left(x, t\right)\right).dx\right| + \left|\int_X \left(\eta_A\left(x, t\right) - \eta_B\left(x, t\right)\right).dx\right| + \left|\int_X \left(\pi_A\left(x, t\right) - \pi_B\left(x, t\right)\right).dt\right|\right\}$$

for $A, B \in TIFS^{(X,T)}$ *and* $t \in T = [t_1, t_2]$. *(see [4], [13], [16], [18], [20], [22], [31], [33], [34], [37], [43], etc.)*

Example 3.1. Let suppose that $X = [0, 4]$ and $T = [0, 5]$. Let define that

$$A\left(T\right) = \left\{\left(\left(x, t\right), \mu_A\left(x, t\right), \eta_A\left(x, t\right)\right); \left(x, t\right) \in X \times T\right\}$$

and

$$B\left(T\right) = \left\{\left(\left(x, t\right), \mu_B\left(x, t\right), \eta_B\left(x, t\right)\right); \left(x, t\right) \in X \times T\right\}$$
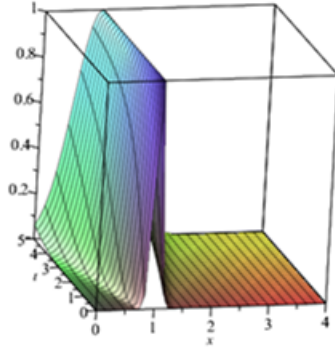
FIGURE 1
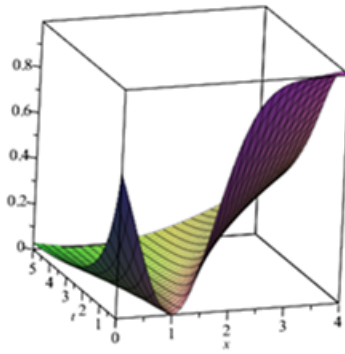
FIGURE 2

where degrees of membership and non-membership are defined as follows respectively $\mu_A(x,t) = e^{-10\frac{(x-1)^2}{t+1}}$ and $\eta_A(x,t) = 1 - e^{-\frac{(x-1)^2}{t+1}}$; $\mu_B(x,t) = \frac{1}{1+e^{-t.(x-2)^2}}$ and $\eta_B(x,t) = \frac{e^{-t.(x-2)^2}}{2+e^{-t.(x-2)}}$ for all $(x,t) \in X \times T$. 3D- graphics of μ_A, η_A, μ_B and η_B are given in Fig.1,2,3,4, respectively.

Fig 1. Graphic of temporal intuitionistic fuzzy membership function μ_A

Fig 2. Graphic of temporal intuitionistic fuzzy non-membership function η_A

Fig 3. Graphic of temporal intuitionistic fuzzy membership function μ_B

Fig 4. Graphic of temporal intuitionistic fuzzy non-membership function η_B

In the following figures, we give changing of distance between A and B obtained by D_1^{***} and D_2^{***} over time in Fig 5. and Fig. 6.

Fig. 5. Changing of distance between A and B obtained by D_1^{***}
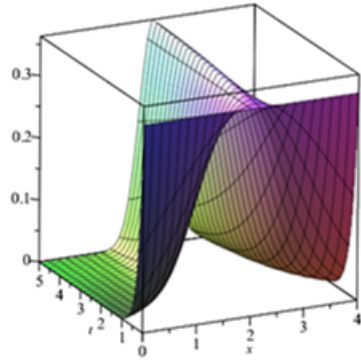
FIGURE 3
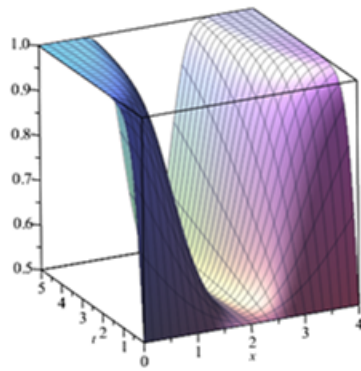
FIGURE 4
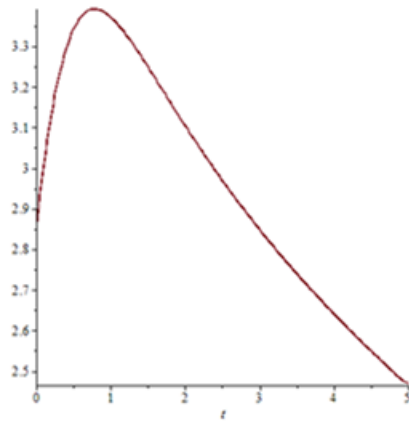
FIGURE 5
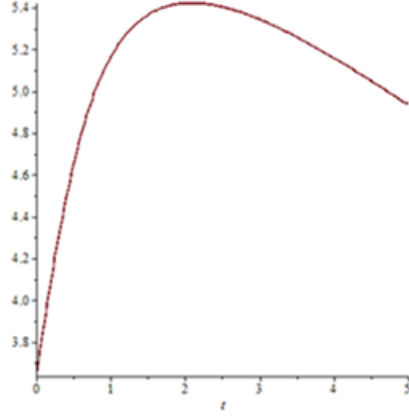
FIGURE 6

Fig. 6. Changing of distance between A and B obtained by D_2^{***}

From these figures, the following overall distances are obtained

$$d_{D_1^{***}}(A, B) = \sup_{t \in T} \{D_1^{***}(A, B)\} = 3,391$$

and

$$d_{D_2^{***}}(A, B) = \sup_{t \in T} \{D_2^{***}(A, B)\} = 5,415$$

respectively. A key issue at this point is D_2^{***} temporal distance measure is more durable and more reliable than D_1^{***} in high degrees of uncertainty. As previously stated in various studies of Szmidt and Kacprzyk for intuitionistic fuzzy sets, in the cases which contains data with high degree of uncertainty, it is obvious that temporal (or overall) distance which obtained with three parameters between temporal intuitionistic fuzzy sets is more accurate than temporal (or overall) distance which obtained with two parameters .

We use aggregation function to generalize the correlation between temporal and overall intuitionistic fuzzy distance as used in [11].

Theorem 3.2. *Let X be a non-empty set and $T = \{t_1, t_2, ..., t_n\}$ be a finite time set. Let suppose that $d_t : TIFS^{(X,T)} \times TIFS^{(X,T)} \to [0, 1]$ is a normal temporal intuitionistic fuzzy sets for $t \in T$ and f is a n-aggregation function without zero divisor. Then, the mapping $d : TIFS^{(X,T)} \times TIFS^{(X,T)} \to [0, 1]$ which defined as:*

$$d(A, B) = f(d_{t_1}(A, B), d_{t_2}(A, B), ..., d_{t_n}(A, B))$$

for $A, B \in TIFS^{(X,T)}$ is a overall intuitionistic fuzzy distance measure.

Proof. Since $d_t : TIFS^{(X,T)} \times TIFS^{(X,T)} \to [0, 1]$ is a intuitionistic fuzzy distance measure on TIFSs for each $t \in T$. $d(A, B) = f(d_{t_1}(A, B), d_{t_2}(A, B), ..., d_{t_n}(A, B))$ is a intuitionistic fuzzy distance measure (see [11]). Due to d_t expresses temporal

distance between A and B for each $t \in T$, d which is obtained by d_t expresses overall distance between A and B. □

Now we give definitions of temporal and overall intuitionistic fuzzy similarity measure in sense of [31]

Definition 3.2. Let X be a non-empty set, T be time set and $s^t : TIFS^{(X,T)} \times TIFS^{(X,T)} \to [0,1]$ be a mapping. If s^t satisfies following conditions for all $A, B, C \in TIFS^{(X,T)}$ and fixed time moment $t \in T$, s_t is called temporal intuitionistic fuzzy similarity measure on $TIFS^{(X,T)}$ at time moment t.

S1. If A is a crisp set, $s^t \left(A, \bar{A} \right) = 0$

S2. $A = B \Leftrightarrow S \left(A, B \right) = 1$ for all $A, B \in TIFS^{(X,T)}$

S3. $s^t \left(A, B \right) = s^t \left(B, A \right)$ for all $A, B \in TIFS^{(X,T)}$

S4. The inequalties $s^t \left(A, C \right) \leq s^t \left(A, B \right)$ and $s^t \left(A, C \right) \leq s^t \left(B, C \right)$ are satisfied for all $A, B, C \in TIFS^{(X,T)}$ which are satisfied $A \subseteq B \subseteq C$

As in the concept of temporal distance measure, similarity measure can be examined in two parts as named temporal and overall. Let us give examples of temporal intuitionistic fuzzy similarity measure can be defined in accordance with this approach. It is clear that these measures are obtained by changing domain set of intuitionistic fuzzy similarity measure which are defined in the literature (see [4, 5, 6, 8, 10, 13, 23, 26, 27, 28, 29, 30, 31, 33, 34, 36, 37, 41, 44, 45]) as $TIFS^{(X,T)}$:

1. $S_1^{t_0} \left(A, B \right) =$

$$1 - \frac{\sum_{i=1}^{n} \left(\left| \mu_A \left(x_i, t_0 \right) - \mu_B \left(x_i, t_0 \right) \right|^{t_0} + \left| \eta_A \left(x_i, t_0 \right) - \eta_B \left(x_i, t_0 \right) \right|^{t_0} \right)^{\frac{1}{t_0}}}{2n}$$

where $X = \{x_1, x_2, ..., x_n\}$, $T = \{t_1, t_2, ..., t_m\}$, $t_0 \in T$ and $A, B \in TIFS^{(X,T)}$,

2. $S_2^{t_{j_0}} \left(A, B \right) =$

$$1 - \frac{\sum_{i=1}^{n} \omega_{(i,j_0)} \left| \left(\left| \mu_A \left(x_i, t_{j_0} \right) - \mu_B \left(x_i, t_{j_0} \right) \right| + \left| \eta_A \left(x_i, t_{j_0} \right) - \eta_B \left(x_i, t_{j_0} \right) \right| \right) \right|}{2n}$$

where $X = \{x_1, x_2, ..., x_n\}$, $T = \{t_1, t_2, ..., x_m\}$, $t_{j_0} \in T$, $A, B \in TIFS^{(X,T)}$ and $\sum_{i=1}^{n} \omega_{(i,j_0)} = 1$ for each $j_0 \in \{1, 2, ..., m\}$ where $\omega_{(i,j)} \in [0,1]$ for each $(i,j) \in \{1, 2, ..., n\} \times \{1, 2, ..., m\}$.

3. $S_3^{t_{j_0}} \left(A, B \right) =$

$$1 - \frac{\sum_{i=1}^{n} \omega_{(i,j_0)} \left(\left| \mu_A \left(x_i, t_{j_0} \right) - \mu_B \left(x_i, t_{j_0} \right) \right| + \left| \eta_A \left(x_i, t_{j_0} \right) - \eta_B \left(x_i, t_{j_0} \right) \right| \right)}{2n}$$

where $X = \{x_1, x_2, ..., x_n\}$, $T = \{t_1, t_2, ..., x_m\}$, $t_{j_0} \in T$, $A, B \in TIFS^{(X,T)}$ and $\sum_{i=1}^{n} \omega_{(i,j_0)} = 1$ for each $j_0 \in \{1, 2, ..., m\}$ where $\omega_{(i,j)} \in [0,1]$ for each $(i,j) \in \{1, 2, ..., n\} \times \{1, 2, ..., m\}$.

4. $S_4^{t_{j_0}}(A,B) =$

$$1 - \frac{\sum\limits_{i=1}^{n} \omega_{(i,j_0)} \left(|\mu_A(x_i,t_{j_0}) - \mu_B(x_i,t_{j_0})| + |\eta_A(x_i,t_{j_0}) - \eta_B(x_i,t_{j_0})| + |\pi_A(x_i,t_{j_0}) - \pi_B(x_i,t_{j_0})| \right)}{2n}$$

where $X = \{x_1, x_2, ..., x_n\}$, $T = \{t_1, t_2, ..., x_m\}$, $t_{j_0} \in T$, $A, B \in TIFS^{(X,T)}$ and $\sum\limits_{i=1}^{n} \omega_{(i,j_0)} = 1$ for each $j_0 \in \{1, 2, ..., m\}$ where $\omega_{(i,j)} \in [0,1]$ for each $(i,j) \in \{1, 2, ..., n\} \times \{1, 2, ..., m\}$.

5. $S_5^{t_0}(A,B) = \frac{1}{n} \sum\limits_{i=1}^{n} \frac{\min\{\mu_A(x_i,t_0),\mu_B(x_i,t_0)\} + \min\{\eta_A(x_i,t_0),\eta_B(x_i,t_0)\}}{\max\{\mu_A(x_i,t_0),\mu_B(x_i,t_0)\} + \max\{\eta_A(x_i,t_0),\eta_B(x_i,t_0)\}}$

where $X = \{x_1, x_2, ..., x_n\}$, $T = \{t_1, t_2, ..., x_m\}$, $t_0 \in T$, $A, B \in TIFS^{(X,T)}$

6. $S_6^{t_0}(A,B) = \frac{\sum\limits_{i=1}^{n} 1 - \frac{1}{2}(|\mu_A(x_i,t_0) - \mu_B(x_i,t_0)| + |\eta_A(x_i,t_0) - \eta_B(x_i,t_0)|)}{n}$

where $X = \{x_1, x_2, ..., x_n\}$, $T = \{t_1, t_2, ..., x_m\}$, $t_0 \in T$, $A, B \in TIFS^{(X,T)}$

7. $S_7^{t_0}(A,B) = \frac{1}{n} \frac{\sum\limits_{i=1}^{n} \min\{\mu_A(x_i,t_0),\mu_B(x_i,t_0)\} + \min\{\eta_A(x_i,t_0),\eta_B(x_i,t_0)\}}{\sum\limits_{i=1}^{n} \max\{\mu_A(x_i,t_0),\mu_B(x_i,t_0)\} + \max\{\eta_A(x_i,t_0),\eta_B(x_i,t_0)\}}$

where $X = \{x_1, x_2, ..., x_n\}$, $T = \{t_1, t_2, ..., x_m\}$, $t_0 \in T$, $A, B \in TIFS^{(X,T)}$

8. $S_8^{t_0}(A,B) = 1 - \frac{1}{2} \left(\max |\mu_A(x_i,t_0) - \mu_B(x_i,t_0)| + \max |\eta_A(x_i,t_0) - \eta_B(x_i,t_0)| \right)$

where $X = \{x_1, x_2, ..., x_n\}$, $T = \{t_1, t_2, ..., x_m\}$, $t_0 \in T$, $A, B \in TIFS^{(X,T)}$

9. $S_9^{t_0}(A,B) = 1 - \frac{\sum\limits_{i=1}^{n} (|\mu_A(x_i,t_0) - \mu_B(x_i,t_0)| + |\eta_A(x_i,t_0) - \eta_B(x_i,t_0)|)}{\sum\limits_{i=1}^{n} (|\mu_A(x_i,t_0) + \mu_B(x_i,t_0)| + |\eta_A(x_i,t_0) + \eta_B(x_i,t_0)|)}$

where $X = \{x_1, x_2, ..., x_n\}$, $T = \{t_1, t_2, ..., x_m\}$, $t_0 \in T$, $A, B \in TIFS^{(X,T)}$

In these definitions, it is seen that the temporal similarity measures are dependent on t the moment with selected temporal intuitionistic fuzzy sets. Since these TIFSs change over time, similarity measures on $TIFS^{(X,T)}$ inevitably change over time. This approach elicits a more spacious work area for applications changed in comparison mechanism. there are many different methods to achieve the overall similarity measure defined from a temporal intuitionistic fuzzy similarity measures. Among these the most significant are defined as follows:

Theorem 3.3. *Let X be non-empty set, T be time set and s^t be a temporal intuitionistic fuzzy similarity measure for each $t \in T$. Then the mapping s defined as $s(A,B) = \max\limits_{t \in T} \{s^t(A,B)\}$ is a overall intuitionistic fuzzy similarity measure.*

More general version of this theorem with Du and Xu's approach [11] can be given as follows:

Theorem 3.4. *: Let X be non-empty set and $T = \{t_1, t_2, ..., t_n\}$ be a time set. Let suppose that the mappings $s^t : TIFS^{(X,T)} \times TIFS^{(X,T)} \to [0,1]$ are temporal intuitionistic fuzzy similarity measure for each $t \in T$ and f is a n-aggregation function without zero divisor . Then, the mapping $s : TIFS^{(X,T)} \times TIFS^{(X,T)} \to [0,1]$ defined as:*

$$s(A,B) = f\left(s^{t_1}(A,B), s^{t_2}(A,B), ..., s^{t_n}(A,B) \right)$$

for all $A, B \in TIFS^{(X,T)}$ is a overall intuitionistic fuzzy similarity measure.

Proof. It can be proven as Theorem 1. □

As noted for temporal intuitionistic fuzzy distance measure, all similarity measures defined for intuitionistic fuzzy sets can be accepted as temporal intuitionistic fuzzy similarity measure for singular time set and they also can be converted into temporal intuitionistic fuzzy similarity measure by selecting the domain set $TIFS^{(X,T)}$. After this process, overall intuitionistic fuzzy similarity measurements indicating the general jurisdiction can be obtained by using some methods such as aggregation function. The most common feature of data used in real-world applications is that they can change over time. In this context, the most remarkable feature is that the uncertainty of the situation will may increase sometimes. As in fuzzy and intuitionistic fuzzy sets, temporal (or overall) intuitionistic fuzzy distance and similarity measures are dual concepts. So, there are several ways to obtain other one from another. In [11], Du and Xu have generalized this relationship by fuzzy negation and aggregation function for intuitionistic fuzzy distance and similarity measures. Their feature is also available in the temporal intuitionistic fuzzy sets. Having not generalized some basic concepts for temporal intuitionistic fuzzy sets is shortcoming in the literature. Some of these concepts which given in [11] are generalized for the temporal intuitionistic fuzzy sets as follows:

Definition 3.3. Let T be a time set. If the mapping $N_t : [0,1] \to [0,1]$ is satisfied following condition for $t \in T$, it is called temporal fuzzy negation at time moment t :

N1. $N_t(0) = 1$, $N_t(1) = 0$

N2. $N_t(b) \le N_t(a)$ for all $a \le b$

if N_t is satisfied

a. $N_t(N_t(a)) = a$ for $t \in T$ and all $a \in [0,1]$, it is called temporal fuzzy strong negation at time moment t,

b. $x = 0 \Leftrightarrow N_t(x) = 1$ for $t \in T$ and all $a \in [0,1]$, it is called temporal fuzzy non-filling negation at time moment t,

c. $x = 1 \Leftrightarrow N_t(x) = 0$ for $t \in T$ and all $a \in [0,1]$, it is called temporal fuzzy non-vanishing negation at time moment t.

The novelty of this definition is that negation will change over time. Adding the time parameters and changing temporal fuzzy negation over time offers unlimited options for obtaining similarity measure from distance measure (or conversely). the temporal fuzzy strong negations can be obtained by adding time parameters to fuzzy negations as follows:

1. $N_{1,\lambda_t}(x) = \frac{1-x}{1+\lambda_t x}$ for $\lambda_t \in (-1, +\infty)$ and $t \in T$,

2. $N_{2,\delta_t}(x) = \sqrt[\delta_t]{1 - x^{\delta_t}}$ for $\delta_t \in (0, +\infty)$ and $t \in T$,

3. $N_{3,\varphi_t}(x) = \begin{cases} 1, & x \le \varphi_t \\ 0, & otherwise \end{cases}$ for $\varphi_t \in (0,1)$ and $t \in T$.

Following theorems are obtained by adding time parameter to theorems which given by Du and Xu [11].

Theorem 3.5. *Let X be non-empty set, T be time set and $A, B \in TIFS^{(X,T)}$. Let suppose that d^t is a temporal distance measure and N_t is a temporal fuzzy*

non-filling negation. Then, the mapping $s_{N_t} : TIFS^{(X,T)} \times TIFS^{(X,T)} \to [0,1]$ defined as $s_{N_t}(A,B) = N_t(d^t(A,B))$ is a temporal intuitionistic fuzzy similarity measure. Conversely, Let suppose that s^t is a temporal intuitionistic fuzzy similarity measure and N_t is a temporal fuzzy non-vanishing negation, then the mapping $d_{N_t} : TIFS^{(X,T)} \times TIFS^{(X,T)} \to [0,1]$ defined as $d_{N_t}(A,B) = N_t(s^t(A,B))$ is a temporal intuitionistic fuzzy distance measure. If N_t is a temporal fuzzy strong negation, the equations $d^t(A,B) = N^t(s_{N^t}(A,B))$ and $s^t(A,B) = N_t(d_{N_t}(A,B))$ are satisfied.

This theorem is preserved with Du and Xu's approach [11] for overall temporal measures as follows.

Theorem 3.6. *Let X be non-empty set, T be time set and $A, B \in TIFS^{(X,T)}$. Let suppose that d is overall intuitionistic fuzzy distance measure and N is a fuzzy non-filling negation. Then, the mapping $s_N : TIFS^{(X,T)} \times TIFS^{(X,T)} \to [0,1]$ defined as $s_N(A,B) = N(d(A,B))$ is an overall intuitionistic fuzzy similarity measure. Conversely, let suppose that s is an overall intuitionistic fuzzy similarity measure and N is a fuzzy non-vanishing negation, then the mapping $d_N : TIFS^{(X,T)} \times TIFS^{(X,T)} \to [0,1]$ defined as $d_N(A,B) = N(s(A,B))$ is a overall intuitionistic fuzzy similarity measure. If N is a fuzzy strong negation, the equations $d(A,B) = N(s_N(A,B))$ and $s(A,B) = N(d_N(A,B))$ are satisfied.*

Another theorem which is given in [11] can be generalized to temporal intuitionistic fuzzy sets as below.

Theorem 3.7. *: Let X be a non empty set and $T = \{t_1, t_2, ..., t_n\}$ be a finite time set. Let suppose that $d^t : TIFS^{(X,T)} \times TIFS^{(X,T)} \to [0,1]$ is a temporal intuitionistic fuzzy distance measure for each $t \in T$, f is a aggregation function without zero divisor and N_t is a temporal fuzzy non-filling negation, then the mapping $s : TIFS^{(X,T)} \times TIFS^{(X,T)} \to [0,1]$ defined as*

$$s(A,B) = f\left(s_{N_{t_1}}(A,B), s_{N_{t_2}}(A,B), ..., s_{N_{t_n}}(A,B)\right)$$

for all $A, B \in TIFS^{(X,T)}$ is a overall intuitionistic fuzzy similarity measure.

Proof. It is clear that each s_{N_t} is a temporal intuitionistic similarity measure for $t \in T$ from Theorem 5. Then, it is obtained that s is a overall intuitionistic fuzzy similarity measure from Theorem 4. $\square$

Now, entropy which is a measure of difference between intuitionistic fuzzy set (or fuzzy sets) and crisp set will be defined for temporal intuitionistic fuzzy sets. the temporal variability encountered in real-world problems changes this difference in a continuous manner. The definition of entropy (as temporal entropy and overall entropy) with Szmidt and Kacprzyk's approach [35] is defined for temporal intuitionistic fuzzy sets as follow:

Definition 3.4. Let X be a non-empty set and T be a time set. If the mapping $e^t : TIFS^{(X,T)} \to [0,1]$ is satisfied the following conditions for $A \in TIFS^{(X,T)}$ and fixed $t \in T$, e^t is called temporal intuitionistic fuzzy entropy on $TIFS^{(X,T)}$.

E1. A is a crisp set for $\Leftrightarrow e^t(A) = 0$,

E2. $e^t(A) = 1 \Leftrightarrow \mu_A(x,t) = \eta_A(x,t)$ for all $x \in X$ and fixed $t \in T$

E3. $e^t(A) \leq e^t(B) \Leftrightarrow$
i. $\mu_A(x,t) \geq \mu_B(x,t)$ and $\eta_A(x,t) \leq \eta_B(x,t)$ for $\mu_B(x,t) \leq \eta_B(x,t)$
ii. $\mu_A(x,t) \leq \mu_B(x,t)$ and $\eta_A(x,t) \geq \eta_B(x,t)$ for $\mu_B(x,t) \geq \eta_B(x,t)$
for all $x \in X$ and fixed $t \in T$
E4. $e^t(A) = e^t(\overline{A})$.
With this definition, the value $e^t(A)$ represents how much is far from being crisp sets at time moment t. If we change the condition E1 with
E1*. A is a fuzzy set $\Leftrightarrow e^t(A) = 0$
, the value $e^t(A)$ represents how much is far from being fuzzy sets at time moment t. Let's call this second measure as type-2 temporal entropy. On the other hand, If the mapping $e : TIFS^{(X,T)} \to [0,1]$ is satisfied the conditions (E1, E2, E3, E4) for each $t \in T$, e is called overall entropy on $TIFS^{(X,T)}$. The mapping $e : TIFS^{(X,T)} \to [0,1]$ defined as $e(A) = \sup_{t \in T} \{e^t(A)\}$ is an overall intuitionistic fuzzy entropy.

Theorem 3.8. *Let X be a non-empty set and $T = \{t_1, t_2, ..., t_n\}$ be a finite time set. Let suppose that $e^t : TIFS^{(X,T)} \to [0,1]$ is temporal intuitionistic fuzzy entropy for $t \in T$ and f is a aggregation function without zero divisor. Then the mapping $e : TIFS^{(X,T)} \to [0,1]$ defined as*

$$e(A) = f\left(e^{t_1}(A), e^{t_2}(A), ..., e^{t_n}(A)\right)$$

for $A \in TIFS^{(X,T)}$ is an overall intuitionistic fuzzy entropy.

Proof. It can be proven as Theorem 1. □

The intuitionistic fuzzy entropies which are defined in [*12, 14, 19, 21, 24, 25, 28, 29, 30, 31, 35, 36, 38, 41, 42, 43, 45*] can be converted into the temporal intuitionistic fuzzy entropy with some minor modifications as follows:

Proposition 3.3. *Let $X = \{x_1, x_2, ..., x_n\}$ be a non-empty set and T is a time set, Then the following mappings are temporal intuitionistic fuzzy entropy on $TIFS^{(X,T)}$*

1. $e_{SK}^t(A) = 1 - \frac{1}{n}\sum_{i=1}^{n}|\mu_A(x_i,t) - \eta_A(x_i,t)|$

2. $e_{VS-1}^t = \dfrac{\sum_{i=1}^{n}\min\{\mu_A(x_i,t),\eta_A(x_i,t)\}+\min\{1-\mu_A(x_i,t),1-\eta_A(x_i,t)\}}{\sum_{i=1}^{n}\max\{\mu_A(x_i,t),\eta_A(x_i,t)\}+\max\{1-\mu_A(x_i,t),1-\eta_A(x_i,t)\}}$

3. $e_{VS-2}^t(A) = \sum_{i=1}^{n}\dfrac{2\mu_A(x_i,t)\eta_A(x_i,t)+\pi_A(x_i,t)^2}{\mu_A(x_i,t)^2+\eta_A(x_i,t)^2+\pi_A(x_i,t)^2}$

4. $e_{Li-1}^t(A) = 1 - \frac{1}{2n}\sum_{i=1}^{n}\left(|\mu_A(x_i,t)-\eta_A(x_i,t)|^3 + |\mu_A(x_i,t)-\eta_A(x_i,t)|\right)$

5. $e^t{}_{GS-1}(A) = \frac{1}{n}\sum_{i=1}^{n}(1-|\mu_A(x_i,t)-\eta_A(x_i,t)|)\frac{1+\pi_A(x_i,t)}{2}$

6. $e^t{}_{Huang-1}(A) = 1 - \frac{1}{n}\sum_{i=1}^{n}\left|\mu_A(x_i,t)^2 - \eta_A(x_i,t)^2\right|$

7. $e^t{}_{Huang-2}(A) = 1 - \sqrt[p]{\frac{1}{n}\sum_{i=1}^{n}\left|\mu_A(x_i,t)^2 - \eta_A(x_i,t)^2\right|^p}$

8. $e^t{}_{Huang-3}(A) = 1 - \sqrt[p]{\frac{1}{n}\sum_{i=1}^{n}|\mu_A(x_i,t) - \eta_A(x_i,t)|^p}$

9. $e^t{}_{Huang-4}(A) = \frac{1}{n}\sum\limits_{i=1}^{n}\frac{1-|\mu_A(x_i,t)-\eta_A(x_i,t)|+\pi_A(x_i,t)}{1+|\mu_A(x_i,t)-\eta_A(x_i,t)|+\pi_A(x_i,t)}$

10. $e^t{}_{Huang-5}(A) =$

$$\frac{1}{n}\sum\limits_{i=1}^{n}1 - |\mu_A(x_i,t)-\eta_A(x_i,t)| + \pi_A(x_i,t)\,|\mu_A(x_i,t)-\eta_A(x_i,t)|$$

11. $e^t{}_{Hung-1}(A) = 1 - \sum\limits_{i=1}^{n}\frac{1}{n}\,|\mu_A(x_i,t)-\eta_A(x_i,t)|$

12. $e^t{}_{Hung-2}(A) = 1 - \sqrt{\sum\limits_{i=1}^{n}\frac{1}{n}\left|\mu_A(x_i,t)-\eta_A(x_i,t)^2\right|}$

13. $e^t{}_{Ye-1}(A) =$

$$\frac{1}{n}\sum\limits_{i=1}^{n}\left(\sin\left(\frac{\pi[1+\mu_A(x_i,t)-\eta_A(x_i,t)]}{4}\right) + \sin\left(\frac{\pi[1-\mu_A(x_i,t)+\eta_A(x_i,t)]}{4}-1\right)\frac{1}{\sqrt{2}-1}\right)$$

14. $e^t{}_{Ye-2}(A) =$

$$\frac{1}{n}\sum\limits_{i=1}^{n}\left(\cos\left(\frac{\pi[1+\mu_A(x_i,t)-\eta_A(x_i,t)]}{4}\right) + \cos\left(\frac{\pi[1-\mu_A(x_i,t)+\eta_A(x_i,t)]}{4}-1\right)\frac{1}{\sqrt{2}-1}\right)$$

15. $e^t{}_{ZL-1} = 1 - \frac{1}{b-a}\int\limits_{a}^{b}|\mu_A(x,t)-\eta_A(x,t)|\,dx\ \text{for } X=[a,b]$

16. $e^t{}_{ZL-1} = \dfrac{\int\limits_{a}^{b}\mu_A(x,t)\wedge\eta_A(x,t)dx}{\int\limits_{a}^{b}\mu_A(x,t)\vee\eta_A(x,t)dx}\ \text{for } X=[a,b]$

The other measure which is closely related with distance measure, similarity measure and entropy in intuitionistic fuzzy sets (and fuzzy set) is inclusion measure (named subsethood measure in some studies). Inclusion measure has been defined for intuitionistic fuzzy sets by Cornelis and Kerre in [9]. This concept is defined for temporal intuitionistic fuzzy sets as follows:

Definition 3.5. Let X be a non-empty set and T be a time set. If the mapping $I_t : TIFS^{(X,T)}\times TIFS^{(X,T)} \to [0,1]$ is satisfied following conditions for fixed $t \in T$, it is named temporal intuitionistic fuzzy inclusion measure on $t \in T$.

I1. $I_t\left(\tilde{1},\tilde{0}\right) = 0$,

I2. $I_t(A,B) = 1 \Leftrightarrow A \subseteq B$

I3. $I_t(C,A) \leq I_t(C,B)$ and $I_t(B,C) \leq I_t(A,C)$ when $A \subseteq B$ and $C \in TIFS^{(X,T)}$

On the other hand, if the mapping $I : TIFS^{(X,T)}\times TIFS^{(X,T)} \to [0,1]$ is satisfied the conditions (I1, I2, I3) for each $t \in T$, I is called overall intuitionistic fuzzy inclusion measure on $TIFS^{(X,T)}$. The mapping $I : TIFS^{(X,T)} \times TIFS^{(X,T)} \to [0,1]$ defined as $I(A,B) = \sup\limits_{t\in T}\{I_t(A,B)\}$ for all $A,B \in TIFS^{(X,T)}$ and $t \in T$ is an overall intuitionistic fuzzy inclusion measure.

The intuitionistic fuzzy inclusion measures which are defined in [44], [45] can be converted into the temporal intuitionistic fuzzy inclusion measures with some minor modifications as well as for other measures.

Proposition 3.4. *Let* $X = \{x_1, x_2, ..., x_n\}$ *be a non-empty set and* T *be a time set. Then the following mappings are temporal intuitionistic fuzzy inclusion measure on* $TIFS^{(X,T)}$.

1. $I_{ZHXL-1}^{t}(A, B) =$

$$1 - \frac{1}{2n} \sum_{i=1}^{n} \{|\mu_A(x_i, t) - \min\{\mu_A(x_i, t), \mu_B(x_i, t)\}| + |\max\{\eta_A(x_i, t), \eta_B(x_i, t)\} - \eta_A(x_i, t)|\}$$

2. $I_{ZHXL-2}^{t}(A, B) =$

$$1 - \sqrt{\frac{1}{2n} \sum_{i=1}^{n} \left\{|\mu_A(x_i, t) - \min\{\mu_A(x_i, t), \mu_B(x_i, t)\}|^2 + |\max\{\eta_A(x_i, t), \eta_B(x_i, t)\} - \eta_A(x_i, t)|^2\right\}}$$

3. $I_{ZDZS-3}^{t}(A, B) =$

$$\begin{cases} 1 & , & \begin{aligned} \mu_B(x_i, t) &= \mu_A(x_i, t) \\ \eta_B(x_i, t) &= \eta_A(x_i, t) \end{aligned} \\ \sum_{i=1}^{n} \frac{1}{2} \left[\frac{\mu_B(x_i,t) - \mu_A(x_i,t) + \eta_B(x_i,t) - \eta_A(x_i,t)}{|\mu_B(x_i,t) - \mu_A(x_i,t)| + |\eta_B(x_i,t) - \eta_A(x_i,t)|} \right] + 1 & , & otherwise \end{cases}$$

The following theorem which is given for the others measures can be given for temporal intuitionistic fuzzy inclusion measure and overall intuitionistic fuzzy inclusion measure as follows:

Theorem 3.9. *Let* X *be a non-empty set and* $T = \{t_1, t_2, ..., t_n\}$ *be a finite time set. Let suppose that the mappings* $I_t : TIFS^{(X,T)} \times TIFS^{(X,T)} \to [0, 1]$ *is a temporal intuitionistic fuzzy inclusion measure for each* $t \in T$ *and* f *is a aggregation function without zero divisor. Then the mapping* $I : TIFS^{(X,T)} \times TIFS^{(X,T)} \to [0, 1]$ *defined as*

$$I(A, B) = f(I_{t_1}(A), I_{t_2}(A), ..., I_{t_n}(A))$$

for all $A, B \in TIFS^{(X,T)}$ *is a overall intuitionistic fuzzy inclusion measure.*

The relationship between these measures are protected as described for the fuzzy and intuitionistic fuzzy sets. Some of these relationships can be generalized for the temporal intuitionistic fuzzy as follows. In this context, these relationships can be proved as in the studies which are given in the reference. Therefore, we will give the following theorems without proof (see more information: *[3, 4, 5, 6, 8, 9, 10,12, 13, 14, 15, 16, 17, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 33, 34, 35, 37, 38, 40, 41, 42, 43, 44, 45, etc.])* .

Theorem 3.10. *Let* X *be a non-empty set and* T *be time set. Let suppose that* s^t *is a temporal intuitionistic fuzzy similarity measure for* $t \in T$. *Then the mapping* $e^t : TIFS^{(X,T)} \to [0, 1]$ *defined as* $e^t(A) = s^t(A, \bar{A})$ *for all* $A \in TIFS^{(X,T)}$ *is a temporal intuitionistic fuzzy entropy*

Theorem 3.11. *Let* X *be a non-empty set and* $T = \{t_1, t_2, ..., t_n\}$ *be finite time set. Let suppose that* s^t *is a temporal intuitionistic fuzzy similarity measure for*

$t \in T$ ve f is a aggregation function without zero divisor. Then the mapping $e : TIFS^{(X,T)} \to [0,1]$ defined as $e\left(A\right) = f\left(s^{t_1}\left(A, \bar{A}\right), s^{t_2}\left(A, \bar{A}\right), ..., s^{t_n}\left(A, \bar{A}\right)\right)$ for all $A \in TIFS^{(X,T)}$ is a overall intuitionistic fuzzy entropy.

Theorem 3.12. *Let X be a non-empty set and T be time set. Let suppose that d^t is a temporal intuitionistic fuzzy distance measure for $t \in T$ and N_t is a temporal fuzzy non-filling negation. Then the mapping $e^t : TIFS^{(X,T)} \to [0,1]$ defined as $e^t\left(A\right) = N_t\left(d^t\left(A, \bar{A}\right)\right)$ for all $A \in TIFS^{(X,T)}$ is a temporal intuitionistic fuzzy entropy.*

Theorem 3.13. *Let X be a non-empty set and $T = \{t_1, t_2, ..., t_n\}$ be finite time set. Let suppose that s^t is a temporal intuitionistic fuzzy similarity measure for $t \in T$, f is a aggregation function without zero divisor and N_t is a temporal fuzzy non-filling negation. The mapping $e : TIFS^{(X,T)} \to [0,1]$ defined as*

$$e\left(A\right) = f\left(N_{t_1}\left(d^{t_1}\left(A, \bar{A}\right)\right), N_{t_2}\left(d^{t_2}\left(A, \bar{A}\right)\right), ..., N_{t_n}\left(d^{t_n}\left(A, \bar{A}\right)\right)\right)$$

for all $A \in TIFS^{(X,T)}$ is a overall intuitionistic fuzzy entropy.

Theorem 3.14. *Let X be a non-empty set and T be time set. Let suppose that I_t is a temporal intuitionistic fuzzy inclusion measure. Then the mapping $E_I^t : TIFS^{(X,T)} \to [0,1]$ defined as $E_I^t\left(A\right) = \left(I_t\left(A \cup \overline{A}, A \cap \overline{A}\right)\right)$ for all $A \in TIFS^{(X,T)}$ is a temporal intuitionistic fuzzy entropy.*

Theorem 3.15. *Let X be a non-empty set and T be time set. . Let suppose that $*$ is a $t-$ norm için I^t is a temporal intuitionistic fuzzy inclusion measure for $t \in T$. Then the mapping $E_I^t : TIFS^{(X,T)} \to [0,1]$ defined as $S_I^t\left(A, B\right) = *\left(I_t\left(A, B\right), I_t\left(B, A\right)\right)$ for all $A, B \in TIFS^{(X,T)}$ is a temporal intuitionistic fuzzy similarity measure.*

As seen from the above theorem, the basic relationship between distance measure, similarity measure, entropy and coverage measure are protected for temporal intuitionistic fuzzy sets as provided in for fuzzy and intuitionistic fuzzy sets. Furthermore, these measures can be separate two groups which are named as temporal and overall for temporal intuitionistic fuzzy sets. It can be done the first instant evaluation and later general evaluation for all data with time-varying nature by this method. Thus, the optimal results can be obtained by various way in the application.

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ON A SOCIAL ECONOMIC MODEL

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Abstract. Distribution of saving for a family sets on a region satisfies Kolmogorov equation given by

$$\frac{\partial u}{\partial t} = -\frac{\partial}{\partial x}\left((c+F)u\right) + \frac{1}{2}\frac{\partial^2}{\partial x^2}\left(bu\right) + f$$

where $u = u(x,t)$ is density distribution of family saving. Boundary condition defined by distribution of minimum saving and total family saving are considered for the model. By the seperation of variables method, eigenvalues and eigenfuncitons of problem are obtained and solution is written. In addition, numerical methods are applied to problem and errors of numerical methods are presented.

1. Introduction

Nonlocal boundary conditions are dealth with some wave,diffusion and any other physical equations [Cannon, Van der Hoek, Ionkin,Kamynin,etc...] Generally these type of problems are solved by numeric methods or reducing point boundary conditions. In this study we consider a family saving model used in economy. This problem is expressed with diffusion equation with integral boundary conditions.

Suppose that $x(t)$ shows saving of a family at time t and satisfy the differential equation

$$(1.1) \qquad dx = F\left(x,t\right)dt + G\left(x,t\right)dX, \quad G \geq 0$$

where X is Markov process, $F(x,t)$ denotes rate of change of the family saving and $G\left(x,t\right)dX$ denotes random change of family income.

For a family set let us assume that equation (1.1) describes the saving of all families by ignoring the dynamic of individual family saving. The density distribution of the saving of families $u(x,t)$ satisfies

$$(1.2) \qquad \frac{\partial u}{\partial t} = -\frac{\partial}{\partial x}\left((c(x,t)+F(x,t))\ u\right) + \frac{1}{2}\frac{\partial^2}{\partial x^2}\left(b(x,t)u\right) + f(x,t)$$

with initial condition

$$(1.3) \qquad u(x,0) = \varphi(x),\ 0 \leq x \leq l$$

and boundary conditions

$$(1.4) \qquad u(0,t) = 0$$

$$(1.5) \qquad \int_0^l x\, u(x,t)\, dx = K(t), \ t \geq 0$$

where $c(x,t)$, $b(x,t)$, $K(t)$ are continuously differentiable functions. $K(t)$ in (1.5) describes total amount of family saving in [0,l] [5].

We will consider special case of problem (1.1)-(1.5) on region
$D = (0 < t < \infty) \times (0 < x < l)$

$$(1.6) \qquad \frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2} + f(x,t)$$

$$(1.7) \qquad u(x,0) = \varphi(x), 0 \leq t \leq T$$

$$(1.8) \qquad u(0,t) = 0$$

$$(1.9) \qquad \int_0^1 x u(x,t) dx = K(t), 0 \leq x \leq l$$

where f(x,t), K(t) ,$\varphi(x)$ are continuously differentible function on region D and a is given constant. Compatibility conditions of this problem is $\int_0^1 \varphi(x)dx = K(0)$.

In order to obtain classical solution of problem (1.6)-(1.9), we transform boundary conditions into homegenous ones by the transformation

$$(1.10) \qquad u(x,t) = v(x,t) + 3K(t)x$$

Carrying out substitution (1.10) in (1.1)-(1.5) problem gives

$$(1.11) \qquad \frac{\partial v}{\partial t} = a^2 \frac{\partial^2 v}{\partial x^2} + F(x,t)$$

$$(1.12) \qquad v(x,0) = \psi(x)$$

$$(1.13) \qquad v(0,t) = 0$$

$$(1.14) \qquad \int_0^1 x v(x,t) dx = 0$$

where $F(x,t) = f(x,t) - 3K'(t)x$ and $\psi(x) = \varphi(x) - 3K(0)x$
This problem has homegenous boundary conditions. Due to linearity, problem can split into two auxilary problem:
i)

$$(1.15) \qquad \frac{\partial v}{\partial t} = a^2 \frac{\partial^2 v}{\partial x^2}$$

$$(1.16) \qquad v(x,0) = \psi(x)$$

$$(1.17) \qquad v(0,t) = 0$$

$$(1.18) \qquad \int_0^1 xv(x,t)dx = 0$$

ii)

$$(1.19) \qquad \frac{\partial v}{\partial t} = a^2 \frac{\partial^2 v}{\partial x^2} + F(x,t)$$

$$(1.20) \qquad v(x,0) = 0$$

$$(1.21) \qquad v(x,t) = 0$$

$$(1.22) \qquad \int_0^1 xv(x,t)dx = 0$$

Integrating both sides of (1.15) with respect to x from 0 to 1 and using integration by parts, integral boundary condition turns into Neumann boundary condition

$$v_x(1,t) - v(1,t) = 0$$

Thus problem (1.15)-(1.18) becomes

$$(1.23) \qquad \frac{\partial v}{\partial t} = a^2 \frac{\partial^2 v}{\partial x^2}$$

$$(1.24) \qquad v(x,0) = \psi(x)$$

$$(1.25) \qquad v(0,t) = 0$$

$$(1.26) \qquad v_x(1,t) - v(1,t) = 0$$

By the Fourier method, Sturm Liouville problem and ODE are ,respectively, obtained as

$$(1.27) \qquad X''(x) + \lambda X(x) = 0$$

$$(1.28) \qquad X(0) = 0$$

$$(1.29) \qquad X'(1) - X(1) = 0$$

and

$$(1.30) \qquad T'(t) + \lambda a^2 T(t) = 0$$

Sturm Liouville problem (1.27)-(1.29)is self adjoint and boundary conditions are regular, morerover strongly regular. Then eigenfunctions of Sturm Liouville problem are Riesz basis on $L^2[0,1]$.

Charecteristic equation of Sturm Liouville problem is

$$\tan k = k$$

Thus the problem has the eigenvalues λ_n , $n = 0,1,2..$ such that $\lambda_0 = 0$ and by using Langrange-Burmann formula

$$k_n = \frac{(2n+1)\pi}{2} - \left(\frac{(2n+1)\pi}{2}\right)^{-1} - \frac{2}{3}\left(\frac{(2n+1)\pi}{2}\right)^{-3} - \frac{13}{15}\left(\frac{(2n+1)\pi}{2}\right)^{-5} - \frac{146}{105}\left(\frac{(2n+1)\pi}{2}\right)^{-7} + O(\frac{1}{n^9})$$

where $\sqrt{\lambda_n} = k_n$.

Corresponding eigenfunctions are obtained by

$$X_0(x) = cx$$
$$X_n(x) = \sin(k_n x), n = 1, 2...$$

Hence solution of problem (1.23)-(1.26) is

$$v_1(x, t) = A_0 x + \sum_{n=1}^{\infty} A_n e^{-a^2 k_n^2 t} \sin(k_n x)$$

where

$$A_0 = 3 \int_0^1 x\psi(x)dx$$

$$A_n = \frac{1}{\frac{1}{2} - \frac{\sin(2k_n)}{4k_n}} \int_0^1 \psi(x) \sin(k_n x)dx, \ \ n = 1, 2...$$

Solution of problem (1.19)-(1.22) can easily obtained by

$$v_2(x, t) = \left[\int_0^t F_0(\tau)d\tau \right] x + \sum_{n=1}^{\infty} \left[\int_0^t F_n(\tau)e^{-a^2 k_n^2(t-\tau)}d\tau \right] \sin(k_n(x))$$

where

$$F_n(\tau) = \int_0^1 F(x, \tau) \sin(k_n x)dx$$

$$F_0(\tau) = \int_0^1 F(x, \tau)x dx$$

2. Numerical Solution

For the numerical solution of problem, we will use the Method of Lines method and Crank-Nicolson method presented in [8],[9] respectively. In both methods, Simpson's rule is used to approximate the integral in (1.18) numerically. We display here a few of numerical results.

Example 1..

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + (x^2 - 2)e^t$$
$$u(x, 0) = x^2$$
$$u(0, t) = 0$$
$$\int_0^1 xu(x, t)dx = \frac{e^t}{4}$$

Exact solution is $u(x, t) = x^2 e^t$. The computed results at various spatial lengths are shown in Table 2. This table exhibits the absolute relative error results for $u(0.5, 0.5)$.

Example 2:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{-2(x^2 + t + 1)}{(t + 1)^3}$$

TABLE 1. Relative Error at u(0.5,0.5) in Example 1

Spatial Length	MOL Method	Crank-Nicolson Method		
h=0.1	1.6291E-5	4.8730E-5		
h=0.05	2.553E-6	1.2398E-5		
h=0.025	3.6733E-7	3.1067E-6		
h=0.0125	4.8533E-8	7.7695E-7		

TABLE 2. Relative Error at u(0.5,0.5) in Example 2

Spatial Length	MOL Method	Crank-Nicolson Method		
h=0.1	1.4572E-4	3.7936E-4		
h=0.05	2.2868E-5	1.0927E-4		
h=0.025	3.3245E-6	2.7647E-5		
h=0.0125	4.468E-7	6.9193E-6		

$$u(x,0) = x^2$$
$$u(0,t) = 0$$
$$\int_0^1 xu(x,t)dx = \frac{1}{4(t+1)^2}$$

Exact solution is $u(x,t) = \frac{x^2}{(t+1)^2}$. The computed results at various spatial lengths are shown in Table 2. This table exhibits the absolute relative error results for $u(0.5, 0.5)$.

We studied a special case of family saving model which is diffusion equation with nonlocal boundary condition. Analytic solution of this problem is obtained. Morever by applying the Method of Lines method [8] and Crank Nicolson [9], numerical solution of problem is found.

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SOME RESULTS ON $S_{\alpha,\beta}$ AND $T_{\alpha,\beta}$ INTUITIONISTIC FUZZY MODAL OPERATORS

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ABSTRACT. In 1999, first Intuitionistic Fuzzy Modal Operators introduced in[2]. Expansion of these operators and new operators defined by different authors[3, 5, 6, 7, 8, 9]. Characteristics of these operators has been studied by several researchers.

In this study, we obtained new results on modal operators which are called $S_{\alpha,\beta}$ and $T_{\alpha,\beta}$.

1. INTRODUCTION

The concept of Intuitionistic fuzzy sets was introduced by Atanassov in 1986 [1], form an extension of fuzzy sets[10] by expanding the truth value set to the lattice $[0,1] \times [0,1]$.

Intuitionistic fuzzy modal operators defined firstly by Atanassov[1, 2]. Then severel extensions of these operators introduced by different authors[2, 8, 5, 6]. Some algebraic and characteristic properties of these operators were studied by several authors.

Definition 1.1. [1] An intuitionistic fuzzy set (shortly IFS) on a set X is an object of the form
$$A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in X\}$$
where $\mu_A(x), (\mu_A : X \to [0,1])$ is called the "degree of membership of x in A ", $\nu_A(x), (\nu_A : X \to [0,1])$ is called the " degree of non- membership of x in A ",and where μ_A and ν_A satisfy the following condition:

$$\mu_A(x) + \nu_A(x) \leq 1, \text{ for all } x \in X.$$

The class of intuitionistic fuzzy sets on X is denoted by $IFS(X)$.

The hesitation degree of x is defined by $\pi_A(x) = 1 - \mu_A(x) - \nu_A(x)$

Definition 1.2. [1]An IFS A is said to be contained in an IFS B (notation $A \sqsubseteq B$) if and only if, for all $x \in X : \mu_A(x) \leq \mu_B(x)$ and $\nu_A(x) \geq \nu_B(x)$.

It is clear that $A = B$ if and only if $A \sqsubseteq B$ and $B \sqsubseteq A$.

Definition 1.3. [1]Let $A \in IFS$ and let $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in X\}$ then the above set is callede the complement of A

$$A^c = \{\langle x, \nu_A(x), \mu_A(x)\rangle : x \in X\}$$

The intersection and the union of two IFSs A and B on X is defined by

$$A \sqcap B = \{\langle x, \mu_A(x) \wedge \mu_B(x), \nu_A(x) \vee \nu_B(x)\rangle : x \in X\}$$

$$A \sqcup B = \{\langle x, \mu_A(x) \vee \mu_B(x), \nu_A(x) \wedge \nu_B(x)\rangle : x \in X\}$$

The notion of Second Type Intuitionistic Fuzzy Modal Operators was firstly introduced by Atanassov as following:

Definition 1.4. [1]Let X be universal and $A \in IFS(X)$ then

(1) $\Box(A) = \{\langle x, \mu_A(x), 1 - \mu_A(x)\rangle : x \in X\}$
(2) $\Diamond(A) = \{\langle x, 1 - \nu_A(x), \nu_A(x)\rangle : x \in X\}$

Definition 1.5. [2]Let X be universal and $A \in IFS(X)$, $\alpha \in [0, 1]$ then

$$D_\alpha(A) = \{\langle x, \mu_A(x) + \alpha\pi_A(x), \nu_A(x) + (1 - \alpha)\pi_A(x)\rangle : x \in X\}$$

Definition 1.6. [2]Let X be universal and $A \in IFS(X)$, $\alpha, \beta \in [0, 1]$ and $\alpha + \beta \leq 1$ then

$$F_{\alpha,\beta}(A) = \{\langle x, \mu_A(x) + \alpha\pi_A(x), \nu_A(x) + \beta\pi_A(x)\rangle : x \in X\}$$

Definition 1.7. [2]Let X be universal and $A \in IFS(X)$, $\alpha, \beta \in [0, 1]$ then

$$G_{\alpha,\beta}(A) = \{\langle x, \alpha\mu_A(x), \beta\nu_A(x)\rangle : x \in X\}$$

Definition 1.8. [2]Let X be universal and $A \in IFS(X)$, $\alpha, \beta \in [0, 1]$ then

(1) $H_{\alpha,\beta}(A) = \{\langle x, \alpha\mu_A(x), \nu_A(x) + \beta\pi_A(x)\rangle : x \in X\}$
(2) $H^*_{\alpha,\beta}(A) = \{\langle x, \alpha\mu_A(x), \nu_A(x) + \beta(1 - \alpha\mu_A(x) - \nu_A(x))\rangle : x \in X\}$
(3) $J_{\alpha,\beta}(A) = \{\langle x, \mu_A(x) + \alpha\pi_A(x), \beta\nu_A(x)\rangle : x \in X\}$
(4) $J^*_{\alpha,\beta}(A) = \{\langle x, \mu_A(x) + \alpha(1 - \mu_A(x) - \beta\nu_A(x)), \beta\nu_A(x)\rangle : x \in X\}$

The simplest One Type Intuitionistic Fuzzy Modal Operators defined in 1999.

Definition 1.9. [2] Let X be a set and $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in X\} \in IFS(X), \alpha, \beta \in [0, 1]$.

(1) $\boxplus A = \left\{ \left\langle x, \frac{\mu_A(x)}{2}, \frac{\nu_A(x)+1}{2} \right\rangle : x \in X \right\}$
(2) $\boxtimes A = \left\{ \left\langle x, \frac{\mu_A(x)+1}{2}, \frac{\nu_A(x)}{2} \right\rangle : x \in X \right\}$

After this definition, in 2001, the extension of these operators were defined as following:

Definition 1.10. [3] Let X be a set and $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in X\} \in IFS(X), \alpha, \beta \in [0, 1]$.

(1) $\boxplus_\alpha A = \{\langle x, \alpha\mu_A(x), \alpha\nu_A(x) + 1 - \alpha\rangle : x \in X\}$
(2) $\boxtimes_\alpha A = \{\langle x, \alpha\mu_A(x) + 1 - \alpha, \alpha\nu_A(x)\rangle : x \in X\}$

The operators $\boxplus_\alpha$ and $\boxtimes_\alpha$ are the extensions of the operators $\boxplus$, $\boxtimes$, resp.
In 2004, Dencheva introduced the second extension of $\boxplus_\alpha$ and $\boxtimes_\alpha$.

Definition 1.11. [8] Let X be a set and $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in X\} \in IFS(X), \alpha, \beta \in [0, 1]$.

(1) $\boxplus_{\alpha,\beta} A = \{\langle x, \alpha\mu_A(x), \alpha\nu_A(x) + \beta\rangle : x \in X\}$ where $\alpha + \beta \in [0,1]$.
(2) $\boxtimes_{\alpha,\beta} A = \{\langle x, \alpha\mu_A(x) + \beta, \alpha\nu_A(x)\rangle : x \in X\}$ where $\alpha + \beta \in [0,1]$.

In 2006, the third extension of the above operators was studied by author . He defined the following operators;

Definition 1.12. [3]Let X be a set and $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in X\} \in IFS(X)$.

(1) $\boxplus_{\alpha,\beta,\gamma}(A) = \{\langle x, \alpha\mu_A(x), \beta\nu_A(x) + \gamma\rangle : x \in X\}$
   where $\alpha, \beta, \gamma \in [0,1], \max\{\alpha, \beta\} + \gamma \leqslant 1$.
(2) $\boxtimes_{\alpha,\beta,\gamma}(A) = \{\langle x, \alpha\mu_A(x) + \gamma, \beta\nu_A(x)\rangle : x \in X\}$
   where $\alpha, \beta, \gamma \in [0,1], \max\{\alpha, \beta\} + \gamma \leqslant 1$.

In 2007, author[5] defined a new operator named $E_{\alpha,\beta}$ and studied some of its properties. This operator as following:

Definition 1.13. [5]Let X be a set and $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in X\} \in IFS(X), \alpha, \beta \in [0,1]$. We define the following operator:

$$E_{\alpha,\beta}(A) = \{\langle x, \beta(\alpha\mu_A(x) + 1 - \alpha), \alpha(\beta\nu_A(x) + 1 - \beta)\rangle : x \in X\}$$

At the same year, Atanassov introduced the operator $\boxdot_{\alpha,\beta,\gamma,\delta}$ which is a natural extension of all these operators in [3].

Definition 1.14. [3]Let X be a set, $A \in IFS(X)$, $\alpha, \beta, \gamma, \delta \in [0,1]$ such that

$$\max(\alpha, \beta) + \gamma + \delta \leqslant 1$$

then the operator $\boxdot_{\alpha,\beta,\gamma,\delta}$ defined by

$$\boxdot_{\alpha,\beta,\gamma,\delta}(A) = \{\langle x, \alpha\mu_A(x) + \gamma, \beta\nu_A(x) + \delta\rangle : x \in X\}$$

In 2008, most general operator $\odot_{\alpha,\beta,\gamma,\delta,\varepsilon,\zeta}$ defined as following:

Definition 1.15. [3]Let X be a set, $A \in IFS(X)$, $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta \in [0,1]$ such that

$$\max(\alpha - \zeta, \beta - \varepsilon) + \gamma + \delta \leqslant 1$$

and

$$\min(\alpha - \zeta, \beta - \varepsilon) + \gamma + \delta \geq 0$$

then the operator $\odot_{\alpha,\beta,\gamma,\delta,\varepsilon,\zeta}$ defined by

$$\odot_{\alpha,\beta,\gamma,\delta,\varepsilon,\zeta}(A) = \{\langle x, \alpha\mu_A(x) - \varepsilon\nu_A(x) + \gamma, \beta\nu_A(x) - \zeta\mu_A(x) + \delta\rangle : x \in X\}$$

In 2010, Çuvalcıoğlu[6] defined a new operator which is a generalization of $E_{\alpha,\beta}$.

Definition 1.16. [6]Let X be a set and $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in X\} \in IFS(X), \alpha, \beta, \omega \in [0,1]$ then

$$Z_{\alpha,\beta}^{\omega}(A) = \{\langle x, \beta(\alpha\mu_A(x) + \omega - \omega.\alpha), \alpha(\beta\nu_A(x) + \omega - \omega.\beta)\rangle : x \in X\}$$

Definition 1.17. [6]Let X be a set and $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in X\} \in IFS(X), \alpha, \beta, \omega, \theta \in [0,1]$ then

$$Z_{\alpha,\beta}^{\omega,\theta}(A) = \{\langle x, \beta(\alpha\mu_A(x) + \omega - \omega.\alpha), \alpha(\beta\nu_A(x) + \theta - \theta.\beta)\rangle : x \in X\}$$

The operator $Z_{\alpha,\beta}^{\omega,\theta}$ is a generalization of $Z_{\alpha,\beta}^{\omega}$, and also, $E_{\alpha,\beta}, \boxplus_{\alpha,\beta}, \boxtimes_{\alpha,\beta}$.
Uni-type intuitionistic fuzzy modal operators introduced by author as following;

Definition 1.18. [7]Let X be a universal, $A \in IFS(X)$ and $\alpha, \beta, \omega \in [0,1]$ then

(1) $\boxplus_{\alpha,\beta}^{\omega}(A) = \{\langle x, \beta(\mu_A(x) + (1-\alpha)\nu_A(x)), \alpha(\beta\nu_A(x) + \omega - \omega\beta)\rangle : x \in X\}$

(2) $\boxtimes_{\alpha,\beta}^{\omega}(A) = \{\langle x, \beta(\alpha\mu_A(x) + \omega - \omega\alpha), \alpha((1-\beta)\mu_A(x) + \nu_A(x))\rangle : x \in X\}$

Definition 1.19. [7]Let X be a set and $A \in IFS(X)$, $\alpha, \beta, \omega, \theta \in [0, 1]$ then

$$E_{\alpha,\beta}^{\omega,\theta}(A) = \left\{ \left\langle \begin{array}{c} x, \beta((1 - (1-\alpha)(1-\theta))\mu_A(x) + (1-\alpha)\theta\nu_A(x) + (1-\alpha)(1-\theta)\omega), \\ \\ \alpha((1-\beta)\theta\mu_A(x) + (1 - (1-\beta)(1-\theta))\nu_A(x) + (1-\beta)(1-\theta)\omega) \end{array} \right\rangle : x \in X \right\}$$

Definition 1.20. [7]Let X be a set, $A \in IFS(X)$ and $\alpha, \beta \in [0, 1]$ then

(1) $B_{\alpha,\beta}(A) = \{\langle x, \beta(\mu_A(x) + (1-\alpha)\nu_A(x)), \alpha((1-\beta)\mu_A(x) + \nu_A(x))\rangle : x \in X\}$

(2) $\boxminus_{\alpha,\beta}(A) = \{\langle x, \beta(\mu_A(x) + (1-\beta)\nu_A(x)), \alpha((1-\alpha)\mu_A(x) + \nu_A(x))\rangle : x \in X\}$

In 2014, new one type intuitionistic fuzzy modal operators were defined in [9].

Definition 1.21. [9]Let X be a set and $A \in IFS(X), \alpha, \beta, \omega \in [0, 1]$ and $\alpha + \beta \leq 1$

(1) $L_{\alpha,\beta}^{\omega}(A) = \{\langle x, \alpha\mu_A(x) + \omega(1-\alpha), \alpha(1-\beta)\nu_A(x) + \alpha\beta(1-\omega)\rangle \, x \in X\}$

(2) $K_{\alpha,\beta}^{\omega}(A) = \{\langle x, \alpha(1-\beta)\mu_A(x) + \alpha\beta(1-\omega), \alpha\nu_A(x) + \omega(1-\alpha)\rangle \, x \in X\}$

As above, we get the following diagram;
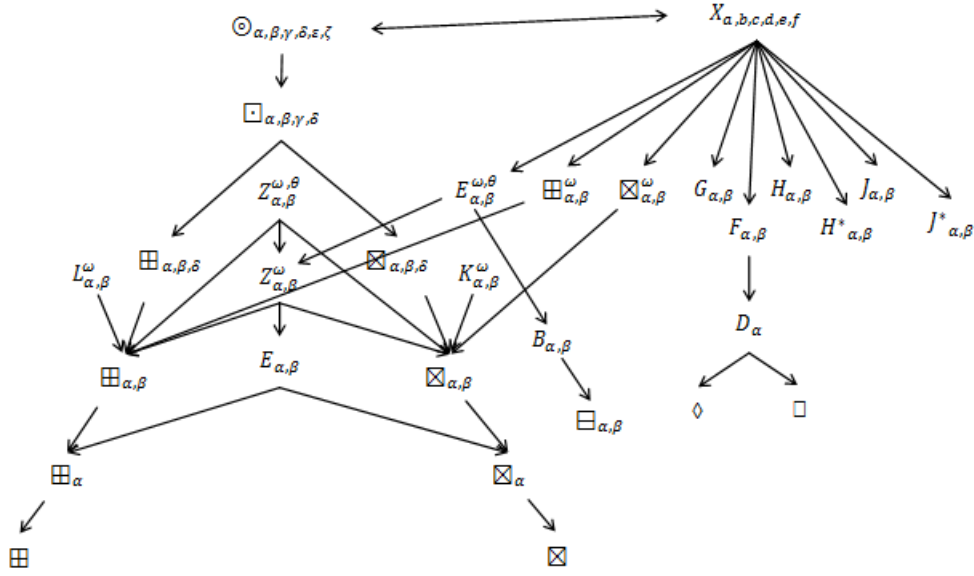
FIGURE 1

The intuitionistic fuzzy modal operator, represented by $\otimes_{\alpha,\beta,\gamma,\delta}$, introduced in 2014 as following;

Definition 1.22. [4]Let X be a set and $A \in IFS(X), \alpha, \beta, \gamma, \delta \in [0,1]$ and $\alpha+\beta \leq 1, \gamma + \delta \leq 1$ then

$$\otimes_{\alpha,\beta,\gamma,\delta}(A) = \{\langle x, \alpha\mu_A(x) + \gamma\nu_A(x), \beta\mu_A(x) + \delta\nu_A(x)\rangle\}$$

2. SOME PROPERTIES OF NEW INTUITIONISTIC FUZZY MODAL OPERATORS

Definition 2.1. Let X be a set and $A \in IFS(X), \alpha, \beta, \alpha + \beta \in [0,1]$.

(1) $T_{\alpha,\beta}(A) = \{< x, \beta(\mu_A(x) + (1-\alpha)\nu_A(x) + \alpha), \alpha(\nu_A(x) + (1-\beta)\mu_A(x)) >:$ $x \in X\}$ where $\alpha + \beta \in [0,1]$.

(2) $S_{\alpha,\beta}(A) = \{< x, \alpha(\mu_A(x) + (1-\beta)\nu_A(x)), \beta(\nu_A(x) + (1-\alpha)\mu_A(x) + \alpha) >:$ $x \in X\}$ where $\alpha + \beta \in [0,1]$.

It is clear that;

$$\begin{aligned} \beta(\mu_A(x) + (1-\alpha)\nu_A(x) + \alpha) + \alpha(\nu_A(x) + (1-\beta)\mu_A(x)) &= (\mu_A(x) + \nu_A(x))(\alpha + \beta - \alpha\beta) + \alpha\beta \\ &\leq \alpha + \beta - \alpha\beta + \alpha\beta \leq 1 \end{aligned}$$

Theorem 2.1. Let X be a set and $A \in IFS(X)$. If $\alpha, \beta, \alpha + \beta \in [0,1]$ then $T_{\alpha,\beta}(A)^c = S_{\alpha,\beta}(A^c)$.

Proof. It is clear from definition. □

Proposition 2.1. Let X be a set and $A \in IFS(X)$. If $\alpha, \beta, \alpha + \beta \in [0,1]$ then

(1) $T_{\beta,\alpha}(A)^c \sqsubseteq T_{\alpha,\beta}(A^c)$
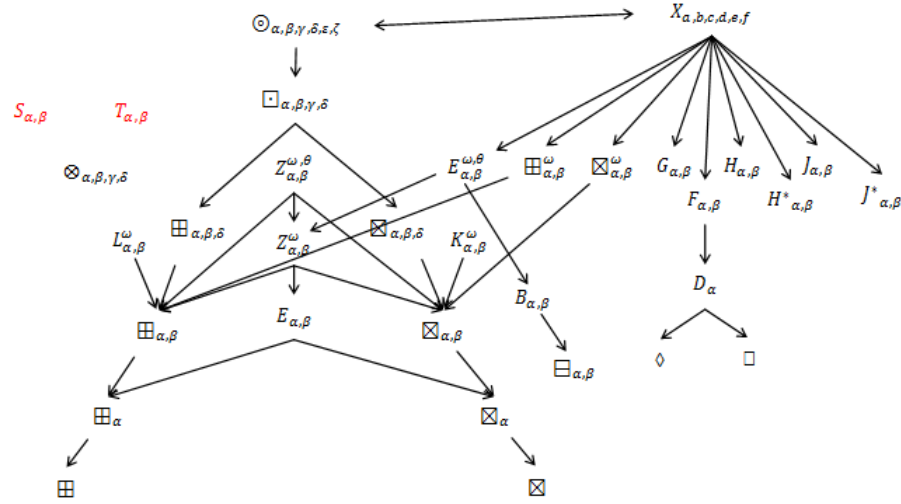
FIGURE 2

(2) $S_{\alpha,\beta}(A^c) \sqsubseteq S_{\beta,\alpha}(A)^c$

Proof. (1)From definition of this operators and complement of an intuitionistic fuzzy set we get that,

$$\beta(\nu_A(x) + (1-\alpha)\mu_A(x)) \leq \beta(\nu_A(x) + (1-\alpha)\mu_A(x) + \alpha)$$

and

$$\alpha(\mu_A(x) + (1-\beta)\nu_A(x) + \beta) \geq \alpha(\mu_A(x) + (1-\beta)\nu_A(x))$$

So, we can say $T_{\beta,\alpha}(A)^c \sqsubseteq T_{\alpha,\beta}(A^c)$.

(2)We can show this inclusion same way. □

Theorem 2.2. *Let X be a set and $A \in IFS(X)$. If $\alpha, \beta, \alpha + \beta \in [0,1]$ and $\beta \leq \alpha$ then*

(1) $T_{\alpha,\beta}(A) \sqsubseteq T_{\beta,\alpha}(A)$
(2) $S_{\beta,\alpha}(A) \sqsubseteq S_{\alpha,\beta}(A)$

Proof. It is clear. □

Theorem 2.3. *Let X be a set and $A, B \in IFS(X)$. If $\alpha, \beta, \alpha + \beta \in [0,1]$ then*

(1) $T_{\alpha,\beta}(A) \sqcap T_{\alpha,\beta}(B) \sqsubseteq T_{\alpha,\beta}(A \sqcap B)$
(2) $T_{\alpha,\beta}(A \sqcup B) \sqsubseteq T_{\alpha,\beta}(A) \sqcup T_{\alpha,\beta}(B)$

Proof. (1) Let $\alpha, \beta \in [0,1]$,

$$
\begin{aligned}
\beta(1-\alpha)\min(\nu_A(x), \nu_B(x)) \quad &\leq \quad \beta(1-\alpha)\max(\nu_A(x), \nu_B(x)) \\
&\Rightarrow \quad \beta\left(\min(\mu_A(x), \mu_B(x)) + (1-\alpha)\min(\nu_A(x), \nu_B(x)) + \alpha\right) \\
&\leq \quad \beta\left(\min(\mu_A(x), \mu_B(x)) + (1-\alpha)\max(\nu_A(x), \nu_B(x)) + \alpha\right)
\end{aligned}
$$

and

$$
\begin{aligned}
\alpha(1-\beta)\max(\mu_A(x),\mu_B(x)) \quad &\geq \quad \alpha(1-\beta)\min(\mu_A(x),\mu_B(x)) \\
&\Rightarrow \quad \alpha\left(\max\left(\nu_A(x),\nu_B(x)\right)+(1-\beta)\max\left(\mu_A(x),\mu_B(x)\right)\right) \\
&\geq \quad \alpha\left(\max(\nu_A(x),\nu_B(x))+(1-\beta)\min(\mu_A(x),\mu_B(x))\right)
\end{aligned}
$$

It is appear from here that $T_{\alpha,\beta}(A) \sqcap T_{\alpha,\beta}(B) \sqsubseteq T_{\alpha,\beta}(A \sqcap B)$.

(2) It can be shown easily. □

Theorem 2.4. *Let X be a set and $A, B \in IFS(X)$. If $\alpha, \beta, \alpha + \beta \in [0,1]$ then*

(1) $S_{\alpha,\beta}(A \sqcup B) \sqsubseteq S_{\alpha,\beta}(A) \sqcup S_{\alpha,\beta}(B)$

(2) $S_{\alpha,\beta}(A) \sqcap S_{\alpha,\beta}(B) \sqsubseteq S_{\alpha,\beta}(A \sqcap B)$

Proof. (1) Let $\alpha, \beta \in [0,1]$,

$$
\begin{aligned}
\alpha(1-\beta)\min(\nu_A(x),\nu_B(x)) \quad &\leq \quad \alpha(1-\beta)\max(\nu_A(x),\nu_B(x)) \\
&\Rightarrow \quad \alpha\left(\max\left(\mu_A(x),\mu_B(x)\right)+(1-\beta)\min\left(\nu_A(x),\nu_B(x)\right)\right) \\
&\leq \quad \alpha\left(\max\left(\mu_A(x),\mu_B(x)\right)+(1-\beta)\max\left(\nu_A(x),\nu_B(x)\right)\right)
\end{aligned}
$$

and

$$
\begin{aligned}
\beta(1-\alpha)\max(\mu_A(x),\mu_B(x)) \quad &\geq \quad \beta(1-\alpha)\min(\mu_A(x),\mu_B(x)) \\
&\Rightarrow \quad \beta\left(\min\left(\nu_A(x),\nu_B(x)\right)+(1-\alpha)\max\left(\mu_A(x),\mu_B(x)\right)+\alpha\right) \\
&\geq \quad \beta\left(\min\left(\nu_A(x),\nu_B(x)\right)+(1-\alpha)\min\left(\mu_A(x),\mu_B(x)\right)+\alpha\right)
\end{aligned}
$$

Thus, $S_{\alpha,\beta}(A \sqcup B) \sqsubseteq S_{\alpha,\beta}(A) \sqcup S_{\alpha,\beta}(B)$.

(2) Can be proved similarly. □

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ON HERMITE-HADAMARD INEQUALITIES FOR GEOMETRIC-ARITHMETICALLY φ-s-CONVEX FUNCTIONS VIA FRACTIONAL NTEGRALS

SÜMEYYE ERMEYDAN AND HÜSEYIN YILDIRIM

ABSTRACT. In this paper, we established integral inequalities of Hermite-Hadamard type involving Riemann-Liouville fractional integrals motivated by the definition of geometric-arithmetically s-convex and v-convex functions. We give equality for fractional integral of φ-convex functions on twice differentiable mapping.

1. INTRODUCTION

Fractional calculus was born in 1695. In the past three hundred years, fractional calculus developed in diverse fields from physical sciences and engineering to biological sciences and economics$[1-8]$. Fractional Hermite-Hadamard inequalities involving all kinds of fractional integrals have attracted by many researches. In $[12]$, Shuang et al. introduced a new concept of geometric-arithmetically s-convex functions and presented interesting Hermite-Hadamard type inequalities for integer integrals of such functions. In $[15]$, Youness introduced a new concept of φ-convex functions. In $[16-17]$, YuMei Liao and colleagues gave Riemann-Lioville Hermite-Hadamard integral inequalities for once and twice differentiable geometric-arithmetically $s-$convex functions. We establish on Hermite-Hadamard inequalities for twice differentiable geometric-arithmetically $\varphi - s-$convex functions via fractional integrals.

2. PRELIMINARIES

In this section, we will give some definitions, lemmas and notations which we use later in this work.

Definition 2.1. ($see\,[3]$) Let $f \in L\,[a, b]$. The Riemann- Liouville fractional integral $\mathrm{J}_{a+}^{\alpha}f$ and $\mathrm{J}_{b-}^{\alpha}f$ of order $\alpha > 0$ with $\alpha \geq 0$ are defined by

$$J_{a+}^{\alpha}f\,(x) = \tfrac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1}f\,(t)\,dt, 0 \leq a < x \leq b$$

and

(2.1) $$J_{b-}^{\alpha}f\,(x) = \tfrac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1}f\,(t)\,dt, 0 \leq a < x \leq b$$

Here Γ is the gamma function.

Definition 2.2. $(see\,[12-16])$ Let $f : I \subseteq \mathbb{R}^+ \to \mathbb{R}^+$and s$\in (0, 1]$. A function $f(x)$ is said to be geometric-arithmetically s-convex on I if for every $x, y \in I$ and t$\in [0, 1]$, we have:

$$(2.2) \qquad f\left(x^t y^{1-t}\right) \leq t^s f(x) + (1-t)^s f(y)$$

Definition 2.3. $(see\,[14])$ The incomplete beta function is defined as follows:

$$(2.3) \qquad B_x(a, b) = \int_0^x t^{a-1}(1-t)^{b-1}\,dt.$$

Where $x \in [0, 1]$, $a, b > 0$.

Definition 2.4. $(see\,[15])$ Let $\varphi : [a, b] \subset \mathbb{R} \to [a, b]$ A function $f : [a, b] \to \mathbb{R}$ is said to be $\varphi - convex$ on $[a, b]$ if, for every $x, y \in [a, b]$ and $t \in [0, 1]$ the following inequality holds:

$$(2.4) \qquad f(t\varphi(x) + (1-t)\varphi(y)) \leq tf(\varphi(x)) + (1-t)f(\varphi(y))$$

Definition 2.5. Let $\varphi : [a, b] \subset \mathbb{R} \to [a, b]$ and $s \in (0, 1]$. A function $f : [a, b] \to \mathbb{R}$ is said to be $\varphi - s - convex$ on $[a, b]$ if, for every $x, y \in [a, b]$ and $t \in [0, 1]$ the following inequality holds:

$$(2.5) \qquad f\left(\varphi(x)^t \varphi(y)^{(1-t)}\right) \leq t^s f(\varphi(x)) + (1-t)^s f(\varphi(y))$$

Lemma 2.1. $(see\,[11])$ For t$\in [0, 1]$, we have

$$(2.6) \qquad (1-t)^n \leq 2^{1-n} - t^n \quad for \quad n \in [0, 1],$$

$$(1-t)^n \geq 2^{1-n} - t^n \quad for \quad n \in [0, \infty).$$

The following inequality was used in the proof directly in$[12]$.

Lemma 2.2. $(see\,[13])$ *for* $t \in [0, 1]$ *and* $x, y > 0$,*we have*

$$(2.7) \qquad tx + (1-t)y \geq y^{1-t}x^t.$$

Lemma 2.3. $(see\,[10])$ *Let* $f : [a, b] \to \mathbb{R}$ *be a twice differentiable mapping on* (a, b) *with* $a < b$. *If* $f'' \in L[a, b]$, *then the following equality for fractional integrals holds:*

$$(2.8) \qquad \begin{array}{l} \frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\left[J_{a+}^\alpha f(b) + J_{b-}^\alpha f(a)\right] \\ = \frac{(b-a)^2}{2}\int_0^1 \frac{1-(1-t)^{\alpha+1}-t^{\alpha+1}}{\alpha+1} f''(ta + (1-t)b)\,dt. \end{array}$$

Lemma 2.4. $(see\,[9])$ *Let* $f : [a, b] \to \mathbb{R}$ *be a twice differentiable mapping on* (a, b) *with* $a < b$. *If* $f'' \in L[a, b]$, *then*

$$(2.9) \qquad \begin{array}{l} \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\left[J_{a+}^\alpha f(b) + J_{b-}^\alpha f(a)\right] - f\left(\frac{a+b}{2}\right) \\ = \frac{(b-a)^2}{2}\int_0^1 m(t)f''(ta + (1-t)b)\,dt. \end{array}$$

where

$$m(t) = \begin{cases} t - \frac{1-(1-t)^{\alpha+1}-t^{\alpha+1}}{\alpha+1}, & t \in \left[0, \frac{1}{2}\right), \\ 1 - t - \frac{1-(1-t)^{\alpha+1}-t^{\alpha+1}}{\alpha+1}, & t \in \left[\frac{1}{2}, 1\right). \end{cases}$$

Lemma 2.5. $(see\,[9])$ *Let* $f : [a, b] \to \mathbb{R}$ *be a differentiable mapping on* (a, b) *with* $a < b$. *If* $f'' \in L[a, b]$, $r > 0$, *then*

$$(2.10) \qquad \begin{array}{l} \frac{f(a)+f(b)}{r(r+1)} + \frac{2}{r+1}f\left(\frac{a+b}{2}\right) - \frac{\Gamma(\alpha+1)}{r(b-a)^\alpha}\left[J_{a+}^\alpha f(b) + J_{b-}^\alpha f(a)\right] \\ = (b-a)^2 \int_0^1 k(t)f''(ta + (1-t)b)\,dt. \end{array}$$

Where

$$k\left(t\right) = \begin{cases} \frac{1-(1-t)^{\alpha+1}-t^{\alpha+1}}{\alpha+1} - \frac{t}{r+1}, & t \in \left[0, \frac{1}{2}\right), \\ \frac{1-(1-t)^{\alpha+1}-t^{\alpha+1}}{\alpha+1} - \frac{1-t}{r+1}, & t \in \left[\frac{1}{2}, 1\right). \end{cases}$$

YuMei Liao and colleagues based on our study, they have provided the following theorems in [16],[17].

Theorem 2.1. *Let $f : [0,b] \to R$ be a differentiable mapping. If $|f'|$ is measurable and $|f'|$ is decreasing and geometric-arithmetically s-convex on $[0,b]$ for some fixed $\alpha \in (0,\infty), s \in (0,1], 0 \leq a < b$, then the following inequality for fractional integrals holds:*

(2.11)
$$\left| \frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}} \left[J_{a^+}^{\alpha} f\left(b\right) + J_{b^-}^{\alpha} f\left(a\right) \right] \right|$$
$$\leq \frac{(b-a)\left(2^{\alpha+s}|f'(b)| - |f'(a)| - |f'(b)|\right)}{2^{\alpha+s+1}\left(\alpha+s+1\right)}$$
$$+ (b-a)|f'(a)| \left[0.5B\left(s+1,\alpha+1\right) - B_{0.5}\left(\alpha+1,s+1\right)\right]$$
$$+ (b-a)|f'(b)| \left[B_{0.5}\left(s+1,\alpha+1\right) - 0.5B\left(a+1,\alpha+1\right)\right].$$

Theorem 2.2. *Let $f : [0,b] \to R$ be a differentiable mapping. If $|f''|$ is measurable and $|f''|$ is decreasing and geometric-arithmetically s-convex on $[0,b]$ for some fixed $\alpha \in (0,\infty), s \in (0,1], 0 \leq a < b$, then the following inequality for fractional integrals holds:*

(2.12)
$$\left| \frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}} \left[J_{a^+}^{\alpha} f\left(b\right) + J_{b^-}^{\alpha} f\left(a\right) \right] \right|$$
$$\leq \frac{(b-a)^2\left(|f''(a)|+|f''(b)|\right)}{2(\alpha+1)} \left(\frac{1}{s+1} - \frac{1}{\alpha+s+1} - B\left(s+1,\alpha+2\right) \right).$$

3. Main Results

Lemma 3.1. *Let I be an interval $a,b \in I$ with $0 \leq a < b$ and $\varphi : I \to \mathbb{R}$ a continuous increasing function. Let $f : [\varphi(a), \varphi(b)] \to \mathbb{R}$ be a twice differentiable mapping on $(\varphi(a), \varphi(b))$. If $f'' \in L[\varphi(a), \varphi(b)]$, then the following equality for fractional integral holds:*

(3.1)
$$\frac{f(\varphi(a))+f(\varphi(b))}{2} - \frac{\Gamma(\alpha+1)}{2(\varphi(b)-\varphi(a))^{\alpha}} \left[J_{\varphi(a)^+}^{\alpha} f\left(\varphi(b)\right) + J_{\varphi(b)^-}^{\alpha} f\left(\varphi(a)\right) \right]$$
$$= \frac{(\varphi(b)-\varphi(a))^2}{2} \int_0^1 \left(\frac{1-(1-t)^{\alpha+1}-t^{\alpha+1}}{\alpha+1} \right) f''\left(t\varphi(a) + (1-t)\varphi(b)\right) dt.$$

Proof. By using Lemma 3 and Definition 5, we have

$$I = \int_0^1 \left(\frac{1-(1-t)^{\alpha+1}-t^{\alpha+1}}{\alpha+1} \right) \left(f''\left(t\varphi(a) + (1-t)\varphi(b)\right)\right) dt$$
$$= \frac{1}{\varphi(b)-\varphi(a)} \left[\int_0^1 \left((1-t)^{\alpha} - t^{\alpha}\right) f'\left(t\varphi(a) + (1-t)\varphi(b)\right) dt \right]$$
$$= \frac{f(\varphi(a))+f(\varphi(b))}{(\varphi(b)-\varphi(a))^2} - \frac{\alpha}{(\varphi(b)-\varphi(a))^2}$$
$$\times \left(\int_0^1 (1-t)^{\alpha-1} f\left(t\varphi(a) + (1-t)\varphi(b)\right) dt \right.$$
$$\left. + \int_0^1 t^{\alpha-1} f\left(t\varphi(a) + (1-t)\varphi(b)\right) dt \right)$$
$$= \frac{f(\varphi(a))+f(\varphi(b))}{(\varphi(b)-\varphi(a))^2} - \frac{\alpha\Gamma(\alpha)}{(\varphi(b)-\varphi(a))^{\alpha+2}}$$
$$\times \left(\int_{\varphi(a)}^{\varphi(b)} \left(\varphi(x) - \varphi(a)\right)^{\alpha-1} f\left(\varphi(x)\right) d\varphi(x) \right.$$
$$\left. + \int_{\varphi(a)}^{\varphi(b)} \left(\varphi(b) - \varphi(x)\right)^{\alpha-1} f\left(\varphi(x)\right) d\varphi(x) \right)$$
$$= \frac{f(\varphi(a))+f(\varphi(b))}{(\varphi(b)-\varphi(a))^2} - \frac{\Gamma(\alpha+1)}{(\varphi(b)-\varphi(a))^{\alpha+2}} \left[J_{\varphi(a)^+}^{\alpha} f\left(\varphi(b)\right) + J_{\varphi(b)^-}^{\alpha} f\left(\varphi(a)\right) \right].$$

If I by multiplying $\frac{(\varphi(b)-\varphi(a))^2}{2}$, it obtain that

$$\frac{(\varphi(b)-\varphi(a))^2}{2} \int_0^1 \left(\frac{1-(1-t)^{\alpha+1}-t^{\alpha+1}}{\alpha+1} \right) f'' t\varphi(a) + (1-t)\varphi(b)\, dt$$
$$= \frac{f(\varphi(a))+f(\varphi(b))}{2} - \frac{\Gamma(\alpha+1)}{2(\varphi(b)-\varphi(a))^\alpha} \left[J_{\varphi(a)^+}^\alpha f(\varphi(b)) + J_{\varphi(b)^-}^\alpha f(\varphi(a)) \right].$$

The proof is done. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Lemma 3.2. *Let I be an interval $a, b \in I$ with $0 \le a < b$ and $\varphi : I \to \mathbb{R}$ a continuous increasing function. Let $f : [\varphi(a), \varphi(b)] \to \mathbb{R}$ be a twice differentiable mapping on $(\varphi(a), \varphi(b))$. If $f'' \in L[\varphi(a), \varphi(b)]$, then the following equality for fractional integral holds:*

(3.2)
$$\frac{\Gamma(\alpha+1)}{2(\varphi(b)-\varphi(a))^\alpha} \left[J_{\varphi(a)^+}^\alpha f(\varphi(b)) + J_{\varphi(b)^-}^\alpha f(\varphi(a)) \right] - f\left(\frac{\varphi(a)+\varphi(b)}{2} \right)$$
$$= \frac{(\varphi(b)-\varphi(a))^2}{2} \int_0^1 m(t) f''(t\varphi(a) + (1-t)\varphi(b))\, dt,$$

where

$$m(t) = \begin{cases} t - \frac{1-(1-t)^{\alpha+1}-t^{\alpha+1}}{\alpha+1}, & t \in \left[0, \frac{1}{2}\right), \\ 1-t - \frac{1-(1-t)^{\alpha+1}-t^{\alpha+1}}{\alpha+1}, & t \in \left[\frac{1}{2}, 1\right). \end{cases}$$

Proof. By using Lemma 4 and Definition 5, we have:

$$\begin{aligned} I &= \int_0^1 m(t) f''(t\varphi(a) + (1-t)\varphi(b))\, dt \\ &= \int_0^{\frac{1}{2}} \left(t - \frac{1-(1-t)^{\alpha+1}-t^{\alpha+1}}{\alpha+1} \right) (f''(t\varphi(a)+(1-t)\varphi(b)))\, dt \\ &\quad + \int_{\frac{1}{2}}^1 \left(1-t - \frac{1-(1-t)^{\alpha+1}-t^{\alpha+1}}{\alpha+1} \right) (f''(t\varphi(a)+(1-t)\varphi(b)))\, dt \\ &= I_1 + I_2. \end{aligned}$$

If use twice the partial integration method for I_1, we have

$$\begin{aligned} I_1 &= \int_0^{\frac{1}{2}} \left(t - \frac{1-(1-t)^{\alpha+1}-t^{\alpha+1}}{\alpha+1} \right) f''(t\varphi(a) + (1-t)\varphi(b))\, dt \\ &= \frac{(\alpha-1+2^{1-\alpha}) f'\left(\frac{\varphi(a)+\varphi(b)}{2} \right)}{2(\alpha+1)(\varphi(a)-\varphi(b))} - \frac{f\left(\frac{\varphi(a)+\varphi(b)}{2} \right)}{(\varphi(a)-\varphi(b))^2} + \frac{\Gamma(\alpha+1)}{(\varphi(a)-\varphi(b))^{\alpha+2}} \\ &\quad \times \left[\int_{\frac{\varphi(a)+\varphi(b)}{2}}^{\varphi(b)} (\varphi(x)-\varphi(a))^{\alpha-1} f(\varphi(x))\, d\varphi(x) \right. \\ &\quad \left. + \int_{\frac{\varphi(a)+\varphi(b)}{2}}^{\varphi(b)} (\varphi(b)-\varphi(x))^{\alpha-1} f(\varphi(x))\, d\varphi(x) \right]. \end{aligned}$$

If use twice the partial integration method for I_2, we have

$$\begin{aligned} I_2 &= \int_{\frac{1}{2}}^1 \left(1-t - \frac{1-(1-t)^{\alpha+1}-t^{\alpha+1}}{\alpha+1} \right) f''(t\varphi(a) + (1-t)\varphi(b))\, dt \\ &= -\left(\frac{(\alpha-1+2^{1-\alpha}) f'\left(\frac{\varphi(a)+\varphi(b)}{2} \right)}{2(\alpha+1)(\varphi(a)-\varphi(b))} \right) - \frac{f\left(\frac{\varphi(a)+\varphi(b)}{2} \right)}{(\varphi(a)-\varphi(b))^2} \\ &\quad + \frac{\Gamma(\alpha+1)}{(\varphi(a)-\varphi(b))^{\alpha+2}} \left[\int_{\varphi(a)}^{\frac{\varphi(a)+\varphi(b)}{2}} (\varphi(x)-\varphi(a))^{\alpha-1} f(\varphi(x))\, d\varphi(x) \right. \\ &\quad \left. + \int_{\varphi(a)}^{\frac{\varphi(a)+\varphi(b)}{2}} (\varphi(b)-\varphi(x))^{\alpha-1} f(\varphi(x))\, d\varphi(x) \right], \end{aligned}$$

by add uping I_1 and I_2, and by multiplying $\frac{(\varphi(b)-\varphi(a))^2}{2}$ with I, it obtain that:

$$\frac{\Gamma(\alpha+1)}{2(\varphi(b)-\varphi(a))^\alpha} \left[J_{\varphi(a)^+}^\alpha f(\varphi(b)) + J_{\varphi(b)^-}^\alpha f(\varphi(a)) \right] - f\left(\frac{\varphi(a)+\varphi(b)}{2} \right)$$
$$= \frac{(\varphi(b)-\varphi(a))^2}{2} \int_0^1 m(t) f''(t\varphi(a) + (1-t)\varphi(b))\, dt.$$

The proof is done.

Lemma 3.3. *Let I be an interval $a, b \in I$ with $0 \leq a < b$ and $\varphi : I \to \mathbb{R}$ a continuous increasing function. Let $f : [\varphi(a), \varphi(b)] \to \mathbb{R}$ be a twice differentiable mapping on $(\varphi(a), \varphi(b))$. If $f'' \in L[\varphi(a), \varphi(b)]$, then the following equality for fractional integral holds:*

(3.3)
$$
\frac{f(\varphi(a)) + f(\varphi(b))}{r(r+1)} + \frac{2}{r+1} f\left(\frac{\varphi(a) + \varphi(b)}{2}\right) - \frac{\Gamma(\alpha+1)}{r(\varphi(b) - \varphi(a))^{\alpha}}
$$
$$
\times \left[J_{\varphi(a)^+}^{\alpha} f(\varphi(b)) + J_{\varphi(b)^-}^{\alpha} f(\varphi(a)) \right]
$$
$$
= (\varphi(b) - \varphi(a))^2 \int_0^1 k(t) f''(t\varphi(a) + (1-t)\varphi(b)) \, dt.
$$

Where

$$
k(t) = \begin{cases} \frac{1 - (1-t)^{\alpha+1} - t^{\alpha+1}}{\alpha+1} - \frac{t}{r+1}, & t \in \left[0, \frac{1}{2}\right), \\ \frac{1 - (1-t)^{\alpha+1} - t^{\alpha+1}}{\alpha+1} - \frac{1-t}{r:+1}, & t \in \left[\frac{1}{2}, 1\right). \end{cases}
$$

□

Proof. By using Definition 5 and Lemma 5, we have

$$
\begin{aligned}
I &= \int_0^1 k(t) f''(t\varphi(a) + (1-t)\varphi(b)) \, dt \\
&= \frac{1}{r(r+1)(\alpha+1)} \\
&\quad \times \int_0^{\frac{1}{2}} \left[(r+1)\left[1 - (1-t)^{\alpha+1} - t^{\alpha+1}\right] - rt(\alpha+1) \right] \\
&\quad \times f''(t\varphi(a) + (1-t)\varphi(b)) \, dt \\
&\quad + \frac{1}{r(r+1)(\alpha+1)} \\
&\quad \times \int_{\frac{1}{2}}^1 \left[(r+1)\left[1 - (1-t)^{\alpha+1} - t^{\alpha+1}\right] - r(1-t)(\alpha+1) \right] \\
&\quad \times f''(t\varphi(a) + (1-t)\varphi(b)) \, dt \\
&= I_1 + I_2.
\end{aligned}
$$

If use twice the partial integration method for I_1, we have

$$
\begin{aligned}
I_1 &= \int_0^{\frac{1}{2}} \left[(r+1)\left[1 - (1-t)^{\alpha+1} - t^{\alpha+1}\right] - rt(\alpha+1) \right] \\
&\quad \times f''(t\varphi(a) + (1-t)\varphi(b)) \, dt \\
&= \left[(r+1)(1 - 2^{-\alpha}) - \frac{r(\alpha+1)}{2} \right] \frac{f'\left(\frac{\varphi(a)+\varphi(b)}{2}\right)}{\varphi(a) - \varphi(b)} \\
&\quad + \frac{r(\alpha+1)f\left(\frac{\varphi(a)+\varphi(b)}{2}\right)}{(\varphi(b)-\varphi(a))^2} + \frac{(\alpha+1)f(\varphi(b))}{(\varphi(b)-\varphi(a))^2} - \frac{(r+1)(\alpha+1)\alpha\Gamma(\alpha)}{(\varphi(b)-\varphi(a))^{\alpha+2}} \\
&\quad \times \left[\int_{\frac{\varphi(a)+\varphi(b)}{2}}^{\varphi(b)} (\varphi(x) - \varphi(a))^{\alpha-1} f(\varphi(x)) \, d\varphi(x) \right. \\
&\quad \left. + \int_{\frac{\varphi(a)+\varphi(b)}{2}}^{\varphi(b)} (\varphi(b) - \varphi(x))^{\alpha-1} f(\varphi(x)) \, d\varphi(x) \right]
\end{aligned}
$$

If use twice the partial integration method for I_2, we have

$$
\begin{aligned}
I_2 &= \int_{\frac{1}{2}}^1 \left[(r+1)\left[1 - (1-t)^{\alpha+1} - t^{\alpha+1}\right] - r(1-t)(\alpha+1) \right] \\
&\quad \times f''(t\varphi(a) + (1-t)\varphi(b)) \, dt \\
&= \left[\frac{r(\alpha+1)}{2} - (r+1)(1 - 2^{-\alpha}) \right] \frac{f'\left(\frac{\varphi(a)+\varphi(b)}{2}\right)}{\varphi(a) - \varphi(b)} \\
&\quad + \frac{r(\alpha+1)f\left(\frac{\varphi(a)+\varphi(b)}{2}\right)}{(\varphi(b)-\varphi(a))^2} + \frac{(\alpha+1)f(\varphi(a))}{(\varphi(b)-\varphi(a))^2} - \frac{(r+1)(\alpha+1)\alpha\Gamma(\alpha)}{(\varphi(b)-\varphi(a))^{\alpha+2}} \\
&\quad \times \left[\int_{\varphi(a)}^{\frac{\varphi(a)+\varphi(b)}{2}} (\varphi(x) - \varphi(a))^{\alpha-1} f(\varphi(x)) \, d\varphi(x) \right. \\
&\quad \left. + \int_{\varphi(a)}^{\frac{\varphi(a)+\varphi(b)}{2}} (\varphi(b) - \varphi(x))^{\alpha-1} f(\varphi(x)) \, d\varphi(x) \right].
\end{aligned}
$$

By add uping I_1 and I_2, and by multipling with $\frac{(\varphi(b)-\varphi(a))^2}{r(r+1)(\alpha+1)}$, it obtain that:

$$\frac{f(\varphi(a))+f(\varphi(b))}{r(r+1)} + \frac{2}{r+1} f\left(\frac{\varphi(a)+\varphi(b)}{2}\right)$$
$$-\frac{\Gamma(\alpha+1)}{r(\varphi(b)-\varphi(a))^\alpha}\left[J^\alpha_{\varphi(a)^+} f(\varphi(b)) + J^\alpha_{\varphi(b)^-} f(\varphi(a))\right]$$
$$= (\varphi(b)-\varphi(a))^2 \int_0^1 k(t) f''(t\varphi(a)+(1-t)\varphi(b))\, dt .$$

The proof is done.

Theorem 3.1. *Let I be an interval $a,b \in I$ with $0 \le a < b$ and $\varphi : I \to \mathbb{R}$ a continuous increasing function. Let $f : [0,\varphi(b)] \to \mathbb{R}$ be a differentiable mapping and $1 < q < \infty$. If $|f''|$ is measurable and $|f''|$ is decreasing and geometric-arithmetically $\varphi - s-$convex on $[0,\varphi(b)]$ for some fixed $\alpha \in (0,\infty)$, $s \in (0,1]$, $0 \le a < b$, then the following inequality for fractional integrals holds:*

$$\frac{f(\varphi(a))+f(\varphi(b))}{2} - \frac{\Gamma(\alpha+1)}{2(\varphi(b)-\varphi(a))^\alpha}\left[J^\alpha_{\varphi(a)^+} f(\varphi(b)) + J^\alpha_{\varphi(b)^-} f(\varphi(a))\right]$$

(3.4)

$$\le \frac{(\varphi(b)-\varphi(a))^2\left(\left|f''(\varphi(a))\right|+\left|f''(\varphi(b))\right|\right)}{2(\alpha+1)}\left(\frac{1}{s+1} - \frac{1}{\alpha+s+1} - B(s+1,\alpha+2)\right).$$

$\square$

Proof. By using Definition 2, Lemma 2 and Lemma 6, we have

$$\left|\frac{f(\varphi(a))+f(\varphi(b))}{2} - \frac{\Gamma(\alpha+1)}{2(\varphi(b)-\varphi(a))^\alpha}\left[J^\alpha_{\varphi(a)^+} f(\varphi(b)) + J^\alpha_{\varphi(b)^-} f(\varphi(a))\right]\right|$$
$$\le \frac{(\varphi(b)-\varphi(a))^2}{2} \int_0^1 \left|\frac{1-(1-t)^{\alpha+1}-t^{\alpha+1}}{\alpha+1}\right| |f''(t\varphi(a)+(1-t)\varphi(b))|\, dt$$
$$\le \frac{(\varphi(b)-\varphi(a))^2}{2(\alpha+1)} \int_0^1 \left(1-(1-t)^{\alpha+1}-t^{\alpha+1}\right) \left|f''\left(\varphi^t(a)+\varphi^{(1-t)}(b)\right)\right|\, dt$$
$$\le \frac{(\varphi(b)-\varphi(a))^2}{2(\alpha+1)} \int_0^1 \left(1-(1-t)^{\alpha+1}-t^{\alpha+1}\right)$$
$$\times \left[t^s |f''\varphi(a)| + (1-t)^s |f''\varphi(b)|\right] dt$$
$$\le \frac{(\varphi(b)-\varphi(a))^2}{2(\alpha+1)} |f''\varphi(a)|$$
$$\times \left[-\int_0^1 t^{\alpha+s+1} dt + \int_0^1 t^s dt - \int_0^1 t^s (1-t)^{\alpha+1}\, dt\right]$$
$$+\frac{(\varphi(b)-\varphi(a))^2}{2(\alpha+1)} |f''\varphi(b)|$$
$$\times \left[-\int_0^1 (1-t)^{\alpha+s+1}\, dt + \int_0^1 (1-t)^s\, dt - \int_0^1 t^{\alpha+1} (1-t)^s\, dt\right]$$
$$\le \frac{(\varphi(b)-\varphi(a))^2}{2(\alpha+1)} \left(|f''\varphi(a)| + |f''\varphi(b)|\right)$$
$$\times \left[-\frac{1}{\alpha+s+1} + \frac{1}{s+1} - B(s+1,\alpha+2)\right].$$

The poof is done.

Theorem 3.2. *Let I be an interval $a,b \in I$ with $0 \le a < b$ and $\varphi : I \to \mathbb{R}$ a continuous increasing function. Let $f : [0,\varphi(b)] \to \mathbb{R}$ be a differentiable mapping and $1 < q < \infty$. If $|f''|^q$ is measurable and $|f''|^q$ is decreasing and geometric-arithmetically $\varphi - s-$convex on $[0,\varphi(b)]$ for some fixed $\alpha \in (0,\infty)$, $s \in (0,1]$, $0 \le a < b$, then the following inequality for fractional integrals holds:*

$$\left|\frac{f(\varphi(a))+f(\varphi(b))}{2} - \frac{\Gamma(\alpha+1)}{2(\varphi(b)-\varphi(a))^\alpha}\left[J^\alpha_{\varphi(a)^+} f(\varphi(b)) + J^\alpha_{\varphi(b)^-} f(\varphi(a))\right]\right|$$

(3.5)

$$\le \frac{(\varphi(b)-\varphi(a))^2 \max\left\{1-2^{1-\alpha},2^{1-\alpha}-1\right\}}{2(\alpha+1)}\left(\frac{\left|f''(\varphi(a))\right|^q+\left|f''(\varphi(b))\right|^q}{s+1}\right).$$

$\square$

Proof. To achieve our aim ,we divide our proof into two cases.

Case 1: $\alpha \in (0,1)$, by using Definition 2 , Hölder's inequality and Lemma 6 ,we have

$$\left| \frac{f(\varphi(a)) + f(\varphi(b))}{2} - \frac{\Gamma(\alpha+1)}{2(\varphi(b)-\varphi(a))^\alpha} \left[J^\alpha_{\varphi(a)^+} f(\varphi(b)) + J^\alpha_{\varphi(b)^-} f(\varphi(a)) \right] \right|$$
$$\leq \frac{(\varphi(b)-\varphi(a))^2}{2} \int_0^1 \left| \frac{1-(1-t)^{\alpha+1}-t^{\alpha+1}}{\alpha+1} \right| \left| f''(t\varphi(a) + (1-t)\varphi(b)) \right| dt$$
$$\leq \frac{(\varphi(b)-\varphi(a))^2}{2(\alpha+1)} \left(\int_0^1 \left| 1-(1-t)^{\alpha+1}-t^{\alpha+1} \right|^p \right)^{\frac{1}{p}}$$
$$\times \left(\int_0^1 \left| f''(t\varphi(a) + (1-t)\varphi(b)) \right|^q dt \right)^{\frac{1}{q}}$$
$$\leq \frac{(\varphi(b)-\varphi(a))^2}{2(\alpha+1)} \left(\int_0^1 \left| 1-(1-t)^{\alpha+1}-t^{\alpha+1} \right|^p \right)^{\frac{1}{p}}$$
$$\times \left(\int_0^1 \left| f''(\varphi^t(a)\varphi^{(1-t)}(b)) \right|^q dt \right)^{\frac{1}{q}}$$
$$\leq \frac{(\varphi(b)-\varphi(a))^2}{2(\alpha+1)} \left(\int_0^1 \left| 1-(1-t)^{\alpha+1}-t^{\alpha+1} \right|^p \right)^{\frac{1}{p}}$$
$$\times \left(\int_0^1 \left[t^s \left| f''(\varphi(a)) \right| + (1-t)^s \left| f''(\varphi(b)) \right| \right] dt \right)^{\frac{1}{q}}$$
$$\leq \frac{(\varphi(b)-\varphi(a))^2}{2(\alpha+1)} \left(\frac{\left| f''(\varphi(a)) \right|^q + \left| f''(\varphi(b)) \right|^q}{s+1} \right)^{\frac{1}{q}} \left(\int_0^1 \left[(1-t)^\alpha + t^\alpha - 1 \right]^p dt \right)^{\frac{1}{p}}$$
$$\leq \frac{(\varphi(b)-\varphi(a))^2}{2(\alpha+1)} \left(\frac{\left| f''(\varphi(a)) \right|^q + \left| f''(\varphi(b)) \right|^q}{s+1} \right)^{\frac{1}{q}} \left(\int_0^1 \left[2^{1-\alpha} - 1 \right]^p dt \right)^{\frac{1}{p}}$$
$$\leq \frac{(\varphi(b)-\varphi(a))^2 \left(2^{1-\alpha} - 1 \right)}{2(\alpha+1)} \left(\frac{\left| f''(\varphi(a)) \right|^q + \left| f''(\varphi(b)) \right|^q}{s+1} \right)^{\frac{1}{q}}.$$

Where $\frac{1}{p} + \frac{1}{q} = 1$.

Case 2: $\alpha \in [1,\infty)$, by using Definition 2, Hölder's inequality and Lemma 6, we have

$$\left| \frac{f(\varphi(a)) + f(\varphi(b))}{2} - \frac{\Gamma(\alpha+1)}{2(\varphi(b)-\varphi(a))^\alpha} \left[J^\alpha_{\varphi(a)^+} f(\varphi(b)) + J^\alpha_{\varphi(b)^-} f(\varphi(a)) \right] \right|$$
$$\leq \frac{(\varphi(b)-\varphi(a))^2}{2(\alpha+1)} \left(\frac{\left| f''(\varphi(a)) \right|^q + \left| f''(\varphi(b)) \right|^q}{s+1} \right)^{\frac{1}{q}}$$
$$\times \left(\int_0^1 \left[1-(1-t)^\alpha - t^\alpha \right]^p dt \right)^{\frac{1}{p}}$$
$$\leq \frac{(\varphi(b)-\varphi(a))^2 \left(1-2^{1-\alpha} \right)}{2(\alpha+1)} \left(\frac{\left| f''(\varphi(a)) \right|^q + \left| f''(\varphi(b)) \right|^q}{s+1} \right)^{\frac{1}{q}}.$$

The proof is done.

Theorem 3.3. *Let I be an interval $a,b \in I$ with $0 \leq a < b$ and $\varphi : I \to \mathbb{R}$ a continuous increasing function. Let $f : [0, \varphi(b)] \to \mathbb{R}$ be a differentiable mapping and $1 < q < \infty$. If $|f''|$ is measurable and $|f'|$ is decreasing and geometric-arithmetically $\varphi - s-$convex on $[0, \varphi(b)]$ for some fixed $\alpha \in (0,\infty), s \in (0,1], 0 \leq a < b$, then the*

following inequality for fractional integrals holds:

$$\left| \frac{\Gamma(\alpha+1)}{2(\varphi(b)-\varphi(a))^\alpha} \left[J^\alpha_{\varphi(a)^+} f(\varphi(b)) + J^\alpha_{\varphi(b)^-} f(\varphi(a)) \right] - f\left(\frac{\varphi(a)+\varphi(b)}{2} \right) \right|$$

(3.6)
$$\leq \frac{(\varphi(b)-\varphi(a))^2 |f''(\varphi(a))|}{2(\alpha+1)}$$
$$\times \left[\frac{\alpha - \alpha 2^{-s-1} - 2^{-s-1}}{1+s} - \frac{\alpha+1}{2+s} + 2B(s+1,\alpha+2) + \frac{1}{\alpha+s+2} \right]$$
$$+ \frac{(\varphi(b)-\varphi(a))^2 |f''(\varphi(a))|}{2(\alpha+1)}$$
$$\times \left[\frac{\alpha 2^{-s-1} + 2^{-s-1} - 1}{1+s} + \frac{1}{\alpha+s+2} + 2B(s+1,\alpha+2) \right].$$

$\square$

Proof. By using Definition 5 and Lemma 7, we have

$$\left| \frac{\Gamma(\alpha+1)}{2(\varphi(b)-\varphi(a))^\alpha} \left[J^\alpha_{\varphi(a)^+} f(\varphi(b)) + J^\alpha_{\varphi(b)^-} f(\varphi(a)) \right] - f\left(\frac{\varphi(a)+\varphi(b)}{2} \right) \right|$$
$$\leq \frac{(\varphi(b)-\varphi(a))^2}{2} \int_0^1 |m(t)| \, |f''(t\varphi(a) + (1-t)\varphi(b))| \, dt$$
$$\leq \frac{(\varphi(b)-\varphi(a))^2}{2} \int_0^1 |m(t)| \, [f''(\varphi^t(a)\varphi^{(1-t)}(b))] \, dt$$
$$\leq \frac{(\varphi(b)-\varphi(a))^2}{2} \int_0^1 |m(t)| \, [t^s |f''(\varphi(a))| + (1-t)^s |f''(\varphi(b))|] \, dt$$
$$\leq \frac{(\varphi(b)-\varphi(a))^2}{2} \int_0^{\frac{1}{2}} \left| t - \frac{1-(1-t)^{\alpha+1}-t^{\alpha+1}}{\alpha+1} \right| [t^s |f''(\varphi(a))| + (1-t)^s |f''(\varphi(b))|] \, dt$$
$$+ \frac{(\varphi(b)-\varphi(a))^2}{2} \int_{\frac{1}{2}}^1 \left| 1 - t - \frac{1-(1-t)^{\alpha+1}-t^{\alpha+1}}{\alpha+1} \right| [t^s |f''(\varphi(a))| + (1-t)^s |f''(\varphi(b))|] \, dt$$
$$\leq \frac{(\varphi(b)-\varphi(a))^2}{2(\alpha+1)} |f''(\varphi(a))| \left[\int_0^{\frac{1}{2}} \left[-t^s + (\alpha+1)t^{s+1} + t^s(1-t)^{\alpha+1} + t^{\alpha+s+1} \right] dt \right.$$
$$+ \int_{\frac{1}{2}}^1 \left[\alpha t^s - (\alpha+1)t^{s+1} + t^s(1-t)^{\alpha+1} + t^{\alpha+s+1} \right] dt \Big] + \frac{(\varphi(b)-\varphi(a))^2}{2(\alpha+1)} |f''(\varphi(b))|$$
$$\times \left[\int_0^{\frac{1}{2}} \left[-(1-t)^s + (\alpha+1)t(1-t)^{s+1} + (1-t)^{\alpha+s+1} + t^{\alpha+s}(1-t)^s \right] dt \right.$$
$$+ \int_{\frac{1}{2}}^1 \left[\alpha(1-t)^s - (\alpha+1)t(1-t)^s + (1-t)^{\alpha+s+1} + t^{\alpha+1}(1-t)^s \right] dt \Big]$$
$$\leq \frac{(\varphi(b)-\varphi(a))^2}{2(\alpha+1)} |f''(\varphi(a))| \left[\frac{\alpha - \alpha 2^{-s-1} - 2^{-s-1}}{1+s} - \frac{\alpha+1}{s+2} + 2B(s+1,\alpha+2) + \frac{1}{\alpha+s+2} \right]$$
$$+ \frac{(\varphi(b)-\varphi(a))^2}{2(\alpha+1)} |f''(\varphi(b))| \left[\frac{\alpha 2^{-s-1} + 2^{-s-1} - 1}{1+s} + 2B(\alpha+2,s+1) + \frac{1}{\alpha+s+2} \right].$$

Theorem 3.4. *Let I be an interval $a,b \in I$ with $0 \leq a < b$ and $\varphi : I \rightarrow \mathbb{R}$ a continuous increasing function. Let $f : [0, \varphi(b)] \rightarrow \mathbb{R}$ be a differentiable mapping and $1 < q < \infty$. If $|f''|$ is measurable and $|f''|^q$ is decreasing and geometric-arithmetically $\varphi - s-$convex on $[0, \varphi(b)]$ for some fixed $\alpha \in (0,\infty), s \in (0,1], 0 \leq a < b$, then the following inequality for fractional integrals holds:*
(3.7)
$$\frac{\Gamma(\alpha+1)}{2(\varphi(b)-\varphi(a))^\alpha} \left[J^\alpha_{\varphi(a)^+} f(\varphi(b)) + J^\alpha_{\varphi(b)^-} f(\varphi(a)) \right] - f\left(\frac{\varphi(a)+\varphi(b)}{2} \right)$$

$$\leq \frac{(\varphi(b)-\varphi(a))^2}{2(\alpha+1)} \left(\frac{|f''(\varphi(a))|^q + |f''(\varphi(b))|^q}{s+1} \right)^{\frac{1}{q}} \left(\frac{(\alpha+1)2^{-p-1} + (\alpha+0.5)^{p+1} - \alpha^{p+1}}{p+1} \right)^{\frac{1}{p}}.$$

Where $\frac{1}{p} + \frac{1}{q} = 1.$

$\square$

Proof. By using Lemma 2, Hölder's inequality and Lemma 7, we have

$$\left| \frac{\Gamma(\alpha+1)}{2(\varphi(b)-\varphi(a))^\alpha} \left[J^\alpha_{\varphi(a)^+} f(\varphi(b)) + J^\alpha_{\varphi(b)^-} f(\varphi(a)) \right] - f\left(\frac{\varphi(a)+\varphi(b)}{2} \right) \right|$$

$$\leq \frac{(\varphi(b)-\varphi(a))^2}{2} \int_0^1 |m(t)| \, |f''(t\varphi(a) + (1-t)\varphi(b))| \, dt$$

$$\leq \frac{(\varphi(b)-\varphi(a))^2}{2} \left(\int_0^1 |m(t)|^p \, dt \right)^{\frac{1}{p}}$$

$$\times \left(\int_0^1 |f''(t\varphi(a) + (1-t)\varphi(b))|^q \, dt \right)^{\frac{1}{q}}$$

$$\leq \frac{(\varphi(b)-\varphi(a))^2}{2} \left(\int_0^1 |m(t)|^p \, dt \right)^{\frac{1}{p}}$$

$$\times \left(\int_0^1 \left| f''\left(\varphi^t(a) \varphi^{(1-t)}(b) \right) \right|^q \, dt \right)^{\frac{1}{q}}$$

$$\leq \frac{(\varphi(b)-\varphi(a))^2}{2} \left(\int_0^1 |m(t)|^p \, dt \right)^{\frac{1}{p}}$$

$$\times \left(\int_0^1 \left[t^s \, |f''(\varphi(a))|^q + (1-t)^s \, |f''(\varphi(b))|^q \right] dt \right)^{\frac{1}{q}}$$

$$\leq \frac{(\varphi(b)-\varphi(a))^2}{2(\alpha+1)} \left(\frac{|f''(\varphi(a))|^q + |f''(\varphi(b))|^q}{s+1} \right)^{\frac{1}{q}}$$

$$\times \left(\int_0^{\frac{1}{2}} \left| t - \frac{1-(1-t)^{\alpha+1}-t^{\alpha+1}}{\alpha+1} \right|^p dt + \int_{\frac{1}{2}}^1 \left| 1 - t - \frac{1-(1-t)^{\alpha+1}-t^{\alpha+1}}{\alpha+1} \right|^p dt \right)^{\frac{1}{p}}$$

$$\leq \frac{(\varphi(b)-\varphi(a))^2}{2(\alpha+1)} \left(\frac{|f''(\varphi(a))|^q + |f''(\varphi(b))|^q}{s+1} \right)^{\frac{1}{q}}$$

$$\dot{u} \times \left((\alpha+1) \int_0^{\frac{1}{2}} t^p dt + \int_{\frac{1}{2}}^1 (\alpha - t + 1)^p \, dt \right)^{\frac{1}{p}}$$

$$\leq \frac{(\varphi(b)-\varphi(a))^2}{2(\alpha+1)} \left(\frac{|f''(\varphi(a))|^q + |f''(\varphi(b))|^q}{s+1} \right)^{\frac{1}{q}} \left(\frac{(\alpha+1)2^{-p-1}+(\alpha+0.5)^{p+1}-\alpha^{p+1}}{p+1} \right)^{\frac{1}{p}}.$$

The proof is done.

Theorem 3.5. *Let I be an interval $a, b \in I$ with $0 \leq a < b$ and $\varphi : I \to \mathbb{R}$ a continuous increasing function. Let $f : [0, \varphi(b)] \to \mathbb{R}$ be a differentiable mapping and $1 < q < \infty$. If $|f''|$ is measurable and $|f''|$ is decreasing and geometric-arithmetically $\varphi - s-$convex on $[0, \varphi(b)]$ for some fixed $\alpha \in (0, \infty), s \in (0, 1], 0 \leq a < b$, then the*

following inequality for fractional integrals holds:

(3.8)

$$
\left| \frac{f(\varphi(a))+f(\varphi(b))}{r(r+1)} + \frac{2}{r+1} f\left(\frac{\varphi(a)+\varphi(b)}{2}\right) - \frac{\Gamma(\alpha+1)}{r(\varphi(b)-\varphi(a))^\alpha} \left[J^\alpha_{\varphi(a)^+} f(\varphi(b)) + J^\alpha_{\varphi(b)^-} f(\varphi(a)) \right] \right|
$$

$$
\leq \frac{(\varphi(b)-\varphi(a))^2}{r(r+1)(\alpha+1)} \max\left\{ \left[r+1-(r+1)2^{-\alpha} \right] \left(\frac{2^{-s-1}\left|f''(\varphi(a))\right|+\left(1-2^{-s-1}\right)\left|f''(\varphi(b))\right|}{s+1} \right) \right.
$$

$$
-r(\alpha+1) \left[\frac{2^{-s-2}(s+1)\left|f''(\varphi(a))\right|+(1-t)^{s+1}(-ts-3t+1)f''(\varphi(a))}{(s+1)(s+2)} \right],
$$

$$
\left. r(\alpha+1) \left[\frac{2^{-s-2}(s+1)\left|f''(\varphi(a))\right|+(1-t)^{s+1}(-ts-3t+1)f''(\varphi(a))}{(s+1)(s+2)} \right] \right\}
$$

$$
+\frac{(\varphi(b)-\varphi(a))^2}{r(r+1)(\alpha+1)} \max\left\{ \left(r+1-(r+1)2^{-\alpha}-r(\alpha+1)\right) \right.
$$

$$
\times \left[\frac{\left(1-2^{-s-1}\right)\left|f''(\varphi(a))\right|-2^{-s-1}\left|f''(\varphi(b))\right|}{s+1} \right]
$$

$$
+r(\alpha+1) \left[\frac{\left(1-2^{-s-1}\right)(s+1)\left|f''(\varphi(a))\right|+2^{-s-2}(s+3)\left|f''(\varphi(b))\right|}{(s+1)(s+2)} \right],
$$

$$
r(\alpha+1) \left[\frac{\left(1-2^{-s-1}\right)\left|f''(\varphi(a))\right|-2^{-s-1}\left|f''(\varphi(b))\right|}{s+1} \right.
$$

$$
\left. \left. -\frac{\left(1-2^{-s-2}\right)(s+1)\left|f''(\varphi(a))\right|+2^{-s-2}(s+3)\left|f''(\varphi(b))\right|}{(s+1)(s+2)} \right] \right\}.
$$

$\square$

Proof. By using Definition 3, Lemma 3 and Lemma 8, we have

$$
\left| \frac{f(\varphi(a))+f(\varphi(b))}{r(r+1)} + \frac{2}{r+1} f\left(\frac{\varphi(a)+\varphi(b)}{2}\right) - \frac{\Gamma(\alpha+1)}{r(\varphi(b)-\varphi(a))^\alpha} \left[J^\alpha_{\varphi(a)^+} f(\varphi(b)) + J^\alpha_{\varphi(b)^-} f(\varphi(a)) \right] \right|
$$

$$
\leq (\varphi(b)-\varphi(a))^2 \int_0^1 |k(t)| \left[t^s \left|f''(\varphi(a))\right| + (1-t)^s \left|f''(\varphi(b))\right| \right] dt
$$

$$
\leq \frac{(\varphi(b)-\varphi(a))^2}{r(r+1)(\alpha+1)} \int_0^{\frac{1}{2}} \left| (r+1)\left[1-(1-t)^{\alpha+1}-t^{\alpha+1}\right] - tr(\alpha+1) \right|
$$

$$
\times \left[t^s \left|f''(\varphi(a))\right| + (1-t)^s \left|f''(\varphi(b))\right| \right] dt
$$

$$
+\frac{(\varphi(b)-\varphi(a))^2}{r(r+1)(\alpha+1)} \int_{\frac{1}{2}}^1 \left| (r+1)\left[1-(1-t)^{\alpha+1}-t^{\alpha+1}\right] - r(\alpha+1)(1-t) \right|
$$

$$
\times \left[t^s \left|f''(\varphi(a))\right| + (1-t)^s \left|f''(\varphi(b))\right| \right] dt
$$

$$
\leq \frac{(\varphi(b)-\varphi(a))^2}{r(r+1)(\alpha+1)} \max\left\{ \left[r+1-(r+1)2^{-\alpha} \right] \left(\frac{2^{-s-1}\left|f''(\varphi(a))\right|+\left(1-2^{-s-1}\right)\left|f''(\varphi(b))\right|}{s+1} \right) \right.
$$

$$
-r(\alpha+1) \left[\frac{2^{-s-2}(s+1)\left|f''(\varphi(a))\right|+(1-t)^{s+1}(-ts-3t+1)f''(\varphi(a))}{(s+1)(s+2)} \right],
$$

$$
\left. r(\alpha+1) \left[\frac{2^{-s-2}(s+1)\left|f''(\varphi(a))\right|+(1-t)^{s+1}(-ts-3t+1)f''(\varphi(a))}{(s+1)(s+2)} \right] \right\}
$$

$$
+\frac{(\varphi(b)-\varphi(a))^2}{r(r+1)(\alpha+1)} \max\left(r+1-(r+1)2^{-\alpha}-r(\alpha+1)\right) \left[\frac{\left(1-2^{-s-1}\right)\left|f''(\varphi(a))\right|-2^{-s-1}\left|f''(\varphi(b))\right|}{s+1} \right.
$$

$$
+r(\alpha+1) \left[\frac{\left(1-2^{-s-1}\right)(s+1)\left|f''(\varphi(a))\right|+2^{-s-2}(s+3)\left|f''(\varphi(b))\right|}{(s+1)(s+2)} \right],
$$

$$
r(\alpha+1) \left[\frac{\left(1-2^{-s-1}\right)\left|f''(\varphi(a))\right|-2^{-s-1}\left|f''(\varphi(b))\right|}{s+1} \right.
$$

$$
\left. -\frac{\left(1-2^{-s-2}\right)(s+1)\left|f''(\varphi(a))\right|+2^{-s-2}(s+3)\left|f''(\varphi(b))\right|}{(s+1)(s+2)} \right]
$$

Where we have used the following inequality:

$$\int_0^{\frac{1}{2}} \left|(r+1)\left[1-(1-t)^{\alpha+1}-t^{\alpha+1}\right]-tr(\alpha+1)\right|$$
$$\times \left[t^s \left|f''(\varphi(a))\right|+(1-t)^s \left|f''(\varphi(b))\right|\right] dt$$
$$\leq \left[(r+1)(1-2^{-\alpha})\right]\left[\frac{2^{-s-1}\left|f''(\varphi(a))\right|+\left(1-2^{-s-1}\right)\left|f''(\varphi(b))\right|}{s+1}\right]$$
$$-r(\alpha+1)\left[\frac{2^{-s-2}\left|f''(\varphi(a))\right|}{s+2}+\frac{1-(s+3)2^{-s-2}\left|f''(\varphi(b))\right|}{(s+2)(s+1)}\right]$$

and

$$\int_0^{\frac{1}{2}} \left|-r-1+(r+1)\left[(1-t)^{\alpha+1}+t^{\alpha+1}\right]+tr(\alpha+1)\right|$$
$$\times \left[t^s \left|f''(\varphi(a))\right|+(1-t)^s \left|f''(\varphi(b))\right|\right] dt$$
$$\leq r(\alpha+1)\left[\frac{2^{-s-2}\left|f''(\varphi(a))\right|}{s+2}+\frac{1-(s+3)2^{-s-2}\left|f''(\varphi(b))\right|}{(s+2)(s+1)}\right],$$

and

$$\int_{\frac{1}{2}}^1 \left|r+1+tr(\alpha+1)-(r+1)\left[(1-t)^{\alpha+1}+t^{\alpha+1}\right]-r(\alpha+1)\right|$$
$$\times \left[t^s \left|f''(\varphi(a))\right|+(1-t)^s \left|f''(\varphi(b))\right|\right] dt$$
$$\leq \left[r+1-(r+1)2^{-\alpha}-r(\alpha+1)\right]\left[\frac{\left(1-2^{-s-1}\right)\left|f''(\varphi(a))\right|-2^{-s-1}\left|f''(\varphi(b))\right|}{s+1}\right]$$
$$+r(\alpha+1)\left[\frac{2^{-s-2}\left|f''(\varphi(a))\right|}{s+2}+\frac{(s+3)2^{-s-2}\left|f''(\varphi(b))\right|}{(s+2)(s+1)}\right],$$

and

$$\int_{\frac{1}{2}}^1 \left|-r-1-tr(\alpha+1)+(r+1)\left[(1-t)^{\alpha+1}+t^{\alpha+1}\right]+r(\alpha+1)\right|$$
$$\times \left[t^s \left|f''(\varphi(a))\right|+(1-t)^s \left|f''(\varphi(b))\right|\right] dt$$
$$\leq r(\alpha+1)$$
$$\times \left[\frac{\left(1-2^{-s-1}\right)\left|f''(\varphi(a))\right|-2^{-s-1}\left|f''(\varphi(b))\right|}{s+1}-\frac{\left(1-2^{-s-2}\right)\left|f''(\varphi(a))\right|}{s+2}-\frac{(s+3)2^{-s-2}\left|f''(\varphi(b))\right|}{(s+2)(s+1)}\right].$$

The proof is done.

Theorem 3.6. *Let I be an interval $a,b \in I$ with $0 \leq a < b$ and $\varphi : I \to \mathbb{R}$ a continuous increasing function. Let $f : [0, \varphi(b)] \to \mathbb{R}$ be a differentiable mapping and $1 < q < \infty$. If $|f''|$ is measurable and $|f''|^q$ is decreasing and geometric-arithmetically $\varphi - s-$convex on $[0, \varphi(b)]$ for some fixed $\alpha \in (0, \infty)$, $s \in (0,1]$, $0 \leq a < b$, then the following inequality for fractional integrals holds:*
(3.9)
$$\left|\frac{f(\varphi(a))+f(\varphi(b))}{r(r+1)}+\frac{2}{r+1}f\left(\frac{\varphi(a)+\varphi(b)}{2}\right)-\frac{\Gamma(\alpha+1)}{r(\varphi(b)-\varphi(a))^\alpha}\left[J^\alpha_{\varphi(a)^+}f(\varphi(b))+J^\alpha_{\varphi(b)^-}f(\varphi(a))\right]\right|$$
$$\leq \frac{(\varphi(b)-\varphi(a))^2}{[r(r+1)(\alpha+1)]^{1+p^{-1}}}\left(\frac{\left|f''(\varphi(a))\right|^q+\left|f''(\varphi(b))\right|^q}{s+1}\right)^{\frac{1}{q}}$$
$$\times \left(\max\left\{\left[(r+1)(1-2^{-\alpha})\right]^{p+1}-\left[\frac{2+r(1-\alpha)-(1+r)2^{-\alpha}}{2}\right]^{p+1},\left[r(\alpha+1)\right]^{p+1}2^{-p-1}\right\}\right.$$
$$\left.+\max\left\{\left[r+1-(r+1)2^{-\alpha}\right]^{p+1}-\left[\frac{2+r(1-\alpha)-(1+r)2^{-\alpha}}{2}\right]^{p+1},\left[\frac{r(\alpha+1)}{2}\right]^{p+1}\right\}\right)^{\frac{1}{p}}.$$

Where $\frac{1}{p}+\frac{1}{q}=1$.

$\square$

Proof. By using Hölder's inequality and Lemma 7,we have

$$\left| \frac{f(\varphi(a))+f(\varphi(b))}{r(r+1)} + \frac{2}{r+1} f\left(\frac{\varphi(a)+\varphi(b)}{2} \right) - \frac{\Gamma(\alpha+1)}{r(\varphi(b)-\varphi(a))^\alpha} \left[J^\alpha_{\varphi(a)^+} f\left(\varphi(b) \right) + J^\alpha_{\varphi(b)^-} f\left(\varphi(a) \right) \right] \right|$$

$$\leq (\varphi(b) - \varphi(a))^2 \int_0^1 |k(t)|\, |f''(t\varphi(a) + (1-t)\varphi(b))|\, dt$$

$$\leq (\varphi(b) - \varphi(a))^2 \left(\int_0^1 |k(t)|^p\, dt \right)^{\frac{1}{p}} \left(\int_0^1 |f''(t\varphi(a) + (1-t)\varphi(b))|^q\, dt \right)^{\frac{1}{q}}$$

$$\leq (\varphi(b) - \varphi(a))^2 \left(\int_0^1 |k(t)|^p\, dt \right)^{\frac{1}{p}} \left(\int_0^1 |f''(\varphi^t(a)\,\varphi^{(1-t)}(b))|^q\, dt \right)^{\frac{1}{q}}$$

$$\leq (\varphi(b) - \varphi(a))^2 \left(\int_0^1 |k(t)|^p\, dt \right)^{\frac{1}{p}} \left(\int_0^1 \left[t^s |f''(\varphi(a))|^q + (1-t)^s |f''(\varphi(b))|^q \right] dt \right)^{\frac{1}{q}}$$

$$\leq (\varphi(b) - \varphi(a))^2 \left(\frac{|f''(\varphi(a))|^q + |f''(\varphi(b))|^q}{s+1} \right)^{\frac{1}{q}} \left(\int_0^1 |k(t)|^p\, dt \right)^{\frac{1}{p}}$$

$$\leq (\varphi(b) - \varphi(a))^2 \left(\frac{|f''(\varphi(a))|^q + |f''(\varphi(b))|^q}{s+1} \right)^{\frac{1}{q}}$$

$$\times \left(\int_0^{\frac{1}{2}} \left| \frac{1-(1-t)^{\alpha+1}-t^{\alpha+1}}{r(\alpha+1)} - \frac{t}{r+1} \right|^p dt + \int_{\frac{1}{2}}^1 \left| \frac{1-(1-t)^{\alpha+1}-t^{\alpha+1}}{r(\alpha+1)} - \frac{1-t}{r+1} \right|^p dt \right)^{\frac{1}{p}}$$

$$\leq \frac{(\varphi(b)-\varphi(a))^2}{[r(r+1)(\alpha+1)]^{1+p-1}} \left(\frac{|f''(\varphi(a))|^q + |f''(\varphi(b))|^q}{s+1} \right)^{\frac{1}{q}}$$

$$\left(\times \max\left\{ [(r+1)[1-2^{-\alpha}]]^{p+1} - \left[\frac{2+r(1-\alpha)-(1+r)2^{-\alpha-1}}{2} \right]^{p+1}, [r(\alpha+1)]^{p+1}\, 2^{-p-1} \right\} \right.$$

$$\left. + \max\left\{ [(r+1)[1-2^{-\alpha}]]^{p+1} - \left[\frac{2+r(1-\alpha)+(1+r)2^{-\alpha-1}}{2} \right]^{p+1}, \left[\frac{r(\alpha+1)}{2} \right]^{p+1} \right\} \right)^{\frac{1}{p}}.$$

Where we have used the following inequalities:

$$\int_0^{\frac{1}{2}} \left| (r+1)\left[1-(1-t)^{\alpha+1}-t^{\alpha+1} \right] - tr(\alpha+1) \right|^p dt$$

$$\leq \frac{[(r+1)[1-2^{-\alpha}]]^{p+1} - \left[\frac{2+r(1-\alpha)-(1+r)2^{-\alpha-1}}{2} \right]^{p+1}}{r(\alpha+1)(p+1)},$$

and

$$\int_0^{\frac{1}{2}} \left| -r-1+(r+1)\left[(1-t)^{\alpha+1}+t^{\alpha+1} \right] + tr(\alpha+1) \right|^p dt$$

$$\leq \frac{[r(\alpha+1)]^{p+1}\, 2^{-p-1}}{r(\alpha+1)(p+1)},$$

and

$$\int_{\frac{1}{2}}^1 \left| (r+1)\left[1-(1-t)^{\alpha+1}-t^{\alpha+1} \right] - r(1-t)(\alpha+1) \right|^p d$$

$$\leq \frac{[(r+1)[1-2^{-\alpha}]]^{p+1} - \left[\frac{2+r(1-\alpha)+(1+r)2^{-\alpha-1}}{2} \right]^{p+1}}{r(\alpha+1)(p+1)},$$

and

$$\int_{\frac{1}{2}}^1 \left| -r-1-tr(\alpha+1)+(r+1)\left[(1-t)^{\alpha+1}+t^{\alpha+1} \right] + r(\alpha+1) \right|^p dt$$

$$\leq \frac{\left[\frac{r(\alpha+1)}{2} \right]^{p+1}}{r(\alpha+1)(p+1)}.$$

The proof is done.

□

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ON A BOUNDARY VALUE PROBLEM WITH RETARDED ARGUMENT IN THE DIFFERENTIAL EQUATION

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Abstract. In this work a boundary value problem on the half axis is studied. The special solution of this boundary value problem is defined. The simplicity of the eigenvalues is shown and it is proven that all positive values of the parameter λ are the eigenvalues of this boundary value problem.

1. Introduction

In this work we study the boundary value problem

$$
(1) \qquad y''(x) + \lambda y(x) + M(x)y(x - \Delta(x)) = 0 \quad (0 \le x < \infty),
$$

$$
(2) \qquad\qquad\qquad y'(0) - hy(0) = 0,
$$

$$
(3) \qquad y(x - \Delta(x)) \equiv \phi(x - \Delta(x)), \ \texttt{if} \ x - \Delta(x) < 0,
$$

$$
(4) \qquad\qquad\qquad \sup_{[0,\infty)} |y(x)| < \infty,
$$

where $M(x)$ and $\Delta(x) \ge 0$ are defined and continuous on the half axis $[0, \infty)$, λ is a real parameter $(-\infty < \lambda < \infty)$, h is an arbitrary real number and $\phi(x)$ is a continuous initial function on the initial set

$$
E_0 = \{x - \Delta(x) : x - \Delta(x) < 0, \ x > 0\} \cup \{0\}
$$

with $\phi(0) = 1$.

The literature for the boundary value problems for differential equations of the second order with retarded arguments begins with [1, 2, 3, 4, 5, 6, 7, 8, 9].

Differential equations with retarded argument, describe processes with aftereffect; they find many applications, particularly in the theory of automatic control, in the theory of self-oscillatory systems, in the study of problems connected with combustion in rocket engines, in a number of problems in economics, biophysics, and many other fields. Equations with retarded argument appear, for example each time when in some physical or technological problem, the force operating at the mass point depends on the velocity and the position of this point, not only at the given instant, but also at some given previous instant.

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The presence of retardations in the system studied often proves to be a result of a phenomenon which essentially influences the course of the process. For example, in automatic control systems the retardation is the time interval which the system requires to react to an input impulse. Various physical applications of such problems can be found in [8].

The rest of this paper is organized as follows. First, the equivalent integral representation for the solution of the boundary value problem (1)-(4) is constructed. Then, the simplicity of the eigenvalues is shown. It is proven that all positive values of the parameter λ are the eigenvalues of the boundary value problem (1)-(4).

2. The special solution

Let $w(x, \lambda)$ be a solution of (1) which satisfies the conditions

$$(5) \qquad\qquad w(0, \lambda) = 1, \ w'(0, \lambda) = h,$$

$$(6) \qquad w\left(x - \Delta(x), \lambda\right) \equiv \phi(x - \Delta(x)), \ \texttt{if} \ x - \Delta(x) < 0.$$

From Theorem I.2.1 (see [8]) it follows that under conditions (5), (6) there exists a unique solution of (1) on the half axis $[0, \infty)$.

Lemma 2.1. *Equation (1) together with the initial conditions (5), (6) are equivalent, for each value of $\lambda > 0$ to the integral equation:*

$$(7) \ \ w(x, \lambda) = \cos sx + \frac{h \sin sx}{s} - \frac{1}{s} \int_0^x M(t) \sin s(x - t) w(t - \Delta(t), \lambda) dt, \ \ (s^2 = \lambda).$$

Proof. If we seek the solution of equation

$$w''(x, \lambda) + \lambda w(x, \lambda) = -q(x) w(x - \Delta(x), \lambda)$$

as

$$w(x, \lambda) = c_1 \cos sx + c_2 \sin sx,$$

by applying the method of variation of parameters we have

$$w(x, \lambda) = \tilde{c}_1 \cos sx + \tilde{c}_2 \sin sx - \frac{1}{s} \int_0^x q(t) \sin s(x - t) w(t - \Delta(t), \lambda) dt.$$

Taking condition (5) into consideration we find

$$\tilde{c}_1 = 1 \quad \texttt{and} \quad \tilde{c}_2 = \frac{h}{s}.$$

$\square$

Before giving the following theorem, it will be useful to keep in mind that; here, the multiplicity of an eigenvalue of a boundary value problem is defined to be the number of linearly independent eigenfunctions corresponding to this eigenvalue.

Theorem 2.1. *The boundary value problem (1)-(4) can have only simple eigenvalues.*

Proof. Let $\tilde{\lambda}$ be an eigenvalue of the boundary value problem (1)-(4) and $\tilde{\varphi}(x, \tilde{\lambda})$ a corresponding eigenfunction. By (2) and (5),

$$W\left\{\tilde{\varphi}\left(0, \tilde{\lambda}\right), w\left(0, \tilde{\lambda}\right)\right\} = \begin{vmatrix} \tilde{\varphi}\left(0, \tilde{\lambda}\right) & 1 \\ \tilde{\varphi}'\left(0, \tilde{\lambda}\right) & h \end{vmatrix} = 0,$$

and according to Theorem II.2.2. in [8] the functions $\tilde{\varphi}(x, \tilde{\lambda})$ and $w(x, \tilde{\lambda})$ are linearly dependent on $[0, \infty)$. Hence it follows that $w(x, \tilde{\lambda})$ is an eigenfunction for the boundary value problem (1)-(4) and all eigenfunctions of this boundary value problem which correspond to the eigenvalue $\tilde{\lambda}$ are pairwise linearly dependent. $\square$

3. Existence Theorem

Theorem 3.1. *Let*

(8)
$$\sup_{t \in E_0} |\phi(x)| = \phi_0 < \infty$$

and in equation (1) let

(9)
$$\int_0^\infty |M(t)|\, dt = M_\infty < \infty.$$

Then all positive values of the parameter λ are eigenvalues of the boundary value problem (1)-(4).

Proof. By (7), if $\lambda > 0$, then

(10)
$$w(x, \lambda) = R_\lambda \sin(st - \psi_\lambda) - \frac{1}{s} \int_0^x M(t) \sin s(x - t) w(t - \Delta(t), \lambda) dt,$$

where

$$R_\lambda = \sqrt{1 + \frac{h^2}{\lambda}}, \quad \cos \psi_\lambda = \frac{1}{R_\lambda}, \quad \sin \psi_\lambda = \frac{h}{sR_\lambda}, \quad (0 \le \psi_\lambda < 2\pi).$$

Let $x_0 \in (0, \infty)$, and $N_\lambda(x_0) = \max\limits_{[0, x_0]} |w(x, \lambda)|$. Evidently, $N_\lambda(x_0) \ge N_\lambda(x)$ $(x_0 \ge x)$ and, from (10), (5), (6), (8) and (9) one of the following inequalities holds:

(11)
$$N_\lambda(x_0) \le R_\lambda + \frac{1}{s} \int_0^{x_0} |M(t)|\, N_\lambda(t) dt$$

or

$$N_\lambda(x_0) \le R_\lambda + \frac{\phi_0}{s} \int_0^{x_0} |M(t)|\, dt \le R_\lambda + \frac{\phi_0 M_\infty}{s}.$$

By Lemma II.3.5 in [8] it follows from (11) that

$$N_\lambda(x_0) \le R_\lambda exp \frac{1}{s} \int_0^{x_0} |M(t)|\, dt \le R_\lambda exp \frac{M_\infty}{s},$$

and for $s > 0$

$$N_\lambda(x_0) \le \max \left\{ R_\lambda exp \frac{M_\infty}{s}; R_\lambda + \frac{\phi_0 M_\infty}{s} \right\} < \infty.$$

The bound obtained is valid for any $\lambda > 0$ and is independent of x_0, proving the theorem. $\square$

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Ω− ALGEBRAS ON CO-HEYTING VALUED SETS

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ABSTRACT. As known, co-Heyting algebras are dual to Heyting algebras. co-Heyting algebra has many studying areas as topos theory, co-intuitionistic logic, linguistics, quantum theory, etc.

In this paper, we studied on co-Heyting valued sets. The co-Heyting valued Ω−algebra, co-Heyting Valued Algebra Homomorphism are defined and some properties of these sets are examined.

1. INTRODUCTION

co- Heyting algebra that is a lattice which dual is Heyting algebra. co-Heyting algebra has applications in many different areas.

Definition 1.1. [1]A Boolean algebra is an algebra $(H, \vee, \wedge, ^-, 0_H, 1_H)$ where $(H, \vee, \wedge, 0_H, 1_H)$ is a distributive lattice and for all $a \in H$,

$$a \wedge \overline{a} = 0 \text{ and } a \vee \overline{a} = 1$$

Definition 1.2. [1]A Heyting algebra is an algebra $(H, \vee, \wedge, \rightarrow, 0_H, 1_H)$ such that $(H, \vee, \wedge, 0_H, 1_H)$ is an lattice and for all $a, b, c \in H$,

$$a \leq b \rightarrow c \Leftrightarrow a \wedge b \leq c$$

$(H, \vee, \wedge, 0_H, 1_H)$ is a Heyting algebra with $\forall a, b \in H$,

$$a \rightarrow b = \bigvee \{c : a \wedge c \leq b, c \in H\}.$$

Proposition 1.1. [4]*An algebra* $(H, \vee, \wedge, \rightarrow, 0_H, 1_H)$ *is a Heyting algebra if and only if* $(H, \vee, \wedge, 0_H, 1_H)$ *is an lattice and the following identities hold for all* $a, b, c \in H$,

(1) $a \rightarrow a = 1$
(2) $a \wedge (a \rightarrow b) = a \wedge b$
(3) $b \wedge (a \rightarrow b) = b$
(4) $a \rightarrow (b \wedge c) = (a \rightarrow b) \wedge (a \rightarrow c)$

Definition 1.3. [1]A co-Heyting algebra is an algebra $(H^*, \vee, \wedge, \hookrightarrow, 0_{H^*}, 1_{H^*})$ such that $(H^*, \vee, \wedge, 0_{H^*}, 1_{H^*})$ is an lattice and for all $a, b, c \in H^*$,

$$a \hookrightarrow b \leq c \Leftrightarrow a \leq b \vee c$$

$(H^*, \vee, \wedge, 0_{H^*}, 1_{H^*})$ is a co-Heyting algebra with $\forall a, b \in H^*$,

$$a \hookrightarrow b = \bigwedge \{c : a \vee c \geq b, c \in H^*\}.$$

A co-Heyting algebra with the ordering reversed will yield a Heyting algebra. The implication operation in this algebra will be $a \rightarrow b = b \hookrightarrow a$.

It is clear that H and H^* are same sets with different order relations. 1_{H^*} and 0_{H^*} are greatest and least elements of H^*, respectively.

Proposition 1.2. *An algebra* $(H^*, \vee, \wedge, \hookrightarrow, 0_{H^*}, 1_{H^*})$ *is a co-Heyting algebra if and only if* $(H^*, \vee, \wedge, 0_{H^*}, 1_{H^*})$ *is an lattice and the following identities hold for all* $a, b, c \in H^*$,

(1) $a \hookrightarrow a = 0$
(2) $a \vee (b \hookrightarrow a) = a \vee b$
(3) $b \vee (b \hookrightarrow a) = b$
(4) $(b \vee c) \hookrightarrow a = (b \hookrightarrow a) \vee (c \hookrightarrow a)$

Proof. (1) $\forall a \in H^*$,

$$a \hookrightarrow a = \bigwedge \{c : a \vee c \geq a, c \in H^*\} = 0$$

(2) From definition it is obtained that,

$$a \vee (b \hookrightarrow a) \geq b \Rightarrow a \vee (b \hookrightarrow a) \geq a \vee b$$

and

$$\begin{aligned}(a \vee b) \vee a \quad &\geq \quad b \Rightarrow a \vee b \geq b \hookrightarrow a \text{ and } a \leq a \vee b \\ &\Rightarrow \quad a \vee b \geq a \vee (b \hookrightarrow a)\end{aligned}$$

(3)$\forall a, b \in H^*$,

$$b \leq a \vee b \Rightarrow b \vee (b \hookrightarrow a) = b$$

(4)$\forall a, b, c \in H^*$,

$$\begin{aligned}(b \quad \hookrightarrow \quad a) \vee (c \hookrightarrow a) \vee a = (a \vee (b \hookrightarrow a)) \vee (a \vee (c \hookrightarrow a)) \geq b \vee c \\ \Rightarrow \quad (b \hookrightarrow a) \vee (c \hookrightarrow a) \geq (b \vee c) \hookrightarrow a\end{aligned}$$

On the other hand, $(b \vee c) \hookrightarrow a \geq (b \hookrightarrow a) \vee (c \hookrightarrow a)$. $\square$

2. co-Heyting Valued Sets

In this section, the concepts of co-Heyting valued set, co-Heyting valued function are defined and some properties of these structures are examined.

Definition 2.1. Let H^* be a complete co-Heyting algebra and X be a universal. H^*- valued set is determined with $[=]$ function

$$[=] : X \times X \rightarrow H^*, [=](a, b) = [a = b]$$

which satisfy the following conditions.

(1) $[a = b] \geq [b = a]$
(2) $[a = b] \vee [b = c] \geq [a = c]$

Let X be a universal. $u \in X, E(u)$ means the degree of existence the elemet u. For H^*-valued sets we will use,

$$E(u) = [u \in X].$$

So, $[u \in X] = [u = u]$.

Definition 2.2. Let A be a $H^*−$valued set. The subset of A is a $s : A \to H^*$ function with following conditions.

(1) $[x \in s] \vee [x = y] \geq [y \in s]$
(2) $[x \in s] \geq [x \in A]$

Definition 2.3. Let $(X, =)$ and $(Y, =)$ are $H^* − valued\ sets$. If $f : X \times Y \to H^*$ function satisfy the following conditions then called $H^*−$valued function and it is shown $f : X \to Y$.

F1 $f(x, y) \geq [x = x] \vee [y = y]$
F2 $[x = x'] \vee f(x, y) \vee [y = y'] \geq f(x', y')$
F3 $f(x, y) \vee f(x, y') \geq [y = y']$
F4 $[x = x] \geq \bigwedge \{f(x, y) : y \in Y\}$

Notation 1: $f(x, y) := [f(x) = y]$

Definition 2.4. Let $(X, =)$ be an $H^*−$valued set. $I : X \times X \to H^*$,$I(x, x') = [x = x']$ function is called unit function.

Definition 2.5. Let $(X, =), (Y, =)$ and $(Z, =)$ are $H^*−$valued sets and $f : X \to Y$, $g : Y \to Z$ are $H^*−$valued functions. For $x \in X, z \in Z$,

$$(g \circ f)(x, z) = \bigwedge \{f(x, y) \vee g(y, z) : y \in Y\}.$$

Proposition 2.1. *Let* $(X, =), (Y, =)$ *and* $(Z, =)$ *are* $H^*−valued\ sets\ and$ $f : X \to Y, g : Y \to Z$ *are* $H^*−valued\ functions.\ The\ function$ $(g \circ f) : X \to Z$ *is a* $H^*−valued\ function.$

Proof. (i)Let $x \in X, z \in Z$,

$$
\begin{aligned}
(g \circ f)(x, z) &= \bigwedge \{f(x, y) \vee g(y, z) : y \in Y\} \\
&\geq \bigwedge \{[x = x] \vee [y = y] \vee [z = z] : y \in Y\} \\
&= [x = x] \vee [z = z] \vee \bigwedge \{[y = y] : y \in Y\} \\
&\geq [x = x] \vee [z = z]
\end{aligned}
$$

(ii)Let $x, x' \in X$ and $z, z' \in Z$,

$$
\begin{aligned}
[x = x'] \vee (g \circ f)(x, z) \vee [z = z'] &= [x = x'] \vee \bigwedge \{f(x, y) \vee g(y, z) : y \in Y\} \vee [z = z'] \\
&= \bigwedge \{[x = x'] \vee f(x, y) \vee g(y, z) \vee [z = z'] : y \in Y\} \\
&= \bigwedge \left\{ \begin{array}{c} [x = x'] \vee f(x, y) \vee [y = y] \\ \vee g(y, z) \vee [z = z'] : y \in Y \end{array} \right\} \\
&\geq \bigwedge \{f(x', y') \vee g(y', z') : y \in Y\} = f(x', y') \vee g(y', z') \\
&\geq \bigwedge \{f(x', y') \vee g(y', z') : y' \in Y\} = (g \circ f)(x', z')
\end{aligned}
$$

(iii)Let $x \in X$ and $z, z' \in Z$,

$$
\begin{aligned}
(g \circ f)(x, z) \vee (g \circ f)(x, z') &= \bigwedge \{f(x, y) \vee g(y, z) : y \in Y\} \vee \bigwedge \{f(x, t) \vee g(t, z') : t \in Y\} \\
&= \bigwedge \{f(x, y) \vee f(x, y) \vee g(y, z) \vee g(y, z') : y \in Y\} \\
&\geq \bigwedge \{[y = y] \vee [z = z'] : y \in Y\} \\
&\geq [z = z']
\end{aligned}
$$

(iv)Let $x \in X$,

$$
\begin{aligned}
\bigwedge \{(g \circ f)(x, z) : z \in Z\} &= \bigwedge \left\{ \bigwedge \{f(x, y) \vee g(y, z) : y \in Y\} : z \in Z \right\} \\
&= \bigwedge \left\{ \bigwedge \{f(x, y) : y \in Y\} : z \in Z \right\} \vee \\
&\quad \bigwedge \left\{ \bigwedge \{g(y, z) : y \in Y\} : z \in Z \right\} \\
&\leq [x = x] \vee \bigwedge \{[y = y] : y \in Y\} = [x = x]
\end{aligned}
$$

$\square$

Definition 2.6. Let $(X, =)$ and $(Y, =)$ are H^*-valued sets and $f : X \to Y$ is H^*-valued function.

(1) f is a monomorphism. $\Leftrightarrow \forall x, x' \in X, y \in Y$,
$$
f(x, y) \vee f(x', y) \geq [x = x']
$$

(2) f is a epimorphism. $\Leftrightarrow \forall y \in Y$,
$$
[y = y'] \geq \bigwedge \{f(x, y) : x \in X\}
$$

Definition 2.7. Let $(X, =)$ be a H^*-valued set. $R : X \times X \to H^*$ is called H^*-valued equivalence relation.$\Leftrightarrow$

R1 $R(x, y) \vee [x = x] = R(x, y), R(x, y) \vee [y = y] = R(x, y)$
R2 $R(x, y) \vee [x = x'] \geq R(x', y), R(x, y) \vee [y = y'] \geq R(x, y')$
R3 $[x = x] \geq R(x, x) :$
R4 $R(x, y) \geq R(y, x)$
R5 $R(x, y) \vee R(y, z) \geq R(x, z)$

Example 2.1. Let $(X, =), (Y, =)$ are H^*-valued sets and $f : X \to Y$ is H^*-valued function.$\forall x_1, x_2 \in X$,
$$
C_f(x_1, x_2) = [f(x_1) = f(x_2)]
$$
function is a H^*-valued equivalence relation on X.

Definition 2.8. Let R be a H^*-valued equivalence relation on X. $d : X \to H^*$ is called equivalence class of $R \Leftrightarrow$

d1 $d(x) \vee R(x, x') \geq d(x')$
d2 $d(x) \vee d(y) \geq R(x, y)$

$d(x)$ is the equivalence class of $x \in X$.

Proposition 2.2. *Let R be a H^*-valued equivalence relation on X and d_1, d_2 are equivalence class of R.*
$$
\begin{aligned}
\bigwedge \{d_1(x) : x \in X\} &= \bigwedge \{d_2(x) : x \in X\} \ \text{and} \\
d_1(x) &\geq d_2(x) \Rightarrow d_1 = d_2
\end{aligned}
$$

Proof. $x_0 \in X$,
$$
\begin{aligned}
d_2(x_0) &= \bigwedge \{d_2(x) \vee d_1(x) : x \in X\} \\
&\geq \bigwedge \{R(x_0, x) \vee d_1(x) : x \in X\} \\
&\geq d_1(x_0)
\end{aligned}
$$

$\square$

Proposition 2.3. *Let* $(X, =), (Y, =)$ *are* H^*- *valued sets and* $f : X \to Y$ *is* H^*-*valued function.* f *is surjective* $\Leftrightarrow \forall y \in Y$,

$$\bigwedge \{[f(x) = y] : x \in X\} = [y = y]$$

3. H^*-Valued $\Omega-$Algebras

Now, let Ω is defined as follows,

$$\Omega = \{\omega : X^n \times X \to H^* : \omega \text{ satisfy F1-F4 conditions}\}$$

It means that, if $\omega \in \Omega$, ω is H^*-valued function. The concept of H^*-valued $\Omega-$algebra can be defined as following;

Definition 3.1. $A = \langle X, \Omega \rangle$ is H^*-valued $\Omega-$algebra. $\Leftrightarrow$ For $\omega \in \Omega$ and $((x_1, x_2, ..., x_n), c) \in X^n \times X$,

$$\bigwedge \left\{ \left\{ \begin{array}{c} \bigvee [x_i \in A] \vee \omega ((x_1, x_2, ..., x_n), d) \\ \vee [c = d] : i = 1, 2, ..., n \end{array} \right\} : d \in X \right\} \geq \omega ((x_1, x_2, ..., x_n), c)$$

Example 3.1. Let $A = \langle X, \Omega \rangle$ be a H^*-valued $\Omega-$algebra.

$$\{\Theta\} : A \to H^*, [x \in \{\Theta\}] = 1_{H^*}$$

is a subset of A.

$$E = \langle \{\Theta\}, \Omega \rangle$$

is a H^*-valued $\Omega-$algebra. E is called trivial H^*-valued $\Omega-$algebra.

Definition 3.2. Let $A = \langle X, \Omega \rangle$ be a H^*-valued $\Omega-$algebra. If $K \subseteq X, B : K \to H^*$ is H^*-valued set, $(B, =) \subseteq (A, =)$ and for all $\omega \in \Omega, \omega \downarrow_B$ satisfy the (1) then B is H^*-valued $\Omega-$subalgebra of A.

Example 3.2. Let $A = \langle X, \Omega \rangle$ be a H^*-valued $\Omega-$algebra. $E = \langle \{\Theta\}, \Omega \rangle$ is H^*-valued $\Omega-$ subalgebra.

Definition 3.3. Let $A = \langle X, \Omega \rangle$ and $B = \langle Y, \Omega \rangle$ are similar H^*-valued $\Omega-$ algebras and $f : A \to B$ be H^*-valued function. f is a H^*-valued $\Omega-$ algebra homomorphism$\Leftrightarrow$

H1 $[x = x] = [f(x) = f(x)]$
H2 $[x = x'] \leq [f(x) = f(x')]$
H3 $f(\omega(x_1, x_2, ..., x_n), y) = \bigvee \{f(x_i, y_i) : y = \omega(y_1, y_2, ..., y_n), f(x_i, y_i) > 0\}$

Example 3.3. Let $A = \langle X, \Omega \rangle$ be a H^*-valued $\Omega-$algebra and $f : E \to A$, $g : E \to A$ are H^*-valued functions. $\forall x, I(\{\Theta\}, x) = 1_{H^*}$ H^*-valued $\Omega-$ algebra homomorphism exist. This homomorphism is unique.

Proposition 3.1. *Let* A, B, C *are similar* H^*-*valued* $\Omega-$ *algebras. If* $f : A \to B$, $g : B \to C$ *are* H^*-*valued* $\Omega-$ *algebra homomorphisms then* $(g \circ f) : A \to C$ *is a* H^*-*valued* $\Omega-$ *algebra homomorphism.*

Proof. Let $x_1, x_2, ..., x_n \in A, z \in C$,

$$(g \circ f)(\omega(x_1, x_2, ..., x_n), z) = \bigwedge \{f(\omega(x_1, x_2, ..., x_n), y) \vee g(y, z) : y \in B\}$$

$$= \bigwedge \left\{ \begin{array}{c} \bigvee \{f(x_i, y_i) : y = \omega(y_1, y_2, ..., y_n), f(x_i, y_i) > 0\} \\ \vee g(y, z) : y \in B \end{array} \right\}$$

$$= \bigwedge \left\{ \begin{array}{c} \bigvee \{f(x_i, y_i) : i = 1, ..., n\} \vee \\ \{g(\omega(y_1, y_2, ..., y_n), z) : y = \omega(y_1, y_2, ..., y_n), f(x_i, y_i) > 0\} \end{array} \right\}$$

$$= \bigwedge \left\{ \begin{array}{c} \bigvee \{f(x_i, y_i) : i = 1, ..., n\} \vee \\ \left\{ \begin{array}{c} \bigvee g(y_i, z_i) : z = \omega(z_1, z_2, ..., z_n), g(y_i, z_i) > 0 : \\ y = \omega(y_1, y_2, ..., y_n), f(x_i, y_i) > 0 \end{array} \right\} \end{array} \right\}$$

$$= \bigwedge \left\{ \bigvee \left\{ \begin{array}{c} f(x_i, y_i) \vee g(y_i, z_i) : y = \omega(y_1, y_2, ..., y_n), f(x_i, y_i) > 0, \\ z = \omega(z_1, z_2, ..., z_n), g(y_i, z_i) > 0 : y_i \in B \end{array} \right\} \right\}$$

$$= \bigvee \left\{ \bigwedge \left\{ \begin{array}{c} f(x_i, y_i) \vee g(y_i, z_i) : y = \omega(y_1, y_2, ..., y_n), f(x_i, y_i) > 0, \\ z = \omega(z_1, z_2, ..., z_n), g(y_i, z_i) > 0 : y_i \in B \end{array} \right\} \right\}$$

$$= \bigvee \left\{ \left\{ \begin{array}{c} (g \circ f)(x_i, z_i) : y = \omega(y_1, y_2, ..., y_n), f(x_i, y_i) > 0, \\ z = \omega(z_1, z_2, ..., z_n), g(y_i, z_i) > 0 \end{array} \right\} : y_i \in B \right\}$$

$$= \bigvee \{(g \circ f)(x_i, z_i) : z = \omega(z_1, z_2, ..., z_n), (g \circ f)(x_i, z_i) > 0, i = 1, ..., n\}$$

$\square$

4. Conclusion

In this paper, we introduced the H^*-valued $\Omega-$algebra and H^*-valued $\Omega-$algebra homomorphism. To study these concepts, firstly we defined H^*- valued set and H^*- valued function. In continuation of this study, injection and projection mappings on H^*-valued $\Omega-$algebra can be defined, isomorphism theorems can be proved.

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SOME CLASSIFICATIONS OF AN X FDK-SPACE

ŞEYDA SEZGEK AND İLHAN DAĞADUR

ABSTRACT. In this paper, we introduce definitions of double wedge and double conull FDK-spaces. Also, we give some important corollary for any double sequence space about these definitions.

1. INTRODUCTION

In [4], the first study on double sequences was examined by Bromwich. And than it was investigated by many authors such as Hardy [6], Moricz [7], Tripathy [16], Başarır and Sonalcan [2]. The notion of regular convergence for double sequences was defined by Hardy [6]. After that both the theory of topological double sequence spaces and the theory of summability of double sequences were studied by Zeltser [17]. The statistical and Cauchy convergence for double sequences were examined by Mursaleen and Edely [8] and Tripathy [15] in recent years. Many recent improvements containing the summability by four dimensional matrices might be found in [10].

Ω denotes the space of all complex valued double sequences which is a vector space with coordinatewise addition and scalar multiplication. Any vector subspace of Ω is called as a double sequence space. The space $\mathcal{M}_u$ of all bounded double sequences is defined by

$$\mathcal{M}_u := \left\{ x \in \Omega \ : \ \|x\|_\infty := \sup_{k,l} |x_{kl}| < \infty \right\},$$

which is a Banach space with the norm $\|.\|_\infty$. In addition we consider the double sequence spaces

$$\Phi := span\{e^{kl} \ : \ k,l \in \mathbb{N}\} = \left\{ x \in \Omega \ : \ \exists N_0 \in \mathbb{N}, \forall (k,l) \in \mathbb{N}^2/[1,N_0]^2 \ : \ x_{kl} = 0 \right\}$$

$$\Phi_1 := \Phi \cup \{e\}$$

$$\mathcal{L}_u := \left\{ x \in \Omega \ : \ \sum_{k,l} |x_{kl}| < \infty \right\}$$

$$\mathcal{BV} := \left\{ x \in \Omega \ : \ \|x\|_{\mathcal{BV}} := \sum_{k,l} |x_{kl} - x_{k+1,l} - x_{k,l+1} + x_{k+1,l+1}| < \infty \right\}$$

$$\mathcal{C}_0 := \left\{ x \in \mathcal{C} \ : \ \forall l \in \mathbb{N} \ : \ \lim_k x_{kl} = 0 \right\}$$

$$\mathcal{C}_c := \left\{ x \in \Omega \ : \ \forall l \in \mathbb{N} : \ \lim_{k,l} x_{kl} \ \text{exist} \right\}.$$

Throughout this paper e denotes the double sequence of ones; (δ^{ij}), $i,j = 1,2,\ldots$, with the one in the (i,j) position.

A subspace E of the vector space Ω is called DK-space, if all the seminorms $r_{kl} : E \to \mathbb{R}, x \mapsto |x_{kl}|$ $(k,l \in \mathbb{N})$ are continuous. An FDK space is a DK-space with a complete, metrizable, locally convex topology. A normable FDK-space is called BDK-space.

2. Main Results

In this section conull (strongly conull) and double wedge (weak double wedge) FDK-spaces are defined and several characterizations are given.

Definition 2.1. If (E, τ) is a FDK-space containing Φ, and $\delta^{ij} \to 0$ in τ, then (E, τ) is called a double wedge space.

Definition 2.2. If (E, τ) is a FDK-space containing Φ, and $\delta^{ij} \to 0$ (weakly) in τ, then (E, τ) is called a weak double wedge space.

Definition 2.3. Let (E, τ) is an FDK-space containing Φ_1. If $\forall f \in E'$,

$$f(e) = \lim_{m,n \to \infty} \sum_{k,l=1}^{m,n} f(\delta^{kl})$$

then (E, τ) is called a conull FDK-space.

Definition 2.4. Let (E, τ) is an FDK-space containing Φ_1. If $\forall f \in E'$,

$$e = \lim_{m,n \to \infty} \sum_{k,l=1}^{m,n} \delta^{kl}$$

then (E, τ) is called a strong conull FDK-space.

Clearly, each strong conull (wedge) FDK-space is a conull (weak wedge) FDK-space and the following diagram is hold:

conull FDK

strong conull FDK   weak wedge FDK

wedge FDK
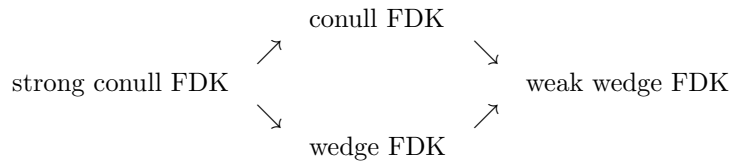
Indeed, let E be a strong conull FDK-space. Then we have $e^{(kl)} \to e$. Hence $P(e^{(kl)} - e) \to 0$ $(k,l \to \infty)$ for the seminorm P in τ. In this case, the following

equation is hold

$$
\begin{aligned}
\delta^{kl} &= e^{(k,l)} - e^{(k-1,l-1)} - \sum_{i=k,j=1}^{k,l} \delta^{ij} - \sum_{i=1,j=l}^{k,l} \delta^{ij} \\
&= e^{(k,l)} - e^{(k-1,l-1)} - \sum_{i=k,j=1}^{k,l} \delta^{ij} - \sum_{i=1,j=l}^{k,l} \delta^{ij} + e^{(k-1,l-1)} - e^{(k-1,l-1)} \\
&= e^{(k,l)} - e^{(k,l-1)} - e^{(k-1,l)} + e^{(k-1,l-1)} + e - e + e - e .
\end{aligned}
$$

So, we have

$$
P\left(\delta^{kl}\right) = P\left(e^{(k,l)} - e^{(k,l-1)} - e^{(k-1,l)} + e^{(k-1,l-1)} + e - e + e - e\right)
$$

$$
\leq P\left(e^{(k,l)} - e\right) + P\left(e^{(k,l-1)} - e\right) + P\left(e^{(k-1,l)} - e\right) + P\left(e^{(k-1,l-1)} - e\right).
$$

It is clearly that since $P\left(\delta^{kl}\right) \to 0$, $k,l \to \infty$, E is double wedge space.

We recall that the $\alpha-$dual of a subset E of Ω is defined to be ([14])

$$
E^{\alpha} := \left\{ x = (x_{kl}) \; : \; \sum_{k,l=1}^{\infty,\infty} |x_{kl}y_{kl}| < \infty \;, \; \forall y = (y_{kl}) \in E \right\}.
$$

Lemma 2.1. *If $z^{(mn)} \in \mathcal{C}_0$, $m,n = 1,2,\ldots$, then there exists $z \in \mathcal{C}_0$ such that*

$$
\lim_{i,j \to \infty} \frac{z_{ij}^{(mn)}}{z_{ij}} = 0 \qquad (m,n = 1,2,\ldots).
$$

Furthermore, for any such z, we have $z^{\alpha} \subseteq \bigcap_{m,n=1}^{\infty,\infty} \{z^{(mn)}\}^{\alpha}$.

Proof. Let $z^{(mn)} \in \mathcal{C}_0$. We can choose two sequences (i_k), (j_l) of positive integers such that

$$
1 = i_0 < i_1 < i_2 < \ldots \qquad 1 = j_0 < j_1 < j_2 < \ldots
$$

and

$$
\max_{\substack{1 \leq m \leq k \\ 1 \leq n \leq l}} |z_{ij}^{(mn)}| < \frac{1}{4^{kl}} \qquad \left(\begin{array}{l} i \geq i_k, \; j \geq j_l \\ k,l = 1,2,\ldots \end{array} \right).
$$

Define $z \in \Omega$ as follows:

$$
z_{ij} = \frac{1}{2^{kl}} \qquad \left(\begin{array}{l} i_k \leq i < i_{k+1}, \; j_l \leq j < j_{l+1} \\ k,l = 0,1,2,\ldots \end{array} \right).
$$

Clearly, $z \in \mathcal{C}_0$ and, fixing m,n

$$
\left| \frac{z_{ij}^{(mn)}}{z_{ij}} \right| < \frac{1}{2^{kl}} \qquad \left(\begin{array}{l} i \geq i_k, \; j \geq j_l \\ k \geq m, \; l \geq n \end{array} \right).
$$

Thus, $\lim_{i,j \to \infty} \frac{z_{ij}^{(mn)}}{z_{ij}} = 0 \;\; \forall m,n.$

Now let $x \in z^{\alpha}$, then

$$
\sum_{i,j=1}^{\infty,\infty} |x_{ij}z_{ij}| < \infty .
$$

Moreover, for $m, n = 1, 2, \ldots$ we have

$$
\begin{aligned}
\sum_{i,j=1}^{\infty,\infty} |x_{ij} z_{ij}^{(mn)}| \ &< \ \sum_{i,j=1}^{\infty,\infty} |x_{ij} z_{ij}| \frac{1}{2^{kl}} \\
&< \ \frac{1}{2^{kl}} \sum_{i,j=1}^{\infty,\infty} |x_{ij} z_{ij}| < \infty.
\end{aligned}
$$

Hence, $x \in \bigcap_{m,n=1}^{\infty,\infty} \{z^{(mn)}\}^{\alpha}$. The proof is completed. $\qquad\square$

$s = (s_m)$, $t = (t_n)$ always denote strictly increasing of nonnegative integers with $s_1 = 0$, $t_1 = 0$. We will be interested in spaces of the form:

$$
m|(s,t)| = \left\{ x \in \Omega \ : \ \sup_{m,n} \sum_{\substack{k=s_m+1 \\ l=t_n+1}}^{s_{m+1},t_{n+1}} |x_{kl}| < \infty \right\}
$$

which becomes a BDK-space under the norm:

$$
x \to \sup_{m,n} \sum_{\substack{k=s_m+1 \\ l=t_n+1}}^{s_{m+1},t_{n+1}} |x_{kl}|.
$$

Theorem 2.1. *Let (E, τ) be an FDK-space. These are equivalent:*
   i) E is a double wedge space,
   ii) E contains z^{α} for some $z \in \mathcal{C}_0$,
   iii) E contains $m|(s,t)|$ for some s, t and the inclusion mapping is compact,
   iv) E contains $\mathcal{L}_u$ and the inclusion mapping is compact.

Proof. $(i \Rightarrow ii)$ Let $\{p_{mn}\}$ be a defining family of seminorms for the topology τ and let

$$
z_{ij}^{(mn)} = p_{mn}(\delta^{ij}) \qquad\qquad (m, n, i, j = 1, 2, \ldots).
$$

Then $z^{(mn)} \in \mathcal{C}_0$, $m, n = 1, 2, \ldots$, since E is a double wedge space. Suppose $y \in \bigcap_{m,n=1}^{\infty,\infty} \{z^{(mn)}\}^{\alpha}$, then for each m, n

$$
\sum_{i,j} |y_{ij} z_{ij}^{(mn)}| < \infty .
$$

Therefore,

$$
\sum_{i,j} |y_{ij} p_{mn}(\delta^{ij})| = \sum_{i,j} p_{mn}(y_{ij} \delta^{ij}) < \infty
$$

is obtained. Since the space E is complete $\sum_{i,j} y_{ij} \delta^{ij}$ converges in (E, τ) to, say x, or $y^{(mn)} \to x$. Thus $y_{ij}^{(mn)} \to x_{ij}$ for each i, j; also we always have $y_{ij}^{(mn)} \to y_{ij}$ for each i, j. Consequently $y = x$; that is

$$
\bigcap_{m,n=1}^{\infty,\infty} \left\{ z^{(mn)} \right\}^{\alpha} \subseteq E .
$$

Choosing z as in Lemma 2.1, (ii) follows.

$(ii \Rightarrow iii)$ Let us choose strictly increasing sequences (s_m), (t_n) of positive integers such that $s_1 = 0$, $t_1 = 0$ and

$$|z_{ij}| \leq \frac{1}{2^{mn}}$$

whenever $i \geq s_m$, $j \geq t_n$, $m, n \geq 2$.

For $x \in m|(s,t)|$ and any positive integers k, l, u, v such that $l \geq k$ and $v \geq u$ we have

$$\sum_{\substack{i=s_k+1 \\ j=t_p+1}}^{s_{l+1},t_{v+1}} |x_{ij}z_{ij}| = \sum_{m=k,n=u}^{l,v} \sum_{\substack{i=s_m+1 \\ j=t_n+1}}^{s_{m+1},t_{n+1}} |x_{ij}z_{ij}| \leq \|x\| \sum_{m=k,n=u}^{l,v} \frac{1}{2^{mn}}.$$

Hence $x \in z^\alpha$. That is; $m|(s,t)| \subseteq z^\alpha \subseteq E$. Also, the inclusion theorem $i : (m|(s,t)|, \|.\|) \to (E,\tau)$ is compact.

$(iii \Rightarrow iv)$ Since $\mathcal{L}_u \subset m|(s,t)|$ always true, the inclusion theorem $i : \mathcal{L}_u \to m|(s,t)|$ is continuous. By hypothesis $\mathcal{L}_u \subset m|(s,t)| \subset E$ and because of $i : (m|(s,t)|, \|.\|) \to (E,\tau)$ is compact $i* : \mathcal{L}_u \to E$ is compact. $\square$

Corollary 2.1. *The intersection of all double wedge FDK-spaces is $\mathcal{L}_u$.*

Proof. Let E be a double wedge FDK-space. Then we have

$$\bigcap E = \bigcap \{z^\alpha \ : \ z \in \mathcal{C}_0\} = \mathcal{C}_0^\alpha.$$

Now we need show that $\mathcal{C}_0^\alpha = \mathcal{L}_u$. Since $\mathcal{C}_0 \subset \mathcal{M}_u$, $\mathcal{M}_u^\alpha \subset \mathcal{C}_0^\alpha$. That is, $\mathcal{L}_u = \mathcal{M}_u^\alpha \subset \mathcal{C}_0^\alpha$ is obtained. Suppose that $x = (x_{kl}) \in \mathcal{C}_0^\alpha$ but $x = (x_{kl}) \notin \mathcal{L}_u$. Then for $y = (y_{kl}) = \frac{1}{k!l!} \in \mathcal{C}_0$ we have

$$\sum |x_{kl}y_{kl}| = \sum |x_{kl}\frac{1}{k!l!}| = e^2 \sum |x_{kl}| = \infty.$$

This means that $x = (x_{kl}) \notin \mathcal{C}_0^\alpha$ which contradicts the hypothesis. So, $x = (x_{kl})$ must be in $\mathcal{L}_u$. That is, $\mathcal{L}_u \subset \mathcal{C}_0^\alpha$. This completes the proof. $\square$

The one to one mapping $S^{(2)}$ of Ω to itself defined by

$$S^{(2)}x = \begin{pmatrix} x_{11} & x_{11}+x_{12} & \cdots & \cdots & \cdots \\ x_{11}+x_{21} & x_{11}+x_{12}+x_{21}+x_{22} & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \sum_{k,l=1}^{m,n} x_{kl} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}$$

$$(S^{(2)})^{-1}x = \begin{pmatrix} x_{11} & x_{12}-x_{11} & \cdots & & \cdots & & \cdots \\ x_{21}-x_{11} & x_{22}-x_{12}-x_{21}+x_{11} & \cdots & & \cdots & & \cdots \\ \cdots & \cdots & \cdots & & \cdots & & \cdots \\ \cdots & \cdots & \cdots & x_{mn}-x_{m,n-1}-x_{m-1,n}+x_{m-1,n-1} & & \cdots \\ \cdots & \cdots & \cdots & & \cdots & & \cdots \end{pmatrix}$$

Theorem 2.2. *i) (E,τ) is strongly conull FDK-space if and only if the space $(S^{(2)})^{-1}(E)$ is a double wedge FDK-space.*

ii) (E,τ) is conull FDK-space if and only if the space $(S^{(2)})^{-1}(E)$ is a weak double wedge FDK-space.

Proof. i) Let (E, τ) is strongly conull FDK-space and $\{P_{mn}\}$ is a set of seminorms on the topology τ. Then a topology with the set of seminorms $\{q_{mn}\}$ make $(S^{(2)})^{-1}(E)$ is a FDK-space such that

$$q_{mn}(x) := P_{mn}(S^{(2)}(x)).$$

Since (E, τ) is strongly conull FDK-space, for all $m, n \in \mathbb{N}$ $P_{mn}\left(e - e^{(mn)}\right) \to 0$. Otherwise,

$$S^{(2)}x = e - e^{(mn)} = \sum_{\substack{k=m+1 \\ l=n+1}}^{\infty,\infty} \delta^{kl} + \sum_{\substack{k=1 \\ l=n+1}}^{m,\infty} \delta^{kl} + \sum_{\substack{k=m+1 \\ l=1}}^{\infty,n} \delta^{kl}$$

is hold. Thus we have

$$
\begin{aligned}
x &= (S^{(2)})^{-1}\left(\sum_{\substack{k=m+1 \\ l=n+1}}^{\infty,\infty} \delta^{kl} + \sum_{\substack{k=1 \\ l=n+1}}^{m,\infty} \delta^{kl} + \sum_{\substack{k=m+1 \\ l=1}}^{\infty,n} \delta^{kl} \right) \\
&= (S^{(2)})^{-1}\begin{pmatrix} 0 & \dots & 0 & 1 & \dots & 1 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & 1 & \dots & 1 \\ 1 & \dots & 1 & 1 & \dots & 1 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 1 & \dots & \dots & \dots & \dots & 1 \end{pmatrix} \\
&= \delta^{m+1,1} + \delta^{1,n+1} - \delta^{m+1,n+1}.
\end{aligned}
$$

From the definition of (q_{mn}) the following equation is obtained:

$$q_{mn}(\delta^{m+1,1} + \delta^{1,n+1} - \delta^{m+1,n+1}) = P_{mn}\left(e - \sum_{k,l=1}^{m,n} \delta^{kl} \right).$$

Since $P_{mn}\left(e - \sum_{k,l=1}^{m,n} \delta^{kl} \right) \to 0$ $(m, n \to \infty)$, we can say $q_{mn}(\delta^{m+1,1} + \delta^{1,n+1} - \delta^{m+1,n+1}) \to 0$ $(m, n \to \infty)$.

In this case, we have $(\delta^{m+1,1} + \delta^{1,n+1} - \delta^{m+1,n+1}) \to 0$ $(m, n \to \infty)$ according to the topology of the space $(S^{(2)})^{-1}(E)$. Since $\delta^{m+1,1} \to 0$ and $\delta^{1,n+1} \to 0$, $\delta^{m+1,n+1} \to 0$ is hold. This gives that $(S^{(2)})^{-1}(E)$ is a double wedge FDK-space.

Now we suppose that $(S^{(2)})^{-1}(E)$ is a double wedge FDK-space. Then we have for all $m, n \in \mathbb{N}$ $q_{mn}(\delta^{m+1,1} + \delta^{1,n+1} - \delta^{m+1,n+1}) \to 0$ $(m, n \to \infty)$. Since

$$q_{mn}(\delta^{m+1,1} + \delta^{1,n+1} - \delta^{m+1,n+1}) = P_{mn}\left(e - \sum_{k,l=1}^{m,n} \delta^{kl} \right),$$

$P_{mn}\left(e - \sum_{k,l=1}^{m,n} \delta^{kl} \right) \to 0$ $(m, n \to \infty)$ is obtained. So E strongly conull FDK-space.

ii) Let (E, τ) is conull FDK-space and let us define the topology of $(S^{(2)})^{-1}(E)$ as the proof of (i). Then $q_{mn}(x) := P_{mn}(S^{(2)}(x))$ and since $(S^{(2)})^{-1}\left(e - \sum_{k,l=1}^{m,n} \delta^{kl} \right) = \delta^{m+1,1} + \delta^{1,n+1} - \delta^{m+1,n+1}$, we have

$$q_{mn}(\delta^{m+1,1} + \delta^{1,n+1} - \delta^{m+1,n+1}) = P_{mn}\left(e - \sum_{k,l=1}^{m,n} \delta^{kl} \right).$$

Because of E is conull FDK space, $P_{mn}\left(e - \sum_{k,l=1}^{m,n} \delta^{kl}\right) \to 0$ (weak) $(m, n \to \infty)$. Hence $\delta^{m+1,1} + \delta^{1,n+1} - \delta^{m+1,n+1} \to 0$ (weak) $(m, n \to \infty)$. Consequently $\delta^{m+1,n+1} \to 0$ (weak) $(m, n \to \infty)$ is obtained. That is, $(S^{(2)})^{-1}(E)$ is a weak double wedge FDK-space. The other hand of the proof is as the proof of (i). $\square$

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A NOTE ON POROSITY CLUSTER POINTS

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Abstract. Porosity cluster points of real valued sequences was defined and studied in [3]. In this paper we give the relation between porosity cluster points and distance function.

1. Introduction

Porosity is appeared in the papers of Denjoy [5], [6], Khintchine [11] and, Dolzenko [7]. It has many applications in theory of free boundaries [10], generalized subharmonic functions [8], complex dynamics [12], quasisymmetric maps [14], infinitesimal geometry [4] and other areas of mathematics.

Let $A \subset \mathbb{R}^+ = [0, \infty)$, then the right upper porosity of A at the point 0 is defined as

$$p^+(A) := \limsup_{h \to 0^+} \frac{\lambda(A, h)}{h}$$

where $\lambda(A, h)$ denotes the length of the largest open subinterval of $(0, h)$ that contains no point of A (for more information look [13]). The notion of right lower porosity of A at the point 0 is defined similarly.

In [1], the notation of porosity which was defined at zero for the subsets of real numbers, has been redefined at infinity for the subsets of natural numbers.

Let $\mu : \mathbb{N} \to \mathbb{R}^+$ be a strictly decreasing function such that $\lim_{n \to \infty} \mu(n) = 0$, (it is called scaling function) and let E be a subset of $\mathbb{N}$.

Upper porosity and lower porosity of the set E at infinity were defined respectively in [1] as follows:

$$(1.1) \qquad \overline{p}_\mu(E) := \limsup_{n \to \infty} \frac{\lambda_\mu(E, n)}{\mu(n)}, \quad \underline{p}_\mu(E) := \liminf_{n \to \infty} \frac{\lambda_\mu(E, n)}{\mu(n)},$$

where

$$\lambda_\mu(E, n) := \sup\{|\mu(n^{(1)}) - \mu(n^{(2)})| : n \le n^{(1)} < n^{(2)}, \ (n^{(1)}, n^{(2)}) \cap E = \emptyset\}.$$

Using the definition of upper porosity, all subsets of natural numbers can be classify as follows: $E \subseteq \mathbb{N}$ is
(i) porous at infinity if $\overline{p}_\mu(E) > 0$;
(ii) strongly porous at infinity if $\overline{p}_\mu(E) = 1$;

(iii) nonporous at infinity if $\overline{p}_\mu(E) = 0$. Throughout this paper, we will consider only the upper porosity of subsets of $\mathbb{N}$.

Let us recall the definition of $\overline{p}_\mu$-convergence of real valued sequences for any scaling function:

Definition 1.1. [2] A sequence $x = (x_n)_{n \in \mathbb{N}}$ is said to be $\overline{p}_\mu$-convergent to l if for each $\varepsilon > 0$,

$$\overline{p}_\mu(A_\varepsilon) > 0,$$

where $A_\varepsilon := \{n \ : \ |x_n - l| \geq \varepsilon\}$. It is denoted by $x \to l(\overline{p}_\mu)$ or $(\overline{p}_\mu - \lim_{n \to \infty} x_n = l)$.

Let $x' = (x_{n_k})$ be a subsequence of $x = (x_n)$ for monotone increasing sequence $(n_k)_{k \in \mathbb{N}}$ and $K := \{n_k : k \in \mathbb{N}\}$, then we abbreviate $x' = (x_{n_k})$ by $(x)_K$.

Definition 1.2. [3]. Let $x = (x_n)$ be a sequence and $(x)_K$ be a subsequence of $x = (x_n)$. If
(i) $\overline{p}_\mu(K) > 0$, then $(x)_K$ is called $\overline{p}_\mu$-thin subsequence of $x = (x_n)$,
(ii) $\overline{p}_\mu(K) = 1$, then $(x)_K$ is called a strongly $\overline{p}_\mu$-thin subsequence of $x = (x_n)$,
(iii) $\overline{p}_\mu(K) = 0$, then $(x)_K$ is a $\overline{p}_\mu$-nonthin (or $\overline{p}_\mu$-dense) subsequence of $x = (x_n)$.

Definition 1.3. [3]. A number α is said to be $\overline{p}_\mu$-limit point of the sequence $x = (x_n)$ if it has a $\overline{p}_\mu$-nonthin subsequence that converges to α.

The set of all $\overline{p}_\mu$-limit points of $x = (x_n)$ is denoted by $L_{\overline{p}_\mu}(x)$.

Definition 1.4. [3]. A number β is said to be a $\overline{p}_\mu$-cluster point of $x = (x_n)$ if for every $\varepsilon > 0$, the set

$$\{n : |x_n - \beta| < \varepsilon\}$$

is nonporous. i.e.,

$$\overline{p}_\mu(\{n : |x_n - \beta| < \varepsilon\}) = 0.$$

For a given sequence $x = (x_n)$; the symbol $\Gamma_{\overline{p}_\mu}(x)$ denotes the set of all $\overline{p}_\mu$-cluster points.

2. Main Results

Some results about the set of $L_{\overline{p}_\mu}(x)$-cluster, and $L_{\overline{p}_\mu}(x)$-limit points of given real valued sequences has been investigated in [3]. In this paper, as a continuation of [3] the same subject will be studied.

Theorem 2.1. *Assume that $x = (x_n)$ is monotone increasing (or decreasing) sequence of real numbers. If $\sup x_n < \infty$ (or $\inf x_n < \infty$), then $\sup x_n \in \Gamma_{\overline{p}_\mu}(x)$ (or $\inf x_n \in \Gamma_{\overline{p}_\mu}(x)$).*

Proof. The proof will be given only for monotone increasing sequences. The other case can be proved by following similar steps. From the definition of supremum for any $\varepsilon > 0$, there exists an $n_0 = n_0(\varepsilon) \in \mathbb{N}$ such that the inequality

$$\sup x_n - \varepsilon < x_{n_0} \leq \sup x_n$$

holds. Since the sequence is monotone increasing, then we have

(2.1) $$\sup x_n - \varepsilon < x_{n_0} < x_n \leq \sup x_n < \sup x_n + \varepsilon$$

for all $n > n_0$.

From (2.1), for any $\varepsilon > 0$ there exists an $n_0 = n_0(\varepsilon) \in \mathbb{N}$ such that following inequality

(2.2) $$|x_n - \sup x_n| < \varepsilon$$

holds for all $n > n_0(\varepsilon)$.

From (2.2), following inclusion

$$\mathbb{N}\setminus\{1, 2, 3, ..., n_0\} \subset \{n : |x_n - \sup x_n| < \varepsilon\}$$

and the inequality

$$\overline{p}_\mu(\{n : |x_n - \sup x_n| < \varepsilon\}) \leq \overline{p}_\mu(\mathbb{N}\setminus\{1, 2, ..., n_0\})$$

hold.

Since $\overline{p}_\mu(\mathbb{N}\setminus\{1, 2, 3, ..., n_0\}) = 0$, then from Lemma 1.1 in [3] we have

$$\overline{p}_\mu(\{n : |x_n - \sup x_n| < \varepsilon\}) = 0.$$

This gives the desired proof. $\square$

Corollary 2.1. *If* $x = (x_n)$ *is a bounded sequence, then* $\sup x_n$ *and* $\inf x_n$ *are belong to* $\Gamma_{\overline{p}_\mu}(x)$.

Let $A, B \subset \mathbb{R}$ and recall the distance between A and B is defined as

$$d(A, B) := \inf\{|a - b| : a \in A, b \in B\}.$$

Theorem 2.2. *Let* $x = (x_n)$ *be a real valued sequence. If* $\Gamma_{\overline{p}_\mu}(x) \neq \emptyset$, *then* $d(\Gamma_{\overline{p}_\mu}(x), x) = 0$.

Proof. Assume $\Gamma_{\overline{p}_\mu}(x) \neq \emptyset$. Let us consider an arbitrary element $y* \in \Gamma_{\overline{p}_\mu}(x)$. Then, for an arbitrary $\varepsilon > 0$ we have

$$\overline{p}_\mu(\{n : |x_n - y*| < \varepsilon\}) = 0.$$

So, the set $\{x_n : |x_n - y*| < \varepsilon\}$ has at least countable number elements of $x = (x_n)$. Let us denote this set by D where $D := \{n_k : |x_{n_k} - y*| < \varepsilon\} \subset \mathbb{N}$. Therefore, we have

$$\begin{aligned}
0 \leq dist(\Gamma_{\overline{p}_\mu}(x), x) &= \inf\{|y - x_n| : y \in \Gamma_{\overline{p}_\mu}(x), n \in \mathbb{N}\} \\
&\leq \inf\{|y* - x_{n_k}| : n_k \in D\} < \varepsilon.
\end{aligned}$$

So, for every $\varepsilon > 0$, we have $0 \leq d(\Gamma_{\overline{p}_\mu}(x), x) < \varepsilon$. $\square$

Theorem 2.3. *Let* $x = (x_n)$ *be a real valued sequence and* $\gamma \in \mathbb{R}$ *be an arbitrary fixed point. If* $d(\gamma, x) \neq 0$, *then* $\gamma \notin \Gamma_{\overline{p}_\mu}(x)$.

Proof. From the hypothesis we have

$$d(\gamma, x) := \inf\{|x_k - \gamma| : k \in \mathbb{N}\} = m > 0.$$

From the assumption the inequality

$$|x_k - \gamma| \geq m$$

holds for all $k \in \mathbb{N}$. It means that the open interval $(\gamma - m, \gamma + m)$ has no elements of the sequence $x = (x_n)$. So, we have

$$\overline{p}_\mu(\{k : |x_k - \gamma| < m\}) = 1.$$

For $0 < \varepsilon < m$ we have

$$\overline{p}_\mu(\{k : |x_k - \gamma| < \varepsilon\}) = 1.$$

So, $\gamma \notin \Gamma_{\overline{p}_\mu}(x)$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

Remark 2.1. If $d(\gamma, x) = 0$, it is not necessary for $\gamma \in \Gamma_{\overline{p}_\mu}(x)$.

Let us consider $(x_n) = (\frac{1}{n})_{n \in \mathbb{N}}$. For $\gamma = \frac{1}{2}$, it is clear that $d(\frac{1}{2}, \frac{1}{n}) = 0$ holds, but $\frac{1}{2} \notin \Gamma_{\overline{p}_\mu}(x) = \{0\}$.

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SECTINON II
ABSTRACTS

Fracral Reconstruction of Dilaton Field

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Abstract

Numerous papers have been presented[1,2,3,4,5] to implement the dynamics of scalar field describing nature of the dark energy by establishing a connection between the pilgrim/new agegraphic/Ricci/ghost/holographic energy density and a scalar field definition. These works showed that the analytical form of potential in terms of the scalar field cannot be obtained due to the complexity of the involved equations. On the other hand, writing a meaningful quantum gravity theory is one of the tough puzzles in modern theoretical physics[6,7]. In the quantum gravity theories, the universe is described as a dimensional flow and one can discuss whether and how these attractive features are connected with the ultraviolet-divergence problem[8]. That's why, such important points motivated us to reconstruct the potential and dynamics of the dilaton scalar field model[9] according to the evolutionary behavior of the extended holographic energy description[10] in fractal geometry.

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The Cauchy Problem for Complex Intuitionistic Fuzzy Differential Equations

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Abstract

In this paper, we discuss the existence of a solution to the Cauchy problem for complex intuitionistic fuzzy differential equations. We first propose definitions of complex intuitionistic fuzzy sets and discuss entailed results which parallel those of complex fuzzy sets.

Keywords : complex intuitionistic fuzzy sets, complex intuitionistic fuzzy differential equations.

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Numerical Solution of Intuitionistic Fuzzy Differential Equations by Runge-Kutta Method of Order Four

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Abstract

This paper presents solution for first order fuzzy differential equation by Runge-Kutta method of order four. This method is discussed in detail and this is followed by a complete error analysis. The accuracy and efficiency of the proposed method is illustrated by solving an intuitionistic fuzzy initial value problem.

Keywords : intuitionistic fuzzy Cauchy problem, Runge-Kutta method of order four.

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Fractional Differential Equations with Intuitionistic Fuzzy Data

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Abstract

The purpose of this paper is to study the existence and uniqueness of solution for fractional differential equation with intuitionistic fuzzy data where the intuitionistic fuzzy fractional derivatives and integral are considered in the Riemann-Liouville sense. Finally we give an example.

Keywords : intuitionistic fuzzy number, Fractional differential equations.

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Solving Second Order Intuitionistic Fuzzy Initial Value Problems with Heaviside Function

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Abstract

In this work, we examined the solution of the following second order intuitionistic fuzzy initial value problem through intuitionistic Zadeh's Extension Principle [17]:

$$y''(x) + \bar{a}_1^i y'(x) + \bar{a}_2^i y(x) = \sum_{j=1}^{r} \bar{b}_j^i g_j(x); \qquad (0.0.1)$$

$$y(0) = \bar{\gamma}_0^i; \qquad (0.0.2)$$

$$y'(0) = \bar{\gamma}_1^i. \qquad (0.0.3)$$

Here $\bar{a}_1^i$, $\bar{a}_2^i$, $\bar{\gamma}_0^i$, $\bar{\gamma}_1^i$ and $\bar{b}_j^i$ (j=1,2,...,r) are intuitionistic fuzzy numbers and $g_i(x)$(i=1,2,...,r) are continuous functions on the interval $[0,\infty)$. We reformulated the approach in [2] and [3] for finding an analytical form of alpha and beta cuts for the solution of intuitionistic fuzzy initial value problem for the second order differential equation with the help of Heaviside step function. Firstly we reformulated the general solution of the crisp differential equation corresponding to Eq. 0.1 and applied intuitionistic Zadeh's Extension Principle to intuitionisticly fuzzify the solution. Then, we obtained the analytical form

of (α, β)-cuts of the solution of the fuzzy initial value problem by using interval operations and Heaviside step function. Finally, we have illustrated some examples by using this algorithm.

Keywords : Intuitionistic Fuzzy Initial Value Problem, Intuitionistc Zadeh's Extension Principle, Heaviside Function.

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Equivalence Among Three 2-Norms on the Space of p-Summable Sequences

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Abstract

There are two known 2-norms defined on the space of p-summable sequences of real numbers. The first 2-norm is a special case of Gähler's formula [Mathematische Nachrichten, 1964], while the second is due to Gunawan [Bulletin of the Australian Mathematical Society, 2001]. The aim of this paper is to define a new 2-norm on ℓ^p and prove the equivalence among these three 2-norms.

Keywords : 2-normed spaces; the space of p-summable sequences; completeness; norm equivalence.

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Some Properties of Soft Mappings on Soft Metric Spaces

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Abstract

In this study we define the soft topology generated by the soft metric and show that every soft metric space is a soft normal space. We also investigate some properties of soft continuous mappings on soft metric spaces and finally we give a few examples of soft contraction mapping on soft metric spaces.

Keywords: soft metric space, soft normal sapce, soft continuous mapping, soft contraction mapping.

204

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Soft Totally Bounded Spaces in Soft Metric Spaces

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Abstract

In this study we define the soft topology generated by the soft metric and show that every soft metric space is a soft normal space. We also investigate some properties of soft continuous mappings on soft metric spaces and finally we give a few examples of soft contraction mapping on soft metric spaces.

Keywords: Soft set, soft metric space, soft sequential compact, soft totally bounded sets.

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Some Generalized Fixed Point Type Theorems on an S-Metric Space

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Abstract

In this talk, we give new contractive mappings on an S-metric space. We investigate some generalizations of the Banach's contraction principle and new fixed point type theorems using the notion of periodic index on an S -metric space.

Keywords: S-metric, Banach's contraction principle, periodic index.

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A New Generalization of Soft Metric Spaces

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Abstract

In this talk, we describe the notion of a soft S-metric as a generalization of a soft metric. We investigate some basic and topological properties of this new metric. Also we give some existence and uniqueness conditions of fixed-point theorems on a complete soft S-metric space. We verify our results with some examples.

Keywords: Soft S-metric space, fixed point, topological properties

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Some Separation Axioms in Fuzzy Soft Topological Spaces

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Abstract

Molodtsov (1999) proposed a completely new concept called soft set theory to model uncertainty, which associates a set with a set of parameters. Pei and Miao (2005) showed that soft sets are a class of special information systems. Later, Maji et al. (2001) introduced the concept of a fuzzy soft set which combines a fuzzy set and a soft set. From then on, many authors have contributed to (fuzzy) soft set theory in the different fields such as algebra, topology and etc. Soft topology is a relatively new and promising domain which can lead to the development of new mathematical models and innovative approaches that will significantly contribute to the solution of complex problems in natural sciences. Separation is an essential part of topology, on which a lot of work has been done. The aim of this work is to generalize some low-level separation axioms in fuzzifying topology and fuzzy topology to the fuzzifying soft topology and fuzzy soft topology by considering parametrization. So, we obtain some fundamental prperties and characterizations of proposed separations.

Keywords: Fuzzy soft set, fuzzy soft topology, separation axiom

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208

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Fuzzy Equilibrium Analysis of a Transportation Network Problem

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Abstract

In this paper, we focused on the solution process of a fuzzy transportation network equilibrium problem. This problem aims to minimize the total travel time of vehicles on traffic flows between specified origin and destination points. The link travel time for a vehicle is taken as a linear function of link flow (the number of vehicles on that link). Thus, the objective function can be formulated in terms of link flows and link travel times in a quadratic form while satisfying the flow conservation constraints. The parameters of this problem are path lengths, number of lanes, average velocity of a vehicle, vehicle-length, clearance, spacing, link capacity and free flow travel time. Considering a road network, path lengths and number of lanes are taken as crisp numbers. The average velocity of a vehicle and the vehicle-length are imprecise in nature, so these are taken as triangular fuzzy numbers. Since the remaining parameters, that are clearance, spacing, link capacity and free flow travel time, are determined by the average velocity of a vehicle and vehicle-length, all of them will be triangular

fuzzy numbers. Finally, the original fuzzy transportation network problem is converted to a fuzzy quadratic programming problem, and it is solved with an existing approach from the literature. A numerical experiment is illustrated.

Keywords: Fuzzy transportation network equilibrium problem, fuzzy quadratic programming, triangular fuzzy numbers.

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A Special Type of Sasakian Finsler Structures on Vector Bundles

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Abstract

Sasakian Finsler structures can be obtained on horizontal and vertical distributions of vector bundles. In this paper, Sasakian Finsler structures satisfying $R(X^H, Y^H)C^* = 0$ on horizontal distribution of vector bundles are examined where R is Riemann curvature tensor, C^* is quasi-conformal curvature tensor and X^H, Y^H are elements of family of vector fields on horizontal distribution. In this regard some structure theorems are examined.

Keywords: Quasi-conformal curvature tensor, Sasakian Finsler structure.

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On Some Properties and Applications of the Quasi-Resolvent Operators of the Infinitesimal Operator of a Strongly Continuous Linear Representation of the Unit Circle Group in a Complex Banach Space

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Abstract

Let A be the infinitesimal operator of a strongly continuous linear representation of the unit circle group on a complex Banach space H. In this talk, the quasi-resolvent operator of A which is denoted by R_λ is defined by the spectrum of A. Some properties and inter relations of operators R_λ are introduced, and by using them, some theorems on existence of periodic solutions to the non-linear equations $\phi(A)x = f(x)$ are stated and proven, where $\phi(A)$ is a polynomial of A and f is a continuous mapping of H into itself.

Existence and Nonexistence for Nonlinear Problems with Singular Potential

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Abstract

Let $\emptyset \subset \mathbb{R}^N$ be a bounded regular domain of $\mathbb{R}^N$ we consider the following class of elliptic problem

$$\begin{cases} -\Delta u = \dfrac{u^q}{d^2} & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $0 < q \leq 2^* - 1$. We investigate the question of existence and nonexistence of positive solutions depending on the range of the exponent q.

Intuitionistic Fuzzy Soft Generalized Superconnected

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Abstract

Shabir and Naz [7] introduced the notion of soft topological spaces which are defined over an initial universe with a fixed set of parameters. They also studied some of basic concepts of soft topological spaces. In the present study, we introduce some new concepts in intuitionistic fuzzy soft topological spaces such

as intuitionistic fuzzy soft generalized superconnected. We also give characterizations and properties of this notion.

Keywords: intuitionistic fuzzy soft set, Intuitionistic fuzzy soft toplogy, intuitionistic fuzzy soft mapping.

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Vietoris Topology in the Context of Soft Set

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Abstract

In the present paper, we study the notion of a Vietoris topology by using soft sets. We obtain some properties related to the first countability of soft Vietoris topology. Then, we focus on second countability of it.

Keywords: Soft set, Soft Vietoris topology, Soft first countability, Soft second countability.

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On Totally Umbilical and Minimal Cauchy Riemannian Lightlike Submanifolds of an Indefinite Kaehler Manifold

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Abstract

In this talk, we survey Cauchy Riemannian lightlike submanifolds of an indefinite Kaehler manifold. Firstly, we mention definition of Cauchy Riemannian (CR, SCR and GCR) lightlike submanifolds of an indefinite Kaehler manifold. Then, we investigate minimal and totally umbilical Cauchy Riemannian lightlike submanifolds and give some examples for these classes.

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Algebraic Properties of Dual Quasi-Quaternions

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Abstract

In this work, we consider the algebra of dual quasi-quaternion and give some algebraic properties of this algebra.

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Fixed Intuitionistic Fuzzy Point Theorem in Hausdorff Intuitionistic Fuzzy Metric Spaces

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Abstract

Atanassov (see [2,3]) introduced and studied the concept of intuitionistic fuzzy sets (i-fuzzy set, for short) and later there has been much progress in the study of i-fuzzy sets by many authors (see [1,4,5,10]). Using the idea of i-fuzzy set, Park [10] defined the notion of intuitionistic fuzzy metric space with the help of continuous t-norms and continuous t-conorms as a generalization of fuzzy metric space due to George and Veeramani [7] and proved some known results of metric spaces for intuitionistic fuzzy metric space. In 2001 Estruch and Vidal [6] introduced the concept of intuitionistic fuzzy mapping (i-fuzzy mapping,for short) and gived an intuitionistic version of Heilpern's mentioned theorem (see [9]). After that, Gregori et al [8] defined Hausdorff intuitionistic fuzzy metric on a family of non-empty compact subsets of a given intuitionistic fuzzy metric space.

In this study we modify concept of Hausdorff intuitionistic fuzzy metric using i-fuzzy sets and obtain fixed i-fuzzy point results for i-fuzzy mappings.

Keywords: i-fuzzy mapping, i-fuzzy point.

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Orthonormal Systems in Spaces of Number Theoritical Functions

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Abstract

In this paper we regard some (for number theory important) examples of set algebras $\mathcal{A}$ on $\mathbb{N}$. In each example we obtain the measure space $\Omega := (\beta\mathbb{N}, \sigma(\overline{\mathcal{A}}), \overline{\delta})$ by the model of Indlekofer which is based on the Stone-Cech compactification of $\mathbb{N}$.

Let $\mathcal{E}(\mathcal{A})$ be the set of simple functions on $\mathcal{A}$ and let $\mathcal{L}^{*\alpha}(\mathcal{A})$ be $\|.\|_\alpha$ the closure of $\mathcal{E}(\mathcal{A})$ with

$$\|f\|_\alpha := \left\{ \limsup_{x \to \infty} \frac{1}{x} \sum_{n \le x} |f(n)|^\alpha \right\}^{\frac{1}{\alpha}}, 1 \le \alpha < \infty.$$

Now our aim was to give a description of a complete orthonormal system for $L^{*2}(\mathcal{A})$ in each regarded case where $L^{*2}(\mathcal{A})$ is denoted the quotient space $\mathcal{L}^{*2}(\mathcal{A})$ modulo null-functions.

Keywords: Stone-Cech compactification, function spaces, complete orthonormal systems.

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A Step Size Strategy for Numerical Integration of the Hurwitz Stable Differential Equation Systems

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Abstract

For the efficiency of the numerical integration of the Cauchy problems, it is not practical to use constant step size. There are some studies in the literature about the variable step size for numerical integration (for example see; [3,4,5]). One of these studies has given in [1,2]. On the region $D = (t, X) : |t - t_0| \leq T, |x_j - x_{j0}| \leq b_j$, in [1,2] the step size strategy for the Cauchy Problem

$X' = AX, X(t_0) = X_0$

has proposed such that the local error $||LE_i|| \leq \delta_L$, where δ_L is the error level that is determined by user. Here $X(t) = (x_j(t))$, $X_0 = (x_{j0})$; $x_{j0} = x_j(t_0)$, $A \in R^{N \times N}$, $X(t)$, X_0 and $b = (b_j) \in R^N$.

In this study, we aimed to develop the step size strategy in [1,2] for the system

$X'^{N \times N}$- Hurwitz stable matrix.

A step size strategy and an algorithm for the Hurwitz systems which calculate the step sizes based on the given strategy and numerical solutions are given. The numerical solutions obtained with the new strategy and algorithm are compared with the results in [1,2]. The given strategy and algorithm are applied to some industrial problems.

Keywords: Variable step size, Hurwitz stable differential equation systems, numerical integration, step size strategy

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Schur Stabilitiy in Floating Point Arithmetic: Systems with Constant Coefficients

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Abstract

The representation of the numbers $\mathbb{F} = \mathbb{F}(\gamma, p_-, p_+, k) = \{0\} \cup \{z | z = \pm \gamma^{p(z)} m_\gamma(z)\}$, which are called floating point numbers [1,9,13,14]. Computers use floating point numbers for computing. These numbers are also called computer numbers or machine numbers [3,10,15,17]. If the results of arithmetic operations are elements of set $\mathbb{F}$, they are directly stored in the memory. Otherwise they are stored with error [4,7,11,12,16,18].

The linear difference equation in order of N as

$y(n + N) = a_N y(n + N - 1) + ... + a_1 y(n)$

can be transform to one order system as

$x(n+1) = Ax(n)$, n-integer number,

where matrix A ($N \times N$ dimension) is the companion matrix. It is well known that the solution of the Cauchy problem

$x(n+1) = Ax(n)$, $x(0) = x_0$

is $x(n) = A^n x_0$ (see, [1,8]). In [2], an algorithm has given which have computed power of the companion matrix.

The matrix $fl(A^n)$ is the computed companion matrix A^n in floating point arithmetic. The effects of floating point arithmetic in the computation of the companion matrix A^n were investigated [5,6]. Error bounds were obtained for $||A^n - fl(A^n)||$, where $A \in M_N(D)$. Additionally, Schur stability were investigated according to floating point arithmetic. The obtained results were supported with numerical examples.

Keywords: floating point arithmetics, difference equation, error analysis, companion matrix.

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220

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The Numerical Solution of Some SIR Epidemic Models with Variable Step Size Strategy

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Abstract

Selection of step size is one of the most important concepts in numerical integration of differential equation systems. Even to use constant step size, it must be investigated how should be selected the step size in the first step of numerical integration. Because, if the selected step size is large in numerical integration, computed solution can diverge from the exact solution. And if the chosen step size is small; calculation time, number of arithmetic operations, the calculation errors start to increase. So, it will be sensible to use small step sizes in the region where the solution changes rapidly and to use bigger step size in the region where the solution changes slowly. So, it is not practical to use constant step size in numerical integration. In literature, step size strategies have been given for the numerical integration. One of these strategies is given for the Cauchy problem

$$X'(t) = AX(t) + \varphi(t, X), X(t_0) = X_0 \qquad (0.0.4)$$

where $A = (a_{ij}) \in R^{N \times N}$, $X \in R^N$ and $\varphi \in C^1([t_0 - T, t_0 + T] \times R^N)$ in [3,4].

Many dynamical system models are represented by non-linear diï¬differential equation systems as in (0.0.4). The epidemic models is one of these systems also attracted attention in recent years (for example see, [1,2,5,6,7]). The classical epidemic model is SIR model.

In this study, we have aimed to investigate the effectiveness of the variable step size strategy for some SIR epidemic models. We have applied the variable step size strategy to the SIR model and its modifications.

Keywords: Step size strategy, variable step size, epidemic model, SIR model, system of non-linear differential equations.

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When is an Archimedean $f-$Algebra Finite Dimensional?

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Abstract

In this note, we give necessary and sufficient conditions for an Archimedean f-algebra to be of finite dimensional. As an application, we give a positive answer to a question raised by Bresar in [1].

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By Calculating for Some Linear Positive Operators to Compare of the Errors in the Approximations

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Abstract

In mathematics,we investigate a Korovkin-type approximation theorem for sequences of positive linear operators on the space of all continuous real valued functions defined on [a,b] in "Approximation theory". In this paper, we get some approximation properties for sequences of positive linear operators constructed by means of the Bernstein operator and give a Korovkin-type approximations properties for them. We research convergence and approximation properties for type generalized Stancu operators and Bernstein operator to give some examples. We also made a comparison between the approximations obtained by them with calculating the errors in the approximations for different continuous functions. Recently, some authors draw graphics of some modified operators and calculating the errors in approximations[1,2]. Figures of these kind of operator are very difficult because these operators have many of properties such that integrals, summations etc. Consequently figures and numerical results verify the theoretical results in the view of different aspects.

Keywords: Approximation,Positive linear operators, Korovkin-type theorem, Comparison, Errors, Figures.

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Intuitionistic Fuzzy Soft Neighborhoods

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Abstract

In this paper, we introduce the concept of intuitionistic fuzzy soft point as a generalization of intuitonistic fuzzy point and study some basic properties. We consider the neighborhood structures of an intuitionistic fuzzy point and generate an intuitonistic fuzzy soft topology by using the systems of neighborhood.

   Keywords: Intuitionistic fuzzy soft set, intuitionistic fuzzy soft point, intuitionistic fuzzy soft topology, neighborhood.

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On Nonlightlike Offset Curves in Minkowski 3-Space

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Abstract

In this study, the offset curves of nonligthlike curves are investigated in three different cases. Then the curvature, torsion and arclength of a given offset curve are expressed in terms of the curvature, torsion of the main curve and constants A and B for each case. Moreover, it is proved that the offset curve constitutes another Bertrand curve.

Keywords: Bertrand Curve, Offset Curve, Minkowski Space.

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Weierstrass Representation, Degree and Classes of the Surfaces in the Four Dimensional Euclidean Space

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Abstract

In this talk, we present on a minimal surface using Weierstrass representation in the four dimensional Euclidean space. We compute implicit equations, degree and class of the surface.

Keywords: 4-space, Weierstrass representation, minimal surface, degree, class.

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A New Type Graph and Their Parameters

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Abstract

The graph theory has been improved fastly since it has applications in different fields of science. In this paper a special algebraic graph has been defined then some parameters of this graph have been studied.

Keywords: Graph, Graph Parameter.

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The Dot Product Graph of Monogenic Semigroup

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Abstract

We can identify S as a cartesian product of finite times a finite semigroup S_M^n which has some elements like $\{0, x, x^2, \ldots , x^n\}$ Let $\Gamma(S)$ be a dot product graph whose vertices are the nonzero elements of S. In this study we are going to analyze some parameters of $\Gamma(S)$.

Keywords: Dot Product, Monogenic Semigroup, Graph.

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Some Number Theoretical Results Related to the Suborbital Graphs for the Congruence Subgroup $\Gamma_0(\frac{n}{h})$

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Abstract

In this work, we study the congruence subgroup $\Gamma_0(\frac{n}{h})$ of the Modular group Γ acting transitively on the subset $\bar{Q}(h)$. From the suborbital graph $F(1,n)$ we obtain some interesting number theoretical results, for instance, for all $n \in \mathbb{N}$, the numbers $n(n-4)b^2 - 4$ are not squares.

Keywords: Graph Theory, Number Theory.

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Schur Stabilitiy in Floating Point Arithmetic: Systems with Periodic Coefficients

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228

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Abstract

The representation of the numbers

$$\mathbb{F} = \mathbb{F}\left(\gamma, p_-, p_+, k\right) = \{0\} \cup \left\{z | z = \pm\gamma^{p(z)} m_\gamma(z)\right\}, \qquad (0.0.5)$$

which are called floating point numbers [2, 13, 17, 18]. Computers use floating point numbers for computing. These numbers are also called computer numbers or machine numbers [6, 14, 19, 21]. If the results of arithmetic operations are elements of set F, they are directly stored in the memory. Otherwise they are stored with error [7, 11, 15, 16, 20, 22].

Let A_n be an N-dimensional periodic matrix (T-period) and difference equation system $x_{n+1} = A_n x_n$. The matrix X_n is called the fundamental matrix of the system, and the matrix X_T is called the monodromy matrix of the system [1, 3, 4, ,5, 12]. The matrix $Y_n = fl(A_{n-1}Y_{n-1})$ is the computed fundamental matrix X_N in floating point arithmetic. Cauchy problem of difference equation system can be written

$$fl(A_{n-1}Y_{n-1}) = Y_n = A_{n-1}Y_n - 1 + \phi_N; Y_0 = I, \qquad (0.0.6)$$

where ϕ_N is computation error of $(A_{n-1}Y_{n-1})$.

The effects of floating point arithmetic in the computation of the fundamental matrix X_N were investigated. Error bounds were obtained for $\|X_N - Y_N\|$, where $A_N \in M_N(D)$. The obtained results were investigated for Schur stability of the system [8, 9, 10]. These results were supported with numerical examples.

Keywords: floating point arithmetic, difference equation, error analysis, Schur stability.

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$\mathcal{I}$-Limit Inferior and $\mathcal{I}$-Limit Superior of Sequences of Sets

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Abstract

In this talk, we extend the concepts of $\mathcal{I}-$Limit superior and $\mathcal{I}-$Limit inferior for reel number sequences to $\mathcal{I}-$Limit superior and $\mathcal{I}-$Limit inferior for sequences of sets, study their certain properties and establish some basic theorems.

Keywords: Statistical convergence, $I-convergence$ and $I^*-convergence$, sequence of sets , Wijsman convergence.

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A Ditopological Fuzzy Structural View of Inverse Systems and Inverse Limits

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Abstract

One of the theories defined to develop some complement-free concepts is the *texture (fuzzy structure) theory* and it was constructed in [1] as a point-based setting for the study of classical sets and fuzzy sets. Add to that, the notion of *ditopology* described on a texture is essentially a topology for which there is no a priori relation between the open and closed sets, and thus ditopological fuzzy structures [1] were conceived as a unified setting for the study of fuzzy topology. Especially, some useful relationships with fuzzy topology may be found in [3].

In the previous study [2], the foundations of a corresponding theory of *inverse (projective) systems* and their limits, called *inverse limits* were laid in the category **ifPTex** of plain textures which are special types of textures, and point functions satisfying a compatibility condition, named w-preserving. Therefore firstly, a detailed analysis of inverse systems and inverse limits was presented in [2] insofar as the category of plain textures is concerned. Evidently, this theory was constituted as an analogue of the inverse system theory in the classical categories **Set**, **Top** and **Rng** in algebra, simultaneously.

As the main theme of this presentation, a suitable theory of inverse systems and their limits is established for some subcategories of the category **ifPDitop** topological over **ifPTex**, whose objects are ditopological fuzzy structures which have plain texturing and morphisms are bicontinuous **ifPTex**-morphisms. In addition, many useful properties of inverse limits in **ifPDitop** are studied via examples in the context of ditopological fuzzy structures, as natural counterparts of the classical cases.

Keywords: Inverse Limit, Fuzzy Topology, Category, Texture, Ditopology.

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On Nullnorms on Bounded Lattices

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Abstract

t-operators and nullnorms were introduced in [9], [7] respectively, which are also generalizations of the notions of t-norms and t-conorms. And then in [10], it is pointed out that nullnorms and t-operators are equivalent since they have the same block structures in $[0,1]^2$. Namely, if a binary operator F is a nullnorm then it is also a t-operator and vice versa.

Definition 0.0.1 *Let $(L, \leq, 0, 1)$ be a bounded lattice. A commutative, associative, non-decreasing in each variable function $F : L^2 \to L$ is called a nullnorm if there is an element $a \in L$ such that $F(x, 0) = x$ for all $x \leq a$, $F(x, 1) = x$ for all $x \geq a$.*

In this study, given a bounded lattice L and a nullnorm on it, taking into account the properties of nullnorms, we investigate an order induced by nullnorms and equivalence relation on bounded lattice. In this way, we obtain that interesting results.

Keywords: Nullnorm, Bounded lattice, Partial order.

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Ideal Version of Weighted Lacunary Statistical Convergence of Sequences of Order α

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Abstract

In this work, we are interested in ideal version of weighted lacunary statistical convergence of sequences of order α and we examine some inclusion relations.

Keywords: I-convergence; I-statistical convergence; weighted lacunary I-statistical convergence of order α; sequence space

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$AK(S)$ and $AB(S)$ **Properties of a K-Space**

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Abstract

A typical sum s on a K- space S has often the representation,

$$s(z) = \lim_{\gamma \in \Gamma} \sum_k u_{\gamma k} z_k, \ z = (z_k) \in S \qquad (0.0.7)$$

where Γ is a directed index set and $u_\gamma = (u_{\gamma k}) \in \phi$, the space of finitely non-zero sequences, for each $\gamma \in \Gamma$. Let a K-space S be equipped with a sum (0.0.7). Then, for each $x = (x_k)$ and $\gamma \in \Gamma$, the sequence $P_\gamma(x) = \sum_k u_{\gamma k} x_k \delta^k$, $(\gamma \in \Gamma)$ is called the γth S-section of x [2]. Here δ^k is the sequence whose kth component is 1 all the others are 0.

If $\lambda \supset \phi$ is a K space, then Boos and Leiger defined the spaces $\lambda_{AB(S)}$ and $\lambda_{AK(S)}$ in [?] as

$$\lambda_{AB(S)} = \{x \in \omega | (P_\gamma(x))_{\gamma \in \Gamma} \text{ is a bounded net in } \lambda\},$$

and

$$\lambda_{AK(S)} = \{x \in (\lambda_{AB(S)} \cap \lambda) | \lim_\gamma P_\gamma(x) \text{ exists in } \lambda\}.$$

In this work, we investigate some properties of these spaces and give some theorems related to the duals.

Keywords: K- spaces, n-th section of a sequence, β-, γ-, f-duality.

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On the Second Homology of the Schützenberger Product of Monoids

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Abstract

For two finite monoids S and T, we prove that the second integral homology of the Schützenberger product $S \Diamond T$ is equal to

$$H_2(S \Diamond T) = H_2(S) \times H_2(T) \times (H_1(S) \otimes_{\mathbb{Z}} H_1(T))$$

as the second integral homology of the direct product of two monoids.

   This is joint work with Hayrullah Ayık and Leyla Bugay.

   (Melek Yağcı   is supported by Ç.Ü. BAP.)

   Keywords: Monoid, Schützenberger product, Second integral homology.

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New Sequence Spaces with Respect to a Sequence of Modulus Functions

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Abstract

In this talk, we introduce the notion of $A^{\mathcal{I}}$-invariant statistical convergence, $A^{\mathcal{I}}$-lacunary invariant statistical convergence with respect to a sequence of modulus functions. We establish some inclusion relations between these spaces under some conditions.

Keywords: Lacunary invariant statistical convergence; Invariant statistical convergency, ideal convergence, modulus function.

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TF-Type Hypersurfaces in 4-Space

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Abstract

We study on translation and factorable hypersurfaces in the four dimensional Euclidean space. We calculate implicit algebraic equations of the hypersurfaces.

Keywords: 4-space, translation hypersurface, factorable hypersurface, algebraic equation.

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A Note on q-Binomial Coefficients

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Abstract

The q-calculus has been developing fast. In the present work we study on a q-extension of binomial coefficients. The infinite sum of q-extension of binomial coefficients is obtained. Then, by using its infinite sum , we obtain Volkenborn integral velue of q-extension of binomial coefficients.

Keywords: p-adic number, Indefinete sum, q-analogue of the binomial coefficients, Volkenborn integral.

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On Statistical Convergence of Sequences of p-Adic Numbers

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Abstract

Let p be a fixed prime number. By $\mathbb{Q}_p$ we denote the field of p-adic numbers, the completion of the rational numbers field $\mathbb{Q}$ with respect to the p-adic norm $|\cdot|_p$. The concept of statistical convergence was introduced by H. Fast (1951) [1] and R. C. Buck (1953) [2] independently for real or complex sequences. This concept was studied by T.Salat (1980) [3], J. A. Fridy (1985) [4] and many authors. We note that the field of p-adic numbers $\mathbb{Q}_p$ is non-Archimedean, means that the ultrametric inequality is valid

$$|x + y|_p \leq \max \left\{ |x|_p, |y|_p \right\}$$

for all $x, y \in \mathbb{Q}_p$. In the present work we define the concept of statistical convergence of sequences for p-adic numbers and give some its properties.

Keywords: p-adic number, statistical convergence of sequence of p-adic numbers, statistical Cauchy sequence of p-adic numbers.

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Lightlike Hypersurface of an Indefinite Kaehler Manifold with a Complex Semi-Symmetric Metric Connection

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Abstract

In this paper, we study lightlike hypersurface of an indefinite Kaehler manifold admitting a complex semi-symmetric metric connection. We get the equations of Gauss and Codazzi. Then, we give some characterizations of lightlike hypersurface in an indefinite complex space form with a complex semi-symmetric metric connection. Finally, we show that the Ricci tensor of lightlike hypersurface of an indefinite Kaehler manifold with complex semi-symmetric metric connection is not symmetric.

Keywords: Lightlike Hypersurface, Indefinite Complex space form, Complex Semi-Symmetric Metric Connection, Levi-Civita connection, Ricci tensor.

Intuitionistic Fuzzy Fractional Evolution Problem

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Abstract

We introduce the generalized intutionistic fuzzy derivative, this concept used in order to give a generalized intuitionistic fuzzy Caputo fractional derivative. And we descuse the intuitionistic fuzzy fractional evolution problem.

Keywords: Generalized intuitionistic fuzzy Hukuhara difference, Generalized intuitionistic fuzzy derivative, generalized intuitionistic fuzzy Caputo-derivative, intuitionistic fuzzy fractional evolution problem.

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A General Tableaux Method for Contact Logics Interpreted over Intervals

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Abstract

In this paper, we focus our attention on tableau methods for contact logics interpreted over intervals on the reals. Contact logics provide a natural framework for representing and reasoning about regions in several areas of computer science such as geological information systems, artificial intelligence and etc. In this paper, we focus our attention on tableau methods for contact logics interpreted over intervals on the reals. Contact logics provide a natural framework for representing and reasoning about regions in several areas of computer science such as geological information systems, artificial intelligence and etc.[1,3,4] . However,

while various tableau methods have been developed for classical logic, modal logics and intuitionistic logic, not much work has been done on tableau methods for contact logics [1,2]. We develop a general tableau method for contact logic interpreted over intervals. In this paper we give sound and complete tableaux-based decision procedures for contact logics. Developing such tableaux-based decision procedures, we obtain new decidability/complexity results.

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Solving Intuitionistic Fuzzy Differential Equations with Linear Differential Operator by Adomain Decomposition Method

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Abstract

In this paper we presented intuitionistic fuzzy differential equation with linear differential operator which can be of any order and it also involves nonlinear functional. So our solution procedure gives the solutions of a large area of problems involving intuitionistic fuzzy differential equations. Adomain decomposition method (ADM) has been used to find the approximate solution. Note that we used ADM which gives solution even for some nonlinear problems that can't be solved by classical methods. We have given two numerical examples and by comparing the numerical results obtain from ADM with the exact solution, we have studied their accuracy.

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Involution Matrices of $\frac{1}{4}-$Quaternions

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Abstract

In this work, we consider the $\frac{1}{4}$−quaternion algebra and give the matrix representations of the involution and anti-involution maps obtained by this algebra.

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Topological Full Groups of Cantor Minimal Systems

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Abstract

In this work, we study topological full groups of Cantor minimal systems. In recent years, this subject has been very popular since it supplies a connection between dynamical systems and group theory. We will investigate the relationship between conjugation of dynamical systems and isomorphism of their topological full groups. Moreover, topological full groups provide the first examples of finitely generated, simple and amenable groups. We will survey the ideas behind the proofs of these facts.

Keywords: Cantor Space, Topological Full Group, Simple Grup, Amenable Group.

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Global Stability to Nonlinear Neutral Differential Equations of First Order

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Abstract

In this paper, we study globally asymptotically stability of zero solution to a nonlinear neutral differential of first order. The technique of the proof involves the fixed point method. By this way, we extend and improve some recent works in the literature.

Keywords: Fixed point theorem, globally asymptotically stability, neutral differential equation, firs order.

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A Study on the Cartesian Product of a Special Graphs

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Abstract

Our main aim in this presentation is to extend these studies over $\Gamma(S_M)$ to the cartesian product. In here, $\Gamma(S_M)$ is a graph of monogenic semigroup $S_M = \{x, x^2, x^3, ..., x^n\}$ with zero. In detail, we will investigate some important graph parameters for the cartesian product of any two (not necessarily different) graphs $\Gamma(S_{M_1})$ and $\Gamma(S_{M_2})$.

   Keywords: Monogenic semigroup graph, Graph product.

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Unit Dual Lorentzian Sphere and Tangent Bundle of Lorentzan Unit 2-Sphere

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Abstract

The purpose of this study is two-fold, firstly to recall some basic concepts and notions of unit dual Lorentzian sphere. Secondly, to define a one-to-one relationship between the unit dual Lorentzian sphere and tangent bundle of Lorentzian unit 2-sphere.

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On the Inverse Problem for a Sturm-Liouville Equation with Discontinuous Coefficient

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Abstract

In this study, it is investigated nonhomogenous boundary value problem.During the solution it is encountered Sturm-Liouville problem with piecewise continuous coefficients and that contained eigenvalue parameter.One transmission condition, which given by as relations between the right and left hand limit of the solution at the point of discontinuity are added to the boundary conditions. We examined some spectral properties of the problem.The numeric solutions of eigenvalues are obtained. According to the spectral datas the inverse problem are researched.

Keywords: Discontinuous Sturm-Liouville Problem, Inverse Problem, Transmission Condition.

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Calculation and Analysis of Electronic Parameters of Electroluminescent Device Cells Through I-V Based Modeling

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250

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Abstract

Light-emitting electrochemical cells (LECs) is one of the simplest kinds of electroluminescent devices. LEC is constituted to an organic single layer structure that was sandwiched between a cathode and an anode. In this study we calculated theoretically of the electronic parameter of LECs device through I-V based modeling. The LEC diode electronic parameters as the ideality factor n and barrier height ϕ_b were obtained using a method developed by Cheung and confirmed by Werner. The net current of a LEC device is due to the thermionic emission and it can be expressed as

$$I = I_0 \exp\left(\frac{q\,(V - IR_s)}{nkT}\right)$$

where V is applied voltage and saturation current I_0 is defined as

$$I_0 = AA^*T^2 \exp\left(-\frac{q\phi_b}{kT}\right).$$

Keywords: LECs device, Electronic parameters, I-V modeling.

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On Some Properties of Sum Spaces

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Abstract

A sum is a continuous linear functional s defined on a K- space $\lambda \supset \phi$ (space of finitely non-zero sequences) such that, $s(z) = \sum_k z_k$, $\forall z = (z_k) \in \phi$. A K-space

λ is called a sum space if and only if $\lambda \supset \phi$ and $\lambda^f = \lambda^\lambda$, where $\lambda^f = \{(f(\delta^k)) : f \in \lambda'\}$ and λ^λ is the set of all sequences x such that $xy \in \lambda$, $\forall y \in \lambda$ [4,6]. Here δ^k is the sequence whose kth component is 1 all the others are 0, $xy = (x_k y_k)$ for $x = (x_k), y = (y_k)$ and λ' is the space of continuous linear functionals on λ.

An FK space $\lambda \supset \phi$ is generalized semiconservative FK space if $\lambda^f \subset \lambda^{\lambda^2}$, where $\lambda^{\lambda^2} = \lambda^{\lambda\lambda} = (\lambda^\lambda)^\lambda$.

In this work, we give some definitions and theorems related with sum spaces and generalized semiconservative FK spaces.

Keywords: FK spaces, $\beta-$ dual, f- dual, Semiconservative FK spaces.

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On q^λ and q_0^λ Invariant Sequence Spaces

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Abstract

Invariant sequence spaces are very helpful for investigations of the duality of sequence spaces. For instance, if the sequence space X satisfies the condition $\ell_\infty.X = X$ then its $\alpha-$, β- and γ- duals are same [4]. Garling [1] investigated B- and B_0- invariant sequence spaces and Buntinas [2] introduced and investigated q- and q_0- invariant sequence spaces and recently, Grosse- Erdmann [3] studied on ℓ_1 invariant sequence spaces.

In this work, we define q^λ and q_0^λ invariant sequence spaces, X with $q^\lambda.X = X$ and $q_0^\lambda.X = X$, respectively. and give some related theorems.

Keywords: K- spaces, λ-boundedness and λ-convergence of a sequence, β-, γ-, f- duality.

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Some New Results on a Graph of Monogenic Semigroup

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Abstract

In [2], it has been defined a new graph $\Gamma(S_M)$ on monogenic semigroups S_M (with zero) having elements $\{0, x, x^2, x^3, ..., x^n\}$. Many researchers have been working on this area after that work, for example [1,3,4]. As a continues study of these studies, in this paper, it will be investigated define some new graph parameters (such as covering number, accessible number, Zagreb indices, ect.) for monogenic semigroup graph $\Gamma(S_M)$.

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Computational Solution of Katugampola Conformable Fractional Differential Equations Via RBF Collocation Method

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Abstract

In conjunction with the development of fractional calculus, conformable derivatives and integrals has been widely used a number of scientic areas. In this talk, we provide a numerical scheme to solve Katugampola conformable fractional differential equations via radial basis function (RBF) collocation technique. In order to confirm our numerical scheme, we present some numerical experiments results.

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Some Integral Inequalities Via Conformable Calculus

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Abstract

The purpose of this talk is making generalization of Gronwall, Volterra and Pachpatte type inequalities for conformable differential equations. Then we provide some upper or lower bound for fractional derivatives and integrals with the help of Katugampola definition for conformable calculus. These results are extensions of some existing Gronwall, Volterra and Pachpatte type inequalities in the previous studies.

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Weighted Ostrowski, Chebyshev and Grüss Type Inequalities for Conformable Fractional Integrals

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Abstract

In this presentation, we have obtained weighted versions of Ostrowski, $\widetilde{C}$ebysev and Grüss type inequalities for conformable fractional integrals. In accordance with this purpose we have used Katugompola type conformable fractional integrals. The present study confirms previous findings and contributes additional evidence that provide the bounds for more general functions.

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Bifurcation and Stability Analysis of a Discrete-Time Predator-Prey Model

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Abstract

We consider discrete-time Leslie Model.We first determine its non-negative fixed point. Later on, we study local stability of the fixed point and determine the conditions on the parameters to show the existence of flip bifurcation by taking the step-size as a bifucation parameter. Analytical results are also supported by some numerical simulation. Moreover, using Center Manifold Theory, we show the existence of flip bfurcation and its properties.

Applications of Hermite-Hadamard Inequalities for $\mathbb{B}$-Convex Functions and $\mathbb{B}^{-1}$-Convex Functions

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Abstract

Hermite-Hadamard Inequality that is expressed in the following form

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(t)\,dt \leq \frac{1}{2}\left(f(a)+f(b)\right)$$

was proven by Hermite in [1] and then, ten years later, Hadamard rediscovered in [2] (for the historical consideration see also [3]). Next, Hermite-Hadamard Inequalities for different kinds of functions were examined in numerous article [4,5].

$\mathbb{B}$-convex functions were introduced and studied in [6]. $\mathbb{B}^{-1}$-convex functions were defined and examined in [7]. Hermite-Hadamard Inequalities for $\mathbb{B}$-convex Functions and $\mathbb{B}^{-1}$-convex Functions were introduced in [8].

In this work, we give the applications of Hermite-Hadamard Inequalities for $\mathbb{B}$-convex functions and $\mathbb{B}^{-1}$-convex functions.

Keywords: Hermite-Hadamard Inequalities, $\mathbb{B}$-convex functions, $\mathbb{B}^{-1}$-convex functions.

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Operations and Extension Principle under T-Intuitionistic Fuzzy Environment

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Abstract

In this paper, we introduce necessary definitions and related theorems on intuitionistic fuzzy T-set theory. In existing crisp set theory, characteristic functions, defined by two valued logic, can take values: zero and one only. In fuzzy set theory, introduced by Zadeh (1965), membership functions can take any value in closed unit interval . And, in intuitionistic fuzzy set theory, introduced by K. T. Atanassov (1986), both membership functions and non-membership functions can take suitable values in closed unit interval . But, we may observe that in those existing theories, we have to assign same membership value unity to elements, even if belongingness of one element is more certain than other to the subsets. In other words, certainly belongingness and more certainly belongingness of elements to subsets of universal sets are treated at par in all these existing theories, including classical crisp set theory. Similarly, in non-membership functions of intuitionistic fuzzy sets, zero is assigned as non-membership values to elements, both not belonging and not belonging certainly to subsets of universal sets. In order to overcome these limitations of existing theories, in 2015, we proposed intuitionistic fuzzy T-set theory, in which real numbers are suitably assigned to membership and non-membership functions. In this paper, we further introduce necessary definitions and related theorems

on intuitionistic fuzzy T-set theory. Those may be considered as generalizations of existing definitions and theorems from existing fuzzy and/or intuitionistic fuzzy set theory. In particular, we have generalized the concepts of extension principle and alpha, beta cut under existing intuitionistic fuzzy environment to T-intuitionistic fuzzy environment. Moreover, we have discussed some associated results under T-intuitionistic fuzzy environment. Finally, conclusions and future research directions are drawn.

Keywords: Intuitionistic fuzzy sets, $T^{(+)}$-characteristic functions, $T^{(-)}$-characteristic functions, intuitionistic Fuzzy T-sets, T-extension principle.

On Generalized Double Statistical Convergence of Order α in Intuitionistic Fuzzy N- Normed Spaces

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Abstract

In the present paper, we introduce and study the notion $\mathcal{I}$-double statistical convergence and ideal λ-double statistical convergence of order α with respect to the intuitionistic fuzzy n-normed space, briefly $IFnNS$, also we examine the relationship between these classes.

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On Feng Qi-Type Integral Inequalities for Conformable Fractional Integrals

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Abstract

In the last few decades, much significant development of integral inequalities had been established. Recall the famous integral inequality of Feng Qi type [1, 2, 3]:

$$\int\limits_a^b (f(t))^{n+2}\, dt \geq \left(\int\limits_a^b f(t) dt \right)^{n+1} \tag{0.0.8}$$

where $f \in C^n (a,b)$, $f^{(i)} \geq 0$, $0 \leq i \leq n$, $f^{(n)} \geq n!$, $n \in \mathbb{R}$.

In this study, we establish the generalized Qi-type inequality involving conformable fractional integrals.

Firstly we give a important integral inequality which is generalized Qi inequality. Finally, we obtain several inequalities related these inequalities using the conformable fractional integral [4,5].

Keywords: Integral Inequalities, Special Functions, Fractional Calculus, Conformable Fractional Integral.

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