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A Study on the Fourth Fundamental Form of the Factorable Hypersurface

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Author's contribution

The sole author designed, analysed, interpreted and prepared the manuscript.

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Abstract

We study the fourth fundamental form of the factorable hypersurface in the four dimensional Euclidean space $\mathbb{E}^4$. We obtain I, II, III, and IV fundamental forms of a factorable hypersurface.

Keywords: Four dimensional space; factorable hypersurface; fourth fundamental form.

1. Introduction

Surfaces and hypersurfaces have been studied by the mathematicians for centuries. We see some papers about factorable surfaces and factorable hypersurfaces such as [1–25].

A factorable hypersurface in $\mathbb{E}^4$ can be parametrized by:

$$\mathbf{x}(u, v, w) = (u, v, w, uvw), \tag{1.1}$$

where $u, v, w \in I \subset \mathbb{R}$.

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In this work, we introduce the fourth fundamental form of the factorable hypersurface in the four dimensional Euclidean space $\mathbb{E}^4$. We give basic notions of four dimensional Euclidean geometry. Moreover, we give fundamental forms I, II, III, and IV of factorable hypersurface.

2 Preliminaries

We give characteristic polynomial of shape operator **S** as follows:

$$P_{\mathbf{S}}(\lambda) = 0 = \det(\mathbf{S} - \lambda I_n) = \sum_{k=0}^{n} (-1)^k \, s_k \lambda^{n-k}, \tag{2.1}$$

where I_n denotes the identity matrix of order n in $\mathbb{E}^{n+1}$. Then, we have curvature formulas

$$\binom{n}{i} \mathfrak{C}_i = s_i,$$

where $\binom{n}{0} \mathfrak{C}_0 = s_0 = 1$ by definition. Therefore, k-th fundamental form of hypersurface M^n is given by

$$\mathrm{I}(\mathbf{S}^{k-1}(X), Y) = \langle \mathbf{S}^{k-1}(X), Y \rangle.$$

So, we obtain

$$\sum_{i=0}^{n} (-1)^i \binom{n}{i} \mathfrak{C}_i \, \mathrm{I}(\mathbf{S}^{k-1}(X), Y) = 0. \tag{2.2}$$

We identify a vector $(\mathrm{a}, \mathrm{b}, \mathrm{c}, \mathrm{d})$ with its transpose in this paper.

Let $\mathbf{M} = \mathbf{M}(u, v, w)$ be an isometric immersion of a hypersurface M^3 in $\mathbb{E}^4$. Inner product of vectors $\vec{x} = (x_1, x_2, x_3, x_4)$ and $\vec{y} = (y_1, y_2, y_3, y_4)$ in $\mathbb{E}^4$ is given by as follows:

$$\langle \vec{x}, \vec{y} \rangle = x_1 y_1 + x_2 y_2 + x_3 y_3 + x_4 y_4.$$

Vector product $\vec{x} \times \vec{y} \times \vec{z}$ of $\vec{x} = (x_1, x_2, x_3, x_4)$, $\vec{y} = (y_1, y_2, y_3, y_4)$, $\vec{z} = (z_1, z_2, z_3, z_4)$ in $\mathbb{E}^4$ is given by as follows:

$$\vec{x} \times \vec{y} \times \vec{z} = \det \begin{pmatrix} e_1 \, e_2 \, e_3 \, e_4 \\ x_1 x_2 x_3 x_4 \\ y_1 y_2 y_3 y_4 \\ z_1 \, z_2 \, z_3 \, z_4 \end{pmatrix}.$$

The Gauss map of a hypersurface **M** is defined by

$$e = \frac{\mathbf{M}_u \times \mathbf{M}_v \times \mathbf{M}_w}{\|\mathbf{M}_u \times \mathbf{M}_v \times \mathbf{M}_w\|},$$

where $\mathbf{M}_u = d\mathbf{M}/du$. For a hypersurface **M** in $\mathbb{E}^4$, we get following fundamental form matrices

$$\mathrm{I} = \begin{pmatrix} E & F & A \\ F & G & B \\ A & B & C \end{pmatrix},$$

$$\mathrm{II} = \begin{pmatrix} L & M & P \\ M & N & T \\ P & T & V \end{pmatrix},$$

$$\mathrm{III} = \begin{pmatrix} X & Y & O \\ Y & Z & R \\ O & R & S \end{pmatrix}.$$

Here, the coefficients of I, II, III are defined by

$$E = \langle \mathbf{M}_u, \mathbf{M}_u \rangle,\ \ F = \langle \mathbf{M}_u, \mathbf{M}_v \rangle,\ \ G = \langle \mathbf{M}_v, \mathbf{M}_v \rangle,\ \ A = \langle \mathbf{M}_u, \mathbf{M}_w \rangle,\ \ B = \langle \mathbf{M}_v, \mathbf{M}_w \rangle,\ \ C = \langle \mathbf{M}_w, \mathbf{M}_w \rangle,$$

$$L = \langle \mathbf{M}_{uu}, e \rangle,\ \ M = \langle \mathbf{M}_{uv}, e \rangle,\ \ N = \langle \mathbf{M}_{vv}, e \rangle,\ \ P = \langle \mathbf{M}_{uw}, e \rangle,\ \ T = \langle \mathbf{M}_{vw}, e \rangle,\ \ V = \langle \mathbf{M}_{ww}, e \rangle,$$

$$X = \langle e_u, e_u \rangle,\ \ Y = \langle e_u, e_v \rangle,\ \ Z = \langle e_v, e_v \rangle,\ \ O = \langle e_u, e_w \rangle,\ \ R = \langle e_v, e_w \rangle,\ \ S = \langle e_w, e_w \rangle,$$

and e is the Gauss map.

3 The Fourth Fundamental Form

We, next, find the fourth fundamental form matrix for a hypersurface $\mathbf{M}(u, v, w)$ in $\mathbb{E}^4$. By using characteristic polynomial $P_{\mathbf{S}}(\lambda) = a\lambda^3 + b\lambda^2 + c\lambda + d = 0$, we have curvature formulas: $\mathfrak{C}_0 = 1$ (by definition),

$$\mathfrak{C}_1 = -\frac{b}{\binom{3}{1} a},\ \ \mathfrak{C}_2 = \frac{c}{\binom{3}{2} a},\ \ \mathfrak{C}_3 = -\frac{d}{\binom{3}{3} a}.$$

Theorem 3.1.

For *any hypersurface* M^3 *in* $\mathbb{E}^4$, *the fourth fundamental form is related by*

$$\mathrm{IV} = 3\mathfrak{C}_1 \mathrm{III} - 3\mathfrak{C}_2 \mathrm{II} + \mathfrak{C}_3 \mathrm{I}. \tag{3.1}$$

Proof. By using $n = 3$ in (2.2) with some computing, we get the fourth fundamental form matrix

Theorem 3.2.

For any hypersurface M^3 *in* $\mathbb{E}^4$, *we get following*

$$\mathrm{IV} = \mathrm{III} \cdot \mathbf{S}.$$

Proof. Using I, II, III, IV, and **S** of (1.1), we get the result.

4 The Fourth Fundamental Form of a Factorable Hypersurface

Next, we obtain the fourth fundamental form of a factorable hypersurface (1.1).

Using the first differentials of (1.1) depends on u, v, w, we have the Gauss map of (1.1):

$$e = \frac{1}{(\det \mathrm{I})^{1/2}} \begin{pmatrix} v\,w \\ u\,w \\ u\,v \\ -1 \end{pmatrix}, \tag{4.1}$$

where $\det \mathrm{I} = u^2v^2 + u^2w^2 + v^2w^2 + 1$. We find the first and the second fundamental form matrices of (1.1), respectively,

$$\mathrm{I} = \begin{pmatrix} v^2w^2+1 & uvw^2 & uv^2w \\ uvw^2 & u^2w^2+1 & u^2vw \\ uv^2w & u^2vw & u^2v^2+1 \end{pmatrix},$$

$$\mathrm{II} = \begin{pmatrix} 0 & -\frac{w}{(\det \mathrm{I})^{1/2}} & -\frac{v}{(\det \mathrm{I})^{1/2}} \\ -\frac{w}{(\det \mathrm{I})^{1/2}} & 0 & -\frac{u}{(\det \mathrm{I})^{1/2}} \\ -\frac{v}{(\det \mathrm{I})^{1/2}} & -\frac{u}{(\det \mathrm{I})^{1/2}} & 0 \end{pmatrix}.$$

Computing $\mathrm{I}^{-1} \cdot \mathrm{II}$, factorable hypersurface (1.1) in $\mathbb{E}^4$ has following shape operator matrix:

$$\mathbf{S} = \begin{pmatrix} \frac{uvw(v^2+w^2)}{(\det \mathrm{I})^{3/2}} & -\frac{w(u^2w^2+1)}{(\det \mathrm{I})^{3/2}} & -\frac{v(u^2v^2+1)}{(\det \mathrm{I})^{3/2}} \\ -\frac{w(v^2w^2+1)}{(\det \mathrm{I})^{3/2}} & \frac{uvw(u^2+w^2)}{(\det \mathrm{I})^{3/2}} & -\frac{u(u^2v^2+1)}{(\det \mathrm{I})^{3/2}} \\ -\frac{v(v^2w^2+1)}{(\det \mathrm{I})^{3/2}} & -\frac{u(u^2w^2+1)}{(\det \mathrm{I})^{3/2}} & \frac{uvw(u^2+v^2)}{(\det \mathrm{I})^{3/2}} \end{pmatrix}.$$

Therefore, we get the third fundamental form matrix using (4.1) of (1.1):

$$\mathrm{III} = \begin{pmatrix} \frac{(v^2+w^2)(v^2w^2+1)}{(\det \mathrm{I})^2} & -\frac{uv(w^4-1)}{(\det \mathrm{I})^2} & -\frac{uw(v^4-1)}{(\det \mathrm{I})^2} \\ -\frac{uv(w^4-1)}{(\det \mathrm{I})^2} & \frac{(u^2+w^2)(u^2w^2+1)}{(\det \mathrm{I})^2} & -\frac{vw(u^4-1)}{(\det \mathrm{I})^2} \\ -\frac{uw(v^4-1)}{(\det \mathrm{I})^2} & -\frac{vw(u^4-1)}{(\det \mathrm{I})^2} & \frac{(u^2+v^2)(u^2v^2+1)}{(\det \mathrm{I})^2} \end{pmatrix}.$$

Then, using Theorem 3.2 on (1.1), we get the fourth quantities of (1.1), i.e. symmetric matrix, as follows

$$\mathrm{IV} = \frac{1}{(\det \mathrm{I})^{7/2}} \begin{pmatrix} \alpha & \beta & \gamma \\ \beta & \delta & \varepsilon \\ \gamma & \varepsilon & \eta \end{pmatrix},$$

where

$$\alpha = 2uvw(v^2w^2+1)(v^2w^2+v^4+w^4-1),$$

$$\beta = -w(-u^2v^4 - u^4v^2 + u^2w^4 + u^4w^2 + v^2w^4 + v^4w^2 + u^2 + v^2 + w^2 + 2u^2v^2w^6 + u^2v^4w^4 + u^4v^2w^4 - u^4v^4w^2),$$

$$\gamma = -v(u^2v^4 + u^4v^2 - u^2w^4 - u^4w^2 + v^2w^4 + v^4w^2 + u^2 + v^2 + w^2 + u^2v^4w^4 + 2u^2v^6w^2 - u^4v^2w^4 + u^4v^4w^2),$$

$$\delta = 2uvw(u^2w^2+1)(u^2w^2+u^4+w^4-1),$$

$$\varepsilon = -u(u^2v^4 + u^4v^2 + u^2w^4 + u^4w^2 - v^2w^4 - v^4w^2 + u^2 + v^2 + w^2 - u^2v^4w^4 + u^4v^2w^4 + u^4v^4w^2 + 2u^6v^2w^2),$$

$$\eta = 2uvw(u^2v^2+1)(u^2v^2+u^4+v^4-1).$$

Corollary 4.1.

A factorable hypersurface (1.1) *in* $\mathbb{E}^4$ *has following relations*

$$\frac{(\det\mathrm{II})(\det\mathrm{III})^2}{(\det\mathrm{I})(\det\mathrm{IV})^2} = \det\mathbf{S} = \mathfrak{C}_3 = \left(\frac{2uvw}{(u^2v^2+u^2w^2+v^2w^2+1)^2}\right)^2.$$

Proof. Using $\mathrm{I}, \mathrm{II}, \mathrm{III}, \mathrm{IV}$, and **S** of (1.1), it is clear.

Corollary 4.2.

A factorable hypersurface (1.1) *in* $\mathbb{E}^4$ *is written by as follows*

$$\mathbf{x}(u,v,w) = \left(u, v, w, -2(\det\mathrm{IV})^{13/6}(\det\mathrm{I})^{1/3}\right).$$

Proof. Using I, IV of (1.1), it is clear.

Corollary 4.3.

A factorable hypersurface (1.1) *in* $\mathbb{E}^4$ *is given by as follows*

$$\mathbf{x}(u,v,w) = \left(u, v, w, \frac{(\det\mathrm{I})^2(\mathfrak{C}_3)^{1/2}}{2}\right).$$

Proof. Using Corollary 4.1, it is clear.

5 Conclusion

Factorable hyper-surfaces have been studied by some authors for years. Results of the factorable hypersurface (1.1) are extended by its fourth quantities in four-space. Moreover, factorable hypersurface (1.1) are given by its quantities $\mathrm{I}, \mathrm{II}, \mathrm{III}, \mathrm{IV}$, and $\mathfrak{C}_3$ in this paper.

Competing Interests

Author has declared that no competing interests exist.

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